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INTRODUCTION

The solution to the problem of multiple scattering of radiation by inhomogene-
ities in the Earth's atmosphere is extremely complex. In fact, no complete analytic
solution is known. The general equation describing the radiative transfer process is
a complex integro-differential equation involving integrals over frequency and angu-
lar directions and differentials along the optical path length. The parametric terms
in the equation are complicated functions of all three variables. Complex boundary
conditions must be included for a complete solution, and these cannot, in general, be
completely or independently specified a priori but are usually expressed in terms of
either the radiation intensity or the flux at both boundaries.

Exact solutions are known for some restricted but extremely useful simplifica-
tions to the transfer problem. An example is a gray medium in which one assumes
isotropic scattering or scattering following the Rayleigh law (Chandrasekhar 1960,
Kourganoff 1963, and Sobolev 1975). These solutions have practical application and
are useful in their own right, but perhaps equally important, they provide benchmark
answers for the evaluation of more approximate numerical and analytical solutions.
Considerable progress has been made and is being made in providing rigorous solutions
to the general problem of anisotropic scattering by using methods of successive
approximation, methods of spherical-harmonic and eigenfunction expansions, and the
classical H-function and X- and Y-functions of Chandrasekhar. These methods fre-
quently result in the generation of auxiliary functions (e.g., the H- or X- and
Y-functions) which are defined by nonlinear integral equations or by singular-linear
integral equations, both of which require additional constraints to ensure unique-
ness. (See the discussion in Irvine 1975 or the excellent review article by Hansen
and Travis 1974.)

A number of numerical techniques have also evolved for solving the complete
radiative transfer equation. These include the adding or doubling method, methods of
increased order of scattering, method of discrete ordinates, the spherical-harmonic-
expansion method, and the Monte Carlo method, to name only a few. (See Irvine 1975
and Hansen and Travis 1974). These techniques have, at least theoretically, the
capability of producing "exact" numerical answers to specific problems involving a
given set of parameters. Although generally conceptually simple, these methods are
frequently rather expensive to implement, requiring large amounts of computer time
for solution. They also have the drawback common to all numerical methods in that a
slight change in an input parameter, an internal parameter, or a boundary condition
requires a completely new run for a solution. Parametric studies with these methods
are expensive, and simple physical relationships are frequently difficult to uncover.

Finally, there is a large group of analytical solutions to specifically formu-
lated (and frequently somewhat artificial) problems which, in theory, have limited
application but which in actual practice give useful and reasonably accurate numer-
ical results over a fairly wide range of parameters. These solutions are generally
posed for one or the other of the two extremes of optical thickness - very thin
atmospheres (single-scattering solutions) or very thick atmospheres (diffusion-type
solutions) - or they are used to make some assumptions concerning the angular distri-
bution of the upward and downward radiation streams (two-stream and Eddington-type
solutions). (See, for example, Liou 1980, Coakley and Chylek 1975, Sagan and Pollack
1967, and Joseph et al. 1976.) Meador and Weaver (1980) have presented an excel-

lent survey of two-stream solutions and have shown that all these solutions have a unified structure. Adamson (1975) has presented an interesting and informative interpretation of the multiple-scattering solution to a two-stream formulation in terms of primary, secondary, etc., scatterings.

The basic difficulty with the above solutions – certainly with the rigorous approach and to a lesser extent with the numerical and approximate methods – is that the reader very quickly becomes enmeshed in a tangle of analysis, algebra, and derivational detail which tends rapidly to obscure the basic physics of the scattering process. Thus, the various terms in the solution become difficult to relate to the physics. No one can argue with the powerful and beautifully rigorous analyses presented in the more advanced texts such as the classic of Chandrasekhar (1960) or the more recent text of Sobolev (1975). But these texts are not light reading, and the mathematical arsenal of the newer researcher may lack the firepower to breach the formidable wall of this rigorous approach. The more intermediate level text of Liou (1980) presents some relief, but because of the inherent mathematical complexity and in order to be a bit more heuristic, a number of fundamental concepts are rather artificially and unsatisfactorily defined as mathematical artifacts with little or no rationale or physical interpretation. Consequently, the reader is left dangling, wondering where some of these things came from and why they were defined the way they were.

The author is one of those who has experienced considerable consternation in attempting to grapple with the more current multiple-scattering literature, and the present paper evolved from an attempt to coagulate these concepts and to transform them into a more intuitive picture of the physical processes which were transpiring. A simple photon tracking procedure was adopted in an attempt to uncover some of the simpler scattering concepts and interrelationships in a purely heuristic way. The equations derived apply to a thin (single-scattering), plane-parallel atmosphere bounded below by a diffusely reflecting, absorbing surface. Polarization effects are neglected. The resulting equations, though not rigorous, have a simple physical interpretation and show good agreement with results obtained from the more precise doubling method for thin atmospheres. The primary focus is on the computation of the plane albedo. A limited number of numerical results are given to indicate the general agreement between the derived results and the doubling method. Some of these results are further dissected to indicate the probabilistic histories of the photons contributing to the albedo after emerging from the top of the atmosphere following sundry scatterings in the atmosphere and reflections from the surface. Neither the approach taken nor the relations derived are intended to be useful to the experienced researcher in radiative transfer theory, but it is hoped that the results might provide the less experienced researcher with some useful insight into the radiative transfer process. In this light, the results are presented as tutorial rather than utilitarian.

OPTICAL CHARACTERIZATION OF THE ATMOSPHERE AND BOUNDARY CONDITIONS

We assume here that, for the purpose of examining the single-scattering process, the optical properties of the atmosphere can be completely specified by the following three parameters (Irvine 1975):

1. The single-scatttering albedo ω_o. (A list of symbols used is given after the references.) This parameter gives the probability that, if a photon interacts with the atmosphere, it will be scattered rather than absorbed.

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2. The asymmetry factor g. This parameter basically controls the relative amount of forward and backward scattering which would result from a group of photons encountering a scattering interaction. Roughly speaking, $0.5(1 - g)$ is the probability of backward scatter and $0.5(1 + g)$ is the probability of a forward scatter. Thus, g is a measure of the degree of anisotropy of the scattering function, and $g = 0$ for completely isotropic scattering.

3. The normal optical thickness of the atmosphere τ_o. This is basically an indication of the effectiveness of the atmosphere in permitting photons to pass through unmolested by an extinction interaction. Specifically, $\exp(-\tau_o)$ is the probability of a photon traveling the optical distance τ_o without either being scattered or absorbed.

In addition to the optical properties of the atmosphere, we must specify two boundary conditions. At the upper boundary, we assume that a number of "solar photons" N_o impinge on and enter the atmosphere in a parallel stream at an angle $\theta_o = \cos^{-1} \mu_o$ to the normal, where $\theta_o = 0°$ refers to photons entering the atmosphere vertically downward.

The surface at the lower boundary is characterized by the surface albedo R. This quantity represents the fraction of the photons which hit the surface that are reflected back up into the atmosphere. The fraction of the photons which hit the surface and are absorbed by it is $1 - R$. It is assumed throughout that the radiation field is monochromatic.

BASIC APPROACH

The basic approach taken here is very similar to that used in developing the adding, or doubling, method used in precise numerical solutions to the radiative transfer problem. (See, for example, Hansen 1969, Hansen and Travis 1974, Paltridge and Platt 1976, or Liou 1980.) When a photon enters the atmosphere, it is assumed that one of three things will happen to it:

1. Nothing at all will happen. It will penetrate through the atmosphere and interact with the surface.

2. It will be absorbed by the atmosphere.

3. It will be scattered either upward and leave the atmosphere or downward and interact with the surface.

The general approach taken here is to evaluate the probabilities of these events taking place under sundry combinations of scatter and reflection.

If a photon is absorbed either by the atmosphere or by the surface, it is lost and need not be considered any further — it is merely counted. Scattered photons, however, must be followed until they are either absorbed or leave the top of the atmosphere.

Scattered photons can, in general, be transmitted in any direction. The probability that a given photon will be scattered into a given infinitesimal solid angle is described mathematically by a probability distribution function, or a phase function. These phase functions can be determined analytically from electromagnetic

theory for simple distributions of spherical particles (Mie theory), but in general these computations are quite complex.

Three highly simplified scattering processes are assumed herein for the scattered photons. Those photons entering the top of the atmosphere (referred to herein as solar photons) and scattered will be assumed to follow a Henyey-Greenstein phase function (Irvine 1975 and Sobolev 1975),

$$p(\mu) = \frac{1 - g^2}{(1 + g^2 - 2g\mu)^{3/2}}$$ (1)

in which g is the asymmetry factor, $\mu = \cos \gamma$, and γ is the scatter angle measured from the forward direction. Azimuthal symmetry is assumed in equation (1), and a positive g produces a forward-peaked scatter. The overall probabilities of forward and backward scattering can be computed from the phase function. This is, in general, not a trivial computation. (See, for example, Coakley and Chylek 1975 and Wiscombe and Grams 1976.) The monodirectional backscattering fraction is given by

$$\beta(\mu_o) = \frac{1}{2} \int_0^1 p(-\mu',\mu_o) \, d\mu'$$ (2)

where (Wiscombe and Grams 1976)

$$p(\mu,\mu') = \frac{1}{\pi} \int_0^\pi p\left[\mu\mu' + (1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2} \cos \theta\right] d\theta$$ (3)

Photons which reach the surface and are reflected off the surface will be assumed to reflect diffusely. That is, the photons will be reflected equally likely in any direction in the upper hemisphere whose plane is the surface.

Finally, those photons which are reflected from the surface and are subsequently scattered by the atmosphere will be assumed to conform to a simple two-stream anisotropic scattering law. The fraction $0.5(1 + g)$ photons will be assumed to scatter forward and will be lost through the top of the atmosphere, and the fraction $0.5(1 - g)$ photons will be scattered downward and again interact with the surface. The rationale for this choice is the atmospheric phase function for scattering is independent of the direction of motion of the photon. That is, if the phase function predicts strongly forward scattering, the scattering will be strongly forward whether the photon is moving upward or downward. For positive g, the Henyey-Greenstein phase function is forward peaked. Each of the photons reflected from the surface will have its own phase function which is sharply forward peaked. But since diffuse reflection is assumed at the surface, the reflected photons will be traveling in all directions from 90° to 0° to the surface normal. Thus, the effective phase function for all these photons considered as an aggregate will be one in which the predominant direction of scatter is vertically upward. The simple law described above will reflect this "two-stream" type of motion.

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Two additional assumptions are made concerning the scattered photons. The incoming solar photons which are scattered will either be scattered upward and leave the atmosphere or scattered downward and interact with the surface. These single-scattered photons will not be subjected to any additional absorption since at each interaction the photon can either be absorbed or scattered, but not both. The photons reflected upward by the surface will be subjected to both scattering and absorption. However, the scattered photons will either escape from the top of the atmosphere or will be scattered downward and interact with the surface again, with no additional absorption in transit. Those photons which are repeatedly reflected upward by the surface and scattered downward by the atmosphere will be followed until they all either escape through the top of the atmosphere or are absorbed by the surface or the atmosphere.

ANALYSIS

In the analysis to follow, a simple bookkeeping procedure is introduced to keep track of all the photons N_o as they are absorbed, scattered, and reflected by the surface. At the end of the analysis, all the photons leaving the top of the atmosphere as well as those absorbed by the atmosphere and by the surface will be counted. Figure 1 may be useful for visualizing the various encounter processes described in this section.

The total number of solar photons reaching the surface unaffected by an extinction interaction is given by

$$N_U = N_o \exp(-\tau_o/\mu_o) = N_o \, T(\theta_o) \tag{4}$$

in which the factor μ_o in the exponential corrects for the total slant path. (See fig. 1.) Therefore, the total number of solar photons which are affected by an extinction process is

$$N_E = N_o - N_o \, T(\theta_o) = N_o \, E(\theta_o) \tag{5}$$

Of these, the number of solar photons absorbed by the atmosphere is

$$N_{EA} = (1 - \omega_o)N_E = (1 - \omega_o)N_o \, E(\theta_o) \tag{6}$$

where ω_o is the single-scattering albedo defined earlier and the number of solar photons undergoing scatter is given by

$$N_{ES} = \omega_o N_E = \omega_o N_o \, E(\theta_o) \tag{7}$$

Let P_u and P_d represent the probability of upward and downward scattering. These will be given later for the Henyey-Greenstein phase function. The number of solar photons scattered upward and lost through the top of the atmosphere is

$$N_{SL} = P_u N_{ES} = P_u \omega_o N_o E(\theta_o) \tag{8}$$

and the number scattered downward to the surface is

$$N_{SD} = P_d N_{ES} = P_d \omega_o N_o E(\theta_o) \tag{9}$$

Therefore, the total number of incident photons which reach the surface on this first pass is

$$N_{G1} = N_U + N_{SD} = N_o [T(\theta_o) + P_d \omega_o E(\theta_o)] \tag{10}$$

Now we examine the fate of these N_{G1} photons. The number of these which are absorbed by the surface is given by

$$N_{G1A} = (1 - R)N_{G1} \tag{11}$$

and, thus, the number of photons reflected back up into the atmosphere is

$$N_{G1R} = RN_{G1} \tag{12}$$

In order to account for the various path lengths through which the diffusely reflected photons travel, we introduce the concept of the diffusivity factor $\beta = 1.66$. This is essentially a weighted mean value of $1/\cos \phi$, where ϕ is the angle from the vertical at which the photons are reflected from the surface. Thus, $1.66\tau_o$ is a weighted mean optical depth for all the photons leaving the surface in all directions from 0° to 90° from the vertical. (See Goody 1964, Paltridge and Platt 1976, or Liou 1980.) Therefore, the number of N_{G1} photons escaping from the top of the atmosphere is given by

$$N_{G1RL} = RN_{G1} \exp(-\beta\tau_o) = RN_{G1} T(\beta) \tag{13}$$

and the number of N_{G1} photons affected by extinction in the atmosphere is

$$N_{G1RE} = RN_{G1} - RN_{G1} T(\beta) = RN_{G1} E(\beta) \tag{14}$$

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Of these,

$$N_{G1RA} = (1 - \omega_o)RN_{G1} E(\beta) \tag{15}$$

are absorbed by the atmosphere on their way out and

$$N_{G1RS} = \omega_o RN_{G1} E(\beta) \tag{16}$$

are scattered. Let the fraction of these reflected photons scattered upward and downward be denoted by G_u and G_d, where

$$G_u = 0.5(1 + g) \tag{17}$$

$$G_d = 0.5(1 - g) \tag{18}$$

Then, the number of N_{G1} photons scattered out through the top of the atmosphere and lost is

$$N_{G1RSL} = G_u \omega_o RN_{G1} E(\beta) \tag{19}$$

and the number scattered downward by the atmosphere and reacting again with the surface is

$$N_{G2} = G_d \omega_o RN_{G1} E(\beta) \tag{20}$$

We quickly repeat this process once more, considering the fate of the N_{G2} photons. The number of N_{G2} photons absorbed by the surface is

$$N_{G2A} = (1 - R)N_{G2} \tag{21}$$

and the number reflected upward is

$$N_{G2R} = RN_{G2} \tag{22}$$

The total number of these lost through the top of the atmosphere is given by

$$N_{G2RL} = RN_{G2} T(\beta) \tag{23}$$

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and the number affected by extinction is

$$N_{G2RE} = RN_{G2} \, E(\beta) \tag{24}$$

Thus, the number of N_{G2} photons absorbed by the atmosphere is

$$N_{G2RA} = (1 - \omega_o)RN_{G2} \, E(\beta) \tag{25}$$

and the number scattered becomes

$$N_{G2RS} = \omega_o RN_{G2} \, E(\beta) \tag{26}$$

Of these scattered photons,

$$N_{G2RSL} = G_u \omega_o RN_{G2} \, E(\beta) \tag{27}$$

are scattered upward and lost through the top of the atmosphere, and

$$N_{G3} = G_d \omega_o RN_{G2} \, E(\beta) \tag{28}$$

are scattered downward and again react with the surface. This process is repeated until all photons are either lost through the top of the atmosphere or are absorbed either by the atmosphere or by the surface.

Now we count up all the photons lost through the top of the atmosphere. We can add equations (8), (13), (19), (23), and (27) to get

$$N_L = P_u \omega_o N_o \, E(\theta_o) + \left[RN_{G1} \, T(\beta) + G_u \omega_o RN_{G1} \, E(\beta) \right] + \left[RN_{G2} \, T(\beta) + G_u \omega_o RN_{G2} \, E(\beta) \right] + \dots$$

$$= P_u \omega_o N_o \, E(\theta_o) + R\left[T(\beta) + G_u \omega_o \, E(\beta) \right]\left[N_{G1} + N_{G2} + \dots \right] \tag{29}$$

where the ellipses indicate the repeating term for each time the photon reaches the surface. By the time we get to equations (20) and (28) we can write

$$N_{G1} + N_{G2} + \dots = N_{G1}\left\{ 1 + \left[G_d \omega_o R \, E(\beta) \right] + \left[G_d \omega_o R \, E(\beta) \right]^2 + \dots \right\}$$

and since $G_d \omega_o A \, E(\beta) < 1$, this can be written in the limit as

$$N_{G1} + N_{G2} + \cdots = \frac{N_{G1}}{1 - G_d \omega_o A \, E(\beta)} \tag{30}$$

Thus, the right-hand side of equation (30) is the total number of photons which ever reach the surface. Some of these interact with the surface many times after repeated reflections and downward scatterings from the atmosphere. Using equation (10) we can write the right-hand side of equation (30) as

$$\frac{N_o \left[T(\theta_o) + P_d \omega_o \, E(\theta_o) \right]}{1 - G_d \omega_o R \, E(\beta)} \tag{31}$$

so that equation (29) can be written in terms of N_o as

$$N_L = P_u \omega_o N_o \, E(\theta_o) + \frac{R \left[T(\beta) + G_u \omega_o \, E(\beta) \right] \left[N_o \, T(\theta_o) + P_d \omega_o N_o \, E(\theta_o) \right]}{1 - G_d \omega_o R \, E(\beta)} \tag{32}$$

We now define the total planar albedo of the atmosphere and the surface $R(\mu_o)$ as the ratio of the total number of photons leaving the atmosphere in any direction to the total number of solar photons entering the atmosphere, or

$$R(\mu_o) = P_u \omega_o \, E(\theta_o) + \frac{R \left[T(\beta) + G_u \omega_o \, E(\beta) \right] \left[T(\theta_o) + P_d \omega_o \, E(\theta_o) \right]}{1 - G_d \omega_o R \, E(\beta)} \tag{33}$$

Similarly, the total number of photons absorbed by the atmosphere alone can be found from summing equations (6), (15), and (25) to get

$$N_A = (1 - \omega_o) N_o \, E(\theta_o) + (1 - \omega_o) R N_{G1} \, E(\beta) + (1 - \omega_o) R N_{G2} \, E(\beta) + \cdots$$

$$= (1 - \omega_o) N_o \, E(\theta_o) + (1 - \omega_o) R \, E(\beta) \left[N_{G1} + N_{G2} + \cdots \right] \tag{34}$$

With equations (30) and (31) this can also be written in terms of N_o as

$$N_A = (1 - \omega_o) \left\{ N_o \, E(\theta_o) + \frac{R \, E(\beta) \, N_o \left[T(\theta_o) + P_d \omega_o \, E(\theta_o) \right]}{1 - G_d \omega_o R \, E(\beta)} \right\} \tag{35}$$

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Define the total atmospheric absorption $A(\mu_o)$ as the ratio of the total number of photons absorbed by the atmosphere to the total number of incoming solar photons, or

$$A(\mu_o) = (1 - \omega_o)\left\{E(\theta_o) + \frac{R\ E(\beta)\left[T(\theta_o) + P_d\omega_o\ E(\theta_o)\right]}{1 - G_d\omega_o R\ E(\beta)}\right\} \qquad (36)$$

Finally, using the conservation of the total number of photons, we can write a pseudo transmission function $T(\mu_o)$ as the fraction of all the solar photons absorbed by the surface, or

$$T(\mu_o) = 1 - R(\mu_o) - A(\mu_o) = \frac{(1 - R)\left[T(\theta_o) + P_d\omega_o\ E(\theta_o)\right]}{1 - G_d\omega_o R\ E(\beta)} \qquad (37)$$

Equations (33), (36), and (37) are the fundamental coefficients describing the reflection, absorption, and transmission characteristics of the atmosphere-surface model selected for this study.

Equation (33) is now simple to interpret physically, especially with the aid of equations which led to its development (eqs. (29) to (32)). The first term, $P_u\omega_o\ E(\theta_o)$, gives the probability that an impinging solar photon will be scattered upward out of the atmosphere before it has a chance to interact with the surface. This quantity can thus be interpreted as the albedo of the atmosphere alone without including the effects of the underlying surface.

The second term is actually the product of two basic probabilities. The term

$$\frac{R\left[T(\theta_o) + P_d\omega_o\ E(\theta_o)\right]}{1 - G_d\omega_o R\ E(\beta)}$$

is the probability that a solar photon will reach the surface, either directly $(T(\theta_o) + P_d\omega_o\ E(\theta_o))$ or after repeated reflections from the surface and downward scatterings by the atmosphere $(R/[1 - G_d\omega_o R\ E(\beta)])$. The direct term is itself the sum of two probabilities - the probability that a solar photon will reach the surface unimpeded by an extinction reaction $(T(\theta_o))$ and the probability that if it is scattered, it will be scattered downward and reach the surface $(P_d\omega_o\ E(\theta_o))$.

The other probability factor in equation (33) is the term $[T(\beta) + G_u\omega_o\ E(\beta)]$, and it gives the probability that a photon which is reflected from the surface will escape through the top of the atmosphere, either passing directly through the atmosphere from below without any further reaction $(T(\beta))$ or scattering upward after interacting with the atmosphere $(G_u\omega_o\ E(\beta))$. Thus, the total albedo is simply the overall probability that an incoming solar photon will survive absorption by the atmosphere or by the surface and will eventually, after repeated reflections and scatterings, leave the top of the atmosphere. The terms in equations (36) and (37) can be interpreted similarly.

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Equation (33) denotes what is commonly referred to as the planar albedo of the atmosphere-surface system. This is the albedo of the system in which all incoming photons have the same direction ($\cos^{-1}\mu_o$) but the outgoing photons can be lost through the top of the atmosphere in any direction. The spherical, or Bond, albedo can be obtained from equation (33) by integrating over all incoming solar directions. The spherical albedo can thus also be interpreted as a diffuse albedo, and it expresses the albedo of the system in which the incoming photons can have any direction in the lower hemisphere and the outgoing photons can have any direction in the upper hemisphere.

One of the standard ways of writing the planar albedo for a finite atmosphere over a reflecting surface is (Cess et al. 1981), in the present notation,

$$R(\mu_o) = S(\mu_o) + \frac{R\bar{T}\,T(\mu_o)}{1 - R\bar{S}} \qquad (38)$$

where $S(\mu_o)$ is the reflectivity of the atmosphere layer alone, $T(\mu_o)$ is the mono-directional transmissivity of the layer, and the quantities $\bar{S}$ and $\bar{T}$ represent the reflectivity and transmissivity for diffuse radiation and are equivalant to the "globally averaged" quantities

$$\left.\begin{aligned} \bar{S} &= 2 \int_0^1 S(\mu)\,\mu\,d\mu \\[2em] \bar{T} &= 2 \int_0^1 T(\mu)\,\mu\,d\mu \end{aligned}\right\} \qquad (39)$$

A direct comparison between equations (33) and (38) immediately displays the physical interpretation of the above defined parameters in terms of the various probabilities described earlier.

NUMERICAL RESULTS

Coakley and Chylek (1975) present some numerical results from their solution for the albedo derived using a more sophisticated analysis based on the equation of radiative transfer compared with some "exact" results from the doubling method. Their calculations are made for two optical thicknesses ($\tau_o = 0.125$ and 0.500) and for two values of surface albedo (R = 0.1 and 0.7). They further considered the scattering to be conservative ($\omega_o = 1$). For the scattering phase function, they used the Henyey-Greenstein function with an asymmetry factor $g = 0.844$, which is representative of clouds of liquid water droplets (Hunt 1971). These same numerical values of parameters are used herein.

The probability of upward (or backward) scattering used in the present calculations was taken from Wiscombe and Grams (1976), in which they present a general algorithm for computing both the planar- and the spherical-backscatter function for

any phase function. For the azimuthally symmetric Henyey-Greenstein phase function, the probability of upward scattering is the ratio of the shaded area of the figure of revolution shown in the sketch to the whole surface area. The ellipsoidal solid is
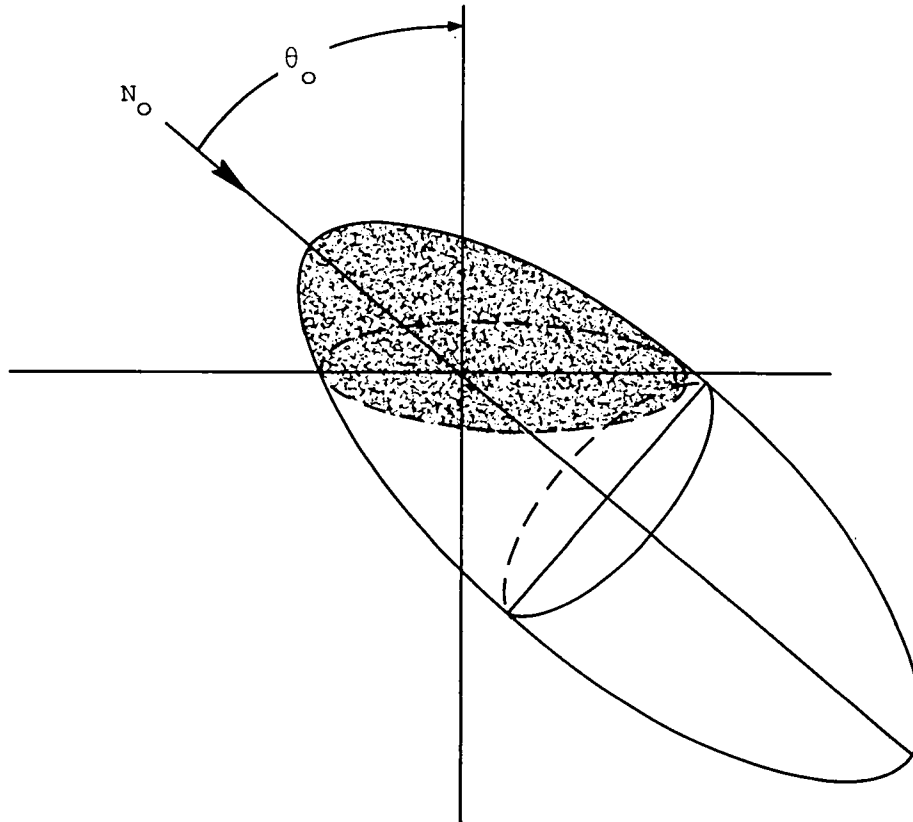
constructed by revolving the Henyey-Greenstein function about its major axis. The probability of upward scattering P_u can be calculated from (Wiscombe and Grams 1976)

$$P_u = 0.5 - \frac{1}{4\pi} \sum_{m=0}^{\infty} (-1m)\frac{\Gamma(m + 0.5)}{\Gamma(m + 2)}(4m + 3)g^{2m+1} P_{2m+1}(\mu) \tag{40}$$

in which Γ is the gamma function, g is the asymmetry factor in the Henyey-Greenstein phase function, and $P_n(\mu)$ is the Legendre polynomial. This equation converges very slowly, especially for $g \approx 1$, and Wiscombe and Grams derived an integral formulation for P_u which they used for their numerical calculations but which will not be given here. The reader is referred to their paper for a thorough discussion of the backscatter problem. For the limiting case of photon entry into the atmosphere vertically ($\mu_o = 1$), equation (1) can be substituted directly into equation (2) and integrated to give

$$P_u = \frac{1 - g}{2g} \frac{1 + g}{\sqrt{1 + g^2}} - 1 \qquad (\mu_o = 1) \tag{41}$$

which agrees with the $\mu = 1.0$ curve of figure 3 of Wiscombe and Grams. For $g = 0.844$, the following table was generated with equation (40):

μ_o	P_u	P_d
0.0	0.500	0.500
.1	.321	.679
.2	.210	.790
.3	.147	.853
.4	.110	.890
.5	.086	.914
.6	.070	.930
.7	.059	.941
.8	.050	.950
.9	.043	.957
1.0	.038	.962

Figure 2 shows a comparison of values calculated from equation (33) with values taken from figures 2 and 3 of Coakley and Chylek (1975). The same conditions were used for both sets of calculations.

For the four cases considered here, the agreement between the present method and the doubling method is very good at all but the shallowest entry angles. The largest errors (about 25 percent) are for the thick atmosphere ($\tau_o = 0.500$) for $R = 0.1$ at $\mu_o = 0.1$, which corresponds to an entry angle of 84°. The errors decrease to less than 1 percent at vertical entry ($\mu_o = 1.0$).

As expected, the agreement between equation (33) and the doubling-method results are very good for the thin atmosphere ($\tau_o = 0.125$), the errors being less than 1 percent for nearly all entry angles. The agreement is only fair for the thicker atmosphere except for nearly vertical entry. Both error sources (shallow entry and thick atmosphere) can be traced to the same factor. Both the shallow entry (larger path length) and the thicker atmosphere present more opportunities for multiple scattering events to occur, and a few additional photons are lost (i.e., contribute to the albedo) through secondary scatterings. These secondary scatterings are considered in the doubling method but are neglected in the present analysis. It should perhaps be noted that, for this highly simplified model of a thin atmosphere, some of the good agreement displayed in figure 2 is to be expected. For thin atmospheres, especially near vertical photon entry, the albedo is dominated by the surface reflection. Hence, large errors in the atmospheric contribution, which is a small part of the albedo here, might not be identified. For the thin atmosphere ($\tau_o = 0.125$), however, the agreement is good even for shallow entry angles, indicating that for this case the atmospheric contribution is accounted for reasonably well. For the present method as well as with any other approximate method, the reader is cautioned not to apply the method to radiative regimes for which the basic assumptions of the method do not apply.

Note further that for both optical thicknesses the planar albedo approaches the value of the surface albedo for nearly vertical entry. This is because for these cases conservative scattering was assumed and, hence, there is no absorption in the atmosphere. Also, as shown later, practically all the photons contributing to the albedo for these cases (over 97 percent) come from solar photons scattered directly by the atmosphere (N_{SL}) and from photons reflected from the surface for the first

time. These photons are either lost directly (N_{G1RL}) or are scattered upward by the atmosphere after the first surface reflection (N_{G1RSL}). The planar albedo is very slightly less than the surface albedo because of the surface absorption of some additional photons during the multiple reflections and the downward scattering taking place after the first reflection.

By inserting the above representative numbers into equation (33), further insight and interpretation of this fundamental result may be attempted. The two probability factors making up equation (33) are plotted separately in figure 3 for the case of conservative scattering ($\omega_o = 1$) so that their individual effects may be more clearly seen and identified. In this figure a surface albedo of R = 0.7 was assumed, and results are shown for two optical thicknesses τ_o = 0.125 and 0.500. The lower curves of figure 3 show the first term of equation (33), $P_u \omega_o E(\theta_o)$, which is the probability that an entry solar photon will be scattered upward and lost before it has a chance to reach the surface. As would be expected, this probability approaches 0.5 for near grazing entry (equal probability of upward and downward scattering) and decreases rapidly as the entry becomes more nearly vertical because of the sharply forward peak produced by the asymmetry factor on the phase function. The thicker atmosphere (τ_o = 0.500) shows a somewhat higher probability of this first scatter event occurring because the denser atmosphere provides more opportunity for an extinction event to occur.

The middle curves show the probability that if a photon reaches the surface, it will be reflected and scattered until it either escapes from the top of the atmosphere or is absorbed by the surface. This is the second term of equation (33). In this case the curves indicate that the denser atmosphere permits fewer photons to escape once they reflect from the surface. This is again reasonable because the denser atmosphere provides more opportunities for a photon to be scattered back downward toward the surface, thus increasing its chances of being absorbed by the surface (or by the atmosphere when $\omega_o < 1$). These two effects are somewhat compensating, as shown by the similarity in the total albedo for the two optical thicknesses (shown in the curves at the top of the figure).

It is apparent from figure 3 that, except for the shallowest entry angles ($\theta_o > 87°$), the bulk of the photons which contribute to the albedo by escaping through the top of the atmosphere are photons which at one time or another have reached the surface and were reflected and scattered about until they were finally lost. This is perhaps more clearly illustrated in figure 4, which shows the lower two pairs of curves of figure 3 rescaled to show the percentage of the total albedo contributed by each of the terms of equation (33). This effect is more pronounced for the thin atmosphere than for the thick one. The thin atmosphere permits more photons to reach the surface in the first place and, correspondingly, allows more of the reflected photons to escape.

Figures 5 and 6 show the relative and accumulative fractions of photons which are scattered directly by the atmosphere, those which reach the surface for the first time, and those which reach the surface for the second time. These are plotted for the thick atmosphere (τ_o = 0.500) with a surface albedo R = 0.7. The labels on the curves are consistent with the equations presented in the body of the text. In figure 6, all but about 3 percent of the photons contributing to the albedo are those which were either directly scattered solar photons (N_{SL}), first-time reflected photons which encounter no extinction on their way out of the atmosphere (N_{G1RL}), or first-time reflected photons which subsequently were scattered upward and lost

(N_{G1RSL}). Of the remaining 3 percent of photons, about 2.9 percent are contributed by the second reflection from the surface, and the remainder are from the subsequent reflections.

Finally, figure 7 shows a comparison of the fractional parameters just discussed for two cases: conservative scattering in which there is no absorption of photons by the atmosphere ($\omega_0 = 1.00$), and scattering in which 95 percent of the photons affected by extinction are scattered and the remaining 5 percent are absorbed in the atmosphere ($\omega_0 = 0.95$). These results are consistent with the intuition developed from preceding arguments. The overall albedo is somewhat lower for the absorbing atmosphere simply because there are fewer photons left to escape after all the reflections and scatterings take place; that is, every time an extinction event is encountered, 5 percent of the participating photons are lost. These losses accumulate through successive events until the differences in planar albedo shown at the top of the figure are realized.

CONCLUDING REMARKS

A simple approach using conservation of photons has been presented for the solution of the scattering of radiation in a thin atmosphere (single scattering) bounded below by a diffusely reflecting, absorbing surface. The equations derived permit a simple physical interpretation as probabilities of the occurrence of various elemental scattering, absorbing, and reflecting events. Numerical results for a thin atmosphere are shown to be in good agreement with more precise results, computed from the doubling method, for all but the shallowest photon entry angles. Similarities in the parametric structure of other thin-atmosphere results can be compared with the various terms of the equations presented herein, thereby permitting the identification and physical interpretation of these terms in light of the probabilistic concepts presented herein.

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SYMBOLS

$A(\mu_o)$ total atmospheric absorption for photons entering the atmosphere at angle $\theta_o = \cos(\theta)$

$E(\beta)$ extinction factor for photons leaving the surface

$E(\theta_o)$ extinction factor for photons entering the atmosphere

G_u, G_d probability of upward or downward atmospheric scattering for photons reflected from the surface

g asymmetry factor

N number of photons

$P_n(\mu)$ Legendre polynomial

P_u, P_d probability of upward or downward atmospheric scattering for photons entering the atmosphere

$p(\mu)$ phase function for scattering

R surface albedo

$R(\mu_o)$ total albedo of atmosphere and surface for photons entering the atmosphere at the angle θ_o

S reflectivity of the atmosphere alone

$T(\beta)$ transmission function for photons diffusely reflected from surface

$T(\theta_o)$ transmission function for photons entering the atmosphere

$T(\mu_o)$ pseudo transmission function for photons absorbed by the surface

β diffusivity factor

$\beta(\mu_o)$ backscatter fraction

Γ gamma function

γ scatter angle measured from the forward direction

θ_o entry angle for solar photons measured from the downward vertical

μ_o $= \cos \theta_o$

τ_o total optical

ϕ scatter angle from the surface measured from the upward vertical

ω_o single-scattering albedo

Subscripts:

A   photon undergoing absorption

D   photon scattered downward

E   photon undergoing extinction

G   photon reaching the surface

L   photon lost through the top of the atmosphere

o   initial or entry value

R   photon reflected from the surface

S   photon scattered into the atmosphere

U   solar photon reaching the surface untouched by extinction

$1,2,\ldots,n$ photon reaching the surface for the first, second, ..., nth time

    A bar over a symbol indicates a globally averaged or diffuse quantity.
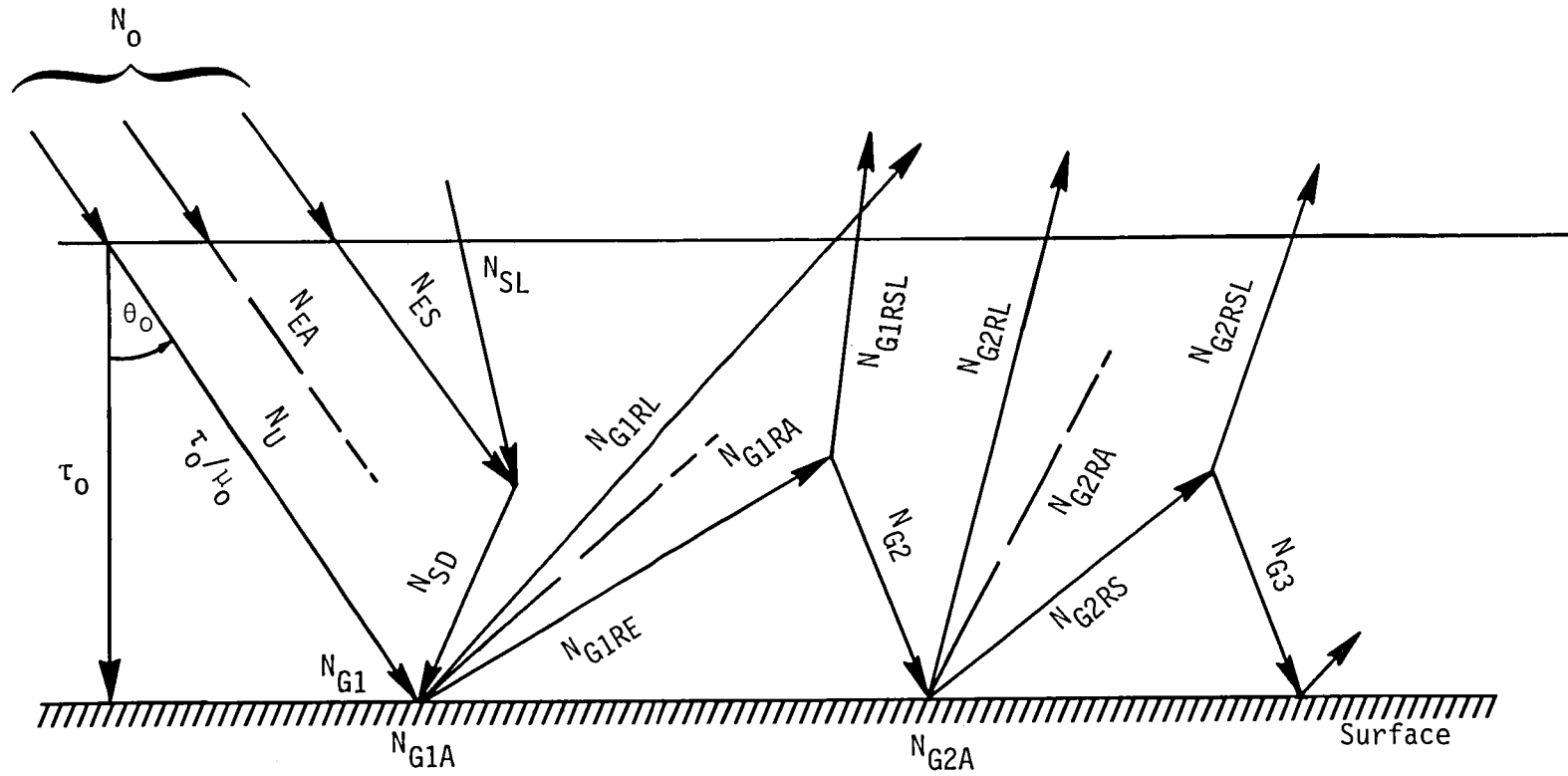
Figure 1.- Reflections and scattering processes considered in the analysis.
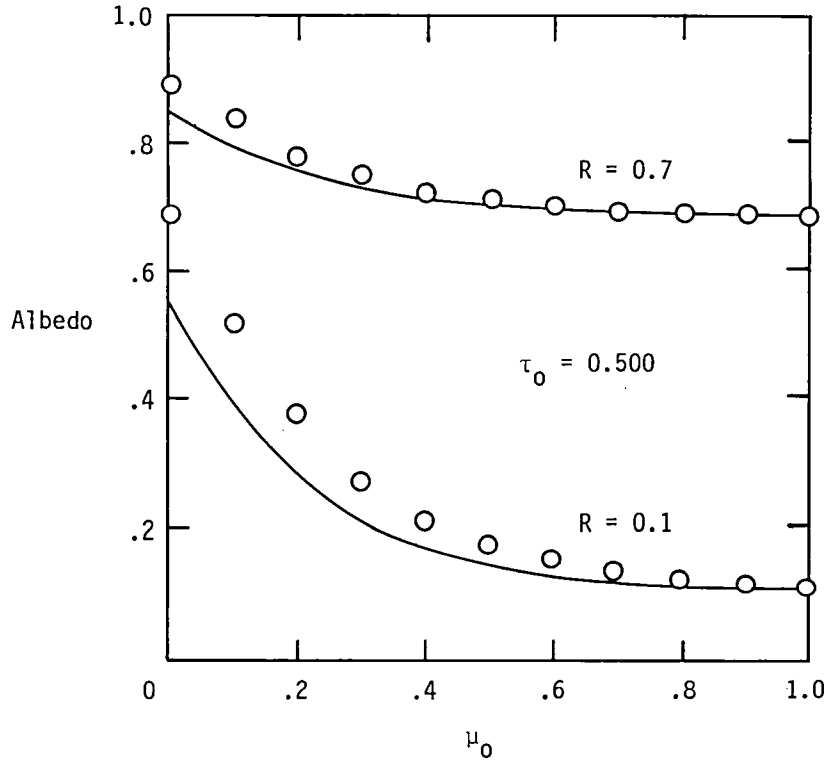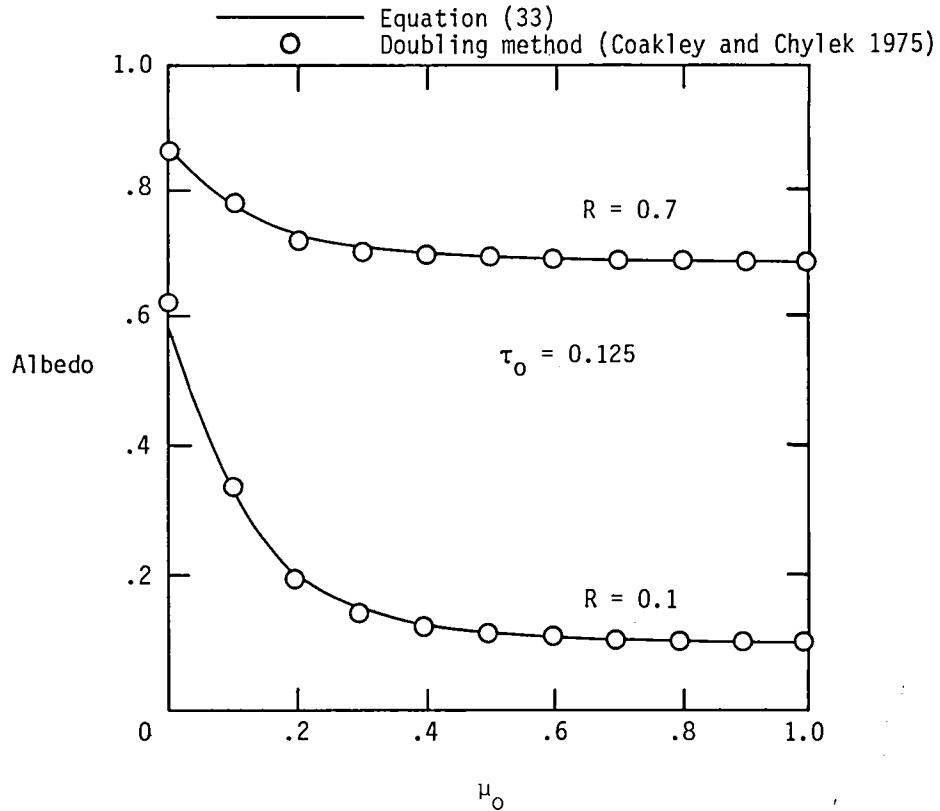
Figure 2.- Spherical albedo versus the cosine of the solar entry angle for atmospheres of two optical thicknesses.
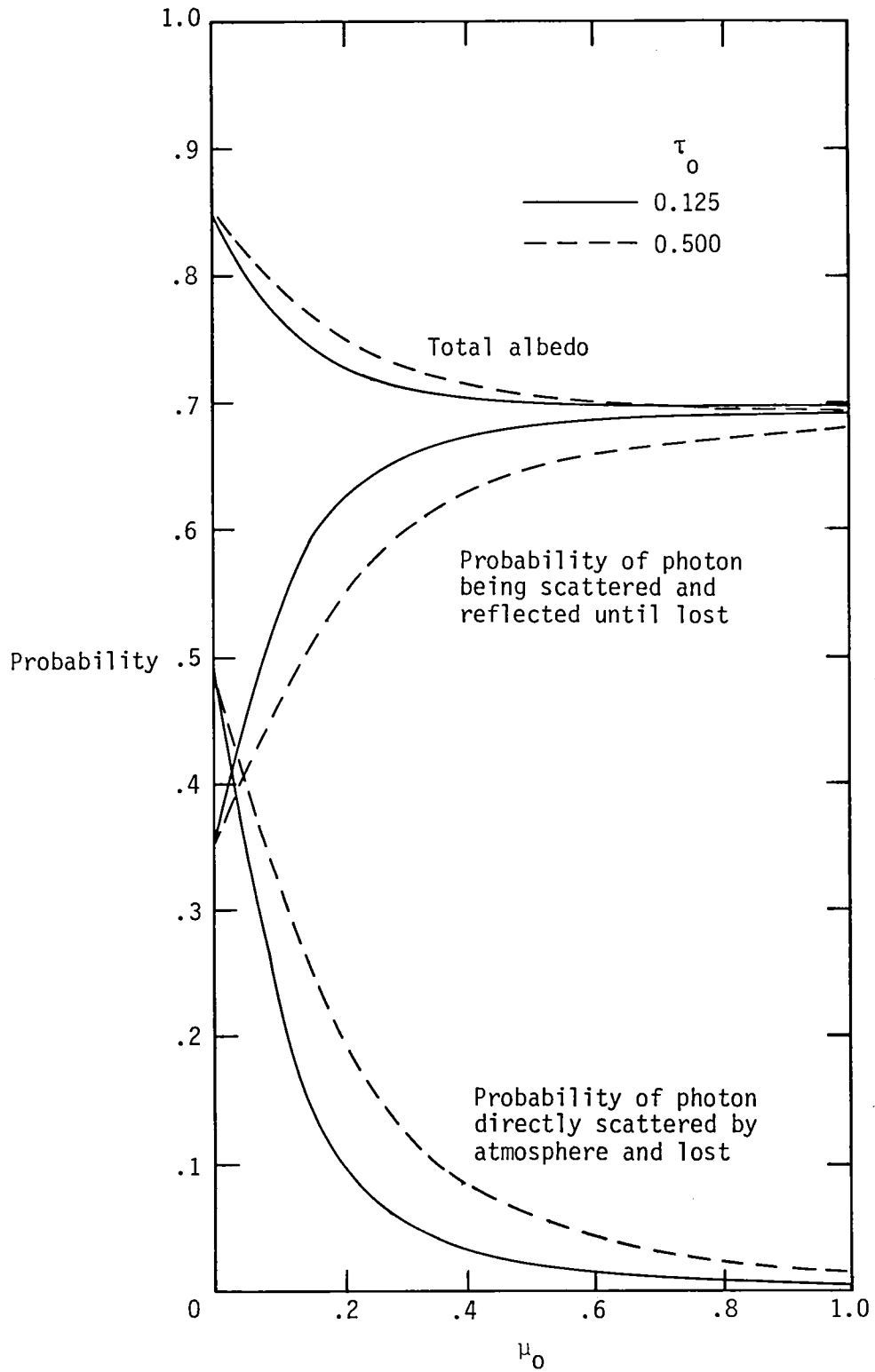
Figure 3.- Individual terms of equation (33) interpreted as event
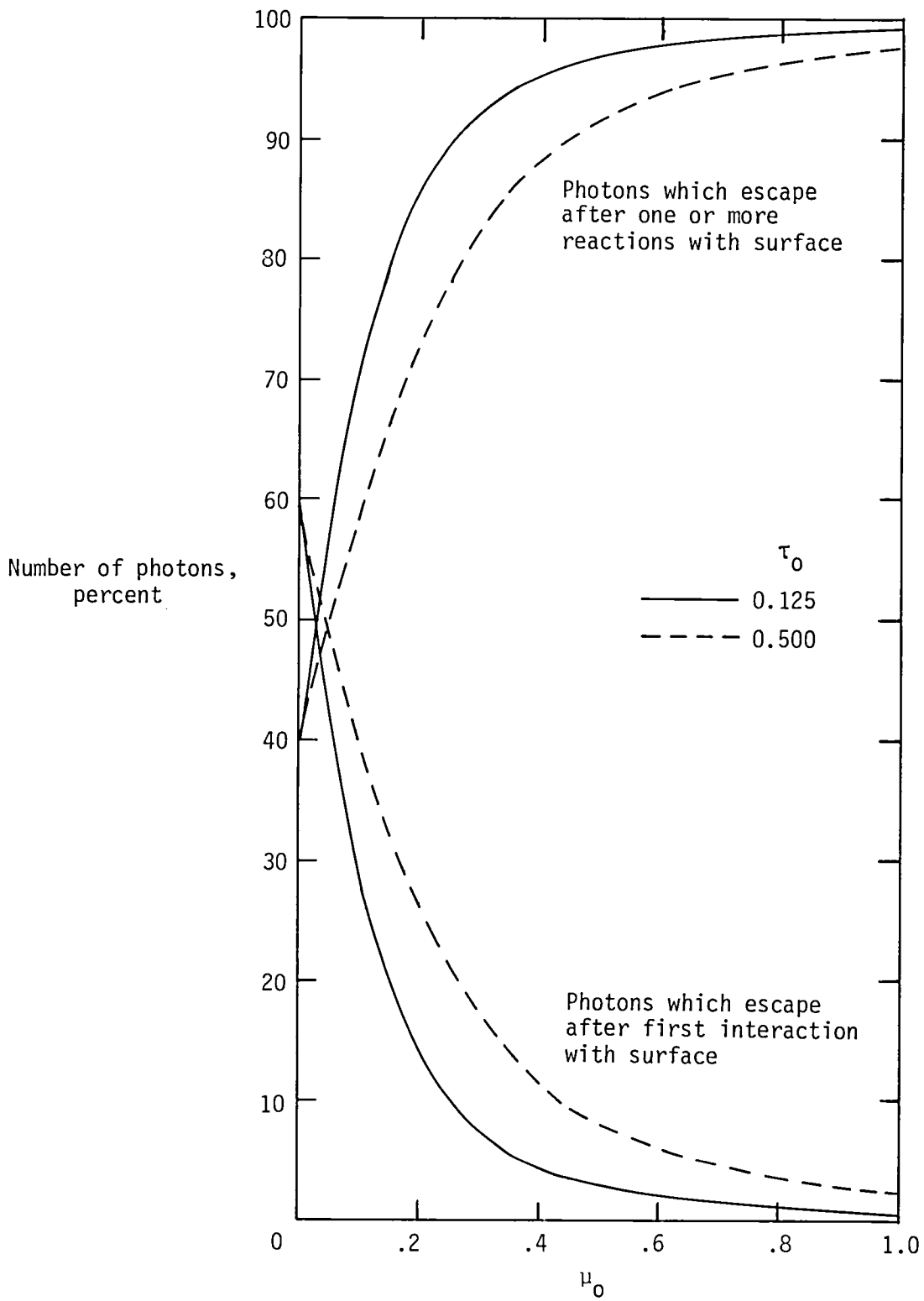probabilities. $R = 0.7$; $\omega_o = 1$.

Figure 4.- Individual terms of equation (33) interpreted as the percentage of photons undergoing the individual events. $R = 0.7$; $\omega_o = 1$.
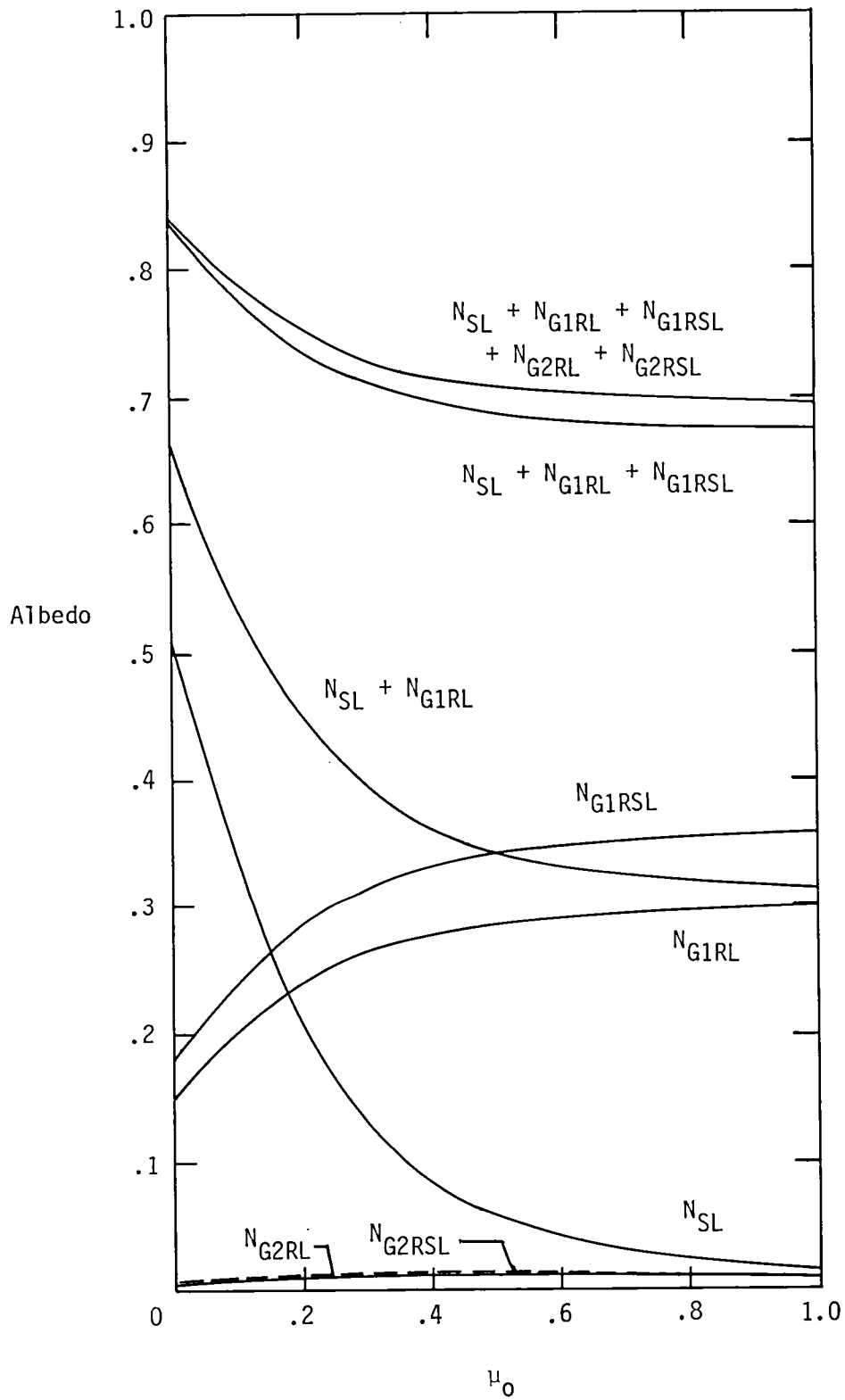
Figure 5.- Contributions to albedo of directly scattered photons N_{SL}, photons which reach the surface once N_{G1}, and photons which reach the surface twice N_{G2}. (See eqs. (8), (13), (19), (23), and (27).) $R = 0.7$; $\tau_o = 0.500$.
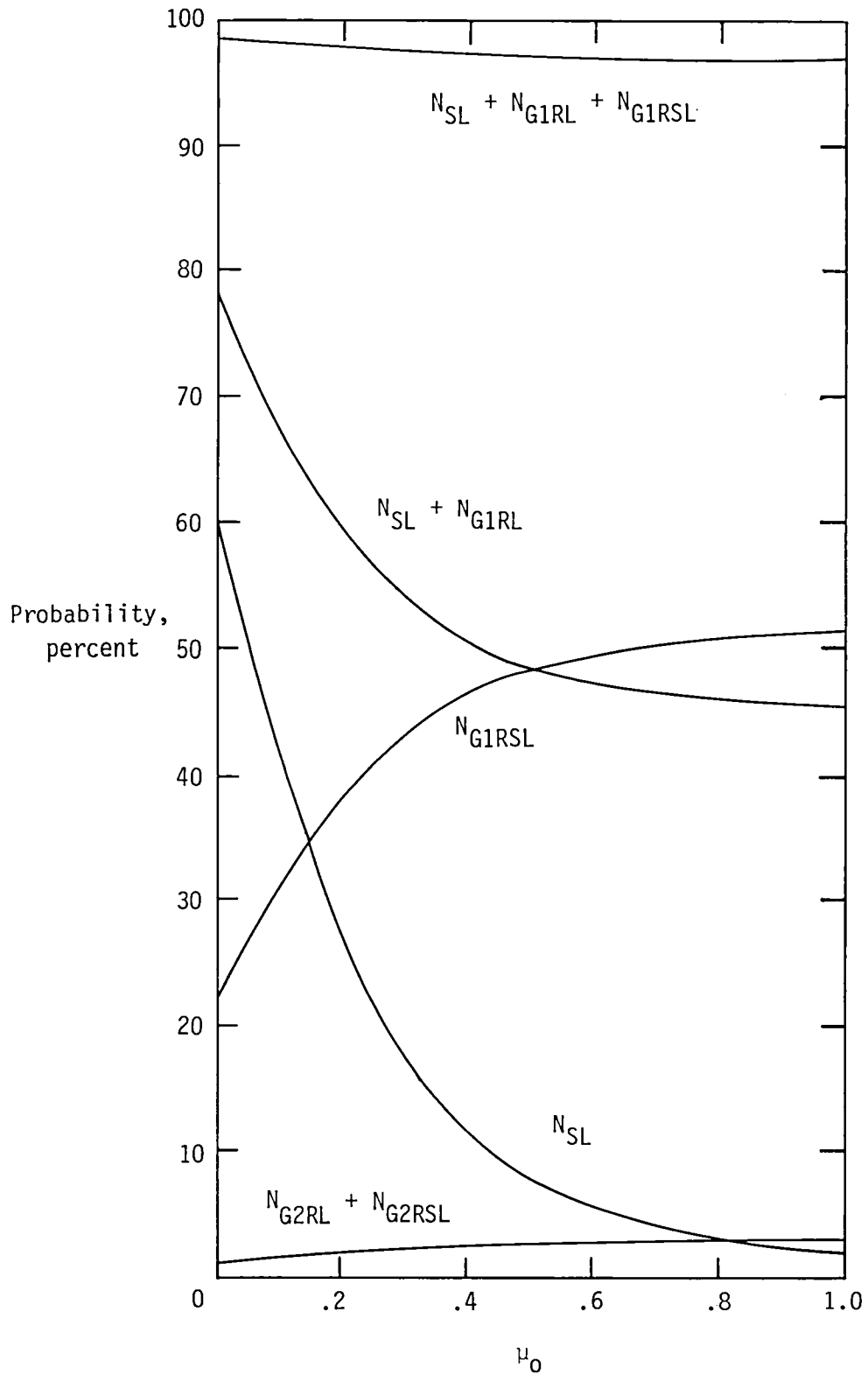
Figure 6.- Percentage of total photons directly scattered N_{SL}, photons which reach the surface once N_{G1}, and photons which reach the surface twice N_{G2}. (See eqs. (8), (13), (19), (23), and (27).) R = 0.7; τ_0 = 0.500.
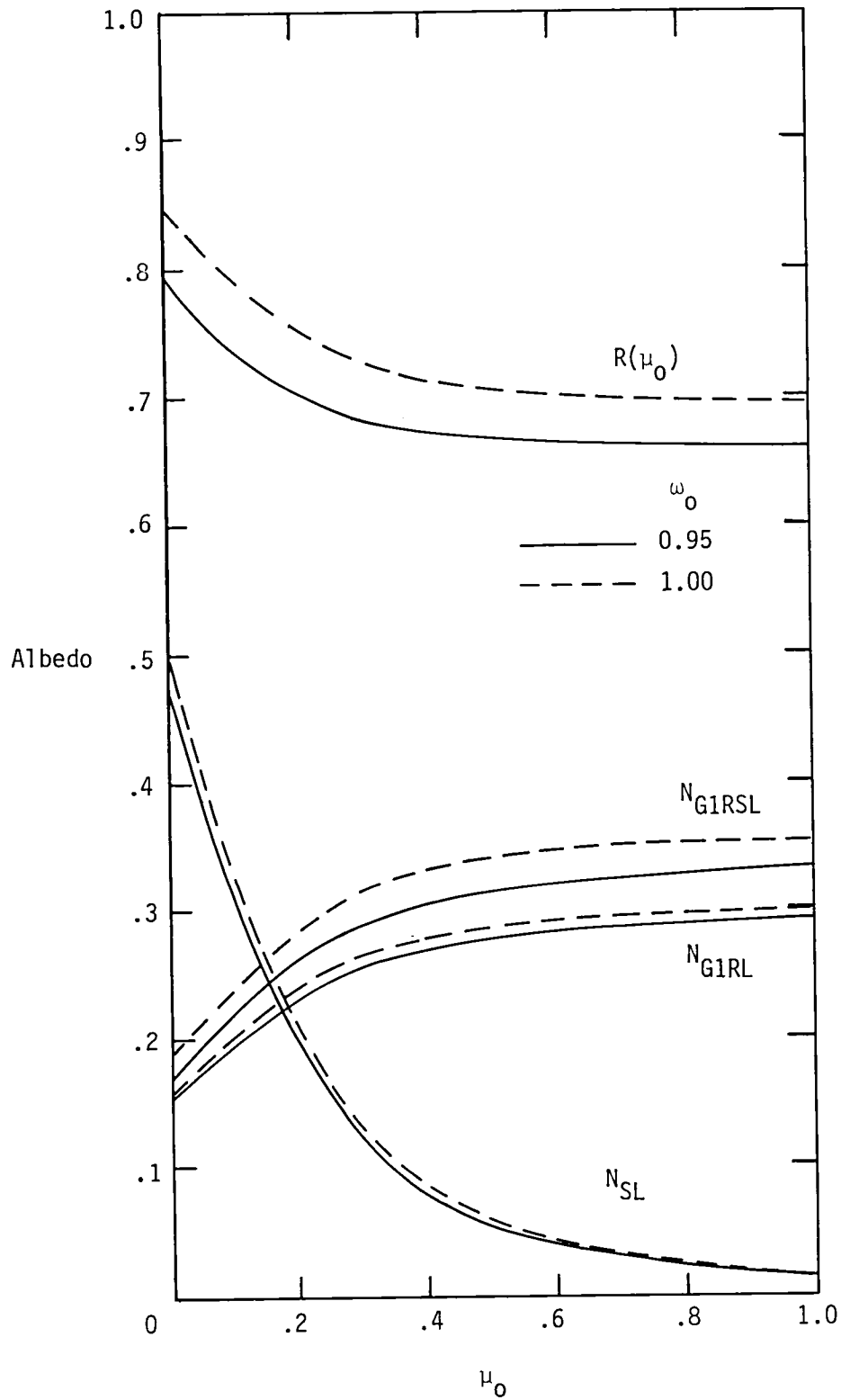
Figure 7.- Effect of weak atmospheric absorption on albedo for photons singly reflected from the surface N_{G1} and directly scattered by the atmosphere (N_{SL}).

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15. Supplementary Notes

16. Abstract

A simple tutorial method, based on a photon tracking procedure, is described to determine the spherical albedo for a thin atmosphere overlying a reflecting surface. This procedure is used to provide a physical structure with which to interpret the more detailed but highly mathematical analyses presented in the literature. The final equations are shown to be in good numerical agreement with more exact solutions for thin atmospheres.

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