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A Structure Function Representation Theorem With Applications to Frequency Stability Estimation

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Random processes with stationary nth differences serve as models for oscillator phase noise. The theorem proved here allows one to obtain the structure function (covariances of the nth differences) of such a process in terms of the differences of a single function of one time variable. In turn, this function can easily be obtained from the spectral density of the process. The theorem is used for computing the variance of two estimators of frequency stability.

I. Introduction

Let the output of a precision oscillator be modelled by $\cos\{2\pi\nu_0[t + x(t)]\}$, where $x(t)$ is a random process representing the "phase time" noise. The most widely used time-domain measure of oscillator stability is the Allan variance, defined by the ensemble average

$$\sigma_A^2(\tau) = \frac{1}{2\tau^2} E\left[\Delta_\tau^2 x(s)\right]^2,\qquad (1)$$

provided that the expectation exists and is independent of the time s. Here, Δ_τ^2 is the backward 2nd difference operator, given by

$$\Delta_\tau^2 f(t) = f(t) - 2f(t - \tau) + f(t - 2\tau).$$

The theorem given here arose from the desire to compute the performance of estimators of $\sigma_A^2(\tau)$. Suppose that $x(t)$ is given on an interval $0 \leqslant t \leqslant T$. Fix $\tau \leqslant T/2$, and set

$$\xi(t) = \frac{1}{\tau\sqrt{2}} \Delta_\tau^2 x(t).$$

A class of unbiased estimators of $\sigma_A^2(\tau)$ is given by

$$V = \int_{2\tau}^{T} \xi^2(t)w(t)dt,\qquad (2)$$

where $w(t)$ is a weighting function (or measure) whose total weight on $[2\tau, T]$ is 1. Two members of this class are treated in Section IV.

Assume further that $\xi(t)$ is a stationary Gaussian process with zero mean and autocovariance function $R_\xi(t)$. Then $\xi^2(t)$ is a stationary process with mean $\sigma_A^2(\tau)$ and autocovariance function $2R_\xi^2(t)$. It follows that

$$EV = E\xi^2(t) = \sigma_A^2(\tau),$$

$$\text{Var } V = \int_{2\tau}^{T}\int_{2\tau}^{T} 2R_\xi^2(s - t)w(s)w(t)dsdt.\qquad (3)$$

Evidently, to compute Var V, we need to obtain $R_{\xi}(t)$ from the model for $x(t)$. Let us suppose, temporarily, that $x(t)$ is stationary, with autocovariance function

$$R_x(t) = \text{Cov}\,[x(s+t), x(s)],$$

not depending on s. A straightforward computation yields $E\Delta_\tau^2 x(s) = 0$, and

$$E\Delta_\tau^2 x(s+t)\Delta_\tau^2 x(s) = \delta_\tau^4 R_x(t), \qquad (4)$$

where $\delta_\tau^4 = \Delta_\tau^2\Delta_{-\tau}^2$, the central 4th difference operator, also given by

$$\delta_\tau^4 f(t) = f(t-2\tau) - 4f(t-\tau) + 6f(t) - 4f(t+\tau) + f(t+2\tau).$$

The left side of (4) is called the 2nd *structure function* of $x(t)$ (Refs. 1, 2). Letting $t = 0$ in (4), we have

$$\sigma_A^2(\tau) = \frac{1}{\tau^2}[3R_x(0) - 4R_x(\tau) + R_x(2\tau)], \qquad (5)$$

as pointed out by Barnes (Ref. 3). Consequently, since $|R_x(t)| \leqslant R_x(0) = \text{Var}\,x$,

$$\sigma_A^2(\tau) \leqslant \frac{8\,\text{Var}\,x}{\tau^2},$$

$$\sigma_A^2(\tau) \sim \frac{3\,\text{Var}\,x}{\tau^2} \quad (\tau \to \infty), \qquad (6)$$

provided $R_x(t) \to 0\ (t \to \infty)$.

This is for stationary $x(t)$. On the other hand, for actual oscillators a behavior like (6) is observed only for *small* τ, below 1 s for quartz crystal oscillators and 100 s for hydrogen masers. As τ increases, the measured $\sigma_A(\tau)$ decreases to a minimum, then stays constant or increases. Of course, since our measurement times are finite, this observation does not "prove" that $x(t)$ is nonstationary. A stationary process with a huge variance and a tiny bandwidth would explain what we see, for we would be looking only at a small piece of the process. If τ were to increase beyond the scale of our observations, then $\sigma_A(\tau)$ would ultimately behave like (6) again. However, if we want to describe the behavior of $x(t)$ on realistic time intervals, a nonobservable low-frequency cutoff only gets in the way, and eventually has to be driven to zero. It is mathematically easier to use a nonstationary model from the very start.

For modelling oscillator phase noise, it is usually sufficient to let $x(t)$ belong to the class of processes whose 2nd differences $\Delta_\tau^2 x(t)$ are stationary. This class includes the stationary processes and those with stationary 1st differences. Such a process has a two-sided "formal" spectral density $S_x(\omega)$, which can have a singularity at $\omega = 0$ that is strong enough to make

$$\int_{-1}^{1} S_x(\omega)d\omega = \infty.$$

Nevertheless, we always have

$$\int_{-\infty}^{\infty} \frac{\omega^4}{1+\omega^4}S_x(\omega)d\omega < \infty \qquad (7)$$

for this class of processes. An example is the power-law spectrum

$$S_x(\omega) = \frac{K}{|\omega|^\beta},$$

where $1 < \beta < 5$.

A rigorous theory of these processes exists (Ref. 1); basically, it shows that one can plunge ahead with the formalism from stationary processes as long as the integrals converge. For example, the transfer function of the operator Δ_τ^2 is $(1 - e^{-i\omega\tau})^2$. Therefore,

$$E\Delta_\tau^2 x(t+s)\Delta_\tau^2 x(s) = \int_{-\infty}^{\infty} e^{i\omega\tau}|1 - e^{-i\omega\tau}|^4$$

$$S_x(\omega)\frac{d\omega}{2\pi} + c^2\tau^4. \qquad (8)$$

The extra term $c^2\tau^4$ comes from a frequency drift component $ct^2/2$ in $x(t)$. Letting $t = 0$, we obtain the Allan variance (1). By this method, the theoretical Allan variance has been evaluated and tabulated (Ref. 4) for $S_x(\omega) = K/|\omega|^k$, k an integer, $0 \leqslant k \leqslant 4$. (For $k \leqslant 1$, a high-frequency cutoff is provided.) Allowing t to be nonzero appears to make (8) more difficult to evaluate. Yet, for our estimation problem, we do need the full covariance function of the process $\Delta_\tau^2 x(t)$. One longs for the simplicity of (4), with $R_x(t)$ given by the simple Fourier integral

$$R_x(t) = \int_{-\infty}^{\infty} e^{i\omega t} S_x(\omega)\frac{d\omega}{2\pi}. \qquad (9)$$

This integral does not exist, however, unless $x(t)$ is stationary (or equal to a quadratic polynomial plus a stationary process).

The theorem to be proved here gives an easily computable replacement for $R_x(t)$, valid for all processes with stationary n^{th} differences. Equation (4) is replaced by

$$E\Delta_r^2 x(s+t)\Delta_r^2 x(s) = \delta_r^4 [2\,\text{Re}\,C(t)] + c^2 r^4, \quad (10)$$

where the (nonunique) function $C(t)$ can be computed by two different methods. Here is the second method: Choose an integer k such that $\omega^k S_x(\omega)$ is integrable near $\omega = 0$. Then let

$$C(z) = I^k \int_0^\infty e^{i\omega z}(i\omega)^k\, S_x(\omega)\frac{d\omega}{2\pi} \quad (11)$$

in the upper half-plane $\text{Im}\,z > 0$. The operator I^k instructs the user to integrate k times with respect to z. One may then allow z to be real.

Formally, all we are doing is differentiating (9) k times and integrating k times. If one does this correctly, one easily gets valid results for all the power-law oscillator noise models. Although Lindsey and Chie (Ref. 2) give a number of formulas that generalize (5), they have to assume that either the phase $x(t)$ or the frequency dx/dt is stationary. For flicker FM or random walk FM noise, these assumptions are false. Lindsey and Chie do hint at the need for distribution theory in this situation. Although our method has obvious connections to the analytic representation of distributions (Ref. 5), we use only the elementary theory of real and analytic functions to arrive at the main result.

To illustrate the theorem, let us consider the noise called *random walk frequency modulation*, defined by $S_x(\omega) = K/\omega^4$. For this noise, $\sigma_A^2(\tau)$ is proportional to τ (as we shall soon see); this kind of Allan variance behavior has been observed in hydrogen maser frequency standards for $\tau > 10^4$ s (Ref. 6). Taking $k = 4$ in (11), we have

$$C(z) = I^4\,\frac{Ki}{2\pi z}$$

$$= \frac{Ki}{2\pi}\frac{z^3 \ln z}{6} \quad (\text{Im}\,z > 0).$$

in which $\ln z$ is to be analytic in $\text{Im}\,z > 0$. The function $(z^3 \ln z)/6$ is just a particular solution to the equation $f^{(4)}(z) = 1/z$. Taking $\ln z = \ln |z| + i\,\text{Arg}\,z$ in the upper half-plane, we get

$$2\,\text{Re}\,C(t) = \begin{cases} 0 & (t > 0) \\ -\dfrac{K}{6}t^3 & (t < 0)\,. \end{cases} \quad (12)$$

This function is nothing like the autocovariance function of a stationary process, yet applying the operator δ_r^4 to it gives the result

$$E\xi(s+t)\,\xi(s) = \frac{1}{2r^2}\,\delta_r^4\,(2\,\text{Re}\,C(t))$$

$$= \frac{K\tau}{12} \cdot \begin{cases} (2-r)^3 - 4(1-r)^3 & (0 \leqslant r \leqslant 1) \\ (2-r)^3 & (1 < r < 2) \\ 0 & (r > 2) \end{cases}$$

$$(13)$$

(where $r = |t|/\tau$), which is a perfectly good autocovariance function. In particular,

$$\sigma_A^2(\tau) = E\xi^2(s) = \frac{K\tau}{3}\,.$$

The result (13) can also be derived by expressing $\xi(t)$ as the output of a filter acting on white noise. The method given here is easier and applies to more general difference operators. Moreover, as Section III shows, all of the power-law spectral models become equally simple. Previously, the odd powers (the "flicker" models) were more difficult to handle than the even powers.

II. The Representation Theorem

A mean-continuous random process $x(t)$ is said to have *stationary n^{th} differences* if, for each real τ, the process $\Delta_r^n x(t)$ is stationary (in the wide sense). For such a process, define the *dc coefficient* c by

$$c = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Delta_1^n x(t)\,dt$$

(in mean square).

For applications, we shall assume that the nth differences of $x(t)$ are ergodic, so that c is nonrandom. It can then be shown that

$$c\tau^n = E\Delta_r^n x(t). \quad (14)$$

The process $x_0(t) = x(t) - ct^n/n!$ equals a polynomial of degree $<n$ plus a mean-zero process. Associated with $x_0(t)$ is a (two-sided) nonnegative *formal spectral density* $S_x(\omega)$, from which all the covariances of the nth differences of $x(t)$ can be obtained. If $x(t)$ is real, then $S_x(-\omega) = S_x(\omega)$.

In connection with these processes, it is convenient to introduce a general real difference operator L that, when applied to a function $f(t)$, gives the result

$$Lf(t) = \sum_\nu a_\nu f(t + t_\nu) \qquad (15)$$

(a finite sum), where the a_ν are real. Its transfer function is the trigonometric polynomial

$$L(i\omega) = \sum_\nu a_\nu e^{i\omega t_\nu} .$$

For example,

$$\Delta_\tau^n f(t) = \sum_{\nu=0}^{n} \binom{n}{\nu}(-1)^\nu f(t - \nu\tau) ,$$

$$\Delta_\tau^n (i\omega) = \left(1 - e^{-i\omega\tau}\right)^n$$

A difference operator L is said to have *order n* if its transfer function satisfies

$$L^{(j)}(0) = \sum_\nu a_\nu t_\nu^j = 0 \quad (0 \leqslant j < n) ,$$

$$L^{(n)}(0) = \sum_\nu a_\nu t_\nu^n \neq 0 .$$

Such an operator annihilates all polynomials of degree $<n$, and reduces the degree of other polynomials by n, as shown by the computation

$$Lt^m = \sum_\nu a_\nu (t + t_\nu)^m = \sum_\nu a_\nu \sum_{j=0}^{m} \binom{m}{j} t_\nu^j t^{m-j}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (16)$$

$$= \sum_{j=0}^{m} \binom{m}{j} L^{(j)}(0) t^{m-j} = \sum_{j=n}^{\infty} \binom{m}{j} L^{(j)}(0) t^{m-j}$$

(Recall that $\binom{m}{j} = 0$ for $j > m$.) In particular, $Lt^n = L^{(n)}(0)$. If

L and M have orders l and m, and $\Lambda = LM$, then Λ has order $l + m$, and

$$\frac{\Lambda^{(l+m)}(0)}{(l+m)!} = \frac{L^{(l)}(0)}{l!} \frac{M^{(m)}(0)}{m!} \qquad (17)$$

If L is given by (15), then the operator L^* (of the same order) is defined by

$$L^* f(t) = \sum_\nu a_\nu f(t - t_\nu) .$$

As we mentioned, an example of order n is $L = \Delta_\tau^n$, for which $L^{(n)}(0) = n!\tau^n$. Another example, for $n = 2$, is the mixed difference operator $\Delta_\tau \Delta_{\tau'}$, which was used for estimating the relative drift rate of a pair of frequency standards (Ref. 7).

We are now set up to give the main result.

Theorem. Let $x(t)$ be a real process with stationary nth differences, nonrandom dc coefficient c, and spectral density $S_x(\omega)$. Let L and M be real difference operators of order n, and let Λ be the difference operator LM^* of order $2n$. Then, the mixed second moments of the processes $Lx(t)$ and $Mx(t)$ can be put into the form

$$E\, Lx(s + t)\, Mx(s) = \Lambda \left[(-1)^n \frac{c^2 t^{2n}}{(2n)!} + 2\,\text{Re}\,C(t) \right] ,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (18)$$

where the (nonunique) function $C(z)$ is analytic in $\text{Im}\,z > 0$, continuous in $\text{Im}\,z \geqslant 0$, and can be obtained by either of the following recipes:

Recipe 1. Choose an integer k between 0 and $2n$ such that

$$\int_0^{-1} \omega^k S_x(\omega)\, d\omega < \infty .$$

Let $C(z) = C_0(z)$, where

$$C_0(z) = \int_0^\infty \left[e^{i\omega z} - \frac{1}{1 + \omega^{2n}} \sum_{j=0}^{k-1} \frac{(i\omega z)^j}{j!} \right] S_x(\omega) \frac{d\omega}{2\pi}$$

$$(\text{Im}\,z \geqslant 0). \qquad\qquad\qquad (19)$$

If $k = 0$, then omit the sum.

Recipe 2. Choose k as in Recipe 1. Define

$$B(z) = \int_0^\infty e^{i\omega z} (i\omega)^k S_x(\omega) \frac{d\omega}{2\pi} \quad (\text{Im } z > 0) . \quad (20)$$

Let $C(z)$ be any kth integral of $B(z)$ on Im $z > 0$ (in other words, any function such that $C^{(k)}(z) = B(z)$). Extend $C(z)$ to the real line by continuity. (This is always possible.)

Remark. The first term in the brackets in (18), when acted upon by Λ, becomes

$$c^2 \frac{L^{(n)}(0)}{n!} \frac{M^{(n)}(0)}{n!} = E[Lx(t)] E[Mx(t)] . \quad (21)$$

Corollary. Let $x(t)$ have stationary nth differences. The structure function

$$D(t, \tau) = E\Delta_\tau^n x(s+t) \Delta_\tau^n x(s)$$

of the process $x(t)$ can be obtained from a function of one variable, namely 2 Re $C(t)$, by

$$D(t; \tau) = c^2 \tau^{2n} + (-1)^n \delta_\tau^{2n} (2 \text{ Re } C(t)). \quad (22)$$

where

$$\delta_\tau^{2n} = (\delta_\tau^2)^n = (-\Delta_\tau \Delta_{-\tau})^n ,$$

the central difference operator of order $2n$.

These results show that 2 Re $C(t)$ contains the same information as $S_x(\omega)$ about the process $x(t)$. We can regard 2 Re $C(t)$ as an analog of the covariance function of a stationary process.

III. Examples for $n = 2$

Oscillator phase noise is often modelled by a linear combination of independent power-law noises with spectra

$$\frac{1}{\omega^4} , \frac{1}{|\omega|^3} , \frac{1}{\omega^2} , \frac{F(\omega/\omega_h)}{|\omega|} , F(\omega/\omega_h) ,$$

where $F(\omega/\omega_h)$ is an integrable low-pass power response function whose one-sided noise bandwidth is $\omega_h/(2\pi)$. Examples

are the *sharp* cutoff (1 for $|\omega| < \omega_h$, 0 elsewhere) and the *exponential* cutoff $\exp(-|\omega|/\omega_h)$. The above noises are called *random walk FM, flicker FM, white FM, flicker PM,* and *white PM.* (The term "white" will be used even if F is not exactly flat near zero frequency.)

The introduction has shown how to apply the representation theorem to random walk FM; let us proceed to the calculation of 2 Re $C(t)$ for the other noises. In order to express the results in the established notation of the frequency and timing literature (Ref. 4), we use the form

$$S_x(\omega) = K_\alpha |\omega|^{\alpha-2} , \quad K_\alpha = \frac{h_\alpha}{2(2\pi)^\alpha} , \quad (23)$$

where a high-frequency cutoff $F(\omega/\omega_h)$ is also applied when $\alpha \geqslant 1$ (PM noises).

First, let $\alpha = 0, -1,$ or -2 (white, flicker, or random walk FM). Using Recipe 2 of the theorem with $k = 2 - \alpha$, we have

$$B(z) = K_\alpha \int_0^\infty e^{i\omega z} i^k \frac{d\omega}{2\pi} = \frac{K_\alpha i^{k+1}}{2\pi z} \quad (\text{Im } z > 0) . \quad (24)$$

Since a kth integral of $1/z$ is $(z^{k-1} \ln z)/(k - 1)!$, we get

$$C(z) = \frac{K_\alpha i^{k+1}}{2\pi} \frac{z^{k-1}}{(k-1)!} \ln z \quad (\text{Im } z \geqslant 0) , \quad (25)$$

in which the branch of $\ln z$ must be analytic in the upper half-plane. We shall let $\ln t$ be real for $t > 0$. Let us examine the three cases separately.

White FM: $\alpha = 0, k = 2.$

$$C(z) = \frac{K_0}{2\pi i} z \ln z ,$$

$$C(t) = \frac{K_0 t}{2\pi i} \cdot \begin{cases} \ln t & (t > 0) \\ \ln|t| + \pi i & (t < 0) , \end{cases}$$

$$2 \text{ Re } C(t) = \frac{h_0}{2} \cdot \begin{cases} 0 & (t \geqslant 0) \\ t & (t < 0) . \end{cases} \quad (26)$$

Flicker FM: $\alpha = -1, k = 3$.

$$C(z) = \frac{K_{-1}}{4\pi} z^2 \ln z,$$

$$2\operatorname{Re} C(t) = \frac{h_{-1}}{2} t^2 \ln |t|. \qquad (27)$$

Random walk FM. $\alpha = -2, k = 4$.

$$C(z) = \frac{K_{-2}i}{2\pi} \frac{z^3}{6} \ln z,$$

$$2\operatorname{Re} C(t) = \frac{h_{-2}\pi^2}{3} \cdot \begin{cases} 0 & (t \geq 0) \\ \\ -t^3 & (t \leq 0). \end{cases} \qquad (28)$$

In a sense, the flicker case is easier than the others because we don't have to keep track of the imaginary part of $\ln z$.

For handling the PM noises, the exponential cutoff is easier to use than the sharp cutoff, and may even be more realistic. Let $S_x(\omega) = K_\alpha |\omega|^{\alpha-2} \exp(-|\omega|/\omega_h)$, where $\alpha = 1$ or 2. Again letting $k = 2 - \alpha$, we get

$$B(z) = \frac{K_\alpha i^{k+1}}{2\pi} \frac{1}{z + i/\omega_h}. \qquad (29)$$

Flicker PM: $\alpha = 1, k = 1$.

$$C(z) = -\frac{K_1}{2\pi} \ln(z + i/\omega_h),$$

$$2\operatorname{Re} C(t) = -\frac{h_1}{8\pi^2} \ln(t^2 + 1/\omega_h^2). \qquad (30)$$

For the sharp high-frequency cutoff, $2\operatorname{Re} C(t)$ turns out to be a cosine integral.

White PM. $\alpha = 2, k = 0$.

$$C(z) = B(z) = \frac{K_2 i}{2\pi} \frac{1}{z + i/\omega_h},$$

$$2\operatorname{Re} C(t) = \frac{h_2 \omega_h}{8\pi^3 (1 + \omega_h^2 t^2)}, \qquad (31)$$

which is just $R_x(t)$, because now $x(t)$ is stationary.

Fractional noises. It is well known (Ref. 4) that if the spectrum satisfies a power law $S_x(\omega) = K_\alpha |\omega|^{\alpha-2}$, where $-3 < \alpha < 1$, then the Allan variance satifies another power law $\sigma_A^2(\tau) = \text{const} \cdot \tau^\mu$, where $\mu = -1 - \alpha$. When Allan variance measurements are made, fractional values of μ sometimes appear over a certain range of τ. Thus, we ought to show how to use the representation theorem for fractional values of α in the range $-3 < \alpha < 1$. We shall do this for $0 < \alpha < 1$, leaving the other cases as an exercise for the reader. Take $k = 1$. Then

$$B(z) = \frac{K_\alpha i}{2\pi} \int_0^\infty e^{i\omega z} \omega^{\alpha-1} d\omega$$

$$= \frac{K_\alpha i}{2\pi} \frac{\Gamma(\alpha)}{(-iz)^\alpha},$$

$$C(z) = i \int B(z) d(-iz) = -\frac{K_\alpha \Gamma(\alpha)}{2\pi(1-\alpha)} (-iz)^{1-\alpha},$$

where the power functions are analytic in the right half-plane and positive on the positive real axis. Then

$$2\operatorname{Re} C(t) = -\frac{h_\alpha \Gamma(\alpha) \sin\left(\frac{1}{2}\pi\alpha\right)}{(2\pi)^{1+\alpha} (1-\alpha)} |t|^{1-\alpha}, \qquad (32)$$

As $\alpha \to 0$, this expression tends to $-h_0|t|/4$, which is indeed a valid representative of $2\operatorname{Re} C(t)$ for $\alpha = 0$.

IV. The Variance of Two Allan Variance Estimators

Let us return to the estimation problem mentioned in the introduction. Recall that phase time $x(t)$ is given for $0 \leq t \leq T$. Therefore, $\xi(t) = \Delta_\tau^2 x(t)/(\tau\sqrt{2})$ is available for $2\tau \leq t \leq T$. Set $m = T/\tau$. Two unbiased estimators of the Allan variance (1) are

$$V_\tau = \frac{1}{m-1} \sum_{j=2}^m \xi^2(j\tau) \qquad (33)$$

(m an integer $\geqslant 2$), called the τ-*overlap* estimator, and

$$V_0 = \frac{1}{-(n-2)} \int_{2r}^{m\tau} \xi^2(t)\,dt \qquad (34)$$

(m real, >2), called the *continuous* estimator, which, although it cannot be achieved in practice, represents a limiting case for a sample time τ_0 much less than τ. The use of such an estimator was suggested by Hor e, Allan, and Barnes (Ref. 8).

Although the τ-overlap estimator has been used for many years, it is reasonable to ask whether the continuous estimator has a smaller variance. In other words, if $\tau_0 \ll \tau$, should we average all the available samples $\xi^2(j\tau_0)$, or should we use only the samples $\xi^2(j\tau)$? Since the data collection time T may be weeks or months, this question is more than academic.

The answer depends both on m and on the spectrum of the phase noise. Assume that $x(t)$ is a Gaussian process with stationary 2nd differences and zero dc component c. If we know $S_x(\omega)$, then we can compute $C(z)$. By the corollary, the autocovariance function of the mean-zero process $\xi(t)$ is

$$R_\xi(t) = \frac{1}{2\tau^2}\,\delta_\tau^4\,(2\,\mathrm{Re}\,C(t))\,. \qquad (35)$$

As we said in the introduction, the autocovariance function of the stationary process $\xi^2(t)$ is $2R_\xi^2(t)$. The means and variances of V_τ and V_0 are now computed straightforwardly:

$$E\,V_\tau \quad E\,V_0 = R_\xi(0) = \sigma_A^2(\tau)\,, \qquad (36)$$

$$\mathrm{Var}\,V_\tau = \frac{2}{(m-1)^2}\sum_{j=-m+2}^{m-2}(m-1-|j|)\,R_\xi^2\,(j\tau)\,, \qquad (37)$$

$$\mathrm{Var}\,V_0 = \frac{2}{T_2^2}\int_{-T_2}^{T_2}(T_2-|t|)\,R_\xi^2\,(t)\,dt\,, \qquad (38)$$

where $T_2 = (m-2)\tau$. (Of course, we recognize that $R_\xi(t)$ is an even function.)

The computations have been carried out for white FM, flicker FM, and random walk FM (see Section III), all of which have been observed in actual oscillators. The flicker FM results

were computed numerically; the others are in closed form, which, however, we shall not give here. For white FM and random walk FM, the τ-overlap formulas agree with those of Lesage and Audoin (Ref. 9); for flicker FM, the τ-overlap numbers agree with Yoshimura's (Ref. 10).

The results are presented in terms of "degrees of freedom," defined for a positive estimator V by

$$\mathrm{d.f.} = \frac{2\,(EV)^2}{\mathrm{Var}\,V}\,.$$

Given d.f., one sometimes uses the appropriate chi-squared distribution for constructing confidence intervals about the estimate (Ref. 8). Whether or not this is done, the d.f. remains a useful figure of merit.

In Fig. 1, for the above three noise types, we plot d.f./$(m-1)$ vs m for V_τ and V_0. For white FM, V_0 is always better than V_τ. For flicker FM, V_0 is better than V_τ except for $m \leqslant 3$. For random walk FM, V_τ is better than V_0 for $m \leqslant 18$. Of course, the smaller values of m are more critical, since d.f. is roughly proportional to $m-1$.

It may seem paradoxical for V_τ to be better than V_0, since V_0 uses all the available data. Both estimators are special cases of (2), however; if one looks for the optimal (minimal variance) estimator of the class (2) (for a given noise type), one will probably find that the optimal weighting function $w(t)$ is nonconstant and almost everywhere nonzero. In other words, one should use all the data, but in a nonuniform way.

V. Concluding Summary

Oscillator stability is usually characterized by the behavior of nth order differences of the phase. The theoretical evaluation, from the phase noise spectrum, of the variances and covariances of these differences involves messy trigonometric integrals, such as (9). The messiness is caused by a $(2n)$th order difference operator tangled up inside the integral. Our representation theorem breaks the integral evaluation into two easy steps. (1) evaluation of a much simpler integral depending only on the noise spectrum; (2) application of that same difference operator to the result of step (1).

In effec., the evaluation of these integrals is uncoupled into two independent operations. In Section III, we tabulated only the result of step (1) (the function called $2\,\mathrm{Re}\,C(t)$) for all the usual power-law oscillator noise models. This short "one-dimensional" table, plus another one-dimensional table of difference operators, can generate a two-dimensional table of results as found, for example, in Ref. 4.

As an application, we examined two Allan variance estimators. Because the representation theorem delivers such simple closed forms for the required autocovariance functions, the computations were quickly executed by a simple BASIC program. The theorem can also be used for evaluating the performance of frequency drift estimators. In general, we get an estimator of frequency drift rate (the dc coefficient c) by operating upon oscillator phase with a second-order difference operation L that need not be of form Δ_τ^2. Here, the full generality of the theorem is needed.

References

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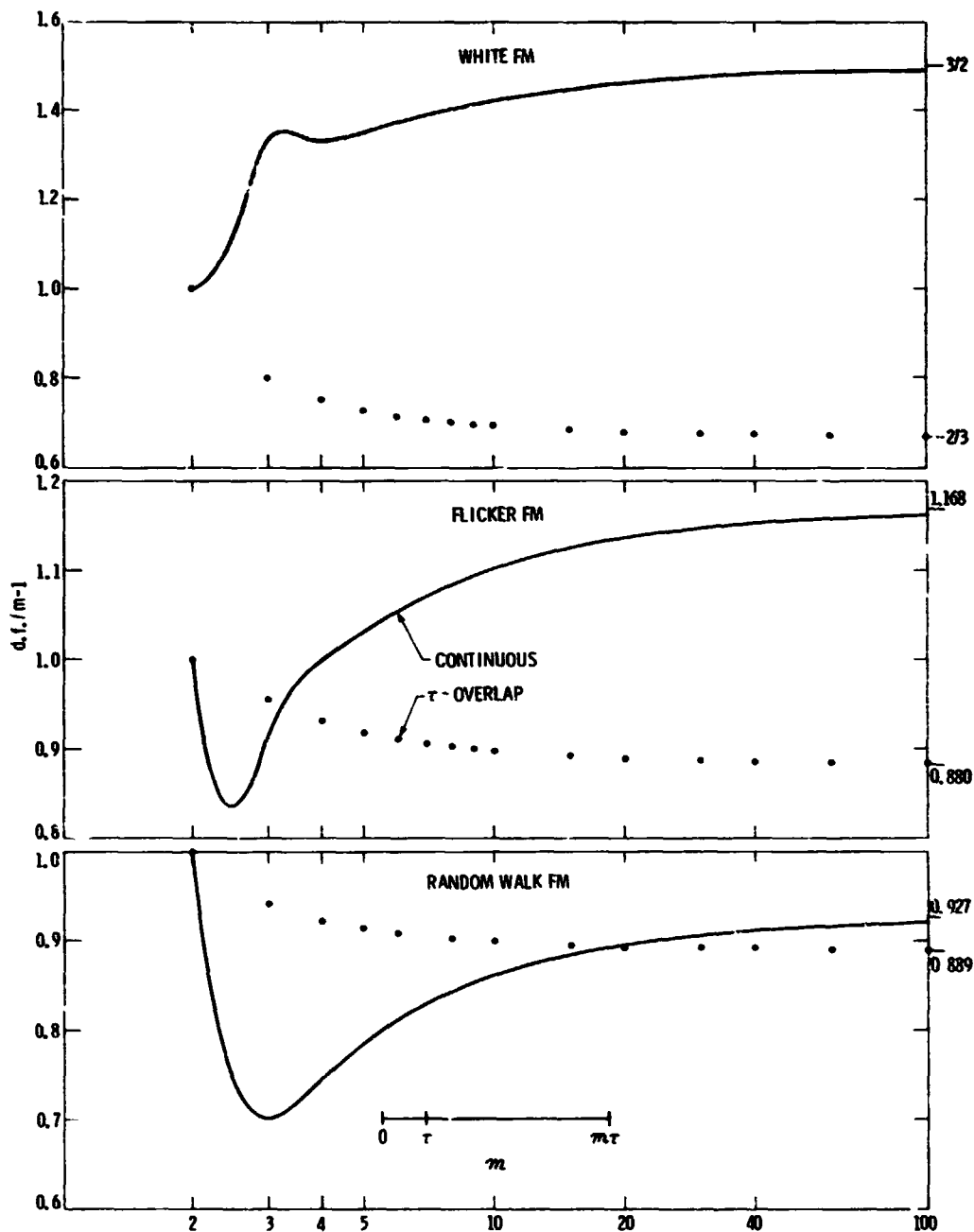
Fig. 1. Degrees of freedom of the τ-overlap and continuous Allan variance estimates for white, flicker, and random-walk frequency modulation. The numbers on the right give the horizontal asymptotes. The observation time is $m\tau$

Appendix

Proof of the Representation Theorem

For convenience, let us first set down an elementary estimate of the Taylor remainder for e^z: If $\text{Re } z \leq 0, k \geq 1$, then

$$\left| e^z - \sum_{j=0}^{k-1} \frac{z^j}{j!} \right| \leq \frac{|z|^k}{k!}. \tag{A1}$$

This can be obtained from the integral form of the remainder.

In analogy with the usual notation for mixed structure functions $D(t; \tau_1, \tau_2)$ (Ref. 1), denote the left side of (18) by $D(t; L, M)$. Without loss of generality, we can assume that the dc coefficient c is zero. Begin with the spectral representation of $D(t; L, M)$ as given by Yaglom:

$$D(t; L, M) = \int_{-\infty}^{\infty} e^{i\omega t} L(i\omega) M(-i\omega) S_x(\omega) \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} \Lambda e^{i\omega t} S_x(\omega) \frac{d\omega}{2\pi}$$

$$= 2 \text{ Re } A(t) . \tag{A2}$$

where

$$A(z) = \int_0^{\infty} \Lambda e^{i\omega z} S_x(\omega) \frac{d\omega}{2\pi} \tag{A3}$$

for $z = t + iu$ ($u \geq 0$), and Λ operates on functions of t. The name of the game is pulling Λ outside the integral. If you do this brutally, the integral usually blows up.

Since the spectral density $S_x(\omega)$ always satisfies

$$\int_0^{\infty} \frac{\omega^{2n}}{1 + \omega^{2n}} S_x(\omega) d\omega < \infty$$

(Ref. 1), the integer k specified in Recipe 1 exists. We can always take $k = 2n$; it is often possible and desirable to use a smaller k. If $k = 0$ works, then $S_x(\omega)$ is integrable, Λ comes outside the integral in (A3), and we are done. In this case, $C(z) = A(z)$, and $x(t)$ is a polynominal plus a stationary process whose autocovariance function is $2 \text{ Re } C(t)$.

Assume $k \geq 1$. Since, for all $u > 0$,

$$\int_0^{\infty} e^{-\omega u} \omega^k S_x(\omega) d\omega < \infty,$$

the function $B(z)$ of Recipe 2 exists and is analytic in $\text{Im } z > 0$. Let us now look at $C_0(z)$ as given in Recipe 1. It will soon be shown that

(a) $C_0(z)$ is continuous on $\text{Im } z \geq 0$ and analytic on $\text{Im } z > 0$;

(b) $C_0^{(k)}(z) = B(z)$ for $\text{Im } z > 0$;

(c) $\Lambda C_0(z) = A(z)$ for $\text{Im } z \geq 0$.

Assuming (a) - (c), we see that $C_0(z)$ is a kth integral of $B(z)$ on $\text{Im } z > 0$; any other k^{th} integral $C(z)$ differs from $C_0(z)$ by at most a polynomial of degree $< k$. Since Λ annihilates all polynomials of degree $< 2n$, properties (a) - (c) hold with $C_0(z)$ replaced by $C(z)$, and the theorem follows from (A2).

To prove (a) - (c) denoted the bracketed expression in (19) by $E(z, \omega)$, which splits into the two parts

$$E_1(z, \omega) = \frac{1}{1 + \omega^{2n}} \left[e^{i\omega z} - \sum_{j=0}^{k-1} \frac{(i\omega z)^j}{j!} \right] ,$$

$$E_2(z, \omega) = \frac{\omega^{2n}}{1 + \omega^{2n}} e^{i\omega z} .$$

Accordingly, $C_0(z)$ splits into two integrals $C_1(z)$ and $C_2(z)$. Since $C_2(z)$ is the Fourier transform of an integrable function, $C_2(z)$ is continuous on $\text{Im } z \geq 0$; on $\text{Im } z > 0$ it is analytic and can be differentiated repeatedly under the integral sign.

Differentiating E_1 repeatedly and applying (A1), we get the bound

$$\frac{\partial^p}{\partial z^p} E_1(z, \omega) \leq \frac{1}{1 + \omega^{2n}} \frac{a^{k-p} \omega^k}{(k - p)!} , \tag{A4}$$

valid for $0 \leq p \leq k$, $\text{Im } z \geq 0$, $|z| \leq a$. The right side of (A4) times $S_x(\omega)$ is integrable. Therefore, the function $C_1(z)$, on the domain $\text{Im } z \geq 0$, can be differentiated k times under the integral sign. In particular, $C_1(z)$ is continuous on this domain. This proves (a), and shows that $C_0(z)$ can be differentiated k times under the integral sign on $\text{Im } z > 0$. Doing so gives (b). Property (c) is true because Λ annihilates polynomials of degree $k - 1$. The theorem is proved.