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Evaluation of Automated Decisionmaking Methodologies and Development of an Integrated Robotic System Simulation

Appendixes B, C, D, E

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Appendix	Title	EVALUATION OF AUTOMATED DECISIONMAKING METHODOLOGIES AND DEVELOPMENT OF AN INTEGRATED ROBOTIC SYSTEM SIMULATION
B	Derivation of Requirements Tool Dynamics	
C	Derivation of Simulation Tool Dynamics	
D	Derivation of Requirements Tool Control Law	
E	Simulation Methodologies	

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FOREWORD

This document covers the work performed on contract NAS1-16759, Evaluation of Automated Decision-Making Methodologies and Development of Integrated Robotic System Simulation, for the Langley Research Center of the National Aeronautics and Space Administration. It was prepared by Martin Marietta Aerospace in accordance with the contract, Part II, Statement of Work.

The final report for this study consists of three volumes:

NASA CR-165975 - Study Results

NASA CR-165976 - Appendix A, Software Documentation

NASA CR-165977 - Appendix B, Derivation of Requirements Tool Dynamics
Appendix C, Derivation of Simulation Tool Dynamics
Appendix D, Derivation of Requirements Tool Control Law
Appendix E, Simulation Methodologies

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Appendix B—Derivation of Requirements Tool Dynamics

1.0 INTRODUCTION

One facet of robotic systems requirements analysis is the analysis of system dynamics. Dynamic analysis is necessary to ensure that the proposed system will be able to both produce the forces and/or torques required for task execution and withstand the stresses and strains of operation. Typically, the dynamic equations of complex systems are time consuming to derive and must be modified and solved repeatedly as system design concepts change. However, in the case of manipulator systems, the entire process of formulating and solving dynamic equations can be automated.

This appendix describes the derivation of a set of dynamic equations that can be used recursively to calculate forces and torques acting at manipulator joints during operation. The equations are valid for any n-link manipulator system with any kind of joints connected in any sequence. The derivation is based on a technique described in Orin, et al (Ref 1).

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2.0 INITIALIZATION

This section sets up the tools that will be used in deriving the dynamic equations. Notation is defined and coordinate systems (as well as the transformations between them) are described.

Consider an n -link manipulator. Define the base to be link 0 and the end effector, or tool, to be link n . Define joint i to be the joint between link $i-1$ and link i . With each joint, associate a right-handed, orthogonal, Cartesian coordinate system; let \bar{X}_i denote the coordinate system associated with joint i . Let \bar{X}_b denote a coordinate system associated with the base.

The location and orientation of \bar{X}_i with respect to link i is fixed. For rotary joints, the origin of \bar{X}_i is placed at the intersection of the joint axis and the link axis of link i . \bar{X}_i is oriented by defining the x axis to point towards the center of gravity (cg) of link i and the y axis to point along the joint axis. The z -axis is defined uniquely by the requirement that \bar{X}_i be right-handed and orthogonal. For roll and sliding joints, the origin of \bar{X}_i is fixed only with respect to \bar{X}_{i-1} , with the x -axis pointing along the link axis, and the y - and z -axes defined to make \bar{X}_i right-handed and orthogonal (Fig. B2-1).

Matrix quantities will be denoted by capital letters; small letters with an underscore will indicate vectors. Hence, T is a matrix, while \underline{t} is a vector. To distinguish magnitude and direction in a vector, the following convention is used: $\underline{t} = t\underline{u}_t$ where t is the magnitude of \underline{t} and \underline{u}_t is a unit vector in the same direction as \underline{t} .

Vectors and matrices may have different representations when expressed in different coordinate systems. A left subscript will be used to denote the coordinate system in which a quantity is expressed. Hence, ${}_i\underline{v}$ denotes a vector expressed in terms of \bar{X}_i ; ${}_j\underline{v}$ denotes the same vector expressed in terms of \bar{X}_j . When no left subscript is written, quantities are assumed to be in base coordinates (i.e., in terms of \bar{X}_b).

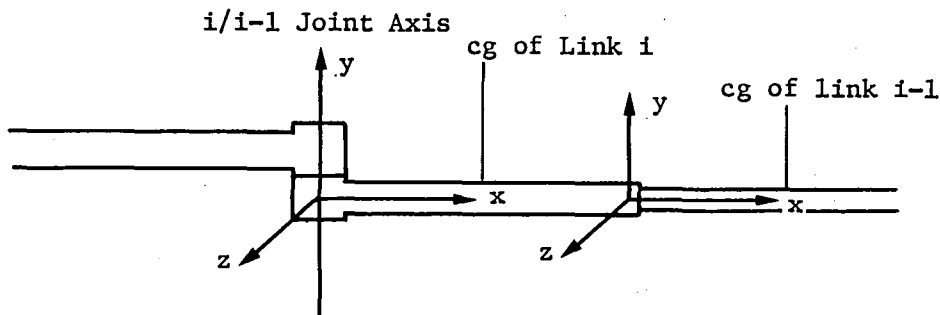


Figure B2-1 Joint Axes Orientation

Transformation matrices and displacement vectors accomplish transformations between coordinate systems. The transformation matrices account for rotational differences between coordinate systems, while the vectors account for translational differences between origins.

Define ${}_jP_i$ to be the matrix that transforms a vector expressed in terms of system \bar{X}_i into terms of a rotated system \bar{X}_j . That is,

$$[B-1] \quad {}_jP_i : \bar{X}_i \rightarrow \bar{X}_j$$

or

$$[B-2] \quad {}_jP_i {}_iV = {}_jV$$

Once again, the transformation matrices only account for rotational differences between coordinate systems. The result, ${}_jV$, can be interpreted as being a vector of equal length as ${}_iV$ and with the same orientation, but originating from \bar{X}_j , rather than \bar{X}_i (Fig. B2-2).

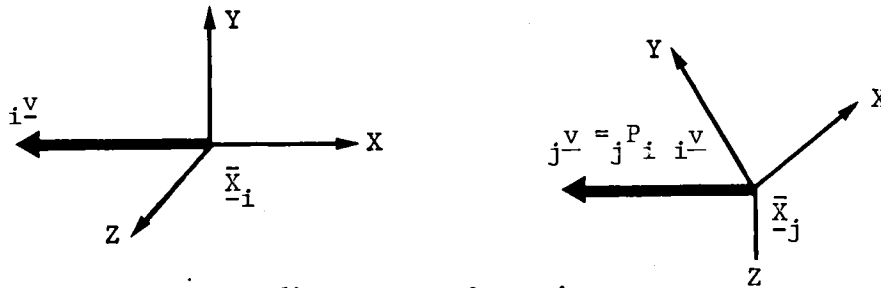


Figure B2-2 Coordinate Transformation

To perform some calculations, the transformation of inertia matrices from one coordinate system to another will be required. Transformations of matrix quantities requires two matrix multiplications. The formula for transforming the inertia matrix of link i , I_i , from the system \bar{X}_i to the system \bar{X}_j is given by

$$[B-3] \quad {}_jI_i = {}_jP_i [{}_iI_i {}_jP_i^{-1}]$$

As the approach to dynamics is developed, certain quantities will be needed in many of the equations. Instead of defining them as required, the following is a list of the vectors, matrices, etc., that will be used in this appendix:

\underline{s}_i = Linear velocity of \bar{X}_i with respect to an inertial coordinate system;

\underline{w}_i = Angular velocity of \bar{X}_i with respect to an inertial coordinate system;

$\underline{\Omega}_i$ = Angular velocity of \bar{X}_i with respect to \bar{X}_{i-1} ;

$\underline{\alpha}_i = \dot{\underline{\omega}}_i$ = Angular acceleration of $\underline{\bar{X}}_i$ with respect to an inertial coordinate system;

$\underline{a}_i = \dot{\underline{s}}_i$ = Linear acceleration of $\underline{\bar{X}}_i$ with respect to an inertial coordinate system;

$\underline{h}_{i,j}$ = Position vector of $\underline{\bar{X}}_j$ with respect to $\underline{\bar{X}}_i$, if $i \neq j$;
 Position vector of the cg of link i with respect to $\underline{\bar{X}}_i$,
 if $i = j$;

\underline{I}_i = Inertia matrix of link i with respect to the cg of link i ;

m_i = Mass of link i ;

\underline{f}_i = Reaction force acting at $\underline{\bar{X}}_i$;

\underline{t}_i = Reaction torque acting at $\underline{\bar{X}}_i$;

\underline{g} = Gravitational acceleration.

The use of a tilde "~" over a vector will denote the 3x3 skew-symmetric matrix of a vector cross product. That is, if $\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

then

$$\underline{\tilde{a}} \underline{b} = \underline{a} \times \underline{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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3.0 A DYNAMICS SOLUTION

The derivation of the dynamic equations of an n-link manipulator system can now be described.

The following application scenario is envisioned. A designer receives a set of requirements for a manipulator arm. He creates an initial design, which he then subjects to the motions that would arise from a typical operation, thereby obtaining a position, velocity, and acceleration profile for each joint. Using these profiles, the reaction forces and torques acting on the system can be computed.

To solve for forces and torques from the motion profiles, a recursive procedure can be used. First, working from \bar{X}_b to \bar{X}_n , the velocities and accelerations with respect to an inertial system will be calculated using the motion profiles as input. (The motion profiles are assumed to contain information on relative values like the angular velocity of \bar{X}_i with respect to \bar{X}_{i-1} .) Once this has been accomplished, the relations $\underline{f} = m \underline{a}$ and $\underline{t} = I \underline{\alpha} + \underline{w} \times I \underline{w}$ can be used to solve for the forces and torques acting at the origin of each coordinate system. If the system joints are single degree of freedom, one of the resultant terms represents an actuator force or torque, and the other terms represent the dynamic reactions acting on the system due to the motion.

To simplify notation, all quantities will be assumed to be in base coordinates, and \bar{X}_b , the base coordinate system, will be assumed to be an inertial system. After the dynamic equations are derived, transformations will be considered.

3.1 COMPUTING INERTIAL MOTION

The equations that will be derived are recursive--the values at \bar{X}_i are calculated from knowledge of values at \bar{X}_{i-1} and from knowledge of the relative motion of \bar{X}_i with respect to \bar{X}_{i-1} .

Figure B3-1 depicts the coordinate systems \bar{X}_{i-1} and \bar{X}_i separated by a single link. In (a), the joint is rotary; in (b), the joint is sliding. If the joints were held fixed, velocities of \bar{X}_i could be calculated by knowing velocities at \bar{X}_{i-1} and the displacement between origins. In particular, the angular velocity of \bar{X}_i would equal that of \bar{X}_{i-1} , while the linear velocity would be a vector sum of the linear velocity at \bar{X}_{i-1} and the angular velocity crossed with the displacement vector $\bar{h}_{i-1,i}$:

$$[B-4] \quad \bar{w}_i = \bar{w}_{i-1}$$

$$[B-5] \quad \bar{s}_i = \bar{s}_{i-1} + \bar{w}_{i-1} \bar{h}_{i-1,i}$$

If the joint between the coordinate systems was free, movement at the joint would contribute to the velocities of the system \bar{X}_i . The rotary joint of (a) would add to angular velocity of \bar{X}_i , while the sliding joint of (b) would contribute to linear velocity. The angular velocity of the rotary joint is Ω_i , and the velocity of the sliding joint is

$$\frac{d}{dt} (h_{i-1,i}) \mu_{h_{i-1,i}} = \dot{h}_{i-1,i} \mu_{h_{i-1,i}}$$

Putting these terms into the equations [B-4] and [B-5] yields

$$[B-6] \quad \bar{w}_i = \bar{w}_{i-1} + \Omega_i$$

$$[B-7] \quad \bar{s}_i = \bar{s}_{i-1} + \bar{w}_{i-1} \bar{h}_{i-1,i} + \dot{h}_{i-1,i} \mu_{h_{i-1,i}}$$

Equations [B-6] and [B-7] can be used repeatedly to calculate the absolute velocities of each of the coordinate systems \bar{X}_i through \bar{X}_n , given the values for the system \bar{X}_b , and the relative velocities at the joints.

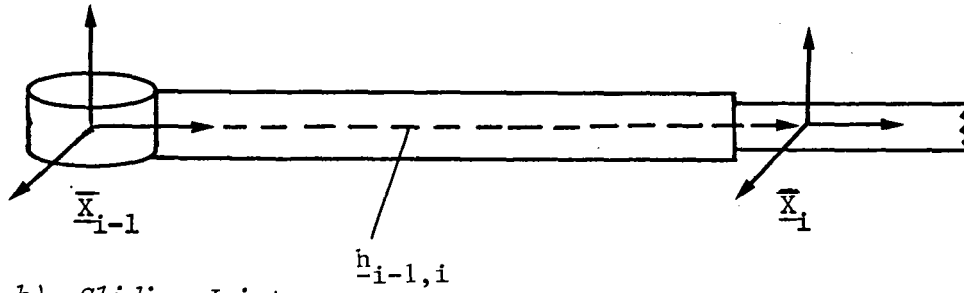
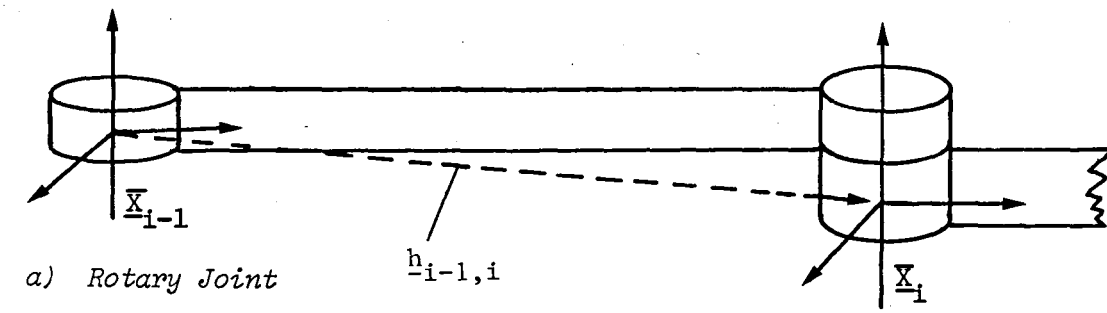


Figure B3-1 Joint Coordinate Systems

Formulas for the angular and linear accelerations of \overline{X}_i can be found from [B-6] and [B-7] using the definitions

$$\underline{\alpha} = \frac{d}{dt}(\underline{w}) \text{ and } \underline{a} = \frac{d}{dt}(\underline{s}).$$

$$\begin{aligned} \underline{\alpha}_i &= \frac{d}{dt} [\underline{w}_i] \\ &= \frac{d}{dt} [\underline{w}_{i-1} + \underline{\Omega}_i] \\ &= \frac{d}{dt} [\underline{w}_{i-1}] + \frac{d}{dt} [\underline{\Omega}_i] \\ &= \underline{\alpha}_{i-1} + \frac{d}{dt} [\underline{\Omega}_i] \underline{\mu}_{\Omega_i} + \underline{\Omega}_i \frac{d}{dt} [\underline{\mu}_{\Omega_i}] \\ &= \underline{\alpha}_{i-1} + \dot{\underline{\Omega}}_i \underline{\mu}_{\Omega_i} + \underline{\Omega}_i (\underline{w}_{i-1} \times \underline{\mu}_{\Omega_i}) \\ \text{[B-8]} \quad \underline{\alpha}_i &= \underline{\alpha}_{i-1} + \dot{\underline{\Omega}}_i \underline{\mu}_{\Omega_i} + \tilde{\underline{w}}_{i-1} \underline{\Omega}_i \\ \\ \underline{a}_i &= \frac{d}{dt} [\underline{s}_i] \\ &= \frac{d}{dt} [\underline{s}_{i-1} + \tilde{\underline{w}}_{i-1} \underline{h}_{i-1,i} + \frac{d}{dt} (\underline{h}_{i-1,i}) \underline{\mu}_{h_{i-1,i}}] \\ &= \frac{d}{dt} [\underline{s}_{i-1}] + \frac{d}{dt} [\underline{w}_{i-1}] \times \underline{h}_{i-1,i} + \underline{w}_{i-1} \times \frac{d}{dt} [\underline{h}_{i-1,i}] \\ &\quad + \frac{d}{dt} [\frac{d}{dt} (\underline{h}_{i-1,i}) \underline{\mu}_{h_{i-1,i}}] \\ &= \underline{a}_{i-1} + \underline{\alpha}_{i-1} \times \underline{h}_{i-1,i} + \underline{w}_{i-1} \times \frac{d}{dt} [\underline{h}_{i-1,i}] \underline{\mu}_{h_{i-1,i}} + \\ &\quad \underline{w}_{i-1} \times \underline{h}_{i-1,i} \frac{d}{dt} [\underline{\mu}_{h_{i-1,i}}] + \frac{d^2}{dt^2} (\underline{h}_{i-1,i}) \underline{\mu}_{h_{i-1,i}} \\ &\quad + \frac{d}{dt} (\underline{h}_{i-1,i}) \frac{d}{dt} (\underline{\mu}_{h_{i-1,i}}) \\ &= \underline{a}_{i-1} + \underline{\alpha}_{i-1} \times \underline{h}_{i-1,i} + \underline{w}_{i-1} \times \frac{d}{dt} (\underline{h}_{i-1,i}) \underline{\mu}_{h_{i-1,i}} + \\ &\quad \underline{w}_{i-1} \times \underline{h}_{i-1,i} (\underline{w}_{i-1} \times \underline{\mu}_{h_{i-1,i}}) + \frac{d^2}{dt^2} (\underline{h}_{i-1,i}) \underline{\mu}_{h_{i-1,i}} \\ &\quad + \frac{d}{dt} (\underline{h}_{i-1,i}) (\underline{w}_{i-1} \times \underline{\mu}_{h_{i-1,i}}) \\ \text{[B-9]} \quad \underline{a}_i &= \underline{a}_{i-1} + \tilde{\underline{\alpha}}_{i-1} \underline{h}_{i-1,i} + 2\tilde{\underline{w}}_{i-1} (\dot{\underline{h}}_{i-1,i} \underline{\mu}_{h_{i-1,i}}) \\ &\quad + \tilde{\underline{w}}_{i-1} (\tilde{\underline{w}}_{i-1} \underline{h}_{i-1,i}) + \ddot{\underline{h}}_{i-1,i} \underline{\mu}_{h_{i-1,i}} \end{aligned}$$

Formulas [B-6] through [B-9] can be used repeatedly to calculate the absolute velocities and accelerations of each of the coordinate systems \bar{X}_i through \bar{X}_n , given the values for the system \bar{X}_b and the relative accelerations at each joint.

If the end effector is holding a body, then a coordinate system can be attached to the body (call it \bar{X}_{n+1}). Velocities and accelerations of the body can be calculated by applying formulas [B-6] through [B-9], using the values calculated for \bar{X}_n .

3.2 COMPUTING REACTIONS

Once velocities and accelerations have been calculated, the values for forces and moments can be obtained.

Assume that the end effector is holding a body. Let \underline{f} and \underline{t} denote the external force and torque acting on the body. Let \bar{X}_{n+1} denote the coordinate system located on the body. Then the resultant of all the forces and torques acting on the origin of \bar{X}_{n+1} are found by using the equations below:

$$[B-10] \quad \underline{f}_{n+1} = m_{n+1} [\underline{a}_{n+1} + \tilde{\alpha}_{n+1} \underline{h}_{n+1,n+1} + \tilde{w}_{n+1} (\tilde{w}_{n+1} \underline{h}_{n+1,n+1}) - \underline{g}] - \underline{f}$$

$$[B-11] \quad \underline{t}_{n+1} = I_{n+1} \underline{\alpha}_{n+1} + \tilde{w}_{n+1} I_{n+1} \underline{w}_{n+1} + \tilde{h}_{n+1,n+1} m_{n+1} [\underline{a}_{n+1} + \tilde{\alpha}_{n+1} \underline{h}_{n+1,n+1} + \tilde{w}_{n+1} (\tilde{w}_{n+1} \underline{h}_{n+1,n+1}) - \underline{g}] - \tilde{h}_{n+1,f} \underline{f} - \underline{t}$$

where $\underline{h}_{n+1,f}$ denotes the vector from the origin of \bar{X}_{n+1} to the point of application of \underline{f} .

The forces and torques on each of the systems \bar{X}_i can be found in a recursive manner, beginning with those at \bar{X}_n and working down to \bar{X}_b using equations similar to formulas [B-6] to [B-9].

Assume that the forces and torques acting on \bar{X}_{i+1} are known and those at \bar{X}_i are sought (Fig. B3-2).

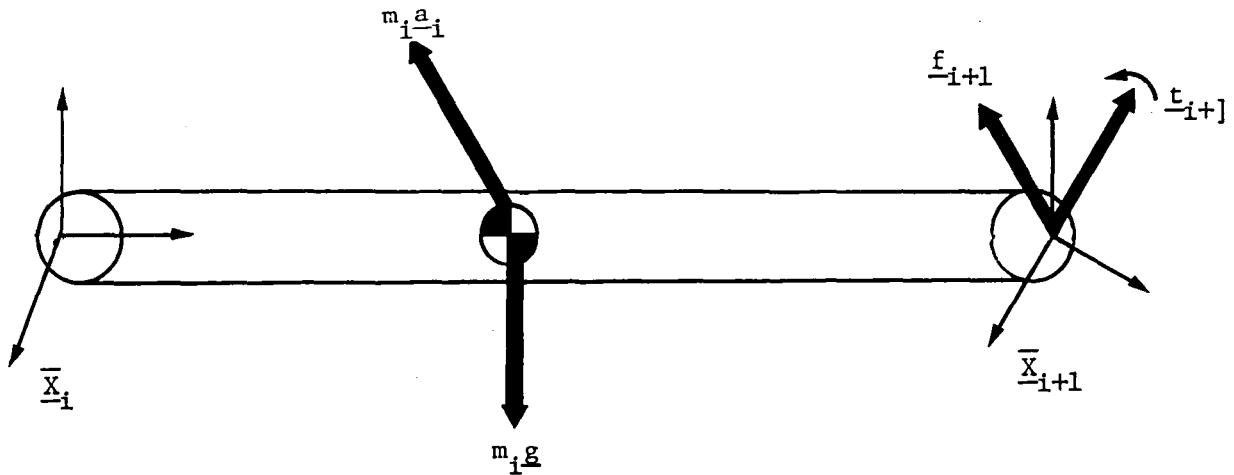


Figure B3-2 Force and Torque Diagram

The values \underline{f}_{i+1} and \underline{t}_{i+1} can be viewed as forces and torques applied to the system \underline{X}_i at a point located by $\underline{h}_{i,i+1}$. Similarly, $m_i \underline{g}$ can be viewed as an applied force and $m_i \underline{a}_i$ as a reaction force (or inertial force) acting at a point located by $\underline{h}_{i,i}$. The forces and torques transmitted to the origin of \underline{X}_i are given by

$$[B-12] \quad \underline{f}_i = \underline{f}_{i+1} + m_i [\underline{a}_i + \tilde{\alpha}_{i-1,i} \underline{h}_{i-1,i} + \tilde{w}_i (\tilde{w}_{i-1,i} \underline{h}_{i-1,i}) - \underline{g}]$$

$$[B-13] \quad \underline{t}_i = \underline{t}_{i+1} + \tilde{h}_{i,i+1} \underline{f}_{i+1} + I_{i-1} \underline{\alpha}_i + \tilde{h}_{i,i} m_i [\underline{a}_i + \tilde{\alpha}_{i-1,i} \underline{h}_{i-1,i} + \tilde{w}_i (\tilde{w}_{i-1,i} \underline{h}_{i-1,i}) - \underline{g}] + \tilde{w}_i I_{i-1} \underline{w}_{i-1}$$

Equations [B-12] and [B-13] are the recursion formulas that can be used to calculate the reaction forces and torques at each joint from the tool back to the base.

Consider \underline{f}_i and \underline{t}_i as each being composed of three constituent vectors directed along the \underline{X}_i coordinate axes. That is,

$$\underline{f}_i = \begin{bmatrix} f_{ix} \\ f_{iy} \\ f_{iz} \end{bmatrix} = \begin{bmatrix} f_{ix} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ f_{iy} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_{iz} \end{bmatrix}$$

and

$$\underline{t}_i = \begin{bmatrix} t_{ix} \\ t_{iy} \\ t_{iz} \end{bmatrix} = \begin{bmatrix} t_{ix} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t_{iy} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t_{iz} \end{bmatrix}$$

If joint i has only a single degree of freedom, then one of the constituent vectors of \underline{f}_i and \underline{t}_i represents the actuator force or torque. The other five constituent vectors represent reactions at the joint due to the manipulator motion and external effects.

3.3 INSERTING TRANSFORMATIONS

To simplify the notation, transformation matrices were not included in the derivations of Sections 3.1 and 3.2. However, in any application, transformation matrices must not be overlooked.

In Section 2.0, lower left subscripts are used to denote the coordinate system a quantity is expressed in. In this case, jP_i is used to represent the transformation matrix from \bar{X}_i to \bar{X}_j . The base coordinate system will be denoted by \bar{X}_b and will be assumed to be an inertial coordinate system.

Equations [B-6] through [B-9] are the equations used to calculate inertial velocities and accelerations, given the relative velocities and accelerations between coordinate systems. These equations are executed recursively, and so quantities that are calculated at one step are used in the next step. By storing the results of transformations performed at each step, transformations need only be applied at succeeding steps to terms that have not already been operated on. Therefore, including transformations only when necessary, equations [B-6] through [B-9] become

$$[B-14] \quad \underline{\dot{w}}_i = \underline{\dot{w}}_{i-1} + {}^P_i[\underline{\dot{\Omega}}_i]$$

$$[B-15] \quad \underline{s}_i = \underline{s}_{i-1} + \underline{\tilde{w}}_{i-1} ({}^P_{i-1}[\underline{h}_{i-1,i}]) + {}^P_{i-1}[(\underline{\ddot{h}}_{i-1,i})_{i-1} \underline{\mu}_{h_{i-1,i}}]$$

$$[B-16] \quad \underline{\alpha}_i = \underline{\alpha}_{i-1} + {}^P_i[(\underline{\dot{\Omega}}_i)_{i-1} \underline{\mu}_{\Omega_i}] + \underline{\tilde{w}}_{i-1} ({}^P_i[\underline{\dot{\Omega}}_i])$$

$$[B-17] \quad \underline{a}_i = \underline{a}_{i-1} + \underline{\tilde{\alpha}}_{i-1} ({}^P_{i-1}[\underline{h}_{i-1,i}]) \\ + 2 \underline{\tilde{w}}_{i-1} ({}^P_{i-1}[(\underline{\ddot{h}}_{i-1,i})_{i-1} \underline{\mu}_{h_{i-1,i}}]) \\ + \underline{\tilde{w}}_{i-1} (\underline{\tilde{w}}_{i-1} ({}^P_{i-1}[\underline{h}_{i-1,i}])) \\ + {}^P_{i-1}[(\underline{\ddot{h}}_{i-1,i})_{i-1} \underline{\mu}_{h_{i-1,i}}]$$

Equations [B-10] and [B-11] for calculating the forces and torques acting on a body held by the tool become

$$[B-18] \quad \underline{f}_{n+1} = m_{n+1} (\underline{a}_{n+1} + \underline{\tilde{\alpha}}_{n+1} {}^P_{n+1}[\underline{h}_{n+1,n+1}] - \underline{g}) \\ - {}^P_{n+1}[\underline{f}]$$

$$[B-19] \quad \underline{t}_{n+1} = ({}^P_{n+1}[\underline{I}_{n+1} {}^P_{n+1}^{-1}]) \underline{b}_{n+1} \\ + \underline{\tilde{w}}_{n+1} ({}^P_{n+1}[\underline{I}_{n+1} {}^P_{n+1}^{-1}]) \underline{w}_{n+1} \\ + {}^P_{n+1}[\underline{h}_{n+1}] \times m_{n+1} (\underline{a}_{n+1} + \underline{\tilde{\alpha}}_{n+1} {}^P_{n+1}[\underline{h}_{n+1,n+1}] \\ - \underline{g}) - {}^P_{n+1}[\underline{h}_{n+1,f} \underline{f}] - {}^P_{n+1}[\underline{t}]$$

and equations [B-12] and [B-13] become

$$[B-20] \quad \underline{b}^f_i = \underline{b}^f_{i+1} + m_i (\underline{b}^a_i + \tilde{\alpha}_i \underline{b}^h_{i,i} - \underline{b}^g)$$

$$[B-21] \quad \underline{b}^t_i = \underline{b}^t_{i+1} + \tilde{h}_{i,i+1} \underline{b}^f_{i+1} + \underline{b}^P_i [\underline{I}_i \underline{b}^{P-1}_i] \underline{b}^\alpha_i \\ + \tilde{h}_{i,i} m_i (\underline{b}^a_i + \tilde{\alpha}_i \underline{b}^h_{i,i} - \underline{b}^g) \\ + \tilde{w}_i (\underline{b}^P_i [\underline{I}_i \underline{b}^{P-1}_i]) \underline{b}^w_i$$

Equations [B-14] through [B-21] form the complete set of equations needed to calculate reaction forces and torques resulting from a given manipulator motion.

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4.0 REFERENCES

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Appendix C—Derivation of Simulation Tool Dynamics

Dynamic simulation of a robotic system requires derivation of the system equations of motion. The derivation can be both difficult and time-consuming. By constraining the system being considered to a specific class of systems, a general set of equations can be derived. The equations of motion for a given system within the class can then be obtained by specifying the values of the coefficients of the general set of equations.

This appendix describes the derivation of the equations of motion for the class of manipulators consisting of rigid links interconnected by rotary joints. In addition, a technique is outlined for reducing the system of equations to eliminate constraint torques. The derivation of the equations and the technique used to reduce the system were both initially described in a paper by W. W. Hooker in 1974 (Ref 1).

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To carry out the derivation of the equations of motion, the notation and terminology to be used must be presented.

The links of the manipulator will be referred to as bodies. The base of the manipulator will be designated body 0, and other bodies will be numbered consecutively from the base. Let P_0 be a fixed reference point on body 0. Let P_i be the point fixed at the center of the joint between body $i-1$ and body i ; P_i can be thought of as the point of contact of body $i-1$ and body i .

The following definitions will be used:

- \underline{T}_i = External torque (not acting through a joint) acting on body i ;
- \underline{T}_i^H = Hinge torque acting on body i through joint i ;
- \underline{F}_i = External force (not hinge force) acting on body i ;
- \underline{F}_i^H = Hinge force acting on body i through joint i ;
- m_i = Mass of body i ;
- m = Total system mass ($m = \sum_{i=1}^n m_i$);
- \underline{R}_{ij} = Vector from P_i to the center of gravity (cg) of body i ;
- \underline{h}_{ij} = $\begin{cases} i \neq j - \text{vector from } P_i \text{ to } P_j, \\ i = j - \text{vector from } P_i \text{ to cg of body } i (= \underline{R}_{ij}); \end{cases}$
- Φ_i = Inertia matrix of body i about its cg;
- Φ_{jk}^i = $\Phi_i - m_i \tilde{R}_{ji} \tilde{R}_{ki}$;
- \underline{w}_i = Angular velocity of body i with respect to an inertial system;
- $\underline{\Omega}_i$ = Relative angular velocity of body i with respect to body $i-1$;
- \underline{H}_i = $\Phi_i \underline{w}_i$ - the angular momentum of body i ;
- $\underline{\psi}_i$ = $\begin{cases} 0 \text{ if } i = 0, \\ \underline{w}_{i-1} \times \underline{\Omega}_i + \underline{\psi}_{i-1}, \text{ otherwise;} \end{cases}$
- \underline{v}_i = $\underline{w}_i \times (\underline{w}_i \times \underline{h}_{ii}) + \underline{\psi}_i \times \underline{h}_{ii} + \sum_{n=0}^{i-1} \underline{w}_n \times (\underline{w}_n \times \underline{h}_{n,n+1}) + \underline{\psi}_n \times \underline{h}_{n,n+1}$;
- \underline{p}_0 = Vector from inertial origin to P_0 ;
- \underline{Y}_i = Vector from P_0 to cg of body i .

The use of a tilde "~" will denote the 3x3 skew-symmetric matrix of a vector cross product. That is,

$$\text{if } \underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \text{ and } \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \text{ then}$$

$$\underline{\tilde{a}} \underline{b} = \underline{a} \times \underline{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The definitions above correspond in most cases to those in Hooker (1974).

Figures C2-1 and C2-2 illustrate some of the definitions.

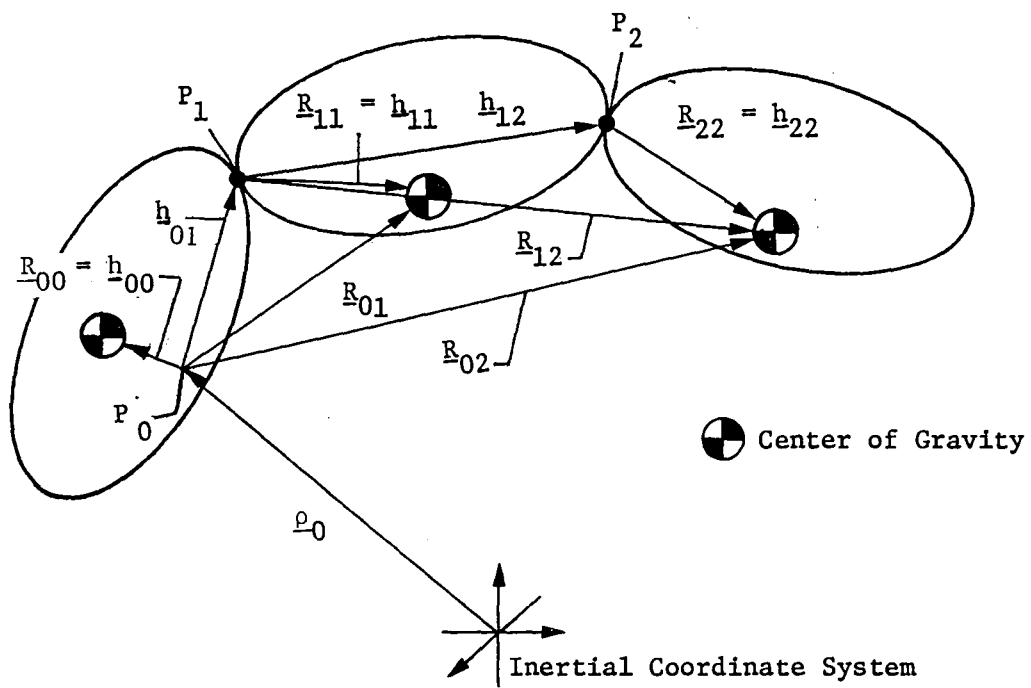


Figure C2-1 Definition of Terms - Position Vectors

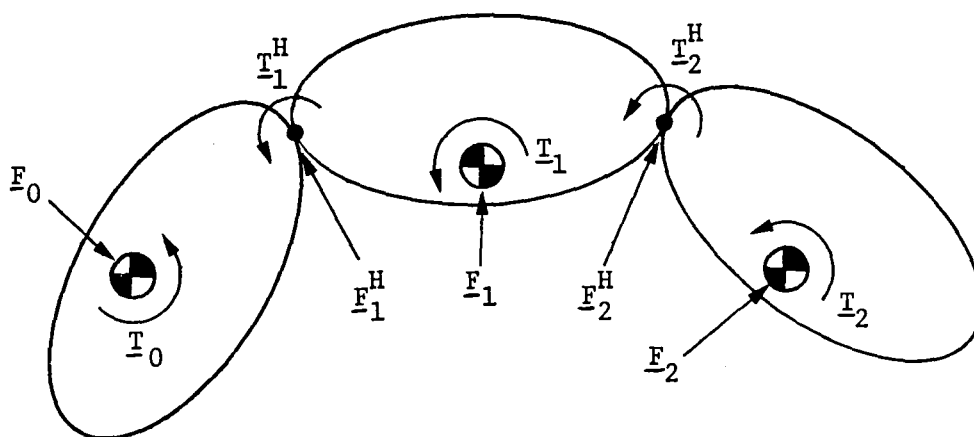


Figure C2-2 Definition of Terms - Forces and Torques

3.0 DERIVATION OF SYSTEM (JOINT 0) EQUATION

3.1 ROTATION ABOUT P_0

We will start by summing the external torques acting on each of the bodies and equating this to the sum of moments about P_0 of each body. The equation is particularly easy to obtain if each body is considered separately.

3.1.1 Body 0 about P_0

The external torques acting on Body 0 about P_0 are: \underline{T}_0 , the external torque; and \underline{F}_0 , the moment about P_0 created by the external force.

The moment about P_0 of body 0 has three components:

- 1) $m_0 \tilde{R}_{00} \ddot{\underline{p}}_0$ - the moment of m_0 times the acceleration of P_0 with respect to the inertial system;
- 2) $m_0 \tilde{R}_{00} \ddot{\underline{Y}}_0$ - the moment of m_0 times the acceleration of the cg of body 0 with respect to P_0 ;
- 3) $\frac{d}{dt}(\phi_0 \underline{w}_0)$ - the change in angular momentum of body 0

Therefore we have

$$[C-1] \quad m_0 \tilde{R}_{00} \ddot{\underline{p}}_0 + m_0 \tilde{R}_{00} \ddot{\underline{Y}}_0 + \frac{d}{dt}(\phi_0 \underline{w}_0) = \underline{T}_0 + \tilde{R}_{00} \underline{F}_0$$

The equation [C-1] is useless unless it contains the quantities known or desired. That is, we need to rewrite equation [C-1] in terms of link lengths, joint velocities, and accelerations.

By definition, an expression for \underline{Y}_0 can be given as

$$\underline{Y}_0 = \underline{h}_{00}$$

so

$$\dot{\underline{Y}}_0 = \frac{d}{dt}(\underline{h}_{00}) = \underline{w}_0 \times \underline{h}_{00}$$

and

$$\ddot{\underline{Y}}_0 = \frac{d}{dt}(\underline{w}_0 \times \underline{h}_{00}) = \dot{\underline{w}}_0 \times \underline{h}_{00} + \underline{w}_0 \times (\underline{w}_0 \times \underline{h}_{00})$$

because all links are rigid and all joints are constrained to rotate only. Furthermore, an expression for the angular velocity is given by

$$\underline{w}_0 = \underline{\Omega}_0, \text{ hence } \dot{\underline{w}}_0 = \dot{\underline{\Omega}}_0 + \underline{\psi}_0 \quad (\underline{\psi}_0 = 0)$$

That being the case, the acceleration can be written as

$$\ddot{\underline{Y}}_0 = \dot{\underline{\Omega}}_0 \times \underline{h}_{00} + \underline{\psi}_0 \times \underline{h}_{00} + \underline{w}_0 \times (\underline{w}_0 \times \underline{h}_{00})$$

where the last terms may be expressed as \underline{v}_0 , i.e.,

$$\underline{\psi}_0 \times \underline{h}_{00} + \underline{w}_0 \times (\underline{w}_0 \times \underline{h}_{00}) = \underline{v}_0$$

In addition, the first term on the right-hand side of the previous equation is

$$\dot{\underline{\Omega}}_0 \times \underline{h}_{00} = -\tilde{R}_{00} \dot{\underline{\Omega}}_0 \text{ since } \underline{h}_{00} = \underline{R}_{00}$$

Combining the above equivalent expressions produces

$$[C-2] \quad m_0 \tilde{R}_{00} \ddot{\underline{Y}}_0 = -m_0 \tilde{R}_{00} \tilde{R}_{00} \dot{\underline{\Omega}}_0 + m_0 \tilde{R}_{00} \underline{v}_0$$

recognizing that

$$\frac{d}{dt}(\phi_0 \underline{w}_0) = \phi_0 \dot{\underline{w}}_0 + \underline{w}_0 \times (\phi_0 \underline{w}_0)$$

and using $\dot{\underline{w}}_0 = \dot{\underline{\Omega}}_0 + \underline{\psi}_0$, and $\phi_0 \underline{w}_0 = \underline{H}_0$ produces

$$[C-3] \quad \frac{d}{dt}(\phi_0 \underline{w}_0) = \phi_0 \dot{\underline{\Omega}}_0 + \phi_0 \underline{\psi}_0 + \underline{w}_0 \times \underline{H}_0$$

Substituting equations [C-2] and [C-3] into [C-1] yields

$$m_0 \tilde{R}_{00} \ddot{\underline{p}}_0 - m_0 \tilde{R}_{00} \tilde{R}_{00} \dot{\underline{\Omega}}_0 + m_0 \tilde{R}_{00} \underline{v}_0 + \phi_0 \dot{\underline{\Omega}}_0 + \phi_0 \underline{\psi}_0 + \underline{w}_0 \times \underline{H}_0 = \underline{T}_0 + \tilde{R}_{00} \underline{F}_0$$

Combining terms using the identity $\phi_0^0 = \phi_0 - m_0 \tilde{R}_{00} \tilde{R}_{00}$ and moving terms to the right-hand side yields

$$[C-4] \quad m_0 \tilde{R}_{00} \ddot{\underline{p}}_0 + \phi_0^0 \dot{\underline{\Omega}}_0 = \underline{T}_0 + \tilde{R}_{00} (\underline{F}_0 - m_0 \underline{v}_0) - \underline{w}_0 \times \underline{H}_0 - \phi_0 \underline{\psi}_0$$

3.1.2 Body 1 about P_0

The external torques acting on body 1 with respect to P_0 are:

- 1) \underline{T}_1 = External torque on body 1;
- 2) $\tilde{R}_{01} \underline{F}_1$ = Moment about P_0 of the mass m_1 times the acceleration of P_0 with respect to the inertial origin.

The moment contribution due to the mass times acceleration of body 1 about P_0 again can be expressed in three terms:

- 1) $m_1 \tilde{R}_{01} \ddot{p}_0$ = Moment about P_0 of the mass m_1 times the acceleration of P_0 with respect to the inertial origin;
- 2) $m_1 \tilde{R}_{01} \ddot{y}_1$ = Moment about P_0 of m_1 times the acceleration of the cg of body 1 with respect to P_0 ;
- 3) $\frac{d}{dt}(\phi_1 \underline{w}_1)$ = Change in angular momentum of body 1.

Again, as in 3.1.1, the moments and torques are set equal to each other.

$$[C-5] \quad m_1 \tilde{R}_{01} \ddot{p}_0 + m_1 \tilde{R}_{01} \ddot{y}_1 + \frac{d}{dt}(\phi_1 \underline{w}_1) = \underline{T}_1 + \tilde{R}_{01} \underline{F}_1$$

And, as in Subsection 3.1.1, the equation above must be expressed in terms of link lengths and joint velocities.

By definition, the position \underline{Y}_1 is given by

$$\underline{Y}_1 = \underline{h}_{01} + \underline{h}_{11}$$

differentiation yields

$$\dot{\underline{Y}}_1 = \underline{w}_0 \times \underline{h}_{01} + \underline{w}_1 \times \underline{h}_{11}$$

and

$$\ddot{\underline{Y}}_1 = \dot{\underline{w}}_0 \times \underline{h}_{01} + \underline{w}_0 \times (\underline{w}_0 \times \underline{h}_{01}) + \dot{\underline{w}}_1 \times \underline{h}_{11} + \underline{w}_1 \times (\underline{w}_1 \times \underline{h}_{11})$$

Inertial angular velocity is given by

$$\underline{w}_0 = \underline{\Omega}_0$$

$$\underline{w}_1 = \underline{\Omega}_0 + \underline{\Omega}_1$$

hence, angular acceleration is given by

$$\dot{\underline{w}}_0 = \dot{\underline{\Omega}}_0 + \underline{\psi}_0$$

$$\begin{aligned} \dot{\underline{w}}_1 &= \frac{d}{dt} (\underline{\Omega}_0 + \underline{\Omega}_1) \\ &= \dot{\underline{\Omega}}_0 + \dot{\underline{\Omega}}_1 + \underline{\Omega}_0 \times \underline{\Omega}_1 \\ &= \dot{\underline{\Omega}}_0 + \dot{\underline{\Omega}}_1 + \underline{\psi}_1 \end{aligned}$$

Substituting in for $\dot{\underline{w}}_0$ and $\dot{\underline{w}}_1$ and rearranging terms yields

$$\begin{aligned} \ddot{\underline{Y}}_1 &= \dot{\underline{\Omega}}_0 \times \underline{h}_{01} + \underline{\psi}_0 \times \underline{h}_{01} + \dot{\underline{\Omega}}_0 \times \underline{h}_{11} + \dot{\underline{\Omega}}_1 \times \underline{h}_{11} + \underline{\psi}_1 \times \underline{h}_{11} \\ &\quad + \underline{w}_0 \times (\underline{w}_0 \times \underline{h}_{01}) + \underline{w}_1 \times (\underline{w}_1 \times \underline{h}_{11}) \end{aligned}$$

Combining terms containing $\dot{\underline{\Omega}}_0$ yields the expression

$$\begin{aligned} \ddot{\underline{Y}}_1 &= \dot{\underline{\Omega}}_0 \times (\underline{h}_{01} + \underline{h}_{11}) + \dot{\underline{\Omega}}_1 \times \underline{h}_{11} + \underline{w}_0 \times (\underline{w}_0 \times \underline{h}_{01}) + \underline{\psi}_0 \times \underline{h}_{01} \\ &\quad + \underline{w}_1 \times (\underline{w}_1 \times \underline{h}_{11}) + \underline{\psi}_1 \times \underline{h}_{11} \end{aligned}$$

By returning to the definitions, the terms on the right-hand side of this equation can be given as

$$\dot{\underline{\Omega}}_0 \times (\underline{h}_{01} + \underline{h}_{11}) = -\tilde{R}_{01}\dot{\underline{\Omega}}_0; \quad \dot{\underline{\Omega}}_1 \times \underline{h}_{11} = -\tilde{R}_{11}\dot{\underline{\Omega}}_1; \quad \text{and}$$

$$\underline{w}_0 \times (\underline{w}_0 \times \underline{h}_{01}) + \underline{\psi}_0 \times \underline{h}_{01} + \underline{w}_1 \times (\underline{w}_1 \times \underline{h}_{11}) + \underline{\psi}_1 \times \underline{h}_{11} = \underline{v}_1$$

Making the substitutions given above, $m_1\tilde{R}_{01}\ddot{\underline{Y}}_1$ can be written as

$$[C-6] \quad m_1\tilde{R}_{01}\ddot{\underline{Y}}_1 = -m_1\tilde{R}_{01}\tilde{R}_{01}\dot{\underline{\Omega}}_0 - m_1\tilde{R}_{01}\tilde{R}_{11}\dot{\underline{\Omega}}_1 + m_1\tilde{R}_{01}\underline{v}_1$$

Substituting the values of \underline{w}_0 and \underline{w}_1 into $\frac{d}{dt}(\phi_1 \underline{w}_1)$ yields

$$\frac{d}{dt}(\phi_1 \underline{w}_1) = \phi_1 \dot{\underline{w}}_1 + \underline{w}_1 \times \phi_0 \underline{w}_1$$

or

$$[C-7] \quad \frac{d}{dt}(\phi_1 \underline{w}_1) = \phi_1 \dot{\underline{\Omega}}_0 + \phi_1 \dot{\underline{\Omega}}_1 + \phi_1 \underline{\psi}_1 + \underline{w}_1 \times \underline{H}_1$$

Substituting equations [C-6] and [C-7] into equation [C-5] produces

$$m_1 \tilde{R}_{01} \ddot{\underline{p}}_0 - m_1 \tilde{R}_{01} \tilde{R}_{01} \dot{\underline{\Omega}}_0 - m_1 \tilde{R}_{01} \tilde{R}_{11} \dot{\underline{\Omega}}_1 + m_1 \tilde{R}_{01} \underline{v}_1 + \phi_1 \dot{\underline{\Omega}}_0 + \phi_1 \dot{\underline{\Omega}}_1 \\ + \phi_1 \underline{\psi}_1 + \underline{w}_1 \times \underline{H}_1 = \underline{T}_1 + \tilde{R}_{01} \underline{F}_1$$

Use of the identities $\phi_{00}^1 = \phi_1 - m_1 \tilde{R}_{01} \tilde{R}_{01}$ and $\phi_{01}^1 = \phi_1 - m_1 \tilde{R}_{01} \tilde{R}_{11}$ and rearrangement of terms yields:

$$[C-8] \quad m_1 \tilde{R}_{01} \ddot{\underline{p}}_0 + \phi_{00}^1 \dot{\underline{\Omega}}_0 + \phi_{01}^1 \dot{\underline{\Omega}}_1 = \underline{T}_1 + \tilde{R}_{01} (\underline{F}_1 - m_1 \underline{v}_1) - \underline{w}_1 \times \underline{H}_1 - \phi_1 \underline{\psi}_1$$

3.1.3 Body 2 about P_0

The external torques on body 2 with respect to P_0 are:

- 1) \underline{T}_2 = External torque on body 2;
- 2) $\tilde{R}_{02} \underline{F}_2$ = Moment about P_0 of the external force acting on body 2.

The moment contribution due to mass times acceleration of body 2 about P_0 can be expressed in three terms:

- 1) $m_2 \tilde{R}_{02} \ddot{\underline{p}}_0$ = Moment about P_0 of m_2 times the acceleration of P_0 with respect to the inertial origin;
- 2) $m_2 \tilde{R}_{02} \ddot{\underline{y}}_2$ = Moment about P_0 of m_2 times the acceleration of the cg of body 2 with respect to P_0 ;
- 3) $\frac{d}{dt}(\phi_2 \underline{w}_2)$ = Change in angular momentum of body 2.

Equating the sums of external torques to the moments of m_2 about P_0 yields:

$$[C-9] \quad m_2 \tilde{R}_{02} \ddot{\underline{p}}_0 + m_2 \tilde{R}_{02} \ddot{\underline{y}}_2 + \frac{d}{dt}(\phi_2 \underline{w}_2) = \underline{T}_2 + \tilde{R}_{02} \underline{F}_2$$

Once again, equation [C-9] must be rewritten to be useful. The equation must be put in terms of link parameters.

As in the expression for body 1, the position of body 2 is given as

$$\underline{y}_2 = \underline{h}_{01} + \underline{h}_{12} + \underline{h}_{22}$$

Differentiating this equation yields the velocity

$$\dot{\underline{Y}}_2 = \underline{w}_0 \times \underline{h}_{01} + \underline{w}_1 \times \underline{h}_{12} + \underline{w}_2 \times \underline{h}_{22}$$

and the acceleration

$$\begin{aligned} \ddot{\underline{Y}}_2 = & \dot{\underline{w}}_0 \times \underline{h}_{01} + \underline{w}_0 \times (\underline{w}_0 \times \underline{h}_{01}) + \dot{\underline{w}}_1 \times \underline{h}_{12} + \underline{w}_1 \times (\underline{w}_1 \times \underline{h}_{12}) \\ & + \dot{\underline{w}}_2 \times \underline{h}_{22} + \underline{w}_2 \times (\underline{w}_2 \times \underline{h}_{22}) \end{aligned}$$

Using the definitions:

$$\underline{w}_0 = \underline{\Omega}_0,$$

$$\underline{w}_1 = \underline{\Omega}_0 + \underline{\Omega}_1,$$

$$\underline{w}_2 = \underline{\Omega}_0 + \underline{\Omega}_1 + \underline{\Omega}_2,$$

the angular acceleration can be written as:

$$\dot{\underline{w}}_0 = \dot{\underline{\Omega}}_0 + \underline{\psi}_0$$

$$\dot{\underline{w}}_1 = \dot{\underline{\Omega}}_0 + \dot{\underline{\Omega}}_1 + \underline{\psi}_1$$

$$\begin{aligned} \dot{\underline{w}}_2 = & \dot{\underline{\Omega}}_0 + \dot{\underline{\Omega}}_1 + \dot{\underline{\Omega}}_2 + \underline{\psi}_1 + \underline{w}_1 \times \underline{\Omega}_2 \\ = & \dot{\underline{\Omega}}_0 + \dot{\underline{\Omega}}_1 + \dot{\underline{\Omega}}_2 + \underline{\psi}_2 \end{aligned}$$

$$\text{where } \underline{\psi}_1 + \underline{w}_1 \times \underline{\Omega}_2 = \underline{\psi}_2$$

Making the above substitutions into the expression for $\ddot{\underline{Y}}_2$ and rearranging terms yields:

$$\begin{aligned} \ddot{\underline{Y}}_2 = & \dot{\underline{\Omega}}_0 \times (\underline{h}_{01} + \underline{h}_{12} + \underline{h}_{22}) + \dot{\underline{\Omega}}_1 \times (\underline{h}_{12} \times \underline{h}_{22}) + \dot{\underline{\Omega}}_2 \times (\underline{h}_{22}) \\ & + \underline{w}_0 \times (\underline{w}_0 \times \underline{h}_{01}) + \underline{\psi}_0 \times \underline{h}_{01} + \underline{w}_1 \times (\underline{w}_1 \times \underline{h}_{12}) + \underline{\psi}_1 \times \underline{h}_{12} \\ & + \underline{w}_2 \times (\underline{w}_2 \times \underline{h}_{22}) + \underline{\psi}_2 \times \underline{h}_{22} \end{aligned}$$

or,

$$\ddot{\underline{Y}}_2 = -\tilde{R}_{02}\dot{\underline{\Omega}}_0 - \tilde{R}_{12}\dot{\underline{\Omega}}_1 - \tilde{R}_{22}\dot{\underline{\Omega}}_2 + \underline{v}_2$$

Therefore, the second term on the left of equation [C-9] can be written as

$$[C-10] \quad m_2 \ddot{\tilde{R}}_{02} \underline{\underline{v}}_2 = -m_2 \ddot{\tilde{R}}_{02} \ddot{\tilde{R}}_{02} \underline{\underline{\dot{\Omega}}}_0 - m_2 \ddot{\tilde{R}}_{02} \ddot{\tilde{R}}_{12} \underline{\underline{\dot{\Omega}}}_1 - m_2 \ddot{\tilde{R}}_{02} \ddot{\tilde{R}}_{22} \underline{\underline{\dot{\Omega}}}_2 + m_2 \ddot{\tilde{R}}_{02} \underline{\underline{v}}_2$$

The expression $\frac{d}{dt}(\phi_2 \underline{\underline{w}}_2)$ can also be simplified as:

$$\frac{d}{dt}(\phi_2 \underline{\underline{w}}_2) = \phi_2 \underline{\underline{\dot{w}}}_2 + \underline{\underline{w}}_2 \times \phi_2 \underline{\underline{w}}_2$$

Substituting in the expressions $\underline{\underline{\dot{w}}}_2 = \underline{\underline{\dot{\Omega}}}_0 + \underline{\underline{\dot{\Omega}}}_1 + \underline{\underline{\dot{\Omega}}}_2 + \underline{\underline{\psi}}_2$ and $\phi_2 \underline{\underline{w}}_2 = \underline{\underline{H}}_2$ into this equation yields:

$$[C-11] \quad \frac{d}{dt}(\phi_2 \underline{\underline{w}}_2) = \phi_2 \underline{\underline{\dot{\Omega}}}_0 + \phi_2 \underline{\underline{\dot{\Omega}}}_1 + \phi_2 \underline{\underline{\dot{\Omega}}}_2 + \phi_2 \underline{\underline{\psi}}_2 + \underline{\underline{w}}_2 \times \underline{\underline{H}}_2$$

The expressions [C-10] and [C-11] can now be substituted into equation [C-9] to produce

$$\begin{aligned} m_2 \ddot{\tilde{R}}_{02} \underline{\underline{\ddot{p}}}_0 - m_2 \ddot{\tilde{R}}_{02} \ddot{\tilde{R}}_{02} \underline{\underline{\dot{\Omega}}}_0 - m_2 \ddot{\tilde{R}}_{02} \ddot{\tilde{R}}_{12} \underline{\underline{\dot{\Omega}}}_1 - m_2 \ddot{\tilde{R}}_{02} \ddot{\tilde{R}}_{22} \underline{\underline{\dot{\Omega}}}_2 + m_2 \ddot{\tilde{R}}_{02} \underline{\underline{v}}_2 \\ + \phi_2 \underline{\underline{\dot{\Omega}}}_0 + \phi_2 \underline{\underline{\dot{\Omega}}}_1 + \phi_2 \underline{\underline{\dot{\Omega}}}_2 + \phi_2 \underline{\underline{\psi}}_2 + \underline{\underline{w}}_2 \times \underline{\underline{H}}_2 = \underline{\underline{T}}_2 + \tilde{R}_{02} \underline{\underline{F}}_2 \end{aligned}$$

Rearranging terms and using the identity $\phi_{0k}^2 = \phi_2 - m_2 \tilde{R}_{02} \tilde{R}_{k2}$ produces the expression

$$[C-12] \quad m_2 \ddot{\tilde{R}}_{02} \underline{\underline{\ddot{p}}}_0 + \phi_{00}^2 \underline{\underline{\dot{\Omega}}}_0 + \phi_{01}^2 \underline{\underline{\dot{\Omega}}}_1 + \phi_{02}^2 \underline{\underline{\dot{\Omega}}}_2 = \underline{\underline{T}}_2 + \tilde{R}_{02} (\underline{\underline{F}}_2 - m_2 \underline{\underline{v}}_2) - \underline{\underline{w}}_2 \times \underline{\underline{H}}_2 - \phi_2 \underline{\underline{\psi}}_2$$

3.1.4 Combining Equations

By carefully examining equations [C-4], [C-8], and [C-12], a pattern for the equations of torques and moments about P_0 can be identified. The general equation of body i is about P_0 is easily seen to be:

$$[C-13] \quad m_i \ddot{\tilde{R}}_{0i} \underline{\underline{\ddot{p}}}_0 + \phi_{00}^i \underline{\underline{\dot{\Omega}}}_0 + \phi_{01}^i \underline{\underline{\dot{\Omega}}}_1 + \dots + \phi_{0i}^i \underline{\underline{\dot{\Omega}}}_i = \underline{\underline{T}}_i + \tilde{R}_{0i} (\underline{\underline{F}}_i - m_i \underline{\underline{v}}_i) - \underline{\underline{w}}_i \times \underline{\underline{H}}_i - \phi_i \underline{\underline{\psi}}_i$$

For a system of $N+1$ bodies, the equation [C-13] can be summed for all bodies i ($i = 0, 1, 2, \dots, N$) to obtain the equation of motion for all bodies with respect to P_0 . In the interest of simplifying notation, define

$$\phi_{0i}^\Sigma = \sum_{n=i}^N \phi_{0i}^n$$

Then the system equation can be written as:

$$[C-14] \quad \sum_{i=0}^N [m_i \ddot{\tilde{R}}_{0i} \underline{\underline{\ddot{p}}}_0] + \phi_{00}^\Sigma \underline{\underline{\dot{\Omega}}}_0 + \phi_{01}^\Sigma \underline{\underline{\dot{\Omega}}}_1 + \dots + \phi_{0N}^\Sigma \underline{\underline{\dot{\Omega}}}_N = \underline{\underline{T}}_0 + \underline{\underline{T}}_1 + \dots + \underline{\underline{T}}_N + \tilde{R}_{00} (\underline{\underline{F}}_0 - m_0 \underline{\underline{v}}_0) + \dots + \tilde{R}_{0N} (\underline{\underline{F}}_N - m_N \underline{\underline{v}}_N) - \underline{\underline{w}}_0 \times \underline{\underline{H}}_0 - \underline{\underline{w}}_1 \times \underline{\underline{H}}_1 - \dots - \underline{\underline{w}}_N \times \underline{\underline{H}}_N - \phi_0 \underline{\underline{\psi}}_0 - \dots - \phi_N \underline{\underline{\psi}}_N$$

4.0 DERIVATION OF JOINT 1 EQUATION

The next step in deriving the complete set of equations of motion is to work outward from P_0 and formulate an equation for each joint. In formulating the equation of motion for joint 1, the moments and torques are considered only for bodies beyond that joint. The hinge torque \underline{T}_1^H must then be considered as an external torque. One way of looking at this is to say that \underline{T}_1^H represents the angular effect of bodies 0 through 1 to the reaction at P_1 .

4.1 ROTATION ABOUT P_1

4.1.1 Body 1 about P_1

The external torques acting on body 1 about P_1 are:

- 1) \underline{T}_1^H
- 2) \underline{T}_1
- 3) $\tilde{R}_{11}\underline{F}_1$ = Moment of the external force \underline{F}_1 about P_1 .

The moment due to mass times acceleration of body 1 about P_1 can be expressed as three terms:

- 1) $m_1\tilde{R}_{11}\ddot{\underline{p}}_0$ = Moment of $m_1\ddot{\underline{p}}_0$ about P_1 ;
- 2) $m_1\tilde{R}_{11}\ddot{\underline{y}}_1$ = Moment of $m_1\ddot{\underline{y}}_1$ about P_1 ;
- 3) $\frac{d}{dt}(\phi_1\underline{w}_1)$ = Change in angular momentum of body 1.

The moments and torques above can be equated, yielding:

$$[C-15] \quad m_1\tilde{R}_{11}\ddot{\underline{p}}_0 + m_1\tilde{R}_{11}\ddot{\underline{y}}_1 + \frac{d}{dt}(\phi_1\underline{w}_1) = \underline{T}_1^H + \underline{T}_1 + \tilde{R}_{11}\underline{F}_1$$

As was the case in Section 3.0, the equation [C-15] must be expressed in terms of the link properties and relative joint velocities. Fortunately, much of the required algebraic manipulations were performed in the preceding section. Thus, we have

$$\ddot{\underline{y}}_1 = -\tilde{R}_{01}\dot{\underline{\Omega}}_0 - \tilde{R}_{11}\dot{\underline{\Omega}}_1 + \underline{v}_1$$

Substituting this into $m_1\tilde{R}_{11}\ddot{\underline{y}}_1$ yields

$$[C-16] \quad m_1\tilde{R}_{11}\ddot{\underline{y}}_1 = -m_1\tilde{R}_{11}\tilde{R}_{01}\dot{\underline{\Omega}}_0 - m_1\tilde{R}_{11}\tilde{R}_{11}\dot{\underline{\Omega}}_1 + m_1\tilde{R}_{11}\underline{v}_1$$

As in Subsection 3.1.2, $\frac{d}{dt}(\phi_1\underline{w}_1)$ can be rewritten as

$$[C-17] \quad \frac{d}{dt}(\phi_1\underline{w}_1) = \phi_1\dot{\underline{\Omega}}_0 + \phi_1\dot{\underline{\Omega}}_1 + \phi_1\underline{\psi}_1 + \underline{w}_1 \times \underline{H}_1$$

Substitution of equations [C-17] and [C-16] in equation [C-15] produces

$$m_1 \ddot{\underline{p}}_0 - m_1 \ddot{\underline{R}}_{11} \ddot{\underline{R}}_{01} \dot{\underline{\Omega}}_0 - m_1 \ddot{\underline{R}}_{11} \ddot{\underline{R}}_{11} \dot{\underline{\Omega}}_1 + m_1 \ddot{\underline{R}}_{11} \underline{v}_1 + \phi_1 \dot{\underline{\Omega}}_0 + \phi_1 \dot{\underline{\Omega}}_1 \\ + \phi_1 \underline{\psi}_1 + \underline{w}_1 \times \underline{H}_1 = \underline{T}_1^H + \underline{T}_1 + \ddot{\underline{R}}_{11} \underline{F}_1$$

If terms are rearranged and the identity $\phi_{kj}^1 = \phi_1 - m_1 \ddot{\underline{R}}_{k1} \ddot{\underline{R}}_{j1}$ is used, the result is

$$[C-18] \quad m_1 \ddot{\underline{R}}_{11} \ddot{\underline{p}}_0 + \phi_{10}^1 \dot{\underline{\Omega}}_0 + \phi_{11}^1 \dot{\underline{\Omega}}_1 = \underline{T}_1^H + \underline{T}_1 + \ddot{\underline{R}}_{11} (\underline{F}_1 - m_1 \underline{v}_1) - \underline{w}_1 \times \underline{H}_1 - \phi_1 \underline{\psi}_1$$

4.1.2 Body 2 about P₁

The external torques acting on body 2 with respect to P₁ are:

- 1) \underline{T}_2
- 2) $\ddot{\underline{R}}_{12} \underline{F}_2$

The moments of body 2 about P₁ are:

- 1) $m_2 \ddot{\underline{R}}_{12} \ddot{\underline{p}}_0$
- 2) $m_2 \ddot{\underline{R}}_{12} \ddot{\underline{Y}}_2$
- 3) $\frac{d}{dt} (\phi_2 \underline{w}_2)$

Equating the external torques to the moment due to the mass times acceleration yields

$$[C-19] \quad m_2 \ddot{\underline{R}}_{12} \ddot{\underline{p}}_0 + m_2 \ddot{\underline{R}}_{12} \ddot{\underline{Y}}_2 + \frac{d}{dt} (\phi_2 \underline{w}_2) = \underline{T}_2 + \ddot{\underline{R}}_{12} \underline{F}_2$$

Note that \underline{T}_2^H is not being considered. This is because if we included it here, then a $-\underline{T}_2^H$ would have to be included in the body 1 equation. The two torque values would then cancel when the equations were combined.

As before, equation [C-19] must be rearranged. From Section 3.0, we have

$$\ddot{\underline{Y}}_2 = -\ddot{\underline{R}}_{02} \dot{\underline{\Omega}}_0 - \ddot{\underline{R}}_{12} \dot{\underline{\Omega}}_1 - \ddot{\underline{R}}_{22} \dot{\underline{\Omega}}_2 + \underline{v}_2$$

Substitution of this equation into $m_2 \ddot{\underline{R}}_{12} \ddot{\underline{Y}}_2$ yields:

$$[C-20] \quad m_2 \ddot{\underline{R}}_{12} \ddot{\underline{Y}}_2 = -m_2 \ddot{\underline{R}}_{12} \ddot{\underline{R}}_{02} \dot{\underline{\Omega}}_0 - m_2 \ddot{\underline{R}}_{12} \ddot{\underline{R}}_{12} \dot{\underline{\Omega}}_1 - m_2 \ddot{\underline{R}}_{12} \ddot{\underline{R}}_{22} \dot{\underline{\Omega}}_2 + m_2 \ddot{\underline{R}}_{12} \underline{v}_2$$

Furthermore, from Section 3.0,

$$[C-21] \quad \frac{d}{dt}(\phi_2 \underline{w}_2) = \phi_2 \dot{\underline{\Omega}}_0 + \phi_2 \dot{\underline{\Omega}}_1 + \phi_2 \dot{\underline{\Omega}}_2 + \phi_2 \underline{\psi}_2 + \underline{w}_2 \times \underline{H}_2$$

[C-21] can be substituted with equation [C-20] into equation [C-19] giving

$$\begin{aligned} m_2 \ddot{\underline{R}}_{12} \underline{\ddot{\rho}}_0 - m_2 \ddot{\underline{R}}_{12} \ddot{\underline{R}}_{02} \dot{\underline{\Omega}}_0 - m_2 \ddot{\underline{R}}_{12} \ddot{\underline{R}}_{12} \dot{\underline{\Omega}}_1 - m_2 \ddot{\underline{R}}_{12} \ddot{\underline{R}}_{22} \dot{\underline{\Omega}}_2 + m_2 \ddot{\underline{R}}_{12} \underline{v}_2 \\ + \phi_2 \dot{\underline{\Omega}}_0 + \phi_2 \dot{\underline{\Omega}}_1 + \phi_2 \dot{\underline{\Omega}}_2 + \phi_2 \underline{\psi}_2 + \underline{w}_2 \times \underline{H}_2 = \underline{T}_2 + \ddot{\underline{R}}_{12} \underline{F}_2 \end{aligned}$$

simplifies the equation to:

$$\begin{aligned} [C-22] \quad m_2 \ddot{\underline{R}}_{12} \underline{\ddot{\rho}}_0 + \phi_{10}^2 \dot{\underline{\Omega}}_0 + \phi_{11}^2 \dot{\underline{\Omega}}_1 + \phi_{12}^2 \dot{\underline{\Omega}}_2 = \underline{T}_2 + \ddot{\underline{R}}_{12} (\underline{F}_2 - m_2 \underline{v}_2) \\ - \underline{w}_2 \times \underline{H}_2 - \phi_2 \underline{\psi}_2 \end{aligned}$$

4.1.3 Combining Equations

By examining the pattern of equations in Section 3.0 and by closely examining equations [C-18] and [C-22], we can see a pattern. The equation of motion for body i about P₁ can be easily written down:

$$\begin{aligned} [C-23] \quad m_1 \ddot{\underline{R}}_{1i} \underline{\ddot{\rho}}_0 + \phi_{10}^i \dot{\underline{\Omega}}_0 + \phi_{11}^i \dot{\underline{\Omega}}_1 + \dots + \phi_{1i}^i \dot{\underline{\Omega}}_i = \underline{T}_i + \ddot{\underline{R}}_{1i} (\underline{F}_i - m_i \underline{v}_i) \\ - \underline{w}_2 \times \underline{H}_2 - \phi_2 \underline{\psi}_2 \end{aligned}$$

The equations of motion about P₁ for all bodies beyond P₁ (that is, for i = 1, 2, ..., N) can now be summed to obtain the joint 1 equation. To simplify notation, let

$$\phi_{1i}^\Sigma = \sum_{n=k}^N \phi_{1i}^n ; k = \max(1, i)$$

Then the joint 1 equation can be written as:

$$\begin{aligned} [C-24] \quad \sum_{i=1}^N [m_i \ddot{\underline{R}}_{1i} \underline{\ddot{\rho}}_0] + \phi_{10}^\Sigma \dot{\underline{\Omega}}_0 + \phi_{11}^\Sigma \dot{\underline{\Omega}}_1 + \dots + \phi_{1N}^\Sigma \dot{\underline{\Omega}}_N \\ = \underline{T}_1^H + \underline{T}_1 + \underline{T}_2 + \dots + \underline{T}_N \\ + \ddot{\underline{R}}_{11} (\underline{F}_1 - m_1 \underline{v}_1) + \dots + \ddot{\underline{R}}_{1N} (\underline{F}_N - m_N \underline{v}_N) \\ - \underline{w}_1 \times \underline{H}_1 - \dots - \underline{w}_N \times \underline{H}_N \\ - \phi_{11} \underline{\psi}_1 - \dots - \phi_{1N} \underline{\psi}_N \end{aligned}$$

5.0 ADDITIONAL JOINT EQUATIONS

From the experience gained in following Sections 2.0 and 3.0, it is apparent that the equations of motion for a regular system of bodies like a manipulator arm have a pattern associated with them. One need only derive equations step by step until the pattern becomes clear, and then write out the rest by induction. Examination of the complete joint equations [C-14] and [C-24] reveal a distinct pattern in the joint equations. Using the convention

$$\phi_{ij}^{\Sigma} = \sum_{n=k}^N \phi_{ij}^n, \quad k = \max(i, j)$$

the equation of motion about joint i can be written:

$$\begin{aligned} \text{[C-25]} \quad & \sum_{n=1}^N [m_n \tilde{R}_{in} \ddot{\theta}_0] + \phi_{i0}^{\Sigma} \dot{\theta}_0 + \phi_{i1}^{\Sigma} \dot{\theta}_1 + \dots + \phi_{iN}^{\Sigma} \dot{\theta}_N \\ & = \underline{T}_i^H + \underline{T}_i + \underline{T}_{i+1} + \dots + \underline{T}_N \\ & \quad + \tilde{R}_{ii}(\underline{F}_i - m_i \underline{v}_i) + \dots + \tilde{R}_{iN}(\underline{F}_N - m_N \underline{v}_N) \\ & \quad - \underline{w}_i \times \underline{H}_i - \dots - \underline{w}_N \times \underline{H}_N \\ & \quad - \phi_{i-1}^{\Sigma} \dot{\theta}_{i-1} - \dots - \phi_{N-1}^{\Sigma} \dot{\theta}_{N-1} \end{aligned}$$

6.0 THE SYSTEM OF EQUATIONS

The set of equations of motion for each joint can now be expressed together in matrix form:

$$\begin{aligned}
 \text{[C-26]} \quad & \begin{bmatrix} \sum_{n=0}^N m_n \tilde{R}_{0n} & \phi_{00}^\Sigma & \phi_{01}^\Sigma & \phi_{02}^\Sigma & \dots & \phi_{0N}^\Sigma \\ \sum_{n=1}^N m_n \tilde{R}_{1n} & \phi_{10}^\Sigma & \phi_{11}^\Sigma & & & \\ \vdots & \vdots & \vdots & & & \\ \sum_{n=N-1}^N m_n \tilde{R}_{N-1,n} & \phi_{N-1,0}^\Sigma & & & & \\ \sum_{n=N}^N m_n \tilde{R}_{Nn} & \phi_{N,0}^\Sigma & \phi_{N1}^\Sigma & \dots & \phi_{NN}^\Sigma \end{bmatrix} \begin{bmatrix} \ddot{\phi}_0 \\ \ddot{\phi}_1 \\ \vdots \\ \ddot{\phi}_{N-1} \\ \ddot{\phi}_N \end{bmatrix} \\
 & = \begin{bmatrix} 0 \\ T_1^H \\ T_2^H \\ \vdots \\ T_N^H \end{bmatrix} + \begin{bmatrix} \sum_{n=0}^N [\underline{T}_n + \tilde{R}_{0n}(\underline{F}_n - m_n \underline{v}_n) - \underline{w}_n \times \underline{H}_n - \phi_n \psi_n] \\ \vdots \\ \sum_{n=1}^N [\underline{T}_n + \tilde{R}_{1n}(\underline{F}_n - m_n \underline{v}_n) - \underline{w}_n \times \underline{H}_n - \phi_n \psi_n] \\ \vdots \\ \sum_{n=N}^N [\underline{T}_n + \tilde{R}_{Nn}(\underline{F}_n - m_n \underline{v}_n) - \underline{w}_n \times \underline{H}_n - \phi_n \psi_n] \end{bmatrix}
 \end{aligned}$$

The matrix equation [C-26] contains $N+1$ equations in $N+2$ unknowns. Several ways exist to make the system square. An additional equation may be derived, or an unknown may be dropped. The latter alternative is best suited for ROBSIM dynamics. For all the situations to be considered during the initial ROBSIM effort, the manipulators will have fixed bases, so $P_0 = 0$. Hence, the first unknown is unnecessary, and we are left with

$$\text{[C-27]} \quad \begin{bmatrix} \phi \end{bmatrix} \begin{bmatrix} \ddot{\phi}_0 \\ \ddot{\phi}_1 \\ \vdots \\ \ddot{\phi}_N \end{bmatrix} = \begin{bmatrix} 0 \\ T_1^H \\ \vdots \\ T_N^H \end{bmatrix} + \begin{bmatrix} E(w) \end{bmatrix}$$

7.0 SOLVING THE SYSTEM OF EQUATIONS

To simplify notation further, denote the system [C-27] as

$$[C-28] \quad A\dot{\underline{\Omega}} = \underline{T} + \underline{E}$$

Vector quantities were used in deriving this system. Therefore, the angular acceleration, $\dot{\underline{\Omega}}$, is given as $\dot{\underline{\Omega}}^t = [\dot{\underline{\Omega}}_0^t, \dot{\underline{\Omega}}_1^t, \dots, \dot{\underline{\Omega}}_N^t]$ where each of the components of $\dot{\underline{\Omega}}$ is a 3x1 vector. Therefore, $\dot{\underline{\Omega}}$, \underline{T} , and \underline{E} each are composed of $N+1$ quantities, each of which is a three-vector, so each has $3N+3$ scalar components. Thus, A is $3N+3 \times 3N+3$. For $N = 7$, solving the system would require inverting a 24×24 matrix. Such a matrix inversion is acceptable.

We have been tacitly assuming that only the $\dot{\underline{\Omega}}$ are unknown and all the quantities on the right hand side of equation C-28 are known. Consider the vector \underline{T} , and examine the component \underline{T}_i^H of \underline{T} . By definition, \underline{T}_i^H represents the torque on body i acting through the joint i . For single degree of freedom joints, one component of \underline{T}_i^H represents the known actuator torque at joint i . The other two components are unknown constraint torques. In order to solve the system, the unknown constraint torques must be eliminated and the size of the system reduced.

The way to eliminate constraint equations is by projecting the system onto the unconstrained axes. For example, suppose we are solving for the motion of a single link around a joint. Then we have a system of three scalar equations in three unknowns, which can be represented as:

$$[C-29] \quad A\underline{x} = \underline{t} + \underline{e}$$

where A is 3×3 and \underline{x} , \underline{t} , and \underline{e} are 3×1 .

Suppose our single joint has only a single degree of freedom - rotation around some axis. Let \underline{g} denote the free axis of rotation. Then the effects of reaction forces and torques on motion around \underline{g} can be computed by projecting all the effects onto the \underline{g} axis. Since the vector dot product is a projection operator, the projection can be done by taking the dot product of the unit vector along \underline{g} , \underline{g}' , with the equation [C-29]. Writing x_n and t_n to represent the free components of \underline{x} and \underline{t} , the resultant scalar equation can be written:

$$[C-30] \quad (\underline{g}'^t A \underline{g}') x_n = t_n + \underline{g}'^t \underline{e}$$

The equation [C-30] is then integrated to obtain a solution.

To solve equation [C-28], let $\underline{g}_1, \underline{g}_2, \dots, \underline{g}_N$ denote unit vectors along the axis of rotation of each joint. Then, to reduce equation [C-28] to solvable size, the projections of the system onto the \underline{g}_i 's must be obtained. The resultant system is:

$$[C-31] \quad (G^t A G) \dot{\underline{\Omega}}_n = \underline{T}_n + G^t \underline{E}$$

where $G = \begin{bmatrix} \underline{g}_1 & & & \\ & \underline{g}_2 & & \\ & & \ddots & \\ 0 & & & \underline{g}_N \end{bmatrix}^t$, and $\dot{\underline{\Omega}}_n, \underline{T}_n$ are the components

of $\dot{\underline{\Omega}}$ and \underline{T} along the free axis.

8.0 LINEARIZED DYNAMICS OF N-LINK SYSTEM

The governing equations of an N-link manipulator system are given by

$$[C-32] \begin{bmatrix} \phi_{00}^{\Sigma 0} \phi_{01}^{\Sigma 1} & \dots & \phi_{0N}^{\Sigma N} \\ \phi_{01}^{\Sigma 1} \phi_{11}^{\Sigma 1} & \dots & \phi_{1N}^{\Sigma N} \\ \vdots & & \vdots \\ \phi_{N0}^{\Sigma N} \phi_{NN}^{\Sigma N} & & \phi_{NN}^{\Sigma N} \end{bmatrix} \begin{bmatrix} \dot{\Omega}_0 \\ \dot{\Omega}_1 \\ \vdots \\ \dot{\Omega}_N \end{bmatrix} = \begin{bmatrix} T_0^H \\ T_1^H \\ \vdots \\ T_N^H \end{bmatrix} + \begin{bmatrix} \sum_{n=0}^N [T_n + \tilde{R}_{on} (F_n - m_n \underline{v}_n) - \underline{w}_n \times \underline{H}_n - \phi_n \underline{\psi}_n] \\ \sum_{n=1}^N [T_n + \tilde{R}_{on} (F_n - m_n \underline{v}_n) - \underline{w}_n \times \underline{H}_n - \phi_n \underline{\psi}_n] \\ \sum_{n=N}^N [T_n + \tilde{R}_{on} (F_n - m_n \underline{v}_n) - \underline{w}_n \times \underline{H}_n - \phi_n \underline{\psi}_n] \end{bmatrix}$$

where the specific parameters have been defined in Section 2.0.
Equation [C-32] can be written in matrix form as

$$[C-33] M(\phi_{ij}) \ddot{\underline{x}} = \underline{T} + \underline{g}(\underline{p}, \underline{f}, \underline{v}, \underline{w}, \underline{\psi})$$

where

$$M = \begin{bmatrix} \phi_{00}^{\Sigma 0} & \dots & \phi_{0N}^{\Sigma N} \\ \vdots & & \vdots \\ \phi_{N0}^{\Sigma N} & \dots & \phi_{NN}^{\Sigma N} \end{bmatrix}; \quad \underline{T} = \begin{bmatrix} T_0^H \\ \vdots \\ T_N^H \end{bmatrix} + \begin{bmatrix} \sum_{n=0}^N T_n \\ \vdots \\ \sum_{n=0}^N T_n \end{bmatrix};$$

$$\ddot{\underline{x}}' = [\dot{\Omega}_0 \quad \dot{\Omega}_1 \quad \dots \quad \dot{\Omega}_N]$$

and the vector, \underline{g} , is : a function of the position vector, \underline{p} ; force vector, \underline{f} ; a vector, \underline{v} , which represents the moment due to centripetal-type force; the absolute angular velocity, \underline{w} , and term, $\underline{\psi}$, which corresponds to cross-coupling effects.

Equation [C-33] can be linearized by letting

$$\begin{aligned}
 \underline{x} &= \underline{x}^* + \delta \underline{x} \\
 \underline{p} &= \underline{p}^* + \delta \underline{p} \\
 \underline{v} &= \underline{v}^* + \delta \underline{v} \\
 [C-34] \quad \underline{w} &= \underline{w}^* + \delta \underline{w} \\
 \underline{\psi} &= \underline{\psi}^* + \delta \underline{\psi} \\
 \underline{M} &= \underline{M}^* + \delta \underline{M} \\
 \underline{T} &= \underline{T}^* + \delta \underline{T}
 \end{aligned}$$

where the * terms correspond to nominal values of the respective terms and the $\delta()$ are perturbations about the nominal. Using equation [C-34], Equation [C-33] is given by

$$[C-35] \quad (\underline{M}^* + \delta \underline{M})(\ddot{\underline{x}}^* + \delta \ddot{\underline{x}}) = \underline{T}^* + \delta \underline{T} + \underline{g}^* + \delta \underline{g}$$

Manipulating Equation [C-35] and keeping only the linear terms yields

$$[C-36] \quad \underline{M}^* \delta \ddot{\underline{x}} + \ddot{\underline{M}} \underline{x}^* = \delta \underline{T} + \delta \underline{g}$$

This equation, in conjunction with Equation [C-34], represents a linear system equivalent to Equation [C-32]. Further clarification of the terms given in Equation [C-36] will now be given.

Consider the second term on the left-hand side of Equation [C-36], i.e.,

$$[C-37] \quad \delta \ddot{\underline{M}} \underline{x}^* = \delta \begin{bmatrix} \phi_{00}^{\Sigma 0} & \cdot & \cdot & \cdot & \phi_{0n}^{\Sigma N} \\ \vdots & & & & \\ \phi_{n0}^{\Sigma N} & \cdot & \cdot & \cdot & \phi_{nn}^{\Sigma N} \end{bmatrix} \begin{bmatrix} \dot{\underline{\Omega}}_0 \\ \vdots \\ \dot{\underline{\Omega}}_N \end{bmatrix}$$

It can be shown that the subinertia matrices are functions of the position vector, \underline{p} , i.e.,

$$\phi_{ij} = f(\underline{p})$$

where

$$\underline{p}' = [\underline{R}_{00} \ \underline{R}_{01} \ \cdot \cdot \cdot \ \underline{R}_{0N}] = [\underline{p}_1 \ \underline{p}_2 \ \cdot \cdot \cdot \ \underline{p}_N]$$

Examination of Figure C-3 further clarifies the definition of these position vectors.

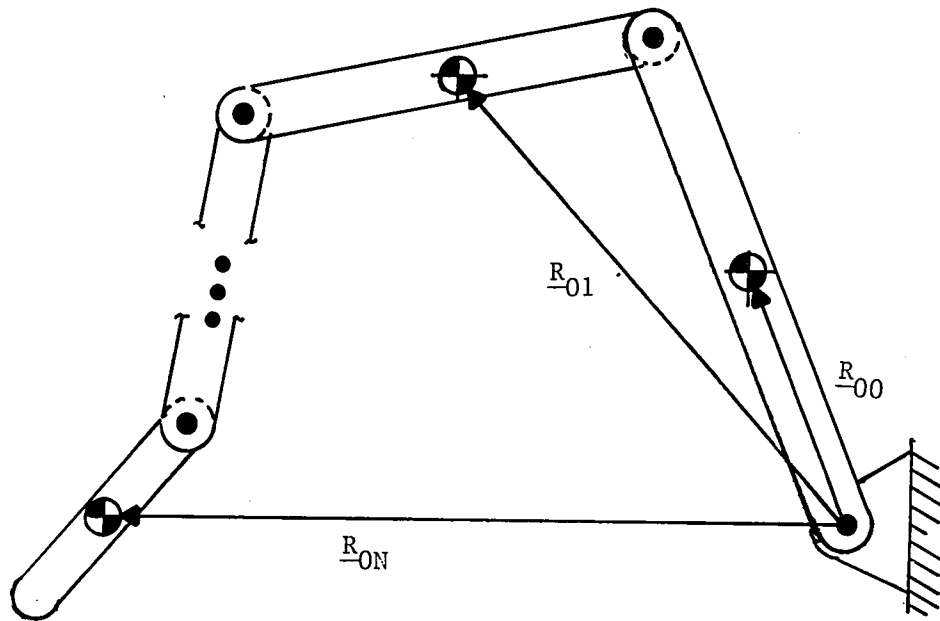


Figure C-3 Position Vector, \underline{p}

At this time, it will be convenient to change notation. That is, let the inertia matrix be given by 3x3 submatrices,

$$[C-38] \dot{M} = \delta \begin{bmatrix} M_{11} & M_{12} & \cdot & \cdot & \cdot & M_{1n} \\ M_{21} & & & & & \cdot \\ \vdots & & & & & \cdot \\ M_{n1} & \cdot & \cdot & \cdot & & M_{nn} \end{bmatrix},$$

where

$$[C-39] \ M_{ij} \triangleq \phi_{ij}^{\Sigma i} \quad i = j = 0, 1, \dots, N$$

Substituting Equation [C-38] into Equation [C-36] yields

$$[C-40] \delta \ddot{\mathbf{M}} \ddot{\mathbf{x}}^* = \begin{bmatrix} \delta M_{11} \ddot{x}_1 + \delta M_{12} \ddot{x}_2 + \dots + \delta M_{1n} \ddot{x}_n \\ \vdots \\ \delta M_{n1} \ddot{x}_1 + \delta M_{n2} \ddot{x}_2 + \dots + \delta M_{nn} \ddot{x}_n \end{bmatrix},$$

and examination of the variation of M_{ij} reveals that

$$[C-41] \delta M_{ij} \ddot{x}_j^* = \delta \begin{bmatrix} \alpha_{11}(p) & \alpha_{12}(p) & \alpha_{13}(p) \\ \alpha_{21}(p) & \alpha_{22}(p) & \alpha_{23}(p) \\ \alpha_{31}(p) & \alpha_{32}(p) & \alpha_{33}(p) \end{bmatrix}_{ij} \ddot{x}_j^*$$

where use has been made of the fact that the subinertia matrices are functions of the position vector, \underline{p} .

Substitution of Equation [C-41] into [C-40] yields equation [C-42]. This equation may be manipulated into [C-43], where the subscript, \underline{p} , refers to the fact that the partial derivative of each element is taken with respect to the position vector, \underline{p} . For example, the notation, α_{ij} , means the partial of the ij component is taken with respect to \underline{p} . The result of this operation is a $1 \times 3n$ row vector that when multiplied by the $3n \times 1$ change in position vector, $\delta \underline{p}$, produces a 1×1 quantity. Also, it should be recognized that the definition of \underline{x} was substituted in its place--see previous definition of \underline{x} .

Equation [C-43] is further expanded to illustrate that [C-40] can be written as a product of a $3n \times 3n$ matrix, $\underline{\Gamma}$, and the $3n \times 1$ position vector, \underline{p} , which produces a $3n \times 1$ vector. Specifically, this is given by

$$\delta \ddot{\mathbf{M}} \ddot{\mathbf{x}} = \underline{\Gamma} \delta \underline{p}.$$

To ease the coding of the previous equation, it is convenient to further decompose this equation. This can be done by partitioning the $3n \times 3n$ matrix $\underline{\Gamma}$ and the perturbed position vector, $\delta \underline{p}$, into its 3×3 submatrices and 3×1 position vectors, $\underline{p}_1, \underline{p}_2, \dots, \underline{p}_n$, respectively.

$\delta \underline{M \ddot{x}} =$

$$\begin{bmatrix}
 \delta \begin{bmatrix} \alpha_{11}(p) & \alpha_{12}(p) & \alpha_{13}(p) \\ \alpha_{21}(p) & \alpha_{22}(p) & \alpha_{23}(p) \\ \alpha_{31}(p) & \alpha_{32}(p) & \alpha_{33}(p) \end{bmatrix}_{11} \underline{\ddot{x}}_1^* & \dots & \delta \begin{bmatrix} \alpha_{11}(p) & \alpha_{12}(p) & \alpha_{13}(p) \\ \alpha_{21}(p) & \alpha_{22}(p) & \alpha_{23}(p) \\ \alpha_{31}(p) & \alpha_{32}(p) & \alpha_{33}(p) \end{bmatrix}_{1n} \underline{\ddot{x}}_n^* \\
 \vdots & & \vdots \\
 \delta \begin{bmatrix} \alpha_{11}(p) & \alpha_{12}(p) & \alpha_{13}(p) \\ \alpha_{21}(p) & \alpha_{22}(p) & \alpha_{23}(p) \\ \alpha_{31}(p) & \alpha_{32}(p) & \alpha_{33}(p) \end{bmatrix}_{n1} \underline{\ddot{x}}_1^* & \dots & \delta \begin{bmatrix} \alpha_{11}(p) & \alpha_{12}(p) & \alpha_{13}(p) \\ \alpha_{21}(p) & \alpha_{22}(p) & \alpha_{23}(p) \\ \alpha_{31}(p) & \alpha_{32}(p) & \alpha_{33}(p) \end{bmatrix}_{nn} \underline{\ddot{x}}_n^*
 \end{bmatrix}$$

[C-42]

$\delta \underline{Mx} =$

$$\left[\begin{array}{c} \left\{ \begin{array}{l} \left[\dot{\Omega}_{0x}^{\alpha}{}_{11p} + \dot{\Omega}_{0y}^{\alpha}{}_{12p} + \dot{\Omega}_{0z}^{\alpha}{}_{13p} \right] \\ \left[\dot{\Omega}_{0x}^{\alpha}{}_{21p} + \dot{\Omega}_{0y}^{\alpha}{}_{22p} + \dot{\Omega}_{0z}^{\alpha}{}_{23p} \right] \\ \left[\dot{\Omega}_{0x}^{\alpha}{}_{31p} + \dot{\Omega}_{0y}^{\alpha}{}_{32p} + \dot{\Omega}_{0z}^{\alpha}{}_{33p} \right] \end{array} \right\}_{11} \delta p \\ \vdots \\ \left\{ \begin{array}{l} \left[\dot{\Omega}_{0x}^{\alpha}{}_{11p} + \dot{\Omega}_{0y}^{\alpha}{}_{12p} + \dot{\Omega}_{0z}^{\alpha}{}_{13p} \right] \\ \left[\dot{\Omega}_{0x}^{\alpha}{}_{21p} + \dot{\Omega}_{0y}^{\alpha}{}_{22p} + \dot{\Omega}_{0z}^{\alpha}{}_{23p} \right] \\ \left[\dot{\Omega}_{0x}^{\alpha}{}_{31p} + \dot{\Omega}_{0y}^{\alpha}{}_{32p} + \dot{\Omega}_{0z}^{\alpha}{}_{33p} \right] \end{array} \right\}_{n1} \delta p \end{array} \right\} + \dots + \left\{ \begin{array}{l} \left[\dot{\Omega}_{NX}^{\alpha}{}_{11p} + \dot{\Omega}_{NY}^{\alpha}{}_{12p} + \dot{\Omega}_{NZ}^{\alpha}{}_{13p} \right] \\ \left[\dot{\Omega}_{NX}^{\alpha}{}_{21p} + \dot{\Omega}_{NY}^{\alpha}{}_{22p} + \dot{\Omega}_{NZ}^{\alpha}{}_{23p} \right] \\ \left[\dot{\Omega}_{NX}^{\alpha}{}_{31p} + \dot{\Omega}_{NY}^{\alpha}{}_{32p} + \dot{\Omega}_{NZ}^{\alpha}{}_{33p} \right] \end{array} \right\}_{1n} \delta p \\ \vdots \\ \left\{ \begin{array}{l} \left[\dot{\Omega}_{NX}^{\alpha}{}_{11p} + \dot{\Omega}_{NY}^{\alpha}{}_{12p} + \dot{\Omega}_{NZ}^{\alpha}{}_{13p} \right] \\ \left[\dot{\Omega}_{NX}^{\alpha}{}_{21p} + \dot{\Omega}_{NY}^{\alpha}{}_{22p} + \dot{\Omega}_{NZ}^{\alpha}{}_{23p} \right] \\ \left[\dot{\Omega}_{NX}^{\alpha}{}_{31p} + \dot{\Omega}_{NY}^{\alpha}{}_{32p} + \dot{\Omega}_{NZ}^{\alpha}{}_{33p} \right] \end{array} \right\}_{nm} \delta p \end{array} \right]$$

[C-43]

[C-44]

$$\delta \underline{\underline{MX}} = \begin{bmatrix} \dot{\Omega}_{OX}^{\alpha}{}^{11} + \dot{\Omega}_{1X}^{\alpha}{}^{12} + \dots + \dot{\Omega}_{NX}^{\alpha}{}^{1n} \\ + \dot{\Omega}_{OY}^{\alpha}{}^{12p} + \dot{\Omega}_{1Y}^{\alpha}{}^{12p} + \dots + \dot{\Omega}_{NY}^{\alpha}{}^{12p} \\ \vdots \\ \dot{\Omega}_{OX}^{\alpha}{}^{n1} + \dot{\Omega}_{1X}^{\alpha}{}^{n2} + \dots + \dot{\Omega}_{NX}^{\alpha}{}^{nn} \\ + \dot{\Omega}_{OY}^{\alpha}{}^{n1p} + \dot{\Omega}_{1Y}^{\alpha}{}^{n2p} + \dots + \dot{\Omega}_{NY}^{\alpha}{}^{nnp} \end{bmatrix} \delta \underline{p}$$

Performing this partitioning will produce the following result,

$$[C-45] \delta \underline{M_x}^{**} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \cdot & \cdot & \cdot & \Gamma_{1n} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \Gamma_{n1} & & & & & \Gamma_{nn} \end{bmatrix} \begin{bmatrix} \delta p_1 \\ \delta p_2 \\ \cdot \\ \cdot \\ \delta p_n \end{bmatrix}$$

where the 3x3 submatrices, Γ , are given by

$$[C-46] \Gamma_{kj} = \sum_{i=1}^n \begin{bmatrix} \alpha_{11x} & \alpha_{12y} & \alpha_{13z} \\ \alpha_{21x} & \alpha_{22y} & \alpha_{23z} \\ \alpha_{31x} & \alpha_{32y} & \alpha_{33z} \end{bmatrix}_{kj}^{P_j} \begin{bmatrix} \ddot{\Omega}_{(i-1)x}^* & 0 & 0 \\ 0 & \ddot{\Omega}_{(i-1)y}^* & 0 \\ 0 & 0 & \ddot{\Omega}_{(i-1)z}^* \end{bmatrix} \\ + \begin{bmatrix} \alpha_{11y} & \alpha_{12z} & \alpha_{13x} \\ \alpha_{21y} & \alpha_{22z} & \alpha_{23x} \\ \alpha_{31y} & \alpha_{32z} & \alpha_{33x} \end{bmatrix}_{ij}^{P_j} \begin{bmatrix} 0 & \ddot{\Omega}_{(i-1)x}^* & 0 \\ 0 & 0 & \ddot{\Omega}_{(i-1)y}^* \\ \ddot{\Omega}_{(i-1)z}^* & 0 & 0 \end{bmatrix} \\ + \begin{bmatrix} \alpha_{11z} & \alpha_{12x} & \alpha_{13y} \\ \alpha_{21z} & \alpha_{22x} & \alpha_{23y} \\ \alpha_{31z} & \alpha_{32x} & \alpha_{33y} \end{bmatrix}_{ij}^{P_j} \begin{bmatrix} 0 & 0 & \ddot{\Omega}_{(i-1)x}^* \\ \ddot{\Omega}_{(i-1)y}^* & 0 & 0 \\ 0 & \ddot{\Omega}_{(i-1)z}^* & 0 \end{bmatrix}$$

where the α matrices correspond to partial derivatives with respect to the x, y, and z components of the position vectors, \underline{p}_j . Equation [C-47] represents the desired form of the second term on the left-hand side of [C-36]. Because the first term of Equation [C-36] is self-explanatory, attention will not be turned toward the right-hand side of [C-36].

The change in torque, given by $\delta \underline{T}$, is the torque to be supplied by the motors which control each manipulator joint. Variations in the vector, \underline{g} , will now be examined.

Since the vector is given to be a function of \underline{p} , \underline{f} , \underline{v} , \underline{w} , and $\underline{\Psi}$, its variation is given by

$$[C-48] \quad \delta \underline{g} = \frac{\partial}{\partial \underline{p}} \delta \underline{p} + \frac{\partial}{\partial \underline{v}} \delta \underline{v} + \frac{\partial}{\partial \underline{w}} \delta \underline{w} + \frac{\partial}{\partial \underline{\Psi}} \delta \underline{\Psi}$$

where it is assumed that \underline{f} is a constant over the interval of interest and therefore does not effect the variation of \underline{g} .

Referring back to the specific definition of \underline{g} from Equation [C-32], it was shown that

$$[C-49] \quad \underline{g} \triangleq \begin{bmatrix} g_{11} \triangleq \sum_{n=0}^n \tilde{R}_{0n} (\underline{F}_n - \underline{m}_{n-n} \underline{v}_n) - \underline{w}_n \times \underline{H}_n - \Phi_{n-n} \underline{\Psi}_n \\ g_{21} \triangleq \sum_{n=1}^n \tilde{R}_{0n} (\underline{F}_n - \underline{m}_{n-n} \underline{v}_n) - \underline{w}_n \times \underline{H}_n - \Phi_{n-n} \underline{\Psi}_n \\ \vdots \\ g_{n1} \triangleq \sum_{n=n}^n \tilde{R}_{0n} (\underline{F}_n - \underline{m}_{n-n} \underline{v}_n) - \underline{w}_n \times \underline{H}_n - \Phi_{n-n} \underline{\Psi}_n \end{bmatrix}$$

Using Equation [C-49], the variation in \underline{g} can be written as

$$[C-50] \quad \delta \underline{g} \triangleq \begin{bmatrix} \delta g_{11} \triangleq \sum_{n=0}^n \delta \left[\tilde{R}_{0n} (\underline{F}_n - \underline{m}_{n-n} \underline{v}_n) - \underline{w}_n \times \underline{H}_n - \Phi_{n-n} \underline{\Psi}_n \right] \\ \vdots \\ \delta g_{n1} \triangleq \sum_{n=n}^n \delta \left[\tilde{R}_{0n} (\underline{F}_n - \underline{m}_{n-n} \underline{v}_n) - \underline{w}_n \times \underline{H}_n - \Phi_{n-n} \underline{\Psi}_n \right] \end{bmatrix}$$

and because the terms inside the summation are identical, $\delta \underline{g}$ may be obtained from examination of the variation in g_{11} , i.e.,

$$[C-51] \quad \delta g_{11} \triangleq A + B + C$$

where

$$A \triangleq \delta \left[\tilde{R}_{0n} (F_n - m_n v_n) \right]$$

$$B \triangleq \delta \left[\underline{w}_n \times \underline{H}_n \right]$$

$$C \triangleq \delta \left[\Phi_{n-n} \Psi \right]$$

Derivation of the A, B, and C contributions to the variation will now be discussed. Consider the expansion of A, i.e.,

$$[C-52] \quad A = \delta \tilde{R}_{0n} (F_n - m_n v_n)^* - m_n \tilde{R}_{0n} \delta v_n$$

or

$$[C-53] \quad A = \delta \tilde{R}_{0n} F_n^* - m_n \delta \tilde{R}_{0n-n} v_n^* - m_n \tilde{R}_{0n}^* \delta v_n$$

It can be shown that

$$[C-54] \quad \delta \tilde{R}_{0n} F_n^* = \left\{ \begin{bmatrix} \beta_{n1x} \\ \beta_{n1y} \\ \beta_{n1z} \end{bmatrix} \begin{bmatrix} f_{nx} & 0 & 0 \\ 0 & f_{ny} & 0 \\ 0 & 0 & f_{nz} \end{bmatrix} \right. \\ + \begin{bmatrix} \beta_{n1y} \\ \beta_{n2z} \\ \beta_{n3x} \end{bmatrix} \begin{bmatrix} 0 & f_{nx} & 0 \\ 0 & 0 & f_{ny} \\ f_{nz} & 0 & 0 \end{bmatrix} \\ \left. + \begin{bmatrix} \beta_{n1z} \\ \beta_{n2x} \\ \beta_{n3y} \end{bmatrix} \begin{bmatrix} 0 & 0 & f_{nx} \\ f_{ny} & 0 & 0 \\ 0 & f_{nz} & 0 \end{bmatrix} \right\} \left[\delta_{pn} \right]$$

where the 3x1 column vectors β_{n1} , β_{n2} , and β_{n3} are defined using,

$$\tilde{R}_{0n} \triangleq \begin{bmatrix} r_{11}^{0n} & r_{12}^{0n} & r_{13}^{0n} \\ r_{21}^{0n} & r_{22}^{0n} & r_{23}^{0n} \\ r_{31}^{0n} & r_{32}^{0n} & r_{33}^{0n} \end{bmatrix}$$

Specifically, the β 's are given by

$$\beta_{n1} \triangleq (r_{11}^{0n} \ r_{21}^{0n} \ r_{31}^{0n})$$

$$\beta_{n2} \triangleq (r_{12}^{0n} \ r_{22}^{0n} \ r_{32}^{0n})$$

$$\beta_{n3} \triangleq (r_{13}^{0n} \ r_{23}^{0n} \ r_{33}^{0n})$$

and

$$\underline{\beta}_{n1s} = \left(\frac{\partial r_{11}^{0n}}{\partial s} \quad \frac{\partial r_{21}^{0n}}{\partial s} \quad \frac{\partial r_{31}^{0n}}{\partial s} \right); s \triangleq x, y, z.$$

In other words, the $\underline{\beta}$'s given in [C-54] correspond to the partial derivatives of the column vectors of the 3x3 matrix \tilde{R}_{0n} with respect to the x, y, and z components of the position vector, \underline{p}_n .

Similarly, the second term of Equation [C-53] is given by

$$\begin{aligned} \text{[C-55]} \quad m_n \tilde{R}_{0n} \underline{v}_n^* &= m_n \left[\begin{array}{c|c|c} \beta_{n1x} & \beta_{n2y} & \beta_{n3z} \end{array} \right] \begin{bmatrix} v_{nx}^* & 0 & 0 \\ 0 & v_{ny}^* & 0 \\ 0 & 0 & v_{nz}^* \end{bmatrix} \\ &+ \left[\begin{array}{c|c|c} \beta_{n1y} & \beta_{n2z} & \beta_{n3x} \end{array} \right] \begin{bmatrix} 0 & v_{nx}^* & 0 \\ 0 & 0 & v_{ny}^* \\ v_{nz}^* & 0 & 0 \end{bmatrix} \\ &+ \left[\begin{array}{c|c|c} \beta_{n1z} & \beta_{n2x} & \beta_{n3y} \end{array} \right] \begin{bmatrix} 0 & 0 & v_{nx}^* \\ v_{ny}^* & 0 & 0 \\ 0 & v_{nz}^* & 0 \end{bmatrix} \left[\delta \underline{p}_n \right] \end{aligned}$$

Continuing on with the derivation of A, Equation [C-53], the last term will now come under consideration. Using the definition of \underline{v}_j from Section 2.0, this term can be written as

$$\begin{aligned} \text{[C-56]} \quad m_n \tilde{R}_{0n} \delta \underline{v}_n &= m_n \tilde{R}_{0n} \delta \left[\sum_{j=0}^{n-1} \left[\underline{w}_j \times (\underline{w}_j \times \underline{h}_{j,j+1}) + \underline{\psi}_j \times \underline{h}_{j,j+1} \right] \right. \\ &\quad \left. + \underline{w}_n \times (\underline{w}_n \times \underline{h}_{nn}) + \underline{\psi}_n \times \underline{h}_{nn} \right] \end{aligned}$$

Propagating the variational operator through the summation yields

$$\begin{aligned}
[C-57] \quad m_n \tilde{R}_{0n}^* \delta v_n = m_n \tilde{R}_{0n} & \left[\sum_{j=0}^{n-1} \left[\delta \underline{w}_j \times (\underline{w}_j \times \underline{h}_{j,j+1}) \right. \right. \\
& + \underline{w}_j \times (\delta \underline{w}_j \times \underline{h}_{j,j+1}) + \underline{w}_j \times (\underline{w}_j \times \delta \underline{h}_{j,j+1}) \\
& + \delta \underline{\psi}_j \times \underline{h}_{j,j+1} + \underline{\psi}_j \times \delta \underline{h}_{j,j+1} \left. \right] \\
& + \delta \underline{w}_n \times (\underline{w}_n \times \underline{h}_{nn}) + \underline{w}_n \times (\delta \underline{w}_n \times \underline{h}_{nn}) \\
& + \underline{w}_n \times (\underline{w}_n \times \delta \underline{h}_{nn}) + \delta \underline{\psi}_n \times \underline{h}_{nn} + \underline{\psi}_n \times \delta \underline{h}_{nn} \left. \right].
\end{aligned}$$

Examination of this expression indicates that by considering the term in the summation, the treatment of terms outside the summation will follow the same manipulative treatment. Therefore, consider the term internal to the summation, i.e., let

$$[C-58] \quad m_n \tilde{R}_{0n} \left[\sum_{j=0}^{n-1} \right] = \underline{a}_n + \underline{b}_n + \underline{c}_n + \underline{d}_n + \underline{e}_n$$

where

$$\underline{a}_n \triangleq \sum_j \delta \underline{w}_j \times \underline{h}_{j,j+1}$$

$$\underline{b}_n \triangleq \sum_j \underline{w}_j \times (\delta \underline{w}_j \times \underline{h}_{j,j+1})$$

$$\underline{c}_n \triangleq \sum_j \underline{w}_j \times (\underline{w}_j \times \delta \underline{h}_{j,j+1})$$

$$\underline{d}_n \triangleq \sum_j \delta \underline{\psi}_j \times \underline{h}_{j,j+1}$$

$$\underline{e}_n \triangleq \sum_j \underline{\psi}_j \times \delta \underline{h}_{j,j+1}$$

The first two expressions, \underline{a} and \underline{b} , can be written in terms of the 3x1 vector, η_j , and the variation in \underline{w}_j as follows:

$$[C-59] \quad \underline{a}_n = \sum_{j=0}^{n-1} \begin{bmatrix} 0 & \eta_{31} & -\eta_{21} \\ -\eta_{31} & 0 & \eta_{11} \\ \eta_{21} & -\eta_{11} & 0 \end{bmatrix}_j \begin{bmatrix} \delta w_{jx} \\ \delta w_{jy} \\ \delta w_{jz} \end{bmatrix}$$

where

$$\eta_{11}^j \triangleq (w_{jy} h_{j,j+1,z})^* - (w_{jz} h_{j,j+1,y})^*$$

$$\eta_{21}^j \triangleq (w_{jz} h_{j,j+1,x})^* - (w_{jx} h_{j,j+1,z})^*$$

$$\eta_{31}^j \triangleq (w_{jx} h_{j,j+1,y})^* - (w_{jy} h_{j,j+1,x})^*$$

Similarly \underline{b} is given by the product of two matrices and a column vector, i.e.,

$$[C-60] \quad \underline{b} = \sum_{j=0}^{n-1} \begin{bmatrix} 0 & w_{jz} & -w_{jy} \\ -w_{jz} & 0 & w_{jx} \\ w_{jy} & -w_{jx} & 0 \end{bmatrix}^* \begin{bmatrix} 0 & h_{j,j+1,z} & -h_{j,j+1,y} \\ -h_{j,j+1,z} & 0 & h_{j,j+1,x} \\ -h_{j,j+1,y} & -h_{j,j+1,x} & 0 \end{bmatrix}^* \begin{bmatrix} \delta w_{jx} \\ \delta w_{jy} \\ \delta w_{jz} \end{bmatrix}$$

To clarify the propagation of the \underline{c} term, consideration of a set of coordinate frames and definition of the $\underline{h}_{j,j+1}$ term become a necessity. Therefore, consider Figure C8-1 to help clarify the definition of $\underline{h}_{j,j+1}$.

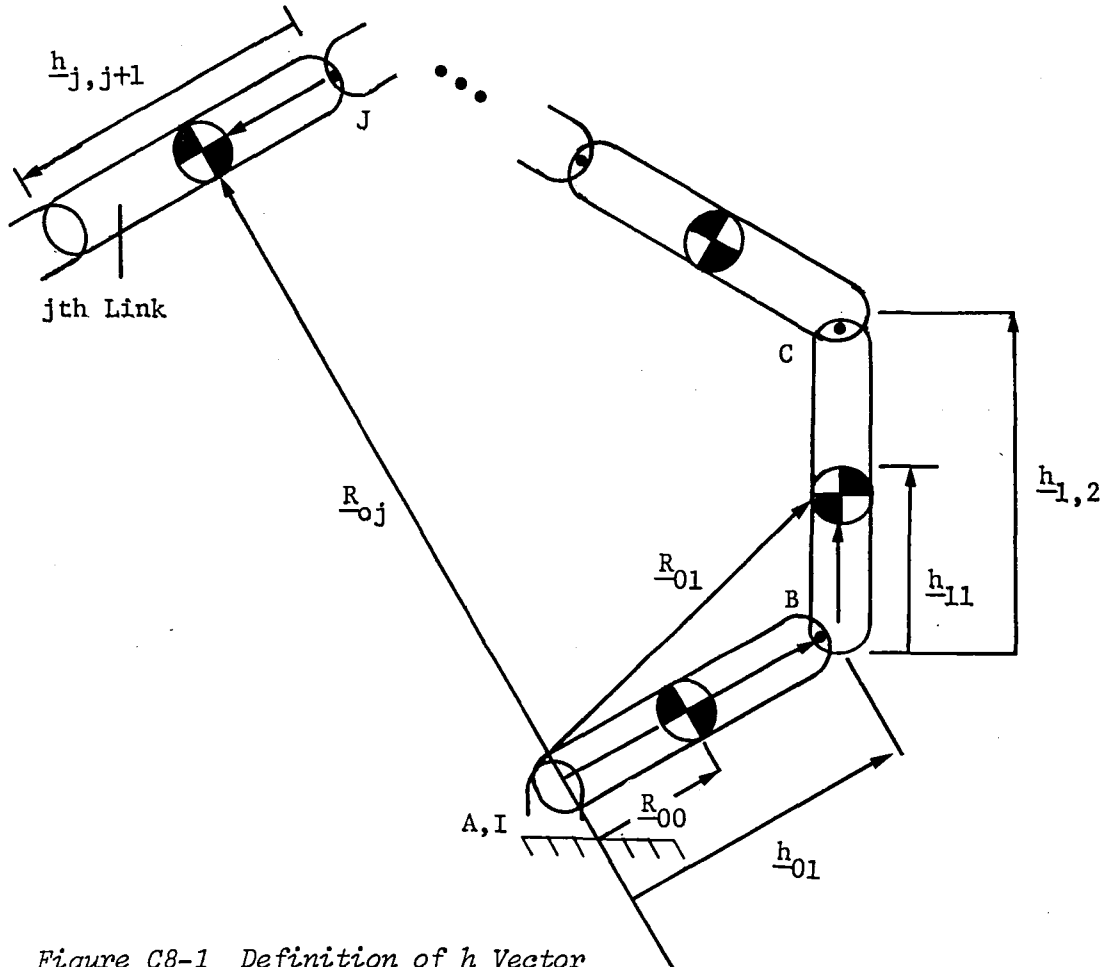


Figure C8-1 Definition of \underline{h} Vector

This figure defines $\underline{h}_{j,j+1}$ as the vector from the origin of the j th link to the origin of the $j+1$ link. Because we have inferred the definition of coordinate frames in the definition, these frames will now be discussed.

Because the equations of motion [C-32] are written with respect to an inertial coordinate frame, this is a natural starting point. Let an inertial coordinate frame be defined by an origin at I; axes x , y , and z ; and unit vectors \vec{i} , \vec{j} , and \vec{k} .

Consider body frames attached to the respective links at the joints. Referring to Figure C8-1, let the origin of the first link be defined as A; axes as x_1 , y_1 , and z_1 , and unit vectors, \vec{e}_1 , \vec{e}_2 , \vec{e}_3 . The coordinate frame of link 2 is defined by an origin at B; its axes as x_2 , y_2 , and z_2 ; unit vectors as \vec{e}_4 , \vec{e}_5 , and \vec{e}_6 . Similarly, the coordinate frame of the j th link is defined by its origin at J; its axes as x_j , y_j and z_j ; unit vectors as \vec{e}_j , \vec{e}_{j+1} and \vec{e}_{j+2} . These coordinate frames are summarized in Table C-1.

Frame	Origin	Axes	
Inertial	I	X Y Z	$\vec{i} \ \vec{j} \ \vec{k}$
Link 1	A	$x_1 \ y_1 \ z_1$	$\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3$
Link 2	B	$y_2 \ y_2 \ y_2$	$\vec{e}_4 \ \vec{e}_5 \ \vec{e}_6$
.	.	.	.
.	.	.	.
.	.	.	.
Link j	J	$x_j \ y_j \ z_j$	$\vec{e}_j \ \vec{e}_{j+1} \ \vec{e}_{j+2}$

Table C-1 Definition of Coordinate Frames

Returning to the discussion of the \underline{c} term of Equation [C-58], it is recognized that the variation of $\underline{h}_{j,j+1}$ must be expressed in the inertial reference frame. However, since this term is conveniently defined in the j th link frame of reference, a transformation from the j th reference frame to the inertial frame of reference is required. This can be expressed as follows:

$$[C-61] \quad \underline{h}_{j,j+1} = T_{IJ} \underline{h}_{j,j+1}_J$$

where the T_{IJ} transformation matrix takes components of \underline{h} expressed in j th reference frame and expressed them in the inertial frame.

To further clarify the notion of the transformation matrix, consider $\underline{h}_{0,1}$ as shown in Figure C8-2. Using this figure, $\underline{h}_{0,1}$ can be expressed in the link 1 reference frame as

$$[C-62] \quad \left. \vec{h}_{01} \right|_A = \left| \vec{h}_{01} \right| \vec{e}_1,$$

and in the inertial reference frame as

$$[C-63] \quad \left. \vec{h}_a \right|_I = T_{IA} \left. \vec{h}_{01} \right|_A$$

$$\text{where} \quad T_{IA} \triangleq \begin{bmatrix} \Psi_0 \\ \Theta_0 \\ \Phi_0 \end{bmatrix}.$$

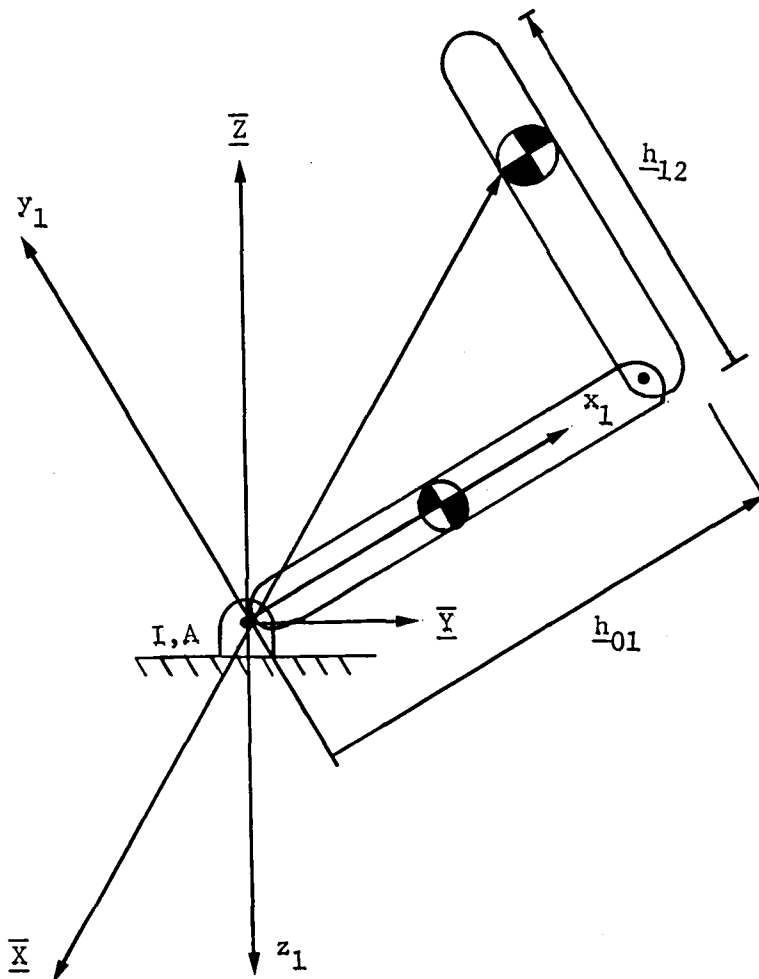


Figure C8-2 Two-Link System

The matrices Ψ_0 , Θ_0 , and Φ_0 correspond to yaw, pitch, and roll of link 1 with respect to the inertial frame. That being the case, these matrices are further defined as:

$$[C-64] \quad \Psi_0 \triangleq \begin{bmatrix} \cos \Psi_0 & -\sin \Psi_0 & 0 \\ \sin \Psi_0 & \cos \Psi_0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \theta_0 \triangleq \begin{bmatrix} \cos \theta_0 & 0 & \sin \theta_0 \\ 0 & 1 & 0 \\ -\sin \theta_0 & 0 & \cos \theta_0 \end{bmatrix}$$

$$\Phi_0 \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi_0 & -\sin \phi_0 \\ 0 & \sin \phi_0 & \cos \phi_0 \end{bmatrix}$$

In a similar fashion, $h_{1,2}$ can be expressed in the inertial frame as

$$\vec{h}_{12} \Big|_I = T_{IB} \vec{h} \Big|_B$$

where

$$T_{IB} = T_{IA} T_{AB}$$

and

$$[C-65] \quad T_{AB} \triangleq \begin{bmatrix} \Psi_1 \end{bmatrix} \begin{bmatrix} \theta_1 \end{bmatrix} \begin{bmatrix} \phi_1 \end{bmatrix}$$

where the matrices Ψ_1 , θ_1 , and ϕ_1 are defined as in Equation [C-64] with exception that these quantities correspond to yaw, pitch, and roll of the second link with respect to the first link.

In a similar fashion $h_{j,j+1}$ can be given as

$$[C-66] \quad \underline{h}_{j,j+1} \Big|_I = T_{IJ} \underline{h}_{j,j+1} \Big|_J$$

where

$$T_{IJ} \triangleq T_{IA} T_{AB} \cdots T_{J-1,J}$$

$$[C-67] \quad T_{IJ} = \prod_{i=0}^{n-1} \begin{bmatrix} \Psi_i \end{bmatrix} \begin{bmatrix} \theta_i \end{bmatrix} \begin{bmatrix} \phi_i \end{bmatrix}$$

Using Equations [C-66] and [C-67], the variation of

\underline{h}_{ij+1} is given by

$$\begin{aligned}
[C-68] \quad \delta \underline{h}_{j,j+1} = & \prod_{i=0}^{n-1} \left\{ \begin{bmatrix} \delta \Psi_i \\ \Theta_i \\ \Phi_i \end{bmatrix} \right. \\
& + \begin{bmatrix} \Psi_i \\ \delta \Theta_i \\ \Phi_i \end{bmatrix} \\
& \left. + \begin{bmatrix} \Psi_i \\ \Theta_i \\ \delta \Phi_i \end{bmatrix} \right\} \underline{h}_{j,j+1} \Big|_J
\end{aligned}$$

Then using Equation [C-66], equation [C-68] can be expressed as

$$\begin{aligned}
[C-69] \quad \delta \underline{h}_{j,j+1} \Big|_I = & \prod_{i=0}^{n-1} \left\{ \begin{bmatrix} t_{1\Phi_i} & t_{2\Theta_i} & t_{3\Psi_i} \end{bmatrix} \begin{bmatrix} h_{j,j+1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right. \\
& + \begin{bmatrix} t_{1\Theta_i} & t_{2\Psi_i} & t_{3\Phi_i} \end{bmatrix} \begin{bmatrix} 0 & h_{j,j+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
& \left. + \begin{bmatrix} t_{1\Psi_i} & t_{2\Phi_i} & t_{3\Theta_i} \end{bmatrix} \begin{bmatrix} 0 & 0 & h_{j,j+1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} \delta \Phi_i \\ \delta \Theta_i \\ \delta \Psi_i \end{bmatrix} \Big|_J
\end{aligned}$$

where \underline{t}_1 , \underline{t}_2 , and \underline{t}_3 are defined to be the column vectors that makeup the \underline{T}_{IJ} transformation matrix. Taking the partial of these vectors with respect to yaw, pitch, and roll is then used to form Equation [C-69].

The expression for \underline{c} from Equation [C-58] can now be written as

$$[C-70] \quad \underline{c} = \sum_{j=0}^{n-1} w_j \delta \underline{h}_{j,j+1} \Big|_I$$

where

$$[C-71] \quad \underline{w}_j = (\underline{w}_j^*) (\underline{w}_j^*)^T$$

and

$$[C-72] \quad \begin{bmatrix} \underline{w}^* \end{bmatrix} \triangleq \begin{bmatrix} 0 & -\underline{w}_{jz} & \underline{w}_{jy} \\ \underline{w}_{jz} & 0 & -\underline{w}_{jx} \\ -\underline{w}_{jy} & \underline{w}_{jx} & 0 \end{bmatrix}^*$$

(Note that 1 denotes transpose and * corresponds to nominal values.)

The next step in the derivation is to expand the d term of Equation [C-58]. Recall that

$$\underline{d} \triangleq \sum_{j=0}^{n-1} \delta \underline{\psi}_j \times \underline{h}_{j,j+1}$$

and since

$$\underline{\psi}_j = \underline{w}_{j-1} \times \underline{\Omega}_j$$

then

$$[C-73] \quad \underline{d} = \sum_{j=1}^{n-1} (\delta \underline{w}_{j-1} \times \underline{\Omega}_j \times \underline{h}_{j,j+1} + \underline{w}_{j-1} \times \delta \underline{\Omega}_j \times \underline{h}_{j,j+1})$$

It should be noted the Equation [C-73] is of the same form as a and b of Equation [C-58]. Since that is the case,

$$[C-74] \underline{d} = \sum_{j=0}^{n-1} \begin{bmatrix} 0 & \epsilon_{31} & -\epsilon_{21} \\ -\epsilon_{31} & 0 & \epsilon_{11} \\ \epsilon_{21} & -\epsilon_{11} & 0 \end{bmatrix}_J \begin{bmatrix} \delta_w(j-1)x \\ \delta_w(j-1)y \\ \delta_w(j-1)z \end{bmatrix} +$$

$$\begin{bmatrix} 0 & w(j-1)z & -w(j-1)y \\ -w(j-1)z & 0 & w(j-1)x \\ w(j-1)y & -w(j-1)x & 0 \end{bmatrix}^* \begin{bmatrix} 0 & h_{j,j+1,z} & -h_{j,j+1,y} \\ -h_{j,j+1,z} & 0 & h_{j,j+1,x} \\ -h_{j,j+1,y} & -h_{j,j+1,x} & 0 \end{bmatrix}^* \begin{bmatrix} \delta\Omega_{jx} \\ \delta\Omega_{jy} \\ \delta\Omega_{jz} \end{bmatrix}$$

where

$$\epsilon_{11}^j \triangleq (\Omega_{jy} h_{j,j+1,z})^* - (\Omega_{jz} h_{j,j+1,y})^*$$

$$\epsilon_{21}^j \triangleq (\Omega_{jz} h_{j,j+1,x})^* - (\Omega_{jx} h_{j,j+1,z})^*$$

$$\epsilon_{31}^j \triangleq (\Omega_{jx} h_{j,j+1,y})^* - (\Omega_{jy} h_{j,j+1,x})^*$$

From the previous discussion the term

$$\underline{e} \triangleq \sum_j \Psi_j \times \underline{h}_{j,j+1}$$

can be written as

$$[C-75] \underline{e} \triangleq \sum_{j=1}^P Q_j \underline{h}_{j,j+1}$$

where

$$[C-76] \quad P_{j-1} \triangleq \begin{bmatrix} 0 & -w_{j-1,z} & w_{j-1,y} \\ w_{j-1,z} & 0 & -w_{j-1,x} \\ -w_{j-1,y} & w_{j-1,x} & 0 \end{bmatrix}$$

and

$$[C-77] \quad Q_j \triangleq \begin{bmatrix} 0 & \Omega_{jz} & -\Omega_{jy} \\ -\Omega_{jz} & 0 & \Omega_{jx} \\ \Omega_{jy} & -\Omega_{jx} & 0 \end{bmatrix}$$

The term $m_n \tilde{R}_{0n} \delta v_n$ can now be summarized as

$$[C-78] \quad m_n \tilde{R}_{0n} \delta \frac{v}{n} = m_n \tilde{R}_{0n} \left[\underline{a} + \underline{b} + \underline{c} + \underline{d} + e \right] \\ + m_n \tilde{R}_{0n} \left[\frac{\delta w}{-n} \times \left(\frac{w}{-n} \times \frac{h}{-nn} \right) \right. \\ + \frac{w}{-n} \times \left(\frac{\delta w}{-n} \times \frac{h}{-nn} \right) + \frac{w}{-n} \times \left(\frac{w}{-n} \times \frac{\delta h}{-nn} \right) \\ \left. + \frac{\delta \Psi}{-n} \times \frac{h}{-nn} + \frac{\Psi}{-n} \times \frac{\delta h}{-nn} \right]$$

Recognizing that the terms in the last set of brackets can be represented similarly to those in this first set, Equation [C-78] can be written as:

$$[C-79] \quad m_n \tilde{R}_{0n} \delta \underline{v}_n = m_n \tilde{R}_{0n} \left\{ \sum_{j=0}^{n-1} \left\{ \left[M_{12}(j,j+1) + M_{14}(j,j+1) \right] \delta \underline{w}_j \right. \right. \\ + \left. \left[M_{15}(j) + M_{18}(j-1,j) \right] \delta h_{j,j+1} \Big|_I \right. \\ + \left. \left[M_{16}(j,j+1) \delta \underline{w}_{j-1} + M_{17}(j-1,j,j+1) \right] \delta \underline{\Omega}_j \right\} \\ + \left[M_{19}(n,n+1) + M_{1,10}(n,n+1) + M_{1,12}(n) \right] \delta \underline{w}_n \\ + \left. M_{1,11}(n) \delta h_{nn} + M_{1,13} \delta \underline{\Omega}_n \right\}$$

where

$$a \triangleq \sum M_{13} \delta \underline{w}_j$$

$$b \triangleq \sum M_{14} \delta \underline{w}_j$$

$$c \triangleq \sum M_{15} \delta h_{j,j+1} \Big|_I$$

$$d \triangleq \sum M_{16} \delta \underline{w}_{j-1} + M_{17} \delta \underline{\Omega}_j$$

$$e \triangleq \sum M_{18} \delta h_{j,j+1} \Big|_I$$

and

$$\delta \underline{w}_n \times (\underline{w}_n \times \underline{h}_{nn}) = M_{19} \delta \underline{w}_n$$

$$\underline{w}_n \times (\delta \underline{w}_n \times \underline{h}_{nn}) = M_{1,10} \delta \underline{w}_n$$

$$\underline{w}_n \times (\underline{w}_n \times \delta h_{nn}) = M_{1,11} \delta h_{nn}$$

$$\delta \underline{\Psi}_n \times \underline{h}_{nn} = M_{1,12} \delta \underline{w}_n + M_{1,13} \delta \underline{\Omega}_n$$

$$\underline{\Psi}_n \times \delta h_{nn} = M_{1,14} \delta h_{nn}$$

The expression for B (see Equation [C-51]) can be obtained by noting that

$$B \triangleq - \delta(\underline{w}_n \times \underline{H}_n)$$

$$= - (\delta \underline{w}_n \times \underline{H}_n + \underline{w}_n \times \delta \underline{H}_n)$$

Using suitable definitions from Section 2.0, this can be written as

$$[C-80] \quad B \triangleq M_{1,15}(n) \delta \underline{w}_n$$

where

$$M_{1,15} = - \left\{ \begin{bmatrix} 0 & v_{31} & -v_{21} \\ -v_{31} & 0 & v_{11} \\ v_{21} & -v_{11} & 0 \end{bmatrix}_n + \begin{bmatrix} 0 & w_{nz} & -w_{ny} \\ -w_{nz} & 0 & w_{nx} \\ w_{ny} & -w_{nx} & 0 \end{bmatrix} \right\}$$

Similarly, C can be written as

$$C \triangleq - \delta \begin{bmatrix} \phi_n & \underline{\psi}_n \end{bmatrix}$$

$$= - \phi_n \delta \underline{\psi}_n$$

which can be expressed as

$$[C-81] \quad C = M_{1,16}(n) \delta \underline{w}_{n-1} + M_{1,17}(n-1) \delta \underline{\Omega}_n$$

where

$$M_{1,16} \triangleq - \phi_n \begin{bmatrix} 0 & \Omega_{nz} & -\Omega_{ny} \\ -\Omega_{nz} & 0 & \Omega_{nx} \\ \Omega_{ny} & -\Omega_{nx} & 0 \end{bmatrix}$$

$$M_{1,17} \triangleq - \phi_n \begin{bmatrix} 0 & \underline{w}_{n-1z} & \underline{w}_{n-1y} \\ -\underline{w}_{n-1z} & 0 & \underline{w}_{n-1x} \\ -\underline{w}_{n-1y} & -\underline{w}_{n-1x} & 0 \end{bmatrix}$$

In summary, the linearization of the right-hand side of Equation [C-36] can be characterized by the linearization of the δg_{11} term given in Equation [C-50]. Combining equations, the appropriate equations for the variation in g_{11} can be expressed as

$$[C-82] \delta g_{11} = \Sigma \left\{ (M_{11} - M_{12}) \delta p_n - \bar{A} \right. \\ \left. + -m_n \tilde{R}_{0n} (M_{19} + M_{1,10} + M_{1,12}) - M_{1,15} \delta w_n \right. \\ \left. + M_{11,11} \delta h_{nn} + [M_{1,13} - M_{1,17}] \delta \Omega_n + M_{1,16} \delta w_{n-1} \right\}$$

where

$$\underline{\underline{A}} = m_n \tilde{R}_{0n} \left[\begin{array}{c} n-1 \\ \Sigma \\ j=0 \end{array} \left\{ [M_{13}(j,j+1) + M_{14}(j,j+1)] \delta w_j \right. \right. \\ \left. + [M_{15}(j) + M_{18}(j-1,j)] \delta h_{j,j+1} \right. \\ \left. + M_{16}(j,j+1) \delta w_{j-1} + M_{17}(j-1,j,j+1) \delta \Omega_j \right\} \left. \right]$$

Using this procedure it can be shown that the N-link case can be written in the general form as a linear $3n \times 3n$ matrix equation

$$[C-83] \underline{M} \ddot{\underline{\theta}} + \underline{C} \dot{\underline{\theta}} + \underline{K} \underline{\theta} = \underline{T} + \underline{D} \underline{p}$$

where

\underline{M} = $3n \times 3n$ Inertia Matrix
 \underline{C} = $3n \times 3n$ Damping Type Matrix
 \underline{K} = $3n \times 3n$ Stiffness Type Matrix
 \underline{T} = Applied Joint Torques
 \underline{D} = $3n \times 3n$ Position Modifier
 \underline{p} = $3n \times 1$ Position Vector

The clarification of Equation [C-83] is easier to obtain using the two link case. This case is presented in the next section.

9.0 LINEARIZATION OF THE TWO-LINK CASE

The equations formulated for the N-link case will now be applied to the two-link configuration. For this case the equations of motion are given to be:

$$\begin{aligned}
 \text{[C-82]} \quad (\phi_{00}^0 + \phi_{00}^1) \dot{\underline{\Omega}}_0 + \phi_{01}^1 \dot{\underline{\Omega}}_1 &= \underline{T}_0^H + \underline{T}_0 + \underline{T}_1 + \tilde{R}_{00} (\underline{F}_0 - m_0 \underline{v}_0) \\
 &\quad + \tilde{R}_{01} (\underline{F}_1 - m_1 \underline{v}_1) - \underline{w}_0 \times \underline{H}_0 \\
 &\quad - \phi_{00}^0 \underline{\Psi} - \underline{w}_1 \times \underline{H}_1 - \phi_{11}^1 \underline{\Psi}
 \end{aligned}$$

$$\text{[C-83]} \quad \phi_{10}^1 \dot{\underline{\Omega}}_0 + \phi_{11}^1 \dot{\underline{\Omega}}_1 = \underline{T}_1^H + \underline{T}_1 + \tilde{R}_{01} (\underline{F}_1 - m_1 \underline{v}_1) - \underline{w}_1 \times \underline{H}_1 - \phi_{11}^1 \underline{\Psi}$$

Using the definition of the inertia dyadics from Section 2.0, these equations can be written as:

$$\begin{aligned}
 \text{[C-84]} \quad (\phi_0 - m_0 \tilde{R}_{00} \tilde{R}_{00} + \phi_1 - m_1 \tilde{R}_{01} \tilde{R}_{01}) \dot{\underline{\Omega}}_0 &+ (\phi_1 - m_1 \tilde{R}_{01} \tilde{R}_{11}) \dot{\underline{\Omega}}_1 \\
 &= \underline{T}_0^H - \underline{T}_0 - \underline{T}_1 + \tilde{R}_{00} (\underline{F}_0 - m_0 \underline{v}_0) + \tilde{R}_{01} (\underline{F}_1 - m_1 \underline{v}_1) \\
 &\quad - \underline{w}_0 \times (\phi_0 \underline{w}_0) - \underline{w}_1 \times (\phi_1 \underline{w}_1) - \phi_1 (\underline{w}_0 \times \underline{\Omega}_1)
 \end{aligned}$$

$$\begin{aligned}
 \text{[C-85]} \quad (\phi_1 - m_1 \tilde{R}_{11} \tilde{R}_{01}) \dot{\underline{\Omega}}_0 &+ (\phi_1 - m_1 \tilde{R}_{11} \tilde{R}_{11}) \dot{\underline{\Omega}}_1 \\
 &= \underline{T}_1^H - \underline{T}_1 - \tilde{R}_{01} (\underline{F}_1 - m_1 \underline{v}_1) - \underline{w}_1 \times (\phi_1 \underline{w}_1) - \phi_1 (\underline{w}_0 \times \underline{\Omega}_1)
 \end{aligned}$$

These equations are now of the form

$$\text{[C-86]} \quad \underline{M}_{\underline{x}}^{\underline{x}} = \underline{T} + \underline{g}$$

where

$$[C-87] \quad M = \begin{bmatrix} \phi_0 - m_0 \tilde{R}_{00} \tilde{R}_{00} & \phi_1 - m_1 \tilde{R}_{01} \tilde{R}_{11} \\ + \phi_1 - m_1 \tilde{R}_{01} \tilde{R}_{01} & \\ \phi_1 - m_1 \tilde{R}_{11} \tilde{R}_{01} & \phi_1 - m_1 \tilde{R}_{11} \tilde{R}_{11} \end{bmatrix}$$

$$[C-88] \quad \underline{T} = \begin{bmatrix} \underline{T}_0^H + \underline{T}_0 + \underline{T}_1 \\ \underline{T}_1^H + \underline{T}_1 \end{bmatrix}$$

$$[C-89] \quad \underline{g} = \begin{bmatrix} \tilde{R}_{00} (\underline{F}_0 - m_0 \underline{v}_0) + \tilde{R}_{01} (\underline{F}_1 - m_1 \underline{v}_1) \\ - \underline{w}_0 \times (\phi_0 \underline{w}_0) - \underline{w}_1 \times (\phi_1 \underline{w}_1) - \phi_1 (\underline{w}_0 \times \underline{\Omega}_1) \\ \tilde{R}_{01} (\underline{F}_1 - m_1 \underline{v}_1) - \underline{w}_1 \times (\phi_1 \underline{w}_1) - \phi_1 (\underline{w}_0 \times \underline{\Omega}_1) \end{bmatrix}$$

and

$$[C-90] \quad \ddot{\underline{x}} = \begin{bmatrix} \dot{\underline{\Omega}}_0 \\ \dot{\underline{\Omega}}_1 \end{bmatrix}$$

Following the discussion for the N-link case [C-86], can be written as

$$[C-91] \quad \delta \underline{M}^* \delta \underline{x} + \delta \underline{M} \underline{x}^* = \delta \underline{T} + \delta \underline{g}$$

where in this case the term involving the perturbed inertia matrix is partitioned into its 3x3 submatrices as

$$[C-92] \quad \delta \underline{M}^* = \begin{bmatrix} \Gamma_{11} & \vdots & \Gamma_{12} \\ \vdots & \ddots & \vdots \\ \Gamma_{21} & \vdots & \Gamma_{22} \end{bmatrix} \delta \underline{p}$$

where

$$[C-93] \quad \delta \underline{p} \triangleq \begin{bmatrix} \underbrace{R_{00x} \ R_{00y} \ R_{00z}}_{\delta \underline{p}_1} \vdots \underbrace{R_{01x} \ R_{01y} \ R_{01z}}_{\delta \underline{p}_2} \end{bmatrix}$$

Using this fact and Equation [C-90] allows the left-hand side of Equation [C-91] to be expressed as

$$[C-94] \quad M^* \delta \underline{x} + \delta M \underline{\dot{x}} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}^* \begin{bmatrix} \delta \underline{\dot{\Omega}}_0 \\ \delta \underline{\dot{\Omega}}_1 \end{bmatrix} + \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{bmatrix} \delta p_1 \\ \delta p_2 \end{bmatrix}$$

Manipulating the equations on the right-hand side of Equation [C-86] can be given as

$$\delta \underline{g} = \begin{bmatrix} \delta g_{11} \\ \delta g_{21} \end{bmatrix}$$

where

$$[C-95] \quad \delta g_{11} = \delta \left[\tilde{R}_{00} (\underline{F}_0 - m_0 \underline{v}_0) + \tilde{R}_{01} (\underline{F}_1 - m_1 \underline{v}_1) - \underline{w}_0 \times (\phi_0 \underline{w}_0) - \underline{w}_1 \times (\phi_1 \underline{w}_1) - \phi_1 (\underline{w}_0 \times \underline{\Omega}_1) \right]$$

and

$$[C-96] \quad \delta g_{21} = \delta \left[\tilde{R}_{01} (\underline{F}_1 - m_1 \underline{v}_1) - \underline{w}_1 \times (\phi_1 \underline{w}_1) - \phi_1 (\underline{w}_0 \times \underline{\Omega}_1) \right]$$

Using the results from the N-link case it can be shown that

$$[C-97] \quad \delta g_{11} = A_1 + B_1 + C_1$$

where

$$A_1 = \delta \left[\tilde{R}_{00} (\underline{F}_0 - m_0 \underline{v}_0) + \tilde{R}_{01} (\underline{F}_1 - m_1 \underline{v}_1) \right]$$

$$B_1 = -\delta \left[\underline{w}_0 \times (\phi_1 \underline{w}_0) + \underline{w}_1 \times (\phi_1 \underline{w}_1) \right]$$

$$C_1 = -\delta \left[\phi_1 (\underline{w}_0 \times \underline{\Omega}_1) \right]$$

Similarly the g_{21} element can be written as

$$\delta g_{21} = A_2 + B_2 + C_2$$

where

$$\begin{aligned}
A_2 &= \delta \left[\tilde{R}_{01} (F_1 - m_1 v_1) \right] \\
B_2 &= -\delta \left[\underline{w}_1 \times (\Phi_1 \underline{w}_1) \right] \\
C_2 &= -\delta \left[\Phi_1 (\underline{w}_0 \times \underline{\Omega}_1) \right]
\end{aligned}$$

Then, because (see N link development)

$$[C-98] \quad \delta \tilde{R}_{0n} \underline{F}_n^* - m_n \delta \tilde{R}_{0n} \underline{v}_n^* = M_{11}(n) \delta p_n - M_{12}(n) \delta p_n$$

the quantity A_1 can be written as

$$\begin{aligned}
A_1 &= \left[M_{11}(1) - M_{12}(1) \right] \delta p_1 + \left[M_{11}(2) - M_{12}(2) \right] \delta p_2 \\
&\quad + \left\{ -m_0 \tilde{R}_{00} \delta \underline{v}_0 - m_1 \tilde{R}_{01} \delta \underline{v}_1 \right\}
\end{aligned}$$

Examination of the last term in brackets yields

$$\begin{aligned}
-m_0 \tilde{R}_{00} \delta \underline{v}_0 - m_1 \tilde{R}_{01} \delta \underline{v}_1 &= -m_0 \tilde{R}_{00} \left[\delta \underline{w}_0 \times \underline{w}_0 \times \underline{h}_{01} \right. \\
&\quad \left. + \underline{w}_0 \times \delta \underline{w}_0 \times \underline{h}_{01} \right. \\
&\quad \left. + \underline{w}_0 \times \underline{w}_0 \times \delta \underline{h}_{01} \right] \\
&\quad - m_1 \tilde{R}_{01} \left[\delta \underline{w}_0 \times \underline{w}_0 \times \underline{h}_{01} \right. \\
&\quad \left. + \underline{w}_0 \times \delta \underline{w}_0 \times \underline{h}_{01} \right. \\
&\quad \left. + \underline{w}_0 \times \underline{w}_0 \times \delta \underline{h}_{01} \right. \\
&\quad \left. + \delta \underline{w}_1 \times \underline{w}_1 \times \underline{h}_{11} \right. \\
&\quad \left. + \underline{w}_1 \times \delta \underline{w}_1 \times \underline{h}_{11} \right. \\
&\quad \left. + \underline{w}_1 \times \underline{w}_1 \times \delta \underline{h}_{11} \right]
\end{aligned}$$

Following the procedure given in the previous section, A_1 can be written as

$$A_1 = \begin{bmatrix} M_{11}(1) & -M_{12}(1) \end{bmatrix} \delta p_1 + \begin{bmatrix} M_{11}(2) & -M_{12}(2) \end{bmatrix} \delta p_2$$

$$- \sum_{n=0}^1 \left[\underline{a}_n + \underline{b}_n + \underline{c}_n \right]$$

where

$$\underline{a}_n = m_n \tilde{R}_{0n} \sum_{j=0}^1 \delta \underline{w}_j \times (\underline{w}_j \times h_{j,j+1})$$

$$\underline{b}_n = m_n \tilde{R}_{0n} \sum_{j=0}^1 \delta \underline{w}_j \times (\delta \underline{w}_j \times h_{j,j+1})$$

$$\underline{c}_n = m_n \tilde{R}_{0n} \sum_{j=0}^1 \underline{w}_j \times (\underline{w}_j \times \delta h_{j,j+1})$$

Following the description given in the N-link linearization, the terms above can be written as

$$\underline{a}_n = m_n \tilde{R}_{0n} \sum_{j=0}^1 \begin{bmatrix} 0 & \eta_{31} & -\eta_{21} \\ -\eta_{21} & 0 & \eta_{11} \\ \eta_{21} & -\eta_{11} & 0 \end{bmatrix}_j \begin{bmatrix} \delta w_{jx} \\ \delta w_{jy} \\ \delta w_{jz} \end{bmatrix}$$

$$\underline{b}_n = m_n \tilde{R}_{0n} \sum_{j=0}^1 \begin{bmatrix} 0 & w_{jz} & -w_{jy} \\ -w_{jz} & 0 & w_{jx} \\ w_{jy} & -w_{jx} & 0 \end{bmatrix} \begin{bmatrix} 0 & h_{j,j+1,z} & -h_{j,j+1,y} \\ -h_{j,j+1,z} & 0 & h_{j,j+1,x} \\ h_{j,j+1,y} & -h_{j,j+1,x} & 0 \end{bmatrix} \begin{bmatrix} \delta w_{jx} \\ \delta w_{jy} \\ \delta w_{jz} \end{bmatrix}$$

$$\underline{c}_n = m_n \tilde{R}_{0n} \sum_{j=0}^1 W_j \delta h_{j,j+1}$$

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where W_j is defined in Equation [C-71].

Expanding these equations yields

$$\underline{a}_0 + \underline{b}_0 = m_0 \tilde{R}_{00} \begin{bmatrix} \eta_0 & \eta_{13} & -\eta_{12} \\ -\eta_{13} & 0 & \eta_{11} \\ \eta_{12} & -\eta_{11} & 0 \end{bmatrix}_0 + \begin{bmatrix} 0 & w_{0z} & -w_{0y} \\ -w_{0z} & 0 & w_{0x} \\ w_{0y} & -w_{0x} & 0 \end{bmatrix} \begin{bmatrix} \delta w_{0x} \\ \delta w_{0y} \\ \delta w_{0z} \end{bmatrix}$$

$$\underline{a}_1 + \underline{b}_1 = m_1 \tilde{R}_{01} \left\{ \begin{bmatrix} 0 & \eta_{13} & \eta_{12} \\ -\eta_{12} & 0 & \eta_{11} \\ \eta_{12} & -\eta_{11} & 0 \end{bmatrix}_1 \right.$$

$$+ \begin{bmatrix} 0 & w_{0z} & -w_{0y} \\ -w_{0z} & 0 & w_{0x} \\ w_{0y} & -w_{0x} & 0 \end{bmatrix} \begin{bmatrix} 0 & h_{01z} & -h_{01y} \\ -h_{01y} & 0 & h_{01x} \\ h_{01y} & -h_{01x} & 0 \end{bmatrix} \left. \begin{bmatrix} \delta w_{0x} \\ \delta w_{0y} \\ \delta w_{0z} \end{bmatrix} \right\}$$

$$+ \begin{bmatrix} 0 & \xi_{13} & -\xi_{12} \\ -\xi_{13} & 0 & \xi_{11} \\ \xi_{12} & -\xi_{11} & 0 \end{bmatrix}$$

$$+ \left\{ \begin{bmatrix} 0 & w_{1z} & -w_{1y} \\ -w_{1z} & 0 & w_{1x} \\ w_{1y} & -w_{1x} & 0 \end{bmatrix} \begin{bmatrix} 0 & h_{11z} & -h_{11y} \\ -h_{11z} & 0 & h_{11x} \\ h_{11y} & -h_{11x} & 0 \end{bmatrix} \begin{bmatrix} \delta w_{1x} \\ \delta w_{1y} \\ \delta w_{1z} \end{bmatrix} \right\}$$

$$\underline{c}_0 + \underline{c}_1 = m_0 \tilde{R}_{00} (\underline{w}_0)(\underline{w}_0)^{-\delta h_{01}} + m_1 \tilde{R}_{01} (\underline{w}_0)(\underline{w}_0)^{-\delta h_{01}} \\ + (\underline{w}_1)(\underline{w}_1)^{-\delta h_{11}}$$

where all terms have been previously defined.

Summarizing up to this point, it has been shown that

$$[C-99] \quad \delta g_{11} = \underline{A}_1 + \underline{B}_1 + \underline{C}_1$$

and that

$$\begin{aligned} \underline{A}_1 = & \left[M_{11}(1) - M_{12}(1) \right] \delta p_1 + \left[M_{11}(2) - M_{12}(2) \right] \delta p_2 \\ & - \left[\underline{a}_0 + \underline{b}_0 + \underline{a}_1 + \underline{b}_1 + \underline{c}_0 + \underline{c}_1 \right] \end{aligned}$$

or that

$$[C-100] \quad \underline{A}_1 = S_{11} \delta p_1 + S_{12} \delta p_2 - S_{13} \delta w_0 - S_{14} \delta w_1 - S_{15} \delta h_{01} - S_{16} \delta h_{11}$$

where

$$S_{11} \triangleq M_{11}(1) - M_{12}(2)$$

$$S_{12} \triangleq M_{11}(2) - M_{12}(2)$$

$$S_{13} \triangleq m_0 \tilde{R}_{00} \left[(\underline{n}_0) + (\underline{w}_0) \right] + m_1 \tilde{R}_{01} \left[(\underline{n}_1) + (\underline{w}_0) (\underline{h}_{01}) \right]$$

$$S_{14} \triangleq m_1 \tilde{R}_{01} \left[(\underline{x}_1) + (\underline{w}_1) (\underline{h}_{11}) \right]$$

Continuing on, the manipulation \underline{B}_1 and \underline{C}_1 into a suitable form will now be conducted. Recall that

$$\underline{B}_1 \triangleq -\delta \left[\underline{w}_0 \times \Phi_{00} \underline{w}_0 + \underline{w}_1 \times \Phi_{11} \underline{w}_1 \right]$$

and because

$$\underline{B}_1 = -\delta w_0 \times \Phi_{00} w_0 - w_0 \times \Phi_{00} \delta w_0 - \delta w_1 \times \Phi_{11} w_1 - w_1 \times \Phi_{11} \delta w_1$$

this can be written as

$$[C-101] \quad \underline{B}_1 = M_{1,15}(0) \delta w_0 + M_{1,15}(1) \delta w_1$$

where

$$M_{1,15}(n) \triangleq - \left\{ \begin{bmatrix} 0 & v_{31} & -v_{21} \\ -v_{31} & 0 & v_{11} \\ v_{21} & -v_{11} & 0 \end{bmatrix}_n + \left[\underline{w}_n \right] \right\}$$

and $[w_n]$ is defined in Equation [C-72].

In this case,

$$\underline{C}_1 \triangleq -\delta \Phi_1(\underline{w}_0 \times \underline{\Omega}_1)$$

which can be manipulated into

$$\underline{C}_1 = -\Phi_1 \left[\delta \underline{w}_0 \times \underline{\Omega}_1 + \underline{w}_0 \times \delta \underline{\Omega}_1 \right]$$

Therefore, following the N-link derivation

$$[C-102] \quad \underline{C}_1 = M_{1,16}^{(1)} \delta \underline{w}_0 + M_{1,17}^{(0)} \delta \underline{\Omega}_1$$

where the M matrices have been previously defined.

Now returning to equation [C-99] and substituting appropriately from [C-100], [C-101] and [C-102], the variation in g_{11} can be written as

$$[C-103] \quad \delta g_{11} = \begin{bmatrix} s_{11} & s_{12} \end{bmatrix} \begin{bmatrix} \delta p_1 \\ \delta p_2 \end{bmatrix} - \begin{bmatrix} s_{13} & s_{14} \end{bmatrix} \begin{bmatrix} \delta \underline{w}_0 \\ \delta \underline{w}_1 \end{bmatrix} - \begin{bmatrix} s_{15} & s_{16} \end{bmatrix} \begin{bmatrix} \delta h_{01} \\ \delta h_{11} \end{bmatrix} \\ + \begin{bmatrix} M_{1,15}^{(0)} & M_{15}^{(1)} \end{bmatrix} \begin{bmatrix} \delta \underline{w}_0 \\ \delta \underline{w}_1 \end{bmatrix} + \begin{bmatrix} M_{1,16}^{(1)} & M_{1,17}^{(0)} \end{bmatrix} \begin{bmatrix} \delta \underline{w}_0 \\ \delta \underline{\Omega}_1 \end{bmatrix}$$

Attention will now be turned to the linearization of the g_{21} element. Because

$$[C-104] \quad \delta \underline{g}_{21} \triangleq \delta \tilde{R}_{01}(\underline{F}_1 - m_1 \underline{v}_1) - \underline{w}_1 \times \Phi \underline{w}_1 - \Phi_1(\underline{w}_0 \times \underline{\Omega}_1)$$

and that

$$\underline{A}_2 = \delta \tilde{R}_{01}(\underline{F}_1 - m_1 \underline{v}_1) - m_1 R_{01} \delta \underline{v}_1 \\ = s_{12} \delta p_2 - m_1 \tilde{R}_{01} \delta \underline{v}_1$$

then

$$\delta \underline{g}_{11} = s_{12} \delta p_2 - m_1 \tilde{R}_{01} \delta \underline{v}_1 - \delta \underline{w}_1 \times \Phi \underline{w}_1 - \Phi_1(\underline{w}_0 \times \underline{\Omega}_1)$$

Now noting that

$$\tilde{m}_1 R_{01} \delta \underline{v}_1 = \underline{a}_1 + \underline{b}_1 + \underline{c}_1$$

$$\delta(\underline{w}_1 \times \underline{\phi}_{w_1}) = M_{1,15}^{(1)} \delta \underline{w}_1$$

$$\delta \left[\underline{\phi}_1 (\underline{w}_0 \times \underline{\Omega}_1) \right] = M_{1,16}^{(1)} \delta \underline{w}_0 + M_{1,17}^{(0)} \delta \underline{\Omega}_1$$

Then the variation in g_{21} is given by

$$\begin{aligned} [C-105] \delta g_{21} = & s_{12} \delta p_2 - \bar{s}_{13} \delta \underline{w}_0 - s_{14} \delta \underline{w}_1 - \bar{s}_{15} \delta \underline{h}_{01} - s_{16} \delta \underline{h}_{11} \\ & + M_{1,15}^{(1)} \delta \underline{w}_1 + M_{1,16}^{(1)} \delta \underline{w}_0 + M_{1,17}^{(0)} \delta \underline{\Omega}_1 \end{aligned}$$

Equation [C-103] can then be written as

$$\begin{aligned} [C-106] \delta \underline{g}_{21} = & \begin{bmatrix} 0 & s_{12} \end{bmatrix} \begin{bmatrix} \delta p_1 \\ \delta p_2 \end{bmatrix} - \begin{bmatrix} \bar{s}_{13} - M_{1,16}^{(1)} & s_{14} - M_{1,15}^{(1)} \end{bmatrix} \begin{bmatrix} \delta \underline{w}_0 \\ \delta \underline{w}_1 \end{bmatrix} \\ & - \begin{bmatrix} \bar{s}_{15} & s_{16} \end{bmatrix} \begin{bmatrix} \delta \underline{h}_{01} \\ \delta \underline{h}_{11} \end{bmatrix} + \begin{bmatrix} 0 & M_{1,17}^{(0)} \end{bmatrix} \begin{bmatrix} \delta \underline{\Omega}_0 \\ \delta \underline{\Omega}_1 \end{bmatrix} \end{aligned}$$

At this point Equations [C-104] and [C-105] can be combined into the vector form

$$\begin{aligned} [C-107] \begin{bmatrix} \delta \underline{g}_{11} \\ \delta \underline{g}_{21} \end{bmatrix} = & \begin{bmatrix} s_{11} & s_{12} \\ 0 & s_{12} \end{bmatrix} \begin{bmatrix} \delta p_1 \\ \delta p_2 \end{bmatrix} \\ & - \begin{bmatrix} \begin{bmatrix} s_{12} - M_{1,15}^{(0)} - M_{1,16}^{(1)} & s_{14} - M_{1,15}^{(1)} \end{bmatrix} \\ \begin{bmatrix} \bar{s}_{13} - M_{1,16}^{(1)} & s_{14} - M_{1,15}^{(1)} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \delta \underline{w}_0 \\ \delta \underline{w}_1 \end{bmatrix} \\ & - \begin{bmatrix} s_{15} & s_{16} \\ \bar{s}_{15} & s_{16} \end{bmatrix} \begin{bmatrix} \delta \underline{h}_{01} \\ \delta \underline{h}_{11} \end{bmatrix} + \begin{bmatrix} 0 & M_{1,17}^{(0)} \\ 0 & M_{1,17}^{(0)} \end{bmatrix} \begin{bmatrix} \delta \underline{\Omega}_0 \\ \delta \underline{\Omega}_1 \end{bmatrix} \end{aligned}$$

This equation can be placed into simpler notation as

$$[C-108] \begin{bmatrix} \delta \underline{g}_{11} \\ \delta \underline{g}_{21} \end{bmatrix} = \begin{bmatrix} U_{11} \\ U_{12} \\ U_{13} \\ U_{14} \end{bmatrix} \begin{bmatrix} \delta \underline{p}_1 \\ \delta \underline{p}_2 \\ \delta \underline{h}_{01} \\ \delta \underline{h}_{11} \end{bmatrix} + \begin{bmatrix} \delta \underline{w}_0 \\ \delta \underline{w}_1 \\ \delta \underline{\Omega}_0 \\ \delta \underline{\Omega}_1 \end{bmatrix}$$

where the matrices, U, have obvious definitions.

At this point additional definitions are used to further decompose [C-108]. For example, it can be shown that (see Equation [C-66] in section 8.0 on N-link linearization)

$$[C-109] \delta \underline{h}_{01} \triangleq \left\{ \begin{bmatrix} t_{1\phi_0} & t_{2\theta_0} & t_{3\psi_0} \end{bmatrix}_{01} \begin{bmatrix} h_{01} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right. \\ + \begin{bmatrix} t_{1\theta_0} & t_{2\psi_0} & t_{3\phi_0} \end{bmatrix}_{01} \begin{bmatrix} 0 & h_{01} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \left. + \begin{bmatrix} t_{1\psi_0} & t_{2\phi_0} & t_{3\theta_0} \end{bmatrix}_{01} \begin{bmatrix} 0 & 0 & h_{01} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} \delta \phi_0 \\ \delta \theta_0 \\ \delta \psi_0 \end{bmatrix}$$

or

$$\delta \underline{h}_{01} = T_{01}(0) \delta \underline{\theta}_0$$

where the 3x3 matrix T_{01} comprises elements internal to the brackets and the 3x1 vector $\delta\theta_0$ is given by

$$\delta\theta_0 \triangleq \begin{bmatrix} \delta\phi_0 & \delta\theta_0 & \delta\psi_0 \end{bmatrix}$$

Similarly, it can be shown that

$$[C-110] \delta h_{11} + T_{11}(0) \delta\theta_0 + T_{11}(1) \delta\theta_1$$

In addition, the fact that w_0 is equal to Ω_0 and that w_1 equals the sum of Ω_0 and Ω_1 allows:

$$[C-111] \delta w_0 = \delta\Omega_0$$

$$[C-112] \delta w_1 = \delta\Omega_0 + \delta\Omega_1$$

Therefore, using Equations [C-109], [C-110], [C-111] and [C-112] in [C-108] yields

$$[C-113] \begin{bmatrix} \delta g_{11} \\ \delta g_{21} \end{bmatrix} = \begin{bmatrix} U_{11} \end{bmatrix} \begin{bmatrix} \delta p_1 \\ \delta p_2 \end{bmatrix} + \begin{bmatrix} U_{12} \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} \delta\dot{\theta}_0 \\ \delta\dot{\theta}_1 \end{bmatrix} \\ - \begin{bmatrix} U_{13} \end{bmatrix} \begin{bmatrix} T_{01}(0) & 0 \\ T_{11}(0) & T_{11}(1) \end{bmatrix} \begin{bmatrix} \delta\theta_0 \\ \delta\theta_1 \end{bmatrix} \\ + \begin{bmatrix} U_{14} \end{bmatrix} \begin{bmatrix} \delta\dot{\theta}_0 \\ \delta\dot{\theta}_1 \end{bmatrix}$$

Where the fact that Ω is defined to be the first derivative of the relative angular position. This can be further compressed to yield

$$\begin{bmatrix} \delta g_{11} \\ \delta g_{21} \end{bmatrix} = \begin{bmatrix} U_{11} \end{bmatrix} \begin{bmatrix} \delta p_1 \\ \delta p_2 \end{bmatrix} + \begin{bmatrix} U_{12} \end{bmatrix} \begin{bmatrix} \delta\dot{\theta}_0 \\ \delta\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} U_{13} \end{bmatrix} \begin{bmatrix} \delta\theta_0 \\ \delta\theta_1 \end{bmatrix}$$

At this juncture the right and left-hand side of Equation [C-42] can be written as

$$\begin{aligned}
 \text{[C-114]} \quad & \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}^* \begin{bmatrix} \ddot{\delta\theta}_0 \\ \ddot{\delta\theta}_1 \end{bmatrix} + \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{bmatrix} \delta p_1 \\ \delta p_2 \end{bmatrix} \\
 & = \delta \underline{T} + \begin{bmatrix} U_{11} \\ U_{12} \end{bmatrix} \begin{bmatrix} \delta p_1 \\ \delta p_2 \end{bmatrix} + \begin{bmatrix} U_{12} \\ U_{13} \end{bmatrix} \begin{bmatrix} \dot{\delta\theta}_0 \\ \dot{\delta\theta}_1 \end{bmatrix} + \begin{bmatrix} U_{13} \\ U_{14} \end{bmatrix} \begin{bmatrix} \delta\theta_0 \\ \delta\theta_1 \end{bmatrix}
 \end{aligned}$$

Combining terms, this equation is given as

$$\begin{aligned}
 \text{[C-115]} \quad & \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \ddot{\delta\theta}_0 \\ \ddot{\delta\theta}_1 \end{bmatrix} - \begin{bmatrix} U_{12} \\ U_{13} \end{bmatrix} \begin{bmatrix} \dot{\delta\theta}_0 \\ \dot{\delta\theta}_1 \end{bmatrix} - \begin{bmatrix} U_{13} \\ U_{14} \end{bmatrix} \begin{bmatrix} \delta\theta_0 \\ \delta\theta_1 \end{bmatrix} \\
 & = \underline{T} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \delta p_1 \\ \delta p_2 \end{bmatrix}
 \end{aligned}$$

where

$$D \triangleq \begin{bmatrix} U_{11} \\ U_{12} \end{bmatrix} - \begin{bmatrix} \Gamma \\ \Gamma \end{bmatrix}$$

This completes the linearization of the two-link case.

10.0 STATE-VARIABLE REPRESENTATION

As discussed in the main body of this report, the simulation framework required for the manipulator dynamics is state variables. This being the case, this section will show how the linearized equations for the two-link case can be placed into the S.V. format. This will be used as a stepping stone to obtain the N-link representation.

Consider the linearized equations of the 2-link case, i.e.,

$$\begin{aligned}
 [C-116] \quad & \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \delta \ddot{\theta}_0 \\ \delta \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \delta \dot{\theta}_0 \\ \delta \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \delta \theta_0 \\ \delta \theta_1 \end{bmatrix} \\
 & = \begin{bmatrix} \delta T_1 \\ \delta T_2 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \delta p_1 \\ \delta p_2 \end{bmatrix}
 \end{aligned}$$

where

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \triangleq - \begin{bmatrix} U_{13} \end{bmatrix} ; \quad \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \triangleq - \begin{bmatrix} U_{13} \end{bmatrix}$$

and the torque terms, δT_1 and δT_2 , correspond to the torques supplied by the servo motors at the respective joints. Equation [C-116] can be rewritten in the form

$$[C-117] \quad M \ddot{\underline{x}}_1 + C \dot{\underline{x}}_1 + K \underline{x}_1 = \underline{t} + D \underline{r}$$

where M, C, and K are 6x6 inertia, damping, and stiffness type matrices; \underline{t} the applied torque vector; \underline{r} the 6x1 position vector; D the 6x6 position modifier; and \underline{x} the 6x1 angular position vector - three for each joint.

Letting

$$[C-118] \quad \dot{\underline{x}}_1 = \underline{x}_2$$

and substituting [C-118] into [C-117] yields a set of 12x1 first-order equations given by

$$[C-119] M \dot{\underline{y}} + K \underline{y} = \underline{a} + B \underline{r}$$

where

$$M \triangleq \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} ; \quad K \triangleq \begin{bmatrix} 0 & I \\ K & C \end{bmatrix}$$

$$\underline{a} \triangleq \begin{bmatrix} 0 \\ \underline{t} \end{bmatrix} ; \quad B \triangleq \begin{bmatrix} 0 \\ D \end{bmatrix}$$

and

$$\underline{y}' \triangleq \begin{bmatrix} \underline{x}'_1 & \underline{x}'_2 \end{bmatrix} = \begin{bmatrix} \underbrace{\phi_0 \ \theta_0 \ \psi_0}_{\underline{x}'_1} \mid \underbrace{\phi_1 \ \theta_1 \ \psi_1}_{\underline{x}'_2} \end{bmatrix}$$

Equation [C-119] can now be discretized by solving the 12 first order equations

$$[C-120] \dot{\underline{y}} = A \underline{y} + E \underline{u}$$

where

$$A \triangleq M^{-1} K ; \quad E \triangleq M^{-1} \begin{bmatrix} I & B \end{bmatrix}$$

and the vector \underline{u} defined as

$$\underline{u}' \triangleq \begin{bmatrix} \underline{a} & \underline{r} \end{bmatrix}$$

The solution to [C-120] is given by the well-known equation

$$[C-121] \underline{y}(t) = \phi(t, t_0) \underline{y}(t_0) + \int_{t_0}^t \phi(\tau, t_0) E(\tau) \underline{u}(\tau) d\tau$$

Assuming \underline{u} is constant over the time interval t_k to t_{k+1} [C-121]
can be written as

$$[C-122] \underline{y}(k+1) = \phi(k+1, k) \underline{y}(k) + \theta(k+1, k) \underline{u}(k)$$

where

$$\theta(k+1, k) \triangleq \int_{t_k}^{t_{k+1}} \phi(\tau, t_k) E(\tau) d\tau$$

The extension of the state-variable technique to the N-link case is clear once it is recognized that [C-117] for the N-link case becomes a $3N$ set of second-order differential equations. Reducing these second-order equations into first order equivalents produces a set of $6N$ equations. Because all that changes in going from a two-link case to a system represented by N-links is the dimensions Equation [C-122] is valid for the Nth link case.

11.0 References

- 1) William W. Hooker - "Equations of Motion for Interconnected Rigid and Elastic Bodies: A Derivation Independent of Angular Momentum", Celestial Mechanics, 11(3): 337-354.

Appendix D--Derivation of Requirements Tool Control Law



This appendix presents in detail one of the many methods by which a human being can interactively control an n-segment manipulator arm, in which segments are connected by rotational joints. The method presented here is known as coordinated rate control, and was first introduced by D. E. Whitney in 1969 (Ref. 2). Since then, the method has been used by other workers and modified. This presentation explains the principles behind the method, as well as some of the details needed to implement it. The coordinated rate control algorithm derived here has been incorporated into ROBSIM to drive the requirements analysis tool described in Chapter IV of the main volume.

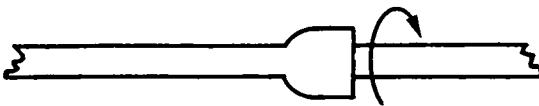
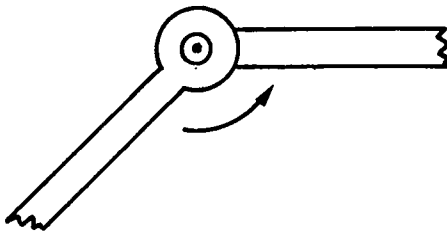
Coordinated rate control was developed to provide a method of manipulator control whereby an operator could input directly details of end-effector control, rather than dealing with the end-effector indirectly, as when controlling individual joints. In coordinated rate control, the operator specifies the rate of end-effector motion (e.g., 2 in./s straight up) and the control algorithm transforms this command into rate commands to individual joints. The resultant of the joint motion is then the end-effector motion that was specified.

The algorithm for calculating joint rates from end-effector rates can be summarized as follows. The movement of every joint contributes to the motion of the end effector. In fact, end-effector motion is the resultant of all the contributions made by the joints. By expressing end-effector velocity as the sum of joint contributions, a system of equations is obtained that relates end-effector rates to joint rates. The system of equations is linear, and can be solved by using techniques from matrix theory.

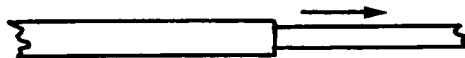
Manipulator joints consist of two general types--rotary and sliding (Fig. D1-1). The development to follow considers arms having only rotary joints, but the logic behind the development is useful for arbitrary arms containing both types of joints.

In addition to end-effector rate to joint rate conversions, this document also discusses some of the accuracy problems with which a control system must contend. Generally, the joint solution derived from a desired end-effector rate is valid only instantaneously. As the joints move, the relationships between links change, and the coefficients of the control equations are also altered. Physical limitations prohibit instantaneous re-calculation and solution of the control equations. Approximations can be made to simplify the system and speed up computations, but the accuracy of the solution is reduced. Some tradeoffs between speed and accuracy are discussed.

Occasionally, the matrix defining the relationship between joint and end-effector velocities may be singular. Singularity would occur, for example, if a motion could be effected in more than one way, or if a motion was not possible. In Section D.2 a solution to this problem is presented.



Rotary:



Sliding:

Figure D1-1 Joint Types

2.0 COORDINATE TRANSFORMATIONS

To perform kinematic analysis of a manipulator system and develop control equations, we must define a variety of coordinate systems and understand the procedures involved in converting from one system to another. This section defines several coordinate systems and develops the transformations between these systems. An example is described.

Let \bar{X}_0 denote the inertial reference frame that is fixed with respect to the manipulator base. Call $\bar{X}_0 = (\bar{x}_0, \bar{y}_0, \bar{z}_0)$ the set of world axes. Let $\bar{X}_e = (\bar{x}_e, \bar{y}_e, \bar{z}_e)$ denote the system of coordinate axes fixed with respect to the manipulator end effector. Call \bar{X}_e the end-effector axes. In addition to end-effector axes, define a set of axes with origins fixed at joint centers:

$\bar{X}_i = (\bar{x}_i, \bar{y}_i, \bar{z}_i)$ is the set of coordinate axes at joint i .

All the coordinate systems are right-handed, orthogonal systems. In the case of joint systems, it is understood that the y axis points towards the terminal end of the manipulator along a line parallel to the terminal link associated with the joint.

Figure D2-1 shows a four-joint manipulator and its coordinate systems. Note, however, that due to space availability, \bar{X}_4 was left out of the figure.

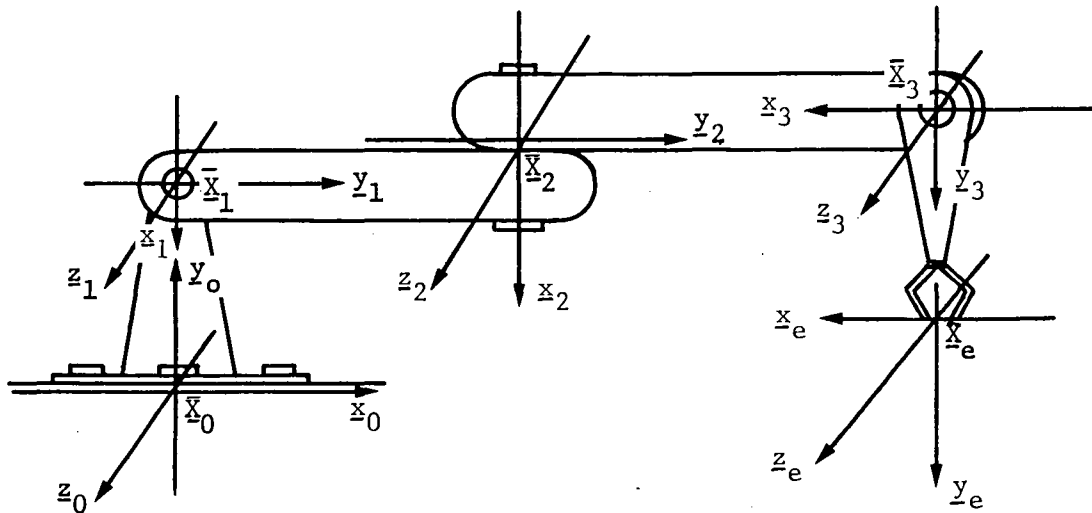


Figure D2-1 Manipulator Coordinate Systems

It is apparent (Fig. D2-1) that the coordinate systems differ from each other in two ways: their origins are displaced, and the axes may be rotated out of alignment. Therefore, if a vector is known with respect to one of the systems, the rotation and translation between systems may have to be considered to write the vector in terms of a second system. The transformation between systems is accomplished by using transformation matrices and displacement vectors.

2.1 TRANSFORMATION MATRICES

An orthogonal matrix can describe the transformation between Cartesian coordinate systems that have been rotated with respect to each other. This orthogonality property is important, because it means that if P is an orthogonal matrix, then $P^{-1} = P^t$. That is, the inverse transformation is just the transpose of the original matrix.

The following is a development of the transformation matrix for a simple, two-dimensional case (Fig. D2-2).

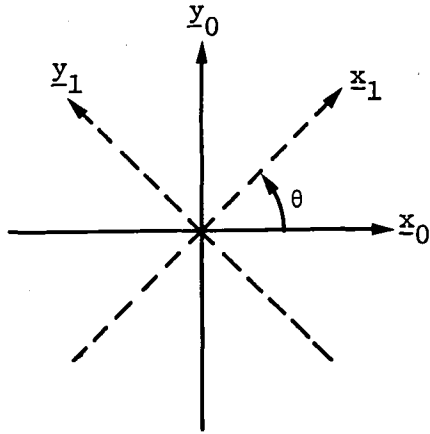


Figure D2-2
Two-Dimensional Coordinate
Transformation

The matrix for the transformation from system \bar{X}_1 to system \bar{X}_0 , when applied to a vector in \bar{X}_1 coordinates, will yield the coordinates of the vector in the \bar{X}_0 system. The matrix can be obtained by writing the coordinates of \bar{x}_1 and \bar{y}_1 in terms of \bar{x}_0 and \bar{y}_0 . That is, if 0P_1 is the matrix describing the transformation from \bar{X} to \bar{X}_0 , then

$${}^0P_1 = \begin{bmatrix} {}^0\bar{x}_1 & : & {}^0\bar{y}_1 \end{bmatrix}$$

where each of ${}^0\bar{x}_1$ and ${}^0\bar{y}_1$ are unit 2-vectors in \bar{X}_0 . So, decomposing \bar{x}_1 , \bar{y}_1 in terms of \bar{x}_0 , \bar{y}_0 , we obtain

$${}^0\bar{x}_1 = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \quad {}^0\bar{y}_1 = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

Thus,

$${}^0P_1 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

And, the reverse transformation, 1P_0 , from \bar{X}_0 to \bar{X}_1 , is given by

$${}^1P_0 = {}^0P_1^t = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Applying trigonometric identities will show that this approach is equivalent to the more familiar approach, in which the entries are the cosines of the angles between coordinate axes (Fig. D2-3).

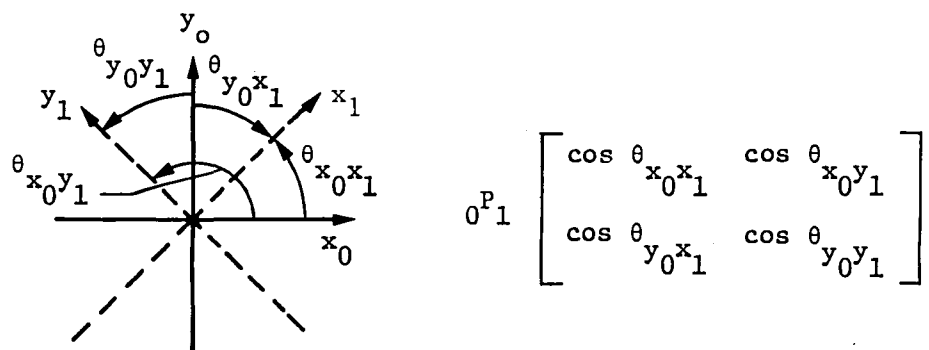


Figure D2-3

Two-Dimensional Coordinate Transformation Matrix

In three dimensions, the concept is easily generalized.

To develop the transformations between coordinate systems on a manipulator arm, we do the following:

- 1) Compute the transformations between adjacent coordinate systems with the manipulator in a reference position. Call this matrix ${}_jP_i^1$.
- 2) Compute the transformation matrix for each joint in terms of the joint motion from reference position as a function of θ_{ij} . Call this matrix ${}_jP_i^2$.
- 3) Then the complete transformation ${}_jP_i$ is given by

$${}_jP_i = ({}_jP_i^1)({}_jP_i^2)$$

Now, a vector given in any manipulator coordinate system can be transformed into any other manipulator system by repeated use of the ${}_jP_i$ matrices. For example, suppose ${}_e\mathbf{v}$ is a vector given in terms of the end-effector axes. (Denote the system a vector is expressed in by a left subscript.) To find ${}_0\mathbf{v}$ apply the sequence of transformation matrices:

$${}_0\mathbf{v} = {}^0P_1 {}^1P_2 \cdots {}^{n-1}P_n {}^nP_e {}_e\mathbf{v}$$

Once again, the inverse transformation would use the transpose of the matrices above.

$$\begin{aligned} e\mathbf{v} &= ({}_0P_1 \cdot \cdot \cdot {}_nPe)^t {}_0\mathbf{v} \\ &= {}_nPe^t \cdot \cdot \cdot {}_0P_1^t {}_0\mathbf{v} \\ &= {}_eP_n \cdot \cdot \cdot {}_1P_0 {}_0\mathbf{v} \end{aligned}$$

2.2 DISPLACEMENT VECTORS

Transformation matrices only account for the rotation between coordinate systems, not for displacement between origins. Therefore, using the transformation

$${}_j\mathbf{v} = {}_jP_i {}_i\mathbf{v}$$

yields a vector ${}_j\mathbf{v}$, of the same length and direction as ${}_i\mathbf{v}$, but originating from the origin of \bar{X}_j (Fig. D2-4).

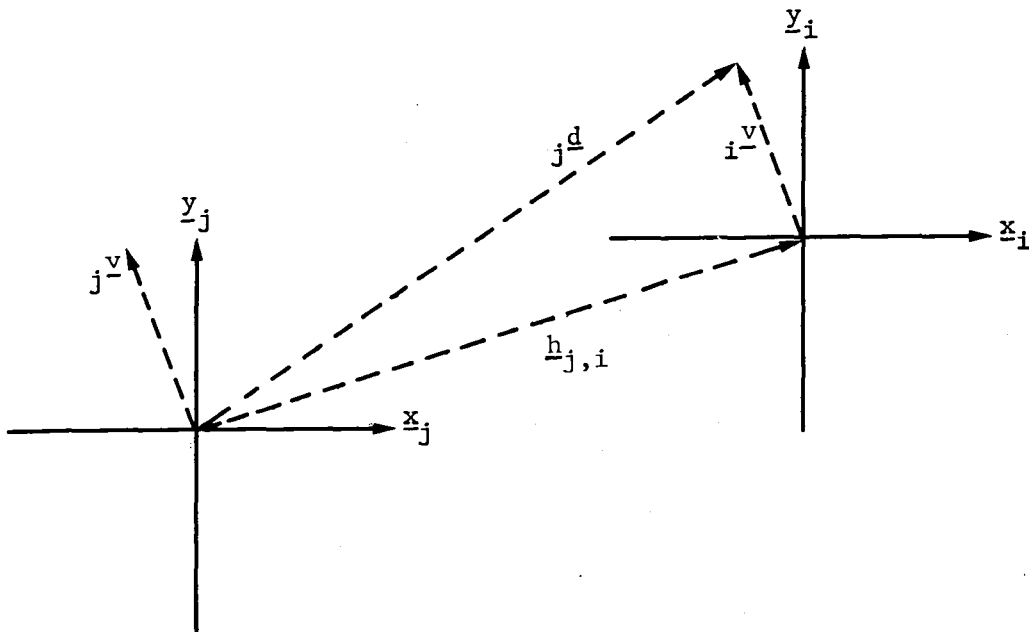


Figure D2-4 Coordinate System Displacement

To obtain a vector representing distance and direction--e.g., from the origin of \bar{X}_1 to the tip of ${}_i\mathbf{v}$ --the displacement between the origins of different coordinate systems must be known.

Let $h_{i,j}$ denote the vector from the origin of \bar{X}_i to the origin of \bar{X}_j . Thus $h_{i,j} = -h_{j,i}$. By applying the parallelogram law for

vectors, the vector from the origin of \bar{X}_j to the tip of ${}_i\mathbf{v}$ can be obtained by adding ${}_j\mathbf{v}$ and ${}_j\mathbf{h}_{i,j}$. That is,

$${}_j\mathbf{d} = {}_j\mathbf{v} + {}_j\mathbf{h}_{j,i}$$

or

$${}_j\mathbf{d} = {}_j\mathbf{P}_{ii}\mathbf{v} + {}_j\mathbf{h}_{j,i}$$

Note that, to be added, vectors must be expressed in the same coordinate system. Using this notation, this means that the lower left subscript of every term must be the same.

Now, the use of transformation matrices and displacement vectors will be combined by means of an example. Figure D2-5 shows a three-joint manipulator. Let ${}_e\mathbf{d}$ denote the location of an object in the end-effector coordinate system. Then the location in world coordinates can be calculated by working backward through each coordinate system until the world axes are reached and ${}_0\mathbf{d}$ is determined.

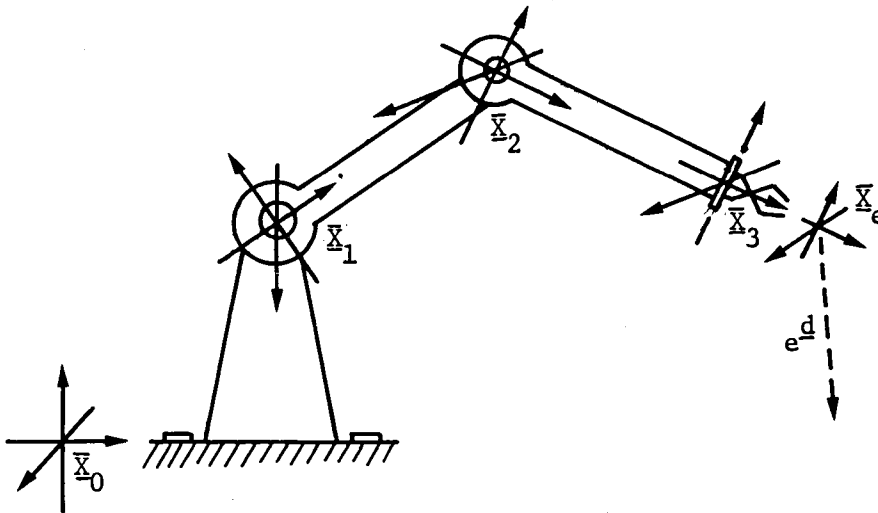


Figure D2-5 Three-Joint Manipulator

To \bar{X}_3 : ${}_3\mathbf{d} = {}_3\mathbf{P}_3\mathbf{d} + {}_3\mathbf{h}_{3,e}$

To \bar{X}_2 : ${}_2\mathbf{d} = {}_2\mathbf{P}_3 [{}_3\mathbf{P}_e {}_e\mathbf{d} + {}_3\mathbf{h}_{3,e}] + {}_2\mathbf{h}_{2,3}$

To \bar{X}_1 : ${}_1\mathbf{d} = {}_1\mathbf{P}_2 [{}_2\mathbf{P}_3 [{}_3\mathbf{P}_e {}_e\mathbf{d} + {}_3\mathbf{h}_{3,e}] + {}_2\mathbf{h}_{2,3}] + {}_1\mathbf{h}_{1,2}$

And, to \bar{X}_0 : ${}_0\mathbf{d} = {}_0\mathbf{P}_1 [{}_1\mathbf{P}_2 [{}_2\mathbf{P}_3 [{}_3\mathbf{P}_e {}_e\mathbf{d} + {}_3\mathbf{h}_{3,e}] + {}_2\mathbf{h}_{2,3}] + {}_1\mathbf{h}_{1,2}] + {}_0\mathbf{h}_{0,1}$

Simplify the notation by multiplying out the quantities in brackets and extending the notation by defining ${}_j\mathbf{P}_i = {}_j\mathbf{P}_{j+1} {}_{j+1}\mathbf{P}_{j+2} \dots {}_{i-1}\mathbf{P}_i$ for $i > j$.

Then,

$${}_0\mathbf{d} = {}_0\mathbf{P}_e {}_e\mathbf{d} + {}_0\mathbf{P}_3 {}_3\mathbf{h}_{3,e} + {}_0\mathbf{P}_2 {}_2\mathbf{h}_{2,3} + {}_0\mathbf{P}_1 {}_1\mathbf{h}_{1,2} + {}_0\mathbf{h}_{0,1}$$

3.0 CONTROL EQUATIONS

In the preceding section, the notation and concepts needed to derive the control equations for the SOS manipulator were described. This section presents a derivation of those equations.

The inputs to the control equations are end-effector rates. Any motion can be described by six rates: velocities in the \underline{x} , \underline{y} , \underline{z} directions, and angular velocity around the \underline{x} , \underline{y} , \underline{z} axes. Desired end-effector rates are input by the operator. The outputs are the individual joint velocities needed to effect the input motion.

The derivation to follow is a means of resolving joint velocities to obtain a resultant end-effector velocity. Equating the resultant to the input yields a set of linear equations of the form

$$D3-1 \quad [\text{Input}] = J \cdot [\text{joint velocities}],$$

which may be solved by inverting J (in some sense) to obtain the joint rates.

The goal in designing a human-operated, interactive control system for a manipulator arm is to minimize the variables the operator must control. A simple control system can be designed so that the operator directly controls the rate of motion of each individual joint. However, this is not a desirable system because to produce a specific motion of the end effector, the operator must somehow anticipate the resultant of a set of joint motions. For one or two joints, this is not difficult, but for a six or seven-joint manipulator, the task is quite complex. For sensitive operations, this kind of control system is not practical.

A better control system allows the operator to control only the end-effector motion. A computer reads the input commands and performs the necessary calculations to produce the required velocities of each joint. Such a control system is practical for most any task. This is precisely the control system to be implemented.

Consider angular rotation of a manipulator end effector. The rotation is about a single axis and has associated with it a velocity. Both the axis and the velocity comprise an angular velocity vector. Let \underline{w}_e denote the angular velocity of the end effector. Then \underline{w}_e can be decomposed into components about the \underline{x} , \underline{y} , and \underline{z} axes of some coordinate system. So, in world axes,

$${}^0\underline{w}_e = \begin{bmatrix} {}^0w_{ex} \\ {}^0w_{ey} \\ {}^0w_{ez} \end{bmatrix}$$

Let \underline{w}_i denote the angular velocity of joint i ; that is, the angular velocity due to the motion of joint i alone. If the angular velocity of the first link is \underline{w}_1 , then the angular velocity of the second link is the sum of \underline{w}_1 and \underline{w}_2 . The angular velocity of the end effector is the sum of the angular velocities produced by each joint. To sum these velocities, they must be expressed in the same coordinate system. This summation requires the use of the transformation matrices from the previous section. Expressing everything in terms of the world axes, we have:

$$D3-2 \quad {}^0\underline{w}_e = {}^0P_1 {}^1\underline{w}_1 + {}^0P_2 {}^2\underline{w}_2 + \dots + {}^0P_7 {}^7\underline{w}_7$$

This single-vector equation yields a system of three linear scalar equations.

To solve a system requires as many equations as the system has unknowns. Consider any one of the joint angular velocity vectors \underline{w}_i . Expressing this vector in terms of the system $\underline{\bar{X}}_i$, we have

$$D3-3 \quad {}^i\underline{w}_i = \begin{bmatrix} {}^i w_{ix} \\ {}^i w_{iy} \\ {}^i w_{iz} \end{bmatrix}$$

and so it appears that each w_i has three unknowns.

However, the joints of the SOS manipulator are single degree of freedom joints. Therefore, all rotations take place around a single axis. By determining the relationship of the system $\underline{\bar{X}}_i$ to the joint axis, the number of unknowns is reduced to one. In general, one of the $\underline{\bar{X}}_i$ axes will coincide with the axis of rotation, so that equation (D3-3) will be of the form

$$D3-4 \quad a) \quad {}^i\underline{w}_i = \begin{bmatrix} {}^i w_{ix} \\ 0 \\ 0 \end{bmatrix} \text{ or } b) \quad {}^i\underline{w}_i = \begin{bmatrix} 0 \\ {}^i w_{iy} \\ 0 \end{bmatrix} \text{ or } c) \quad {}^i\underline{w}_i = \begin{bmatrix} 0 \\ 0 \\ {}^i w_{iz} \end{bmatrix}$$

depending on which axis of $\underline{\bar{X}}_i$ is in alignment with the joint axis.

Thus, the system of equations (D3-2) in actuality contains at most seven unknowns. To obtain the remaining control equations, and to completely determine end-effector motion, consider end-effector translational velocity.

Translational velocity can be represented as a three-vector, the components of which represent the magnitude of the velocities along the \underline{x} , \underline{y} , and \underline{z} axes. Let \underline{v}_e denote end-effector translational velocity. Then \underline{v}_e becomes

$$\underline{v}_e = \begin{bmatrix} v_{ex} \\ v_{ey} \\ v_{ez} \end{bmatrix}$$

To see how rotational joints can be used to produce translational motion, consider the rotating bar (Fig. D3-1).

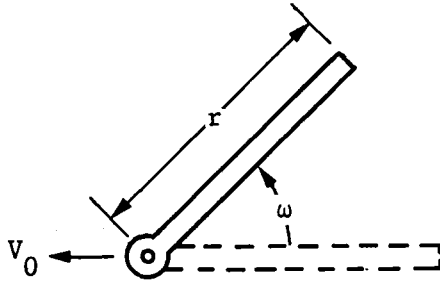


Figure D3-1 Rotating Bar

The bar is rotating with an angular velocity of ω , and is r units long. However, at any instant of time, the motion is translational, along a line tangent to the curved path. The velocity at any given point is proportional to its distance from the center of rotation. A formula from mechanics gives the exact relationship as

$$\underline{v} = \underline{\omega} \times \underline{r} + \underline{v}_0$$

That is, at any instant, translational velocity is equal to the vector cross product of angular velocity and the radius vector, plus the velocity of the origin.

Thus, the rotational motion at each joint produces an instantaneous translational motion of the connecting joint. Let \underline{r}_i denote the radius vector from the i^{th} joint to the origin of \underline{x}_e . Then $\underline{\omega}_i \times \underline{r}_i$ is the vector cross product or translational velocity of the end effector that is produced by the angular velocity vector of joint i . Therefore, summing all such components yields the total translational velocity of the end effector:

$$\underline{v}_e = {}^0P_1 (\underline{\omega}_1 \times \underline{r}_1) + \dots + {}^0P_7 (\underline{\omega}_7 \times \underline{r}_7)$$

Note that equation (D3-6) applies only to arms that have no sliding joints.

This single vector equation yields three linear, scalar equations. Because the matrices jP_i and the vectors \underline{r}_i are functions of the joint positions, the only unknowns are the single nonzero components of the vectors $\underline{\omega}_i$.

Combining equations (D3-2) and (D3-6) into a single equation:

$$D3-7 \quad \begin{bmatrix} 0^V_e \\ \dots \\ 0^W_e \end{bmatrix} = \begin{bmatrix} 0^P_1 ({}_1\vec{w}_1 \times {}_1\vec{r}_1) + \dots + 0^P_7 ({}_7\vec{w}_7 \times {}_7\vec{r}_7) \\ \dots \\ 0^P_1 {}_1\vec{w}_1 + \dots + 0^P_7 {}_7\vec{w}_7 \end{bmatrix}$$

The vector equation above yields six linear, scalar equations in seven unknowns. These are not enough equations to completely solve the system. There are ways of arriving at an answer by introducing another constraint (e.g., minimum energy expenditures), hence yielding a seventh equation. However, the configuration of the SOS manipulator offers another alternative.

Examining the drawing of the SOS arm (Fig. D3-2) reveals that shoulder joints control elbow-position orientation. It is possible to select one of the joints to be controlled separately. In this case, the value of \vec{w}_i for the shoulder joint selected is an input, rather than an unknown, and can, therefore, be eliminated from the control equations. This leaves six equations in six unknowns, so the system can be solved uniquely.

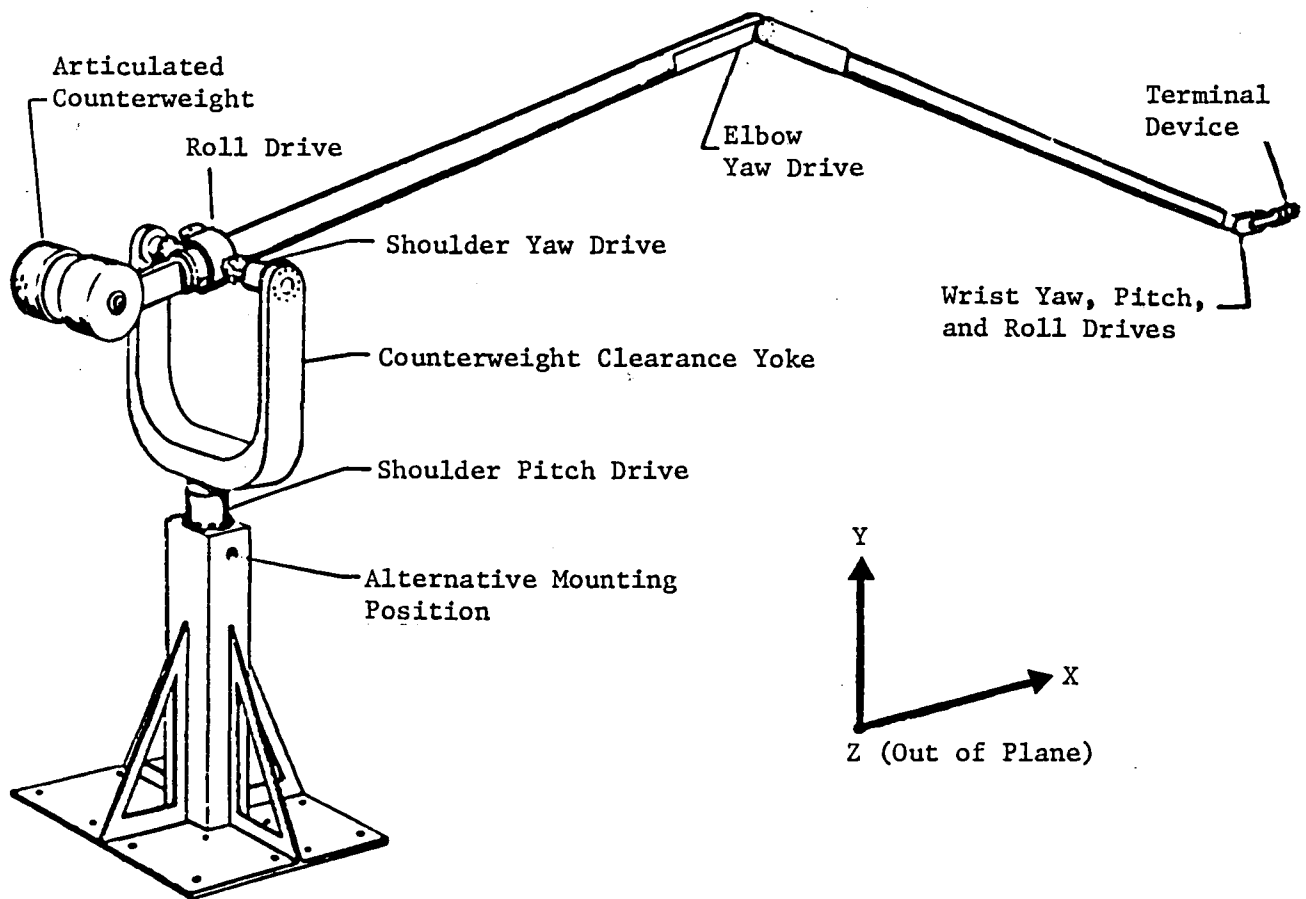


Figure D3-2 The SOS Manipulator

For the sake of notation, let joint 1 be selected to be the joint controlled directly by the operator. Then the control equation (D3-7) takes the form:

[illegible]

Evaluating the cross products and performing the internal matrix multiplications, a matrix equation is obtained of the form:

$$D-3-9 \quad \begin{bmatrix} 0^v_e \\ \vdots \\ 0^w_e \end{bmatrix} = J \begin{bmatrix} 2^w_{2j} \\ \vdots \\ 7^w_{7k} \end{bmatrix}$$

where w_{nj} is the angular velocity of joint n , and j represents the axis of rotation (because joints have one degree of freedom, the rotation will be around a single-coordinate axis). In the case that joint n revolves around the x -axis, $w_{ny} = w_{nz} = 0$, therefore w_{nx} is the unknown that must be calculated.

Once the system of equations is in world coordinates, it is necessary to find only the matrix inverse (or pseudo-inverse) of J to solve the system.

First, evaluate terms of the form $0^{P_i} (i \underline{w}_i \times i \underline{r}_i)$:

$$D-3-10 \quad (i^w_i \times i^r_i) = \begin{bmatrix} i^w_{ix} \\ i^w_{iy} \\ i^w_{iz} \end{bmatrix} \times \begin{bmatrix} i^r_{ix} \\ i^r_{iy} \\ i^r_{iz} \end{bmatrix}$$

$$= \begin{vmatrix} i & j & k \\ i^w_{ix} & i^w_{iy} & i^w_{iz} \\ i^r_{ix} & i^r_{iy} & i^r_{iz} \end{vmatrix}$$

$$= \begin{bmatrix} i^w_{iy} & i^r_{iz} & -i^w_{iz} & i^r_{iy} \\ i^w_{iz} & i^r_{ix} & -i^w_{ix} & i^r_{iz} \\ i^w_{ix} & i^r_{iy} & -i^w_{iy} & i^r_{ix} \end{bmatrix}$$

$$D-3-11 \quad {}^0P_i \begin{pmatrix} {}_i\vec{w}_i \times {}_i\vec{r}_i \end{pmatrix} = \begin{bmatrix} {}_0i^P_{11} & {}_0i^P_{12} & {}_0i^P_{13} \\ {}_0i^P_{21} & {}_0i^P_{22} & {}_0i^P_{23} \\ {}_0i^P_{31} & {}_0i^P_{32} & {}_0i^P_{33} \end{bmatrix} \begin{bmatrix} {}_i^w_{iy} & {}_i^r_{iz} - {}_i^w_{iz} & {}_i^r_{iy} \\ {}_i^w_{iz} & {}_i^r_{ix} - {}_i^w_{ix} & {}_i^r_{iz} \\ {}_i^w_{ix} & {}_i^r_{iy} - {}_i^w_{iy} & {}_i^r_{ix} \end{bmatrix}$$

$$= \begin{bmatrix} ({}_0i^P_{13} \quad {}_i^r_{iy} - {}_0i^P_{12} \quad {}_i^r_{iz}) \quad {}_i^w_{ix} \\ ({}_0i^P_{23} \quad {}_i^r_{iy} - {}_0i^P_{22} \quad {}_i^r_{iz}) \quad {}_i^w_{ix} \\ ({}_0i^P_{33} \quad {}_i^r_{iy} - {}_0i^P_{32} \quad {}_i^r_{iz}) \quad {}_i^w_{ix} \end{bmatrix}$$

$$+ ({}_0i^P_{11} \quad {}_i^r_{iz} - {}_0i^P_{13} \quad {}_i^r_{ix}) \quad {}_i^w_{iy} \\ + ({}_0i^P_{21} \quad {}_i^r_{iz} - {}_0i^P_{23} \quad {}_i^r_{ix}) \quad {}_i^w_{iy} \\ + ({}_0i^P_{31} \quad {}_i^r_{iz} - {}_0i^P_{33} \quad {}_i^r_{ix}) \quad {}_i^w_{iy}$$

$$\begin{bmatrix} + ({}_0i^P_{12} \quad {}_i^r_{ix} - {}_0i^P_{11} \quad {}_i^r_{iy}) \quad {}_i^w_{iz} \\ + ({}_0i^P_{22} \quad {}_i^r_{ix} - {}_0i^P_{21} \quad {}_i^r_{iy}) \quad {}_i^w_{iz} \\ + ({}_0i^P_{32} \quad {}_i^r_{ix} - {}_0i^P_{31} \quad {}_i^r_{iy}) \quad {}_i^w_{iz} \end{bmatrix}$$

In actual practice, the vector ${}^0P_i ({}_i\vec{w}_i \times {}_i\vec{r}_i)$ will have three terms, instead of nine. The reason is that the i th joint will have only one degree of freedom. Hence, two of the factors w_{ix} , w_{iy} , and w_{iz} will be zero, eliminating six terms from equation (D3-11) above.

Now evaluate terms of the form ${}^0P_i \dot{w}_i$:

$$\begin{aligned}
 \text{D-3-12 } {}^0P_i \dot{w}_i &= \begin{bmatrix} {}^0i^P_{11} & {}^0i^P_{12} & {}^0i^P_{13} \\ {}^0i^P_{21} & {}^0i^P_{22} & {}^0i^P_{23} \\ {}^0i^P_{31} & {}^0i^P_{32} & {}^0i^P_{33} \end{bmatrix} \begin{bmatrix} \dot{w}_{ix} \\ \dot{w}_{iy} \\ \dot{w}_{iz} \end{bmatrix} \\
 &= \begin{bmatrix} {}^0i^P_{11} \dot{w}_{ix} + {}^0i^P_{12} \dot{w}_{iy} + {}^0i^P_{13} \dot{w}_{iz} \\ {}^0i^P_{21} \dot{w}_{ix} + {}^0i^P_{22} \dot{w}_{iy} + {}^0i^P_{23} \dot{w}_{iz} \\ {}^0i^P_{31} \dot{w}_{ix} + {}^0i^P_{32} \dot{w}_{iy} + {}^0i^P_{33} \dot{w}_{iz} \end{bmatrix}
 \end{aligned}$$

Again, since two of the components \dot{w}_{ix} , \dot{w}_{iy} , and \dot{w}_{iz} will be zero, the vector ${}^0P_i \dot{w}_i$ will have only three terms.

So, the final form of J is:

$$\text{D-3-13 } J_{\dot{w}} = \begin{bmatrix} {}^02^P_{1k} {}^2r_{2j} - {}^02^P_{1j} {}^2r_{2k} & \cdots & {}^07^P_{1k} {}^7r_{7j} - {}^07^P_{1j} {}^7r_{7k} \\ {}^02^P_{2k} {}^2r_{2j} - {}^02^P_{2j} {}^2r_{2k} & \cdots & {}^07^P_{2k} {}^7r_{7j} - {}^07^P_{2j} {}^7r_{7k} \\ {}^02^P_{3k} {}^2r_{2j} - {}^02^P_{3j} {}^2r_{2k} & \cdots & {}^07^P_{3k} {}^7r_{7j} - {}^07^P_{3j} {}^7r_{7k} \\ & {}^02^P_{1i} & \cdots & {}^07^P_{1i} \\ & {}^02^P_{2i} & \cdots & {}^07^P_{2i} \\ & {}^02^P_{3i} & \cdots & {}^07^P_{3i} \end{bmatrix} \begin{bmatrix} \dot{w}_{2i} \\ \dot{w}_{3i} \\ \dot{w}_{4i} \\ \dot{w}_{5i} \\ \dot{w}_{6i} \\ \dot{w}_{7i} \end{bmatrix}$$

where, for each column, one of the following three cases holds:

- | | | |
|---------------|---------------|---------------|
| 1) $i = x, 1$ | 2) $i = z, 3$ | 3) $i = y, 2$ |
| $j = y, 2$ | $j = x, 1$ | $j = z, 3$ |
| $k = z, 3$ | $k = y, 2$ | $k = x, 1$ |

Which of the three choices is to be used depends upon the joint axis of rotation.

All that remains is determining the solution to the matrix rate equations.

4.0 SOLVING MATRIX EQUATIONS

A matrix equation with as many equations as unknowns has three possible classes of solutions: (1) one unique solution, (2) infinitely many solutions, and (3) no solution. If a unique solution exists, it can be found by inverting the matrix J . If many solutions exist, some criteria must be developed by which to select the best solution. If no solution exists, then the closest approximation to a solution must be found.

One technique of matrix inversion is the simple procedure generally taught in general math courses--Gaussian elimination. The procedure for inverting a nonsingular matrix M is as follows:

- 1) Form the augmented matrix $(M : I) = A$;
- 2) Use elementary row operations to change M into the identity matrix. These operations are
 - a) Interchange the position of rows,
 - b) Multiply a row by a constant,
 - c) Add a row times a constant to another row.

These steps yield an augmented matrix of the form $B = (I : M')$, instead of $A = (M : I)$. It can be shown that $M^{-1} = M'$.

In general, a linear system $M\mathbf{x} = \mathbf{b}$ is solved by finding M^{-1} , then $\mathbf{x} = M^{-1}\mathbf{b}$. However, using Gaussian elimination, it is not necessary to explicitly find M^{-1} .

It can be shown that the performance of elementary row operations is equivalent to multiplication by a matrix:

$$(M : I) \longrightarrow \text{row operations} \longrightarrow (I : M^{-1})$$

is equivalent to:

$$(M : I) \longrightarrow P(M : I) = (PM : PI) = (I : M^{-1})$$

Furthermore, it can be shown that $P = M^{-1}$. Therefore,

$$P(M : \mathbf{b}) = (PM : P\mathbf{b}) = (I : M^{-1}\mathbf{b}) = (I : \mathbf{x})$$

This means that the solution $\mathbf{x} = M^{-1}\mathbf{b}$ can be found by row-reducing the matrix $(M : \mathbf{b})$ to the form $(I : \mathbf{x})$.

4.1 THE PSEUDO-INVERSE

Not all matrix equations have unique solutions. It is possible for a system to have no solution, or for the system to have an infinite number of solutions. As might be expected, each of the above alternatives leads to a different approach to solving the system.

If a system of equations has no solution, then a best or closest system solution must be estimated. Of course, the best possible estimate is desired. One criterion for a best estimate is minimal difference between estimated and true output. That is, if $A\mathbf{x} = \mathbf{b}$ is the original system (\mathbf{x} is unknown) and \mathbf{x}' is an approximation with $A\mathbf{x}' = \mathbf{b}'$, then we seek to minimize $|\mathbf{b} - \mathbf{b}'|$. We call the approximation that produces the minimum difference the "least squares" solution, or "least error" solution.

The situation that has no solution often arises in "overdetermined" systems, in which there are more equations than unknowns describing the system.

If A is of full rank--i.e., if all the columns of A are linearly independent--then the least-squares solution, $\bar{\mathbf{x}}$, of $A\mathbf{x} = \mathbf{b}$ is given by

$$D4-1 \quad \bar{\mathbf{x}} = (A^t A)^{-1} A^t \mathbf{b}$$

If a system of equations has an infinite number of solutions, then some criterion must be derived to select a solution. If the criterion is that of minimal vector length in the Euclidean sense, we call the solution selected the "least effort" solution. That is, if $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots)$ all satisfy $A\mathbf{x} = \mathbf{b}$, then the least effort solution is $\hat{\mathbf{x}}$ where

$$D4-2 \quad \hat{\mathbf{x}} = \min_j |\mathbf{x}_j|$$

The situation of infinite solutions often arises in "underdetermined" systems in which there are fewer equations than unknowns. In this case, there are not enough constraints to completely determine the system. The underdetermined system can be represented by $A\mathbf{x} = \mathbf{b}$ where A is $n \times k$; \mathbf{x} is $k \times 1$; \mathbf{b} is $n \times 1$; and $n < k$. If A is of full rank, then the solution of least length, $\hat{\mathbf{x}}$, or the least effort solution, is given by

$$D4-3 \quad \hat{\mathbf{x}} = A^t (AA^t)^{-1} \mathbf{b}$$

The pseudo-inverse is a matrix that combines all the above cases--as well as that of the unique solution--into a single matrix formula. Let A^+ denote the pseudo-inverse of a matrix A . Then, for $A\mathbf{x} = \mathbf{b}$,

- 1) $A^+ = A^{-1}$ if A is square and nonsingular;
- 2) $A^+ = (A^t A)^{-1} A^t$ if A is $n \times k$ of full rank ($n > k$);
- 3) $A^+ = A^t (AA^t)^{-1}$ if A is $n \times k$ of full rank ($n < k$).

In addition, if A is not of full rank, then A^+ provides the least-error solution of least length.

The pseudo-inverse can be found by row-reducing the matrix A (Gaussian elimination) to a reduced row-echelon form.

If A is $n \times k$ of rank r , this reduction proceeds to the point where,

- 1) The first r rows of A are nonzero;
- 2) The last $n - r$ rows of A are identically zero;
- 3) The first r linearly independent columns of A are the first r columns of the identity matrix.

e.g., $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are reduced row-echelon forms;

$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 5 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are not reduced row-echelon forms.

Given a reduced row-echelon form of A , then A can be written as $A = CD$, where C is the $n \times r$ matrix consisting of the first r linearly independent columns of A (these are in the same positions as the columns of the identity matrix in the reduced row-echelon form), and D is the $r \times k$ matrix consisting of the non-zero rows of the row-echelon form of A . The equation $A = CD$ is called a rank factorization of A .

If $A = CD$ is a rank factorization, then it is easy to find A^+ :

D4-4 $A^+ = D^t (DD^t)^{-1} (C^t C)^{-1} C^t$

A procedure for solving the matrix problem

D4-5 $\begin{bmatrix} 0^v_e \\ \vdots \\ 0^w_e \end{bmatrix} = J \begin{bmatrix} 2^w_{2j} \\ \vdots \\ 7^w_{7k} \end{bmatrix}$

can now be outlined.

First, formulate the problem as if J were nonsingular and proceed with row-reduction to find the inverse. If row-reduction produces one or more rows of zeroes, then J is singular, and the pseudo-inverse must be found. Use the row-reduced submatrix K to find a rank factorization of K . This, in turn, is used to find the pseudo-inverse of K , which can be used to solve the system.

The pseudo-inverse of K may be used instead of the pseudo-inverse of J , provided the row-reduction matrix operations are used on $\begin{bmatrix} \underline{v} \\ \underline{w} \end{bmatrix}$. Consider the system

D4-6 $A\underline{x} = \underline{b}$

where A and \underline{b} are known. Solve for \underline{x} . This is completely analogous to the original equation (D3-9).

Form the augmented matrix and proceed with Gaussian elimination:

$$D4-7 \quad [A \vdots I]$$

If A is singular, row-reduce matrix A as far as possible to obtain matrix K . The augmented matrix becomes

$$D4-8 \quad [K \vdots E]$$

where E is the matrix operator that transforms A to K :

$$D4-9 \quad E[A \vdots I] = [EA \vdots EI] = [K \vdots E]$$

Solving the system using the pseudo-inverse of K is equivalent to solving the original system if we replace \underline{b} by $E\underline{b}$:

$$D4-10 \quad \hat{\underline{x}} = K^+ E\underline{b} = (EA)^+ E\underline{b} = A^+ E^+ E\underline{b} = A^+ \underline{b}, \text{ since } E^+ E = I.$$

so

$$D4-11 \quad \hat{\underline{x}} = A^+ \underline{b}, \text{ or } \hat{\underline{x}} = K^+ E\underline{b}$$

Using the pseudo-inverse of K is easier computationally because the formula for K^+ reduces to

$$D4-12 \quad K^+ = G^t (GG^t)^{-1} F^t$$

where $K = FG$ is a rank factorization of K .

To see this, if K is a row-reduced matrix, then the first r linearly independent columns are only the first r columns of the identity matrix. Hence,

$$D4-13 \quad F = \left\{ \begin{bmatrix} I \\ \vdots \\ 0 \end{bmatrix} \right\} \begin{matrix} n \text{ rows} \\ n-r \text{ rows} \end{matrix}$$

and

$$D4-14 \quad (F^t F)^{-1} = \left([I \vdots 0] \begin{bmatrix} I \\ \vdots \\ 0 \end{bmatrix} \right)^{-1} = (I \cdot I + 0 \cdot 0)^{-1} = I^{-1} = I$$

D4-15 Thus,

$$(F^t F)^{-1} F^t = I F^t = F^t$$

which yields the simplified formula for K^+ .

A flow diagram for the problem is presented in Figure D4-1.

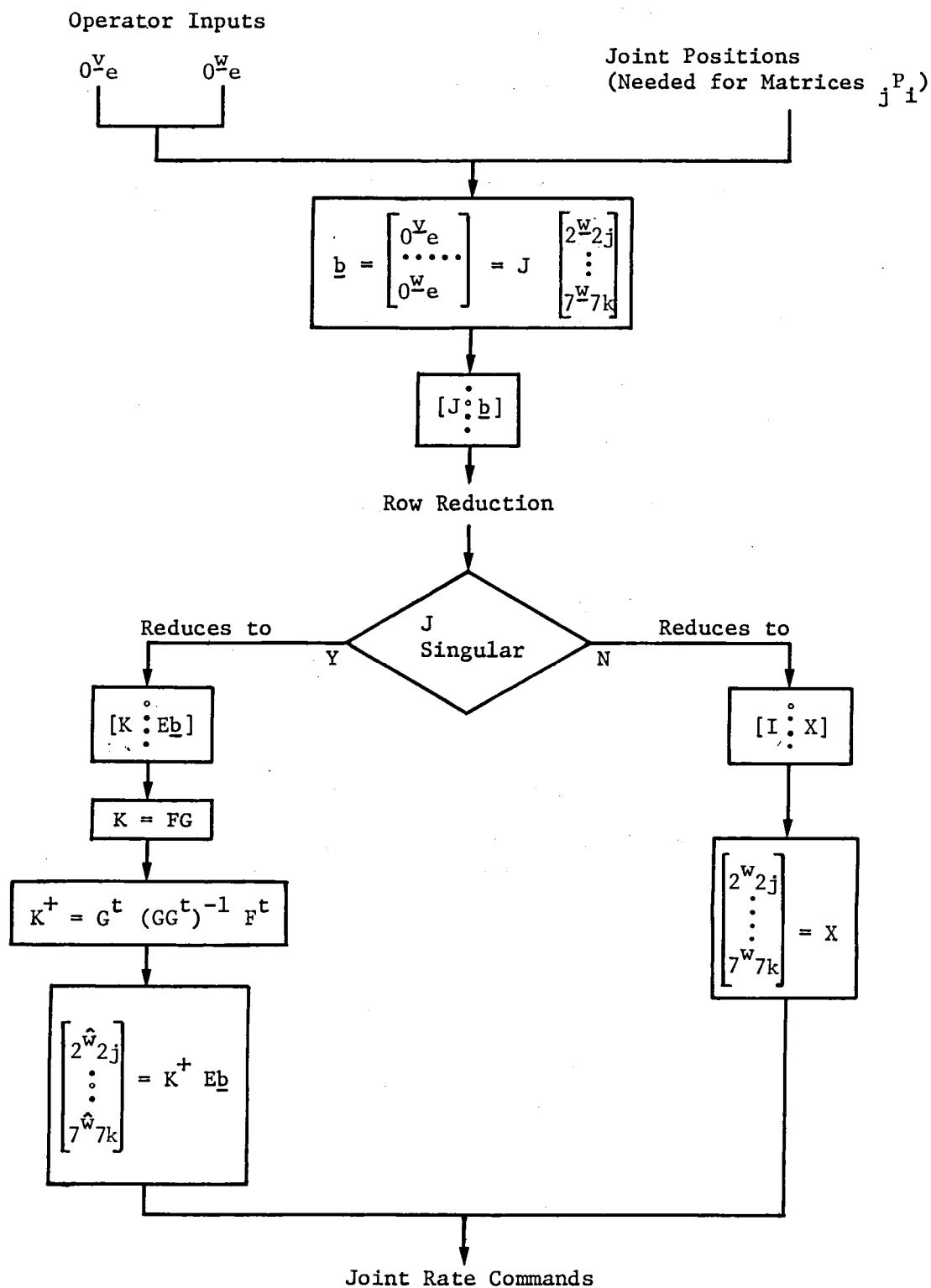


Figure D4-1 Joint Rate Algorithm Flow Diagram

For a more detailed description of the development and the properties of the pseudo-inverse, see Subsection D7 of this report.

The solution obtained for the matrix problem is only a instantaneous solution. That is, it remains correct only in a finite interval of time or distance from the position and instant in which it was calculated. This is true for several reasons:

- 1) Because the joints of the SOS manipulator are rotational, the end-effector motion will generally be only approximately linear. In fact, the end-effector path will usually be curved.
- 2) The values used in computing the J matrix depend on joint positions, which change as the manipulator moves. Therefore, the values of J (and hence J^{-1}) are constantly changing.

To produce accurate motion, the matrix problem must be solved repeatedly. Frequently computed joint commands produce more accurate motion. Therefore, part of the problem in calculating the system is to solve it as rapidly as possible. This section briefly explores various techniques that can be used to speed up the solution of the matrix problem.

The method originally employed for use in the SOS manipulator was to separate the translational and rotational control problems, and solve each using a different set of joints.

The method implemented was to use the shoulder roll joint as separate operator input. The remaining three joints of the arm below the wrist were used to affect translational motion. Translational commands were input, producing a 3 x 3 control matrix that could easily be inverted to find the needed joint rates.

The angular velocity of the end effector produced by these joints was then calculated and subtracted from the input angular velocity commands to obtain the net angular velocity. The three wrist joints were used to affect this net velocity, requiring the solution of a second 3 x 3 matrix.

The flow chart for this control system is depicted in Figure D5-1.

Using this scheme, it is only necessary to invert small matrices, so that the entire solution process is accelerated. A problem with this technique is that while rotational effects of shoulder and elbow joint motion are considered in solving the rotation problem, the translational effects of wrist joint motion are not considered in solving the translation problem.

A second option is similar to the first, but feeds all results back into succeeding calculations. Briefly stated, the translation commands are input, and any translation due to wrist joint motion is subtracted from them. This operation yields the net translation. The shoulder and elbow joints are used to produce the net translation, requiring the inversion of a 3 x 3 matrix.

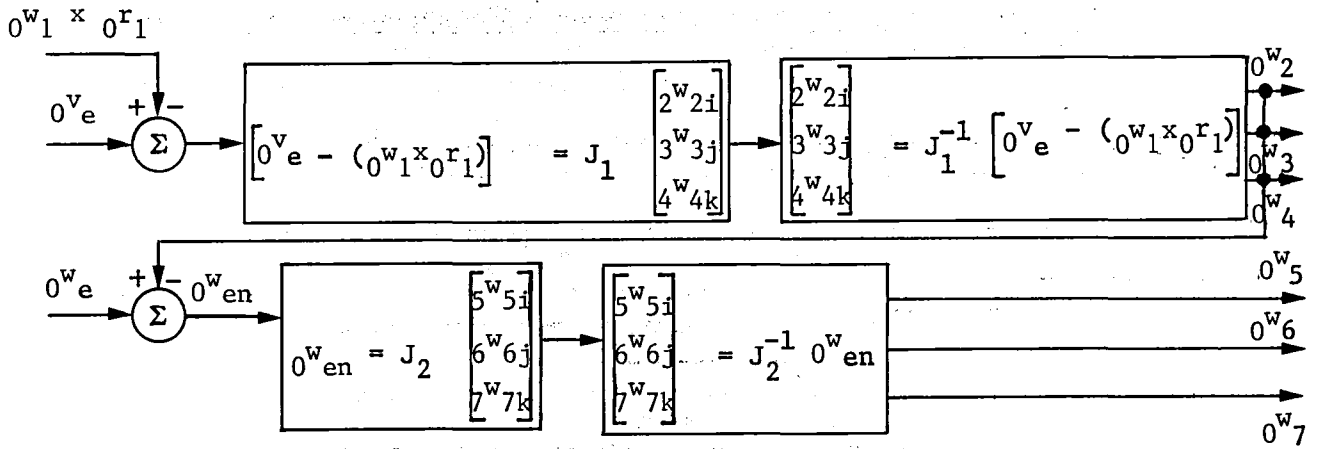


Figure D5-1 Separate Translational and Rotational Control Flow

The shoulder and elbow joint motions produce an end-effector rotation. This rotation is subtracted from the input commands to yield a net rotation. The wrist joints are used to effect the rotation, again requiring the inversion of a 3×3 matrix.

A flow diagram for this system is shown in Figure D5-2.

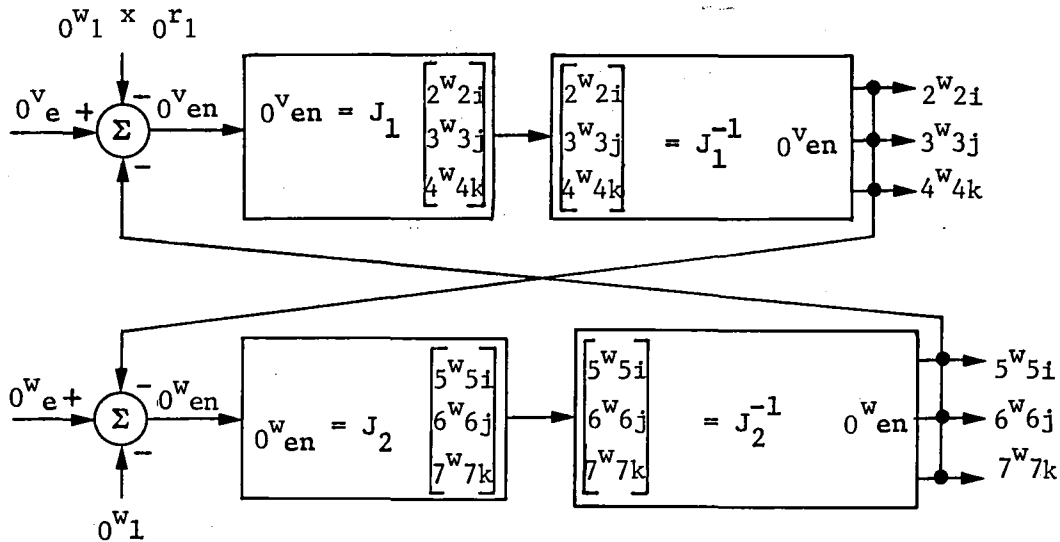


Figure D5-2 Combined Translational and Rotational Control Flow

This scheme accounts for the effects of each set of joints on both translation and rotation, but a problem may remain. It is not possible to exclude the case of an infinite loop in compensation. That is, it may be possible that a rotation $\Delta\theta$ may be necessary to compensate for a translation Δs , but to compensate for $-\Delta\theta$, $-\Delta s$ must be implemented, which means $-\Delta\theta$ is necessary. This matter must be solved in some way before such a scheme is implemented.

The problem considered in this document is that of interactive control of a robotic manipulator. The goal was to derive a method whereby manipulator end-effector motion can be directly controlled by human operators without having to consider the motion of each individual joint.

The solution was derived using the concept of coordinated rate control. End-effector motions are input as rates; therefore, there are six inputs--three translational rates and three rotational rates, corresponding to the three components of translational velocity and angular velocity vectors. These end-effector rates are used to produce the necessary joint rates, thus effecting the desired motion.

The transformation from end-effector rates to joint rates involves two concepts:

- 1) The resultant velocity is the vector sum of the constituent velocities.
- 2) The resultant angular velocity is the vector sum of the constituent angular velocities.

These concepts yielded the two equations:

$$6.1 \quad {}^0\mathbf{v}_e = {}^0P_2({}^2\mathbf{w}_2 \times {}^2\mathbf{r}_2) + \dots + {}^0P_7({}^7\mathbf{w}_7 \times {}^7\mathbf{r}_7)$$

and

$$6.2 \quad {}^0\mathbf{w}_e = {}^0P_2 {}^2\mathbf{w}_2 + \dots + {}^0P_7 {}^7\mathbf{w}_7$$

which were combined (after multiplying out terms) to yield the matrix equation

$$\begin{bmatrix} {}^0\mathbf{v}_e \\ \dots \\ {}^0\mathbf{w}_e \end{bmatrix} = J \begin{bmatrix} {}^2\mathbf{w}_{2j} \\ \vdots \\ {}^7\mathbf{w}_{7k} \end{bmatrix}$$

The matrix J may be singular, or nonsingular. This means that, given \mathbf{v}_e and \mathbf{w}_e , there may be one, more than one, or no vector $[{}^1\mathbf{w}_1, {}^2\mathbf{w}_2, \dots, {}^7\mathbf{w}_7]^t$ that produces the specific output motion. Hence, a versatile way of solving the system was sought. This solution method was the pseudo-inverse, and an algorithm was outlined (and a flow chart drawn) for computing the pseudo-inverse of an arbitrary matrix J .

Finally, timing considerations were investigated, yielding two methods of approximating the exact solution. Both methods require less calculation time using the pseudo-inverse than the exact method.

7.0 SOLUTION OF GENERAL LINEAR SYSTEMS

This section sketches a theoretical development of the matrix pseudo-inverse, and is intended for the interested reader who is unfamiliar with the concept of "solving" singular linear systems. A complete theoretical development can be found in (Ref. 5). The development proceeds by analyzing the problems of obtaining best solutions to linear systems with more equations than unknowns, then systems with fewer equations than unknowns. The tools developed to solve these systems are then combined, in a natural way, to yield the pseudo-inverse, the tool used to solve general linear systems.

7.1 OVERDETERMINED SYSTEMS

Consider the matrix equation

$$D7.1-1 \quad A\mathbf{x} = \mathbf{b}$$

where:

A is $n \times k$, known, $k < n$;
 \mathbf{b} is $n \times 1$, known;
 \mathbf{x} is $k \times 1$, unknown.

Note that, in applying matrix multiplication, the first element of \mathbf{x} multiplies every element in the first column of A . Similarly, the second element of \mathbf{x} multiplies every element in the second column of A . In fact, if the i^{th} column of A is denoted $A_{.i}$, then the equation above can be rewritten

$$D7.1-2 \quad A_{.1}x_1 + A_{.2}x_2 + \dots + A_{.k}x_k = \mathbf{b}$$

This formulation shows that, for the system to have a solution, \mathbf{b} must be a linear combination of the columns of A , or, \mathbf{b} must be an element of $CS(A)$, the column space of A .

If \mathbf{b} is not contained in $CS(A)$, a solution $\bar{\mathbf{x}}$ is desired such that the minimum error occurs between the desired system state \mathbf{b} and the system state $\bar{\mathbf{b}}$ produced by $\bar{\mathbf{x}}$. The Euclidean distance, $|\mathbf{b} - \bar{\mathbf{b}}|$, must be minimized to produce the best estimate for \mathbf{x} , denoted as $\bar{\mathbf{x}}$. The $\bar{\mathbf{b}}$ that minimizes $|\mathbf{b} - \bar{\mathbf{b}}|$ is the projection of \mathbf{b} on the column space of A .

Consider an element of an n -dimensional vector space V . $CS(A)$, the column space of A , is k -dimensional, with $k < n$. The Gram-Schmidt theorem says that the basis of $CS(A)$ can be extended using vectors orthogonal to those in the basis of $CS(A)$ to form a basis for the n -dimensional vector space V . Therefore, V can be considered as the sum of two vector subspaces-- $CS(A)$ and $CS(A)^\perp$ ($V = CS(A) + CS(A)^\perp$). The vector \mathbf{b} can now be written as the sum of two vectors, α contained in $CS(A)$, and β contained in $CS(A)^\perp$.

Since $\underline{b} = \underline{\alpha} + \underline{\beta}$, let \underline{b}_A be the projection of \underline{b} into $CS(A)$.

$$\text{Now, } \underline{b} - \underline{b}_A = \underline{\alpha} + \underline{\beta} - \underline{b}_A$$

$$\text{So, } |\underline{b} - \underline{b}_A|^2 = |\underline{\alpha} + \underline{\beta} - \underline{b}_A|^2 = |\underline{\alpha} - \underline{b}_A|^2 + |\underline{\beta}|^2$$

because $\underline{\alpha}$ and $\underline{\beta}$ are orthogonal. The vector \underline{b}_A is contained in $CS(A)$, so changing \underline{b}_A can only affect the first term, which is minimized when $\underline{\alpha} = \underline{b}_A$; i.e., when \underline{b}_A is the component of \underline{b} in $CS(A)$. Now that it has been demonstrated that \underline{b}_A minimizes $|\underline{b} - \underline{b}_A|$, the original vector equation can be reformulated.

$$\text{Recall } A\underline{x} = \underline{b}, \text{ and } A\underline{\bar{x}} = \underline{b}_A$$

$$\text{where } A_{.1}x_1 + A_{.2}x_2 + A_{.k}x_k = \underline{b}$$

D7.1-3

$$\text{and } A_{.1}\bar{x}_1 + \dots + A_{.k}\bar{x}_k = \underline{b}_A$$

D7.1-4

The vector $\underline{\bar{x}}$ consists of the components of \underline{b}_A in the basis of $CS(A)$. The components of $\underline{\bar{x}}$ can be obtained by taking the inner product of \underline{b} with the basis of $CS(A)$. Let $i \in \{1, 2, \dots, k\}$. Then for $\underline{b} = \underline{b}_A + \underline{b}_A'$

$$0 = \underline{b}_A' \cdot A_{.i}$$

D7.1-5

$$= (\underline{b} - \underline{b}_A) \cdot A_{.i}$$

$$= \underline{b} \cdot A_{.i} - \underline{b}_A \cdot A_{.i}$$

therefore,

$$\text{D7.1-6 } \underline{b}_A \cdot A_{.i} = \underline{b} \cdot A_{.i}$$

but

$$\text{D7.1-7 } \underline{b}_A = \bar{x}_1 A_{.1} + \bar{x}_2 A_{.2} + \dots + \bar{x}_k A_{.k} = \sum_{j=1}^k \bar{x}_j A_{.j}$$

thus,

$$\text{D7.1-8 } \underline{b}_A \cdot A_{.i} = \left(\sum_{j=1}^k \bar{x}_j A_{.j} \right) \cdot A_{.i}$$

$$= \sum_{j=1}^k \bar{x}_j (A_{.j} \cdot A_{.i})$$

Recall that, for any vectors $\underline{\alpha}$ and $\underline{\beta}$ from the same space,

$$\text{D7.1-9 } \underline{\alpha} \cdot \underline{\beta} = \underline{\beta} \cdot \underline{\alpha} = \underline{\alpha}^t \underline{\beta}$$

Thus, the dot product in D7.1-8 can be rewritten:

$$\begin{aligned} \text{D7.1-10 } \underline{b}_A \cdot A_{.i} &= \sum_{j=1}^k \bar{x}_j (A_{.j}^t A_{.i}) \\ &= \bar{x}_1 A_{.1}^t A_{.i} + \dots + \bar{x}_k A_{.k}^t A_{.i} \end{aligned}$$

or, in vector/matrix form:

$$\text{D7.1-11 } \underline{b}_A \cdot A_{.i} = [A_{.1}^t A_{.i}, \dots, A_{.k}^t A_{.i}] \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_k \end{bmatrix}$$

Now, taking the dot product over all the vectors $A_{.i}$:

$$\text{D7.1-12 } \begin{bmatrix} \underline{b}_A \cdot A_{.1} \\ \vdots \\ \underline{b}_A \cdot A_{.k} \end{bmatrix} = \begin{bmatrix} A_{.1}^t A_{.1} & \dots & A_{.1}^t A_{.k} \\ \vdots & & \vdots \\ A_{.k}^t A_{.1} & \dots & A_{.k}^t A_{.k} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_k \end{bmatrix} = A^t A \underline{\bar{x}}$$

Noting that $\underline{b}_A \cdot A_{.i} = A_{.i} b_A = A_{.i}^t b_A$, equation D7.1-12 becomes

$$\text{D7.1-13 } A^t A \underline{\bar{x}} = \begin{bmatrix} \underline{b}_A \cdot A_{.1} \\ \vdots \\ \underline{b}_A \cdot A_{.k} \end{bmatrix} = A^t \underline{b}$$

The matrix $A^t A$ is $k \times k$, made up of the inner product of k linearly independent vectors. It is therefore nonsingular. Equation D7.1-12 can therefore be solved for $\underline{\bar{x}}$:

$$\text{D-7.1-14 } \underline{\bar{x}} = (A^t A)^{-1} A^t \underline{b}$$

The expression $(A^t A)^{-1} A^t$ is the general form of the matrix operation used to obtain a least-squares solution of $A \underline{x} = \underline{b}$, where A is $n \times k$ of rank k , $k \leq n$.

7.2 UNDERDETERMINED SYSTEMS

Consider the matrix equation

$$\text{D7.2-1 } A \underline{x} = \underline{b}$$

where

$A = n \times k$, known, full rank;

$\underline{x} = k \times 1$, unknown;

$\underline{b} = n \times 1$, known.

If $k > n$, then the matrix equation contains fewer equations than unknowns. Thus, fewer constraints exist to completely specify a unique solution, and thus the system is described as underdetermined.

Due to the lack of sufficient constraints, the system has an infinite number of solutions. These solutions are said to be in a hyperplane, (also called an affine space), which is a vector subspace that has been displaced from the origin.

Formally, a hyperplane can be defined by a vector added to every element of a subspace. That is, let U be a subspace, and \underline{q} be a vector that is not contained in U . Then \underline{q} and U define a hyperplane H by:

$$D7.2-2 \quad H = \underline{q} + U$$

This definition is shown intuitively in Figure D7-1 by considering the two-dimensional case. A system with two unknowns but only one constraint equation produces an infinite number of solutions, all of which lie on a line displaced from the origin. The line containing all the solutions is a one-dimensional hyperplane.

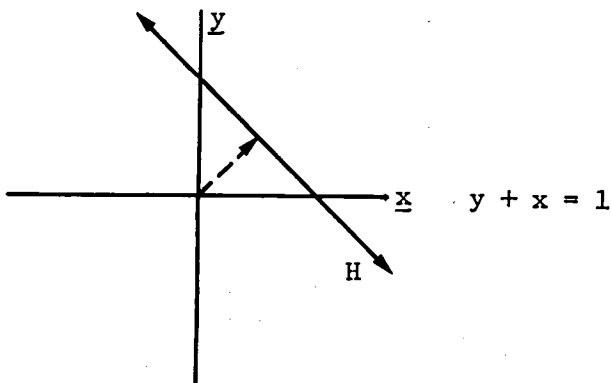


Figure D7-1 One-Dimensional Hyperplane

One criterion for selecting a solution from the infinite set is to choose the vector in the set that has the shortest Euclidean length. By examining the two-dimensional case (above), the shortest vector in the set is seen to be the vector from the origin that is perpendicular to the line.

In general, the solution sought in the underdetermined case is the vector from the origin perpendicular to the hyperplane. This solution is referred to as the "least-effort" solution.

Consider the matrix equation (D7.2-1) in terms of linear combinations of columns of A . In this expanded form, the equation is written as

$$D7.2-3 \quad A_{.1}x_1 + \dots + A_{.k}x_k = \underline{b}$$

Thus, \underline{b} can be rewritten as the sum of n vectors. Because each of the k vectors is n -dimensional, $n \leq k$, the vectors $A_{.i}$ are not linearly independent. (There can be a maximum of n vectors in a set of linearly independent vectors of dimension n .)

Therefore, there exists a vector \underline{y} such that

$$D7.2-4 \quad A_{.1}y_1 + \dots + A_{.k}y_k = 0$$

Any vector \underline{y} such that $A\underline{y} = 0$ is a member of the null space of A [written as $NS(A)$].

Let \underline{x} be a solution vector of $A\underline{x} = \underline{b}$. Now, if \underline{y} is contained in $NS(A)$, then $(\underline{x} + \underline{y})$ is also a solution. This is proved as follows:

$$\begin{aligned} A(\underline{x} + \underline{y}) &= A\underline{x} + A\underline{y} \\ D7.2-5 \quad &= \underline{b} + 0 \\ &= \underline{b} \end{aligned}$$

Every solution of $A\underline{x} = \underline{b}$ can be written as

$$D7.2-6 \quad \underline{x} = \underline{x}_p + \underline{x}_n$$

where \underline{x}_p is a solution of $A\underline{x} = \underline{b}$, and \underline{x}_n is an element of $NS(A)$. Let H be the set of all solutions of $A\underline{x} = \underline{b}$. Then, if \underline{x}_p is a given solution,

$$D7.2-7 \quad H = \underline{x}_p + NS(A)$$

That is, the solutions to the underdetermined system $A\underline{x} = \underline{b}$ lie in a hyperplane that is the null space of A translated from the origin.

As discussed above, the least-effort solution to an underdetermined system is the shortest vector from the origin to the hyperplane. This shortest vector is intuitively seen (as above) to be perpendicular to the hyperplane.

Given any solution, \underline{x}_p , of the equation $A\underline{x} = \underline{b}$, the perpendicular can be computed. The perpendicular is the projection of \underline{x}_p onto the space perpendicular to $NS(A)$. To see this, consider the two-dimensional case shown in Figure D7-2.

The solution, \underline{x} , is the sum of two vectors perpendicular to each other. The least-effort solution is therefore the vector component perpendicular to $NS(A)$ that is contained in every solution.

Let $NS(A)^\perp$ represent the vector space perpendicular to $NS(A)$.

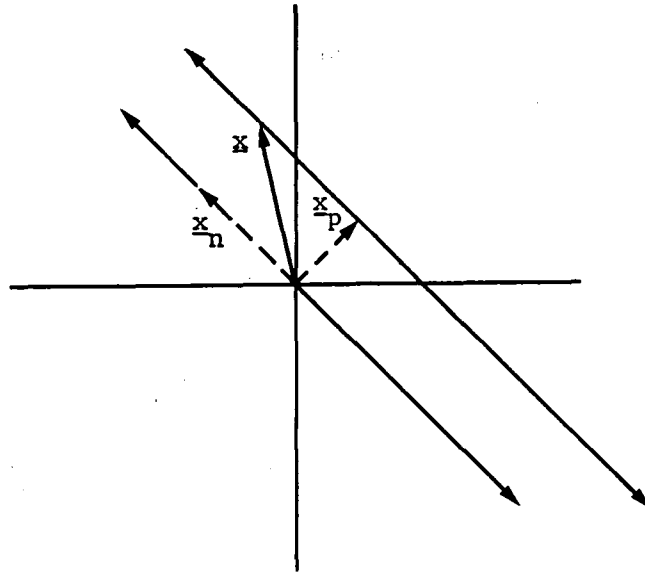


Figure D7-2
Two-Dimensional Underdetermined System

Lemma: The columns of A^t form a basis for $NS(A)^\perp$

Proof:

Let \underline{x}_n be in $NS(A)$. Then, if $\underline{y}^t \underline{x} = 0$, the vector \underline{y} is an element of $NS(A)^\perp$.

$$D7.2-8 \quad A \underline{x}_n = 0$$

can be rewritten as

$$D7.2-9 \quad A_{.1}x_1 + \dots + A_{.k}x_k = 0$$

or

$$A_{11}x_1 + \dots + A_{1k}x_k = 0$$

$$D7.2-10 \quad \vdots$$

$$A_{n1}x_1 + \dots + A_{nk}x_k = 0$$

but,

$$D7.2-11 \quad \begin{aligned} 0 &= A_{i1}x_1 + \dots + A_{ik}x_k = [A_{i1}, \dots, A_{ik}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= A_{i.} \underline{x} \\ &= (A_{.i})^t \underline{x} \end{aligned}$$

So every column of A^t , when taken in an inner product with \underline{x} , yields 0. Therefore, every column of A^t lies in $NS(A)^\perp$. Since there are n such columns and A is of full rank, all the n columns are linearly

independent. And, since the rank of $NS(A)$ is $k-n$, the rank of $NS(A)'$ is n . So, the columns of A^t are linearly independent and therefore form a basis for the n -dimensional space $NS(A)'$.

The least-effort solution of equation (D7.2-1) can now be found. Let $\hat{\underline{x}}$ denote the least-effort solution. Then $\hat{\underline{x}}$ is a linear combination of the columns at A^t :

$$\hat{\underline{x}} = c_1 A^t_{\cdot 1} + c_2 A^t_{\cdot 2} + \dots + c_n A^t_{\cdot n}$$

$$D7.2-12 \quad \hat{\underline{x}} = A^t \underline{c}$$

Now

$$D7.2-13 \quad A \hat{\underline{x}} = \underline{b}$$

or,

$$D7.2-14 \quad A(A^t \underline{c}) = \underline{b}$$

so

$$D7.2-15 \quad (AA^t) \underline{c} = \underline{b}$$

and

$$D7.2-16 \quad \underline{c} = (AA^t)^{-1} \underline{b}$$

But \underline{c} is not the vector sought. The least-effort solution is $\hat{\underline{x}} = A^t \underline{c}$. Multiplying both sides of the equation (D7.2-16) by A^t , yields

$$D7.2-17 \quad \hat{\underline{x}} = A^t \underline{c} = A^t (AA^t)^{-1} \underline{b}$$

The expression $A^t (AA^t)^{-1}$ is the general form of the matrix operator used to obtain the least effort solution of $A \underline{x} = \underline{b}$, where A is $n \times k$, $n \leq k$.

7.3 GENERAL SYSTEMS

The solution to general classes of matrix equations becomes the combination of two solutions: (1) solutions to the overdetermined problem and (2) solutions to the underdetermined problem.

Consider $A \underline{x} = \underline{b}$

where

A is $n \times k$, known;
 \underline{x} is $k \times 1$, unknown;
 \underline{b} is $n \times 1$, known.

For the general class of problems, n can be greater than k , or, vice versa, n can be less than k , or n can be equal to k .

Assume A is of the rank r , $r \leq \min(n, k)$. Suppose A is rank-factored into

$$7.3-1 \quad A = CD$$

where each submatrix C , D is full rank. Then $A = CD$ is called a "rank-factorization" of A .

Then the equation to be solved may be rewritten as

$$7.3-2 \quad CD\mathbf{x} = \mathbf{b}$$

Let $\mathbf{z} = D\mathbf{x}$; then $C\mathbf{z} = \mathbf{b}$.

C is $n \times r$ and full rank, so the system is overdetermined. The solution to such an overdetermined system is given by

$$D7.3-3 \quad \mathbf{z} = (C^t C)^{-1} C^t \mathbf{b}$$

However, $\mathbf{z} = D\mathbf{x}$ where D is $r \times k$, $r \leq k$, and full rank so that the system $\mathbf{z} = D\mathbf{x}$ underdetermined. The solution to an underdetermined system is given by

$$D7.3-4 \quad \hat{\mathbf{x}} = D^t (DD^t)^{-1} \mathbf{z}.$$

The complete solution of the general matrix problem is the combination of equations (D7.3-3) and (D7.3-4). This solution is given by

$$D7.3-5 \quad \hat{\mathbf{x}} = D^t (DD^t)^{-1} (C^t C)^{-1} C^t \mathbf{b}$$

The pseudo-inverse of matrix A is denoted as A^+ :

$$D7.3-6 \quad A^+ = D^t (DD^t)^{-1} (C^t C)^{-1} C^t$$

It can be shown that A^+ yields the least-effort solution to the least-error problem. If A is nonsingular, it can be shown that $A^+ = A^{-1}$.

Lemma: $A^+ = A^{-1}$ for A nonsingular.

Proof:

We need to show $A^+A = AA^+ = I$. We will demonstrate only $A^+A = I$, since the proof of the second part is virtually identical.

$$\begin{aligned} A^+A &= D^t (DD^t)^{-1} (C^t C)^{-1} C^t CD \quad \text{since } A = CD \\ &= D^t (DD^t)^{-1} (C^t C)^{-1} (C^t C) D \\ &= D^t (DD^t)^{-1} D \quad \text{since } (C^t C)^{-1} (C^t C) = I \end{aligned}$$

Because A is non-singular, and C , D are of full rank. (by definition of rank-factorization), then C , D must also be non-singular. Therefore

$$\begin{aligned}
 A^+A &= D^t(DD^t)^{-1}D \\
 &= D^t(D^t)^{-1}D^{-1}D \quad \text{since } (AB)^{-1} = B^{-1}A^{-1} \\
 &\quad \text{for } A, B \text{ non-singular} \\
 &= I
 \end{aligned}$$

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Appendix E—Simulation Methodologies



APPENDIX E--SIMULATION METHODOLOGIES

The analysis of any physical system must begin by characterizing the specific process to be modeled. This characterization must consider (1) the system type, (2) governing equations of the process, and (3) system questions to be answered. Table E-1 shows these notions.

Table E-1 Process Characterization

- Types of Systems

- Deterministic, i.e., Noise-Free System
- Stochastic, i.e., Systems with Noise

- System Description

- Ordinary Differential Equations
 - Partial Differential Equations
 - Algebraic Equations
- } Linear or
} Nonlinear

- System Questions to be Answered

- Solution to Governing Equations
- Estimation
- Identification
- Control

Systems may be typed as being either deterministic or stochastic. A deterministic system is defined as one that incorporates no uncertainty and the stochastic type as one that includes uncertainty in the model. True systems may be represented by physical laws expressed by partial differential equations (PDE), ordinary differential equations (ODE), and algebraic equations (AE). Hence, the governing equations of the process must be given by PDE, ODE, AE, or a combination of the three. To answer the system questions, one must solve the governing equations. In the case of the deterministic type system, the solution may be easily obtained; however, the stochastic system presents additional concerns.

The solution of stochastic systems may be obtained by defining and solving the estimation, identification, and control (EIC) problems

associated with the given process. This study illustrates the need for solving the EIC problems as they pertain to simulation of a manipulator system.

Open-loop methods (OLM) and closed-loop methods (CIM) are considered. OLMs are described generally by systems that lack feedback from the operation that is occurring. An OLM is presented in Figure E-1.

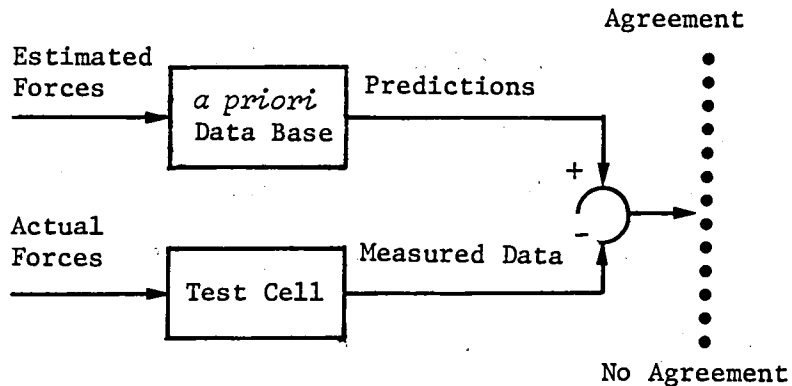


Figure E-1 Open-Loop Method

The open-loop methodology in Figure E-1 uses an *a priori* data base to provide predictions of outputs, which are compared with measured data from the test cell. Results can range from complete agreement to no agreement at all. In addition, the confidence level of results of a given test is typically low until a large statistical data base indicates the attributes of the samples. Typically, as one begins to accumulate test data, the predictions are adjusted to accommodate these data. This "knob tweaking" is usually conducted by methods that are far from being mathematically rigorous.

Because OLMs have considerable shortcomings, the need for a more systematic approach has become evident in recent years. Modern system theory provides such an approach based on CIMs that form the bases for most applications where feedback is used to control the system. The CIM is shown in Figure E-2 by using adaptive procedures that provide a model with quantified confidence levels. Modern system theory recognizes the potential differences in predictions and results that can be attributed to forcing function uncertainty, incorrect estimates of constituents in the governing equations, and the possibility that the model order is insufficient to describe these systems. The adaptive processor design attempts to accommodate these uncertainties and to systematically drive the difference between predictions and test data to a minimum.

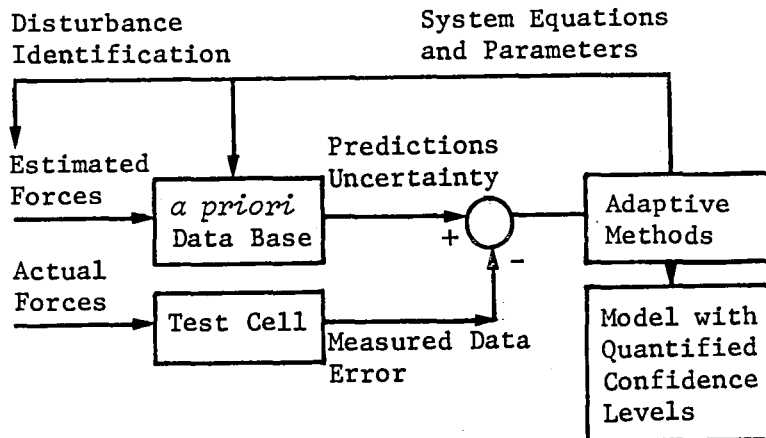


Figure E-2 Closed-Loop Method

The theoretical aspects required to obtain the systematic results of the CLM are introduced by presenting the thesis: "Control of the error that results from the comparison of data obtained from the true system to that produced by an a priori data base will result in:

- 1) Obtaining a classification set, which implies the system can be modeled (this, in turn, implies that the system equations can be solved);
- 2) Or denying the classification set, which implies that the system cannot be modeled."

Acceptance of this thesis implies that the solution of the control problem is imperative to model the system.

The term "classification set" has been introduced and needs further clarification.

The notion of a set is generally defined in mathematical terms. In particular, this set, C , may be given as the set that contains the following elements: independent variables, dependent variables, coefficients of the governing equations, and the sensor output. Mathematically this set can be written as:

$$[E-1] \quad C \triangleq \left\{ \begin{array}{llll} \text{INDEPENDENT} & \text{DEPENDENT} & \text{COEFFICIENTS} & \text{SENSOR} \\ \text{VARIABLES} & \text{VARIABLES} & \text{OF} & \text{OUTPUT} \\ & & \text{EQUATIONS} & \end{array} \right\}$$

Because this definition for the classification set holds for the propagation of a signal through any system, the elements of this set will be obtained from the equations that govern the evolution of the ROBSIM control signal. Equation [E-2] shows the relationship of estimation, identification, and control to the classification set. For example,

a solution to the estimation problem will yield estimates of the independent and dependent variables. Similarly, the solution to control encompasses the solution of both estimation and identification, in addition to providing an estimate of sensor output.

$$[E-2] \quad C \triangleq \left\{ \underbrace{\begin{array}{cc} \text{INDEPENDENT} & \text{DEPENDENT} \\ \text{VARIABLES} & \text{VARIABLES} \end{array}}_{\text{ESTIMATION}} \quad \underbrace{\begin{array}{c} \text{COEFFICIENTS} \\ \text{OF} \\ \text{EQUATIONS} \end{array}}_{\text{IDENTIFICATION}} \quad \text{SENSOR OUTPUT} \right\}$$

CONTROL

Equation [E-1] presents the classification set in terms of theory. Members of this set can also be defined in terms of simulation of a manipulator system in the following way:

$$[E-3] \quad C \triangleq \left\{ \underbrace{\begin{array}{cc} \text{REFERENCE} & \text{CONTROL AND} \\ \text{INPUT} & \text{STATE} \\ & \text{VARIABLES} \end{array}}_{\text{ESTIMATION}} \quad \underbrace{\begin{array}{c} \text{DYNAMIC,} \\ \text{CONTROL AND} \\ \text{MEASUREMENT} \\ \text{MODIFIERS} \end{array}}_{\text{IDENTIFICATION}} \quad \text{UNCERTAINTIES} \quad \text{SENSOR OUTPUT} \right\}$$

CONTROL

where the corresponding EIC relationships are shown.

Before further clarification of the classification set is feasible, the basic elements of the physical process must be considered. To do this for the OLM, a lower layer of Figure E-1 must be produced. This second layer is shown in Figure E-3. From this figure, two basic elements may be defined as being (1) the subsystem under evaluation, and (2) a measurement device. The subsystem under investigation produces a measurable set, M, and the measurement device acting on this set produces the output, Z. In addition to illustrating the basic elements, Figure E-3 shows the basis for their physical makeup.

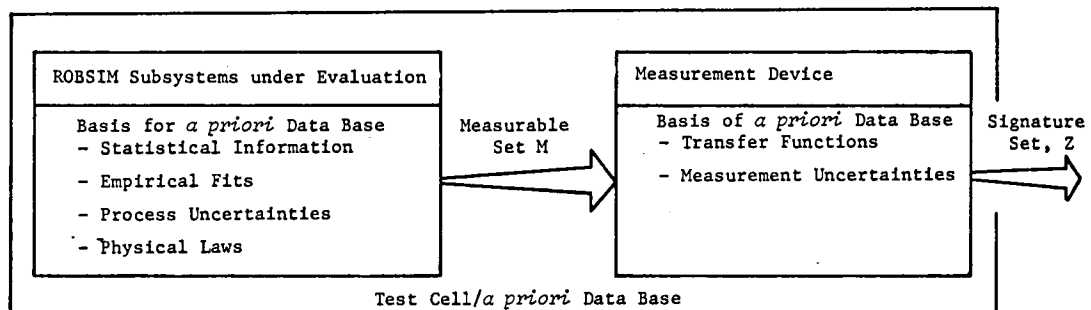


Figure E-3 Second Layer of Physical Process

However, the description of the process does not allow total definition of the classification set. For example, the only elements of the classification set that are illustrated are the measurable set, M , and the signature set, Z . In addition, the measurable set contains only some of the dependent variables, none of the independent variable, and none of the coefficient set.

This fact requires a further analysis of the layers to completely define the notion of the classification set. Unfortunately, the third layer (Fig. E-4) is still not sufficient to completely identify this set. However, additional element members can be defined. Specifically, some members of the observable set, O , are distinguishable, as well as all of the independent variables (i.e., internal and external stimuli). A further reduction is required to obtain the complete dependent variable and coefficient sets. This juncture requires a description of the specific problem of interest. Because this is the case, further clarification of the classification set will be postponed until the clarification can be applied to a specific example.

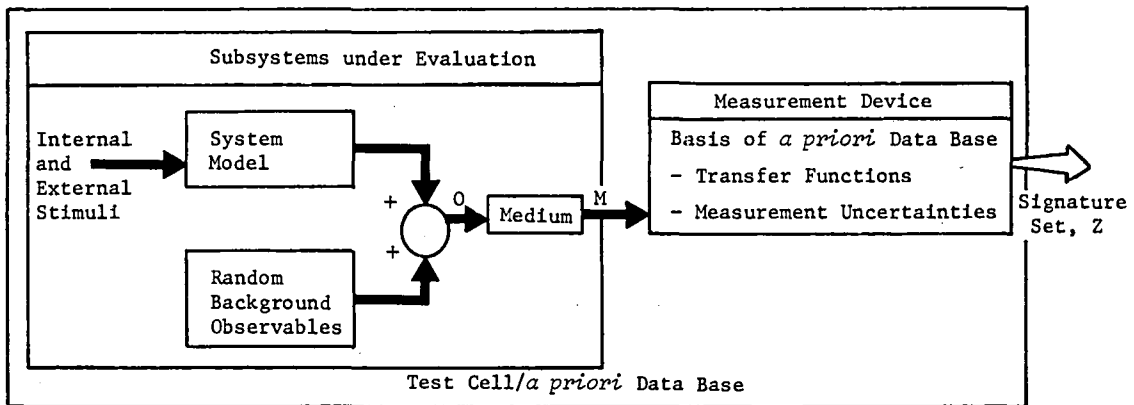


Figure E-4 Third Layer Mechanization

The definition of the system level problems involves terms such as state variable, control vector, and observations. For a definition of these terms, consider the fourth-layer schematic of Figure E-1 as shown by Figure E-5.

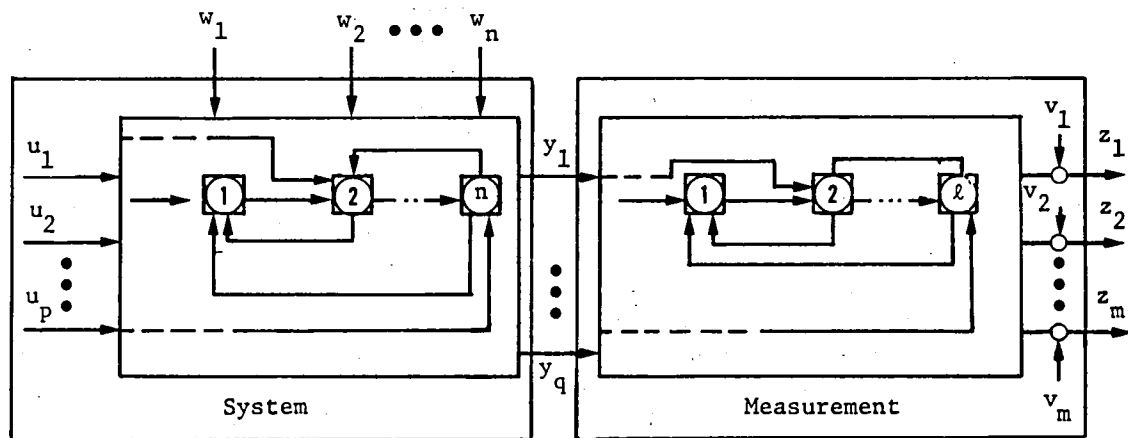


Figure E-5 Fourth Layer a priori Model

The figure represents a typical system in terms of its characteristics-- independent and dependent variables. Dependent variables are defined as state variables and independent variables are designated as the control variables.

The location of these quantities is internal to block 1 through n . In Figure E-5, the independent variables are designated by the p multi-inputs, u . The dependent variables, x , are imbedded in the system components designated by blocks 1 through n . Definition of the system state vector would be incomplete without accounting for background uncertainties. The uncertainties are represented by the w terms illustrated in Figure E-5. Note that the measurable set (i.e., outputs of the system) generally does not consist of all dependent or state variables.

The sensor system acts on the measurable set. This action takes place dynamically and statically. Specifically, these measurables enter into the sensing device and are subjected to phase and amplitude distortion (i.e., dynamics) and ultimately sensed as a voltage (i.e., static). Hence, dynamics of the sensor must be accounted for in addition to dynamics of the system. In this sense, dynamics attributable to the sensor are typically augmented to the state vector with dimension n to produce a system state of dimension $n+l$. Because sensing devices have errors associated with them, the signature produced by the sensor will be corrupted by this uncertainty. This is designated by the quantity, v (Fig. E-5). Addition of the noise quantity produces a signature set designated by the term, z .

To solve the EIC problems associated with simulation of a manipulator system, it is imperative that the system be modeled by using a state-variable (SV) format.

The SV framework was motivated by: (1) solutions to the estimation, identification, and control problems that require a SV representation; (2) control mobility and data processing/extraction algorithms that can be derived from solutions to the EIC problems; (3) use of SV that do not limit design to the time domain (i.e., state variables can be expressed in the frequency domain, thus allowing classical design).

To illustrate this concept, consider Figure E-6, which represents a simplified block diagram of the simulation tool elements.

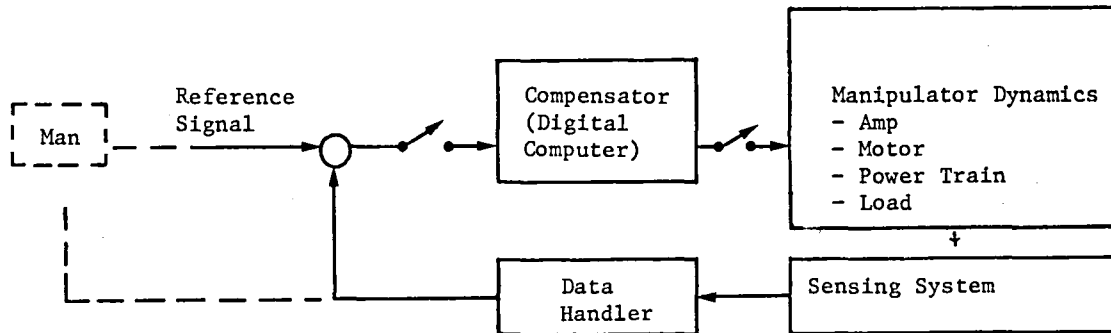


Figure E-6 Simulation Tool Simplified Elements

Because this effort is to mathematically model the manipulator system such as that depicted in Figure E-6, this system must be represented in a mathematical format. Figure E-7a represents simplified elements of the simulator tool in a mathematical context. Note that the system as configured will have discrete and continuous components. Because systems of this type are difficult to analyze, it is convenient to discretize the entire system. Figure E-7b illustrates an equivalent discrete system.

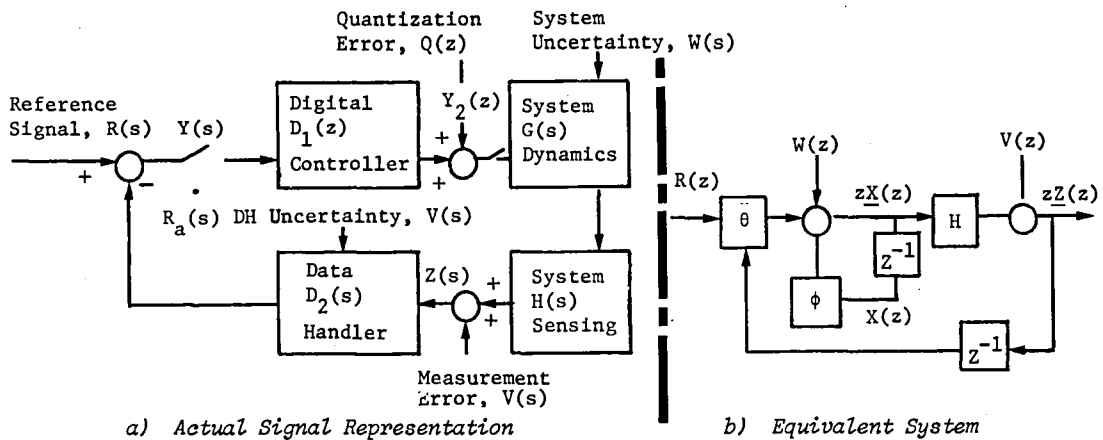


Figure E-7 Simulation Tool Element Representations

To further justify the use of SV as the framework of the simulation, Figures E-8a and E-8b illustrate that design and analysis of the system are not limited to either the time or frequency domain. (These figures show the system equivalence in both domains.)

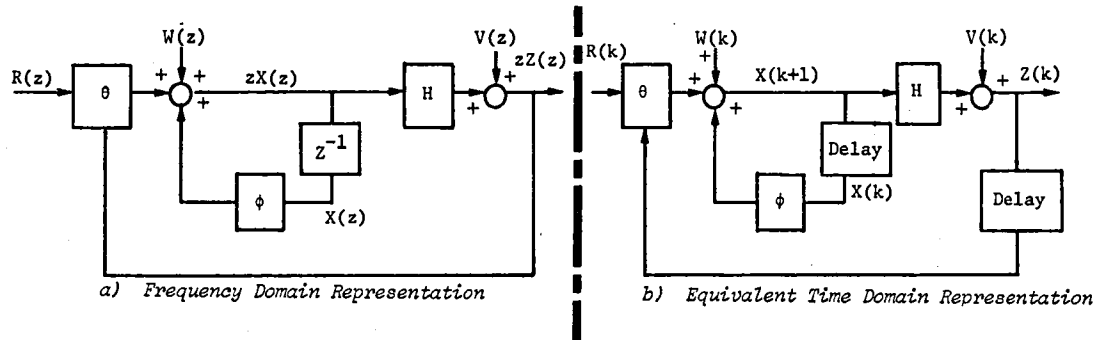


Figure E-8 Equivalent Systems

As a further justification of the SV framework, consider an example of the problem: Suppose that it is desirable to have the manipulator follow a prescribed reference trajectory given by $r(1)$, $r(2)$, ..., $r(n)$. Assume that certain members of the classification set are known. Specifically they are the reference input, random uncertainty, all measurement modifiers, and certain members of the dynamic and control modifiers.

To solve the control problem it is imperative to eliminate the uncertainty of the system. To do this, estimates of the control and state variables are required. Having obtained these results, the identification problem can be solved. Solution of the identification problem will produce an optimal digital controller for the system.

Consider the system given by Figure E-7b. The classification set for this configuration is given as

			DYNAMIC, CONTROL AND MEASUREMENT MODIFIERS		
	REFERENCE INPUT	CONTROL AND STATE VARIABLE		UNCERTAINTIES	SENSOR OUTPUT
[E-4]					
C \triangleq {	<u>r</u>	<u>x</u> σ_{x_i}	ϕ θ H	σ_w σ_v	<u>z</u>
		ESTIMATION	IDENTIFICATION		
	CONTROL				

where the roles of estimation and identification are labeled. Given the set of assumptions, it should be noted that the identification problem is limited to solving uncertain parameters in ϕ and θ -- specifically those elements that pertain to the digital compensator. Figure E-9 shows how this process is mechanized, and the use of the Kalman filter to produce estimates of the state vector, \hat{x} , and propagation of the uncertainty in the estimate, $\sigma_{\hat{x}}$; and describes how the performance measure, J , is used to update parameters in ϕ and θ .

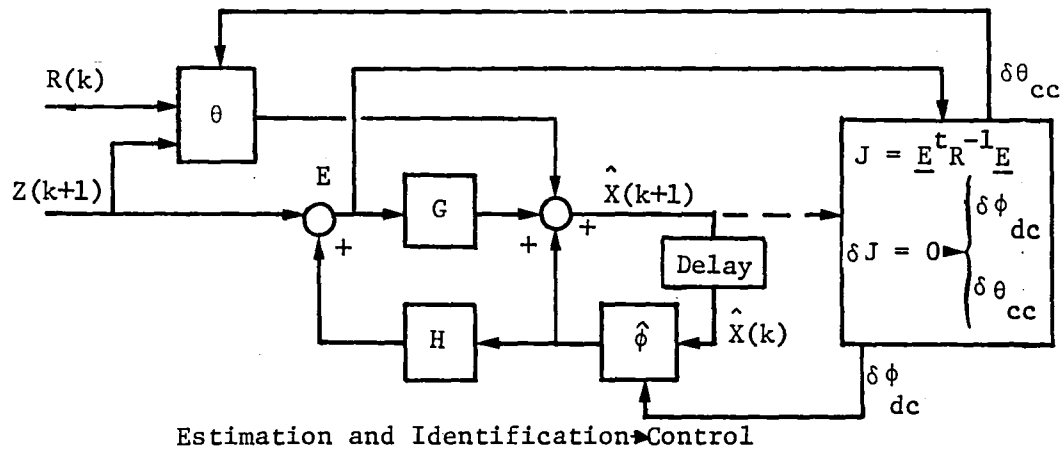


Figure E-9 Control Mechanization

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16. Abstract This report consists of Appendices B, C, D, and E to NASA Contractor Report NASA CR-165975. Appendix B describes the derivation of a set of dynamics equations that can be used recursively to calculate forces and torques acting at the joints of an n-link manipulator given the manipulator joint rates. The equations are valid for any n-link manipulator system with any kind of joints connected in any sequence. Appendix C describes the derivation of the equations of motion for the class of manipulators consisting of n rigid links interconnected by rotary joints. A technique is outlined for reducing the system of equations to eliminate constraint torques. Appendix C also contains the derivation of the linearized dynamics equations for an n-link manipulator system. The general n-link linearized equations are then applied to a two-link configuration. Appendix D describes the coordinated rate control algorithm used to compute individual joint rates when given end effector rates. Appendix E presents a short discussion of simulation methodologies.					
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