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STRUCTURE OF FIELD ROTATING DISTURBANCES IN WARM PLASMA

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Comprehensive technical information on the methodology and results of this research is contained in the preprint "Field Rotating Disturbances in Warm Plasma", a copy of which is attached, and which is to be considered part of this final technical report. This manuscript has been submitted to the Journal of Fluid Mechanics.

FIELD ROTATING DISTURBANCES IN WARM PLASMA

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ABSTRACT

A model in which thermal effects are simulated through use of a multibeam plasma distribution function is developed and investigated to see if solutions which take an initially uniform magnetized plasma to a new uniform state with a different field orientation are possible. The momentum conservation integrals are found to admit two classes of such solutions, but only one class exhibits appropriate asymptotic behavior. Extensive numerical integrations have failed to demonstrate the existence of the desired solutions.

1.0 INTRODUCTION

At present there exists no theory which describes the structure of large amplitude plasma disturbances in which the magnetic field is rotated while plasma flow remains parallel to the field both far upstream and far downstream from the disturbance. Jump conditions appropriate to such structures are known from MHD theory, and include true rotational discontinuities in which only the field direction changes, intermediate waves in which plasma flow goes from super to sub-alfvenic and which appear nonevolutionary in MHD (Jeffrey and Taniuti, 1964), and mixed structures which occur in anisotropic plasmas and which exhibit characteristics of both shocks and rotational discontinuities (Hudson, 1970).

Work with large amplitude disturbances having the desired upstream properties has been carried out by Montgomery (1959), Saffman (1961), Kellog (1964), and Kakutani (1966), and special cases are summarized by Tidman and Krall (1971). These authors used similar cold plasma models, and found solutions representing infinite wave trains as well as solitary waves in which plasma and magnetic field parameters are identical on either side of the disturbance. In no cases were field rotating solitary waves found, and it is relatively easy to show that such waves are not possible in cold plasma. The closest solutions to those desired are Saffman's "quasishocks", in which an initially uniform plasma undergoes a disturbance after which plasma and field parameters wander ergodically about certain mean values, but never approach the uniform conditions sought here.

Interest in field rotating waves has been rekindled by the availability of magnetic field data from spacecraft crossing both the terrestrial and Jovian magnetopauses. During times when the solar field is approximately antiparallel to the planetary field, the magnetopause resembles a layer through which plasma may flow and across which the transverse magnetic field is rotated. Many magnetic field signatures are remarkably laminar and resemble in polarization and overall shape the solitons of Saffman and others, if somehow only half the soliton solution could be isolated. The available data and their relation to current theory have been summarized by Sonnerup (1977).

The present paper pursues a suggestion by Sonnerup that thermal effects which might permit field rotating layers could be introduced into the theory by assuming a multibeam plasma distribution function. This work shows that the momentum conservation conditions, which do not generally admit field rotating solutions in the cold plasma case, do admit two classes of such solutions when thermal effects are included. Asymptotic analysis shows that uniform solutions appropriate to the downstream conditions are capable of growth for one class but not the other.

2.0 EQUATIONS OF THE MULTIBEAM THEORY

The analysis is performed for a one dimensional case in which all variables are functions of z only. Assume the plasma to consist of N beams of ions and a single beam of electrons, with the quasineutrality condition satisfied. The electrons are assumed to follow the magnetic field, while the electric force on the ions is neglected. Although these latter two assumptions can be relaxed, they do not appear to affect qualitatively the results obtained for the waves being sought here. Finally, there is assumed to be no current in the z direction. The equations describing the system are then:

$$n_k V_{zk} = n_{0k} V_{z0k} \qquad \text{CONTINUITY} \tag{1}$$

$$\sum n_{0k} \frac{V_{z0k}}{V_{zk}} = n_0 \frac{V_{z0}}{V_z} \qquad \text{QUASINEUTRALITY} \tag{2}$$

$$\frac{dB_x}{dz} = \frac{4\pi e}{c}\left(\sum n_{0k} V_{yk} \frac{V_{z0}}{V_{zk}} - n_0 V_y \frac{V_{z0}}{V_z}\right) \tag{3}$$

AMPERE'S LAW

$$\frac{dB_y}{dz} = -\frac{4\pi e}{c}\left(\sum n_{0k} V_{xk} \frac{V_{z0k}}{V_{zk}} - n_0 V_x \frac{V_{z0}}{V_z}\right) \tag{4}$$

$$B_z = \text{CONSTANT} \tag{5}$$

$$\frac{d\underset{\sim}{V}_k}{dz} = \frac{e}{McV_{zk}}\left(\underset{\sim}{V}_k \times \underset{\sim}{B}\right) \qquad \text{ION MOTION} \tag{6}$$

$$\frac{V_x}{B_x} = \frac{V_y}{B_y} = \frac{V_z}{B_z} \qquad \text{ELECTRON MOTION} \tag{7}$$

$$\sum n_{ok} V_{zok} = n_o V_{zo} \qquad \text{NO CURRENT ALONG } z \tag{8}$$

Here $\underline{V}$ and n are the electron velocity and density, $\underline{V}_k$ and n_k the velocity and density of the k^{th} ion beam, M the ion mass, e the elementary charge, $\underline{B}$ the magnetic field, and c the speed of light. Quantities with the subscript $_o$ represent values at some chosen initial point. All sums run from 1 to N, where N is the number of beams. These equations are obvious generalizations of the usual two fluid model. However, they can be shown to follow rigorously from the Vlasov-Maxwell equations in the case of a multibeam distribution function when looping orbits (ie, negative values of V_z) are not permitted (Jones 1977).

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Introducing equations (7) and (8) into equation (3) gives

$$\frac{dB_x}{dz} = \frac{4\pi e}{c} \sum n_{ok} V_{zok} \left(\frac{V_{yk}}{V_{zk}} - \frac{B_y}{B_z} \right). \tag{9}$$

On the other hand, the x component of the ion motion equation may be written

$$\frac{dV_{xk}}{dz} = \frac{eB_z}{Mc} \left(\frac{V_{yk}}{V_{zk}} - \frac{B_y}{B_z} \right).$$

Multiplying this equation by $Mn_{ok}V_{zok}$ and summing over k gives

$$M \sum n_{ok} V_{zok} \frac{dV_{xk}}{dz} = \frac{eB_z}{c} \sum n_{ok} V_{zok} \left(\frac{V_{yk}}{V_{zk}} - \frac{B_y}{B_z} \right). \tag{10}$$

Comparing equations (9) and (10) allows an integral of the system to be written:

$$M \sum n_{ok} V_{zok} V_{xk} - \frac{B_z B_x}{4\pi} = P_x , \tag{11}$$

where P_x is a constant of the motion. An identical procedure gives an integral associated with the y component:

$$M \sum n_{0k} V_{z0k} V_{yk} - \frac{B_z B_y}{4\pi} = P_y . \tag{12}$$

The z component is handled in a similar but not identical manner, yielding:

$$M \sum n_{0k} V_{z0k} V_{zk} + \frac{B^2}{8\pi} = P_z . \tag{13}$$

Equations (11) - (13) state that the flux of momentum is constant.

The ion motion equation (6) and the momentum equations (11) - (13) constitute 3N+3 equations in the 3N+2 unknowns B_x, B_y, and the V_k's. One equation is redundant and can be used for checking numerical results. In practice, equation (13) is best suited to this purpose.

Because the ions feel only the magnetic force, their velocity magnitudes remain constant. Thus N additional integrals can be formed expressing constancy of the terms $V^2_{xk} + V^2_{yk} + V^2_{zk}$. Under the no loop condition there is no ambiguity in the sign of V_{zk}, so these integrals can be used to eliminate V_{zk} altogether.

3.0 UNIFORM FIELD SOLUTION

The large amplitude waves being sought would connect two regions in which the flow and field are asymptotically uniform and aligned with each other. In the single beam case (N=1) the only way in which such uniformity can occur is with the single velocity vector parallel to the magnetic field. Then the derivatives in equation (5) vanish and the solution is truly uniform. In this single beam case it is easily seen that the momentum equations admit no large changes in field orientation except for the special case $P_x = P_y = 0$, where an exact field reversal appears possible. In this case, however, all derivatives vanish identically and there can be no wave.

In the multibeam system, however, a situation in which individual beam velocities are not parallel to the field may nevertheless produce a net flow along a uniform field. In this case the velocity components perpendicular to the field represent a sort of quasirandom distribution which simulates the effect of a nonzero ion temperature. To quantify this situation, let all N beams have the same velocity magnitude, and let each velocity vector make the same angle with the field. Call the tangent of this angle γ, so that γ is the ratio of perpendicular to parallel velocity components. γ^2 is thus a measure of the "temperature".

When N=1 a nonzero γ violates the uniformity condition, and in fact gives rise to the infinite wavetrains of Saffman (1961). When N>1, it is reasonable to look for solutions in which the N beams carry the same particle flux, have their velocity vectors spaced as regularly as possible about the field, and rotate about the field at the larmor frequency. With all beams carrying the same flux, the quantity $n_{ok}V_{zok}$ appearing in equations (11) - (13) is independent of k, so that in a uniform field region those equations take the forms:

$$\sum V_{xk} = \text{CONSTANT} \tag{14}$$

$$\sum V_{yk} = \text{CONSTANT} \tag{15}$$

$$\sum V_{zk} = \text{CONSTANT} . \tag{16}$$

Furthermore, with a uniform field, the z component of equation (6) can be multiplied by V_{zk}, summed over k, and integrated to give:

$$\sum V_{zk}^{2} = \text{CONSTANT}. \tag{17}$$

In the two beam case equations (16) and (17) constitute two equations in two unknowns, which require that $V_{z1} = V_{z2}$. This can only happen in a uniform field if the velocities lie exactly parallel to the field, so the uniform field two beam solution collapses to the one beam cold plasma case. With N>2 this constraint is lifted, so that N=3 is the simplest case in which a nontrivial uniform field solution can exist.

Consider the particles of each beam to follow helical trajectories differing only in phase. It is necessary to write equations for these trajectories as functions of z when the particles spiral about a field which does not lie along the z axis. The situation is shown in figure 1. Only a single trajectory is shown, which passes through the origin. However, all space is filled with trajectories of particles of this beam, and the trajectories are all in phase at a given z, not at a given distance from a plane perpendicular to the field. This situation is necessary because of the condition that the system be one dimensional, with all variables functions of z only. Any wavefronts existing in nonuniform field regions lie perpendicular to the z axis.

The trajectories can be described easily in a coordinate system x′, y, z′ in which z′ lies along **B**. In this system the trajectory of figure 1 is:

$$Y_k = R_L \cos(\omega t_k - \delta_k) \tag{18}$$

$$X_k' = R_L \sin(\omega t_k - \delta_k) \tag{19}$$

$$Z_k' = V_{\parallel} t_k , \tag{20}$$

where $R_L = McV_\perp/eB$ is the larmor radius, $\omega = eB/Mc$ the larmor frequency, $V_\perp$ and $V_{//}$ the velocity components perpendicular and parallel to $\underline{B}$, and δ_k a phase angle for the k^{th} beam. t_k is a time parameter proportional to the position of the guiding center along the field. Because the larmor orbits are not perpendicular to the z axis, V_{zk} changes with orbital phase, and t_k is not a simple function of z. Performing a rotation gives the parameterized trajectory in the x, y, z system:

$$Y_k = R_L \cos(\omega t_k - \delta_k) \tag{21}$$

$$X_k = R_L \cos\theta \sin(\omega t_k - \delta_k) + V_{//} t_k \sin\theta \tag{22}$$

$$Z_k = V_{//} t_k \cos\theta - R_L \sin\theta \sin(\omega t_k - \delta_k), \tag{23}$$

where θ is the angle between the field and the z axis. Differentiating equations (21) - (23) gives the velocities:

$$V_{yk} = -\gamma V_{\parallel} \sin(\omega t_k - \delta_k) \tag{24}$$

$$V_{xk} = \gamma V_{\parallel} \cos\theta \cos(\omega t_k - \delta_k) + V_{\parallel} \sin\theta \tag{25}$$

$$V_{zk} = V_{\parallel} \cos\theta - \gamma V_{\parallel} \sin\theta \cos(\omega t_k - \delta_k), \tag{26}$$

where, again, $\gamma = V_{\perp} / V_{\parallel}$. Although these velocities were derived for particles whose guiding center passed through the origin, they would describe the entire velocity field as a function of z if the relation between t_k and z were known.

Equation (23) provides the desired relation, but is unfortunately transcendental. Analytic progress can be made by assuming $\gamma \ll 1$ and expanding in powers of γ. Because the difference between $V_{\perp}$ and V involves terms of order γ^2, it is necessary to carry terms of this order if results differing from the cold plasma theory are to be obtained. Since terms involving t_k in equations (24) - (26) are already multiplied by γ, it is only necessary to solve equation (22) to first order in γ. To do so, define a time parameter $t = z/(V_{\parallel} \cos\theta)$ which is strictly proportional to z and write $t_k = t + \Delta t_k$, where Δt_k is assumed shall compare to one. Then noting that $R_L = \gamma V_{\parallel} / \omega$, and letting $\beta = \tan\theta$, equation (23) becomes

$$\omega t_k = \omega t + \omega \Delta t_k - \gamma\beta \sin(\omega t - \delta_k + \omega \Delta t_k) .$$

Simplifying and retaining only terms of order γ, this is

$$\omega \Delta t_k = \gamma\beta \sin(\omega t - \delta_k) . \tag{27}$$

Inserting equation (27) into equations (24) - (26) gives the velocity solutions to second order in γ:

$$V_{yk} = -\gamma V_{\parallel} \sin(\omega t - \delta_k)\left\{ 1 + \gamma\beta \cos(\omega t - \delta_k)\right\} \tag{28}$$

$$V_{xk} = V_{\parallel} \sin\theta \left\{ 1 + \frac{\gamma}{\beta}\cos(\omega t - \delta_k) - \gamma^2 \sin^2(\omega t - \delta_k)\right\} \tag{29}$$

$$V_{zk} = V_{\parallel} \cos\theta \left\{ 1 - \gamma\beta \cos(\omega t - \delta_k) + \gamma^2\beta^2 \sin^2(\omega t - \delta_k)\right\} . \tag{30}$$

These are explicit functions of z because $t = z/(V_\parallel \cos\theta)$. With uniform phasing ($\zeta_k = 2\pi k/n$), it is straightforward to verify that, to terms of order $\gamma^2\beta^2$, these equations constitute a uniform field solution of equations (6), (11) - (13). The inclusion of β in the smallness parameter means the field angle cannot approach too closely to $\pi/2$. In the process, the following useful results emerge:

$$n_{0k} V_{z0k} = \frac{n_0}{N} V_\parallel \cos\theta , \qquad (31)$$

$$\sum V_{xk} = N V_\parallel \sin\theta \left(1 - \tfrac{1}{2}\gamma^2\right), \qquad (32)$$

$$\sum V_{zk} = N V_\parallel \cos\theta \left(1 + \tfrac{1}{2}\gamma^2\beta^2\right) . \qquad (33)$$

where $n_o = \sum n_{ok}$ is the total initial ion density.

Before proceeding, it is interesting to note that this solution is not exactly uniform. There are small wiggles in the field, but in the three beam theory they are already of higher order than γ^2. As N is increased, comparison of equations (28) through (30) with a numerical solution of the transcendental equation (23) shows that nonuniformities in the field drop off rapidly with increasing N. Also, the field vectors are not equally and rigidly spaced around the field, but wiggle back and forth in relation to one another. This is necessary to keep the flux constant and along $\underset{\sim}{B}$ as individual V_{zk}'s, and hence densities, vary.

4.0 FIELD ROTATION CONDITIONS

A particular uniform field solution is determined completely by the parameters θ, $V_{\parallel}$, γ, n_o, and B_z. Of these n_o, being an initial condition, and B_z are strictly constant even through a nonuniform region. The other three might vary throughout a nonuniform region and emerge changed but again constant to characterize a different uniform field solution. This possibility is governed in part by the equations:

$$M \sum n_{0k} V_{z0k} V_{xk} - \frac{B_z B_x}{4\pi} = P_x \tag{11}$$

$$M \sum n_{0k} V_{z0k} V_{zk} + \frac{B^2}{8\pi} = P_z \tag{13}$$

$$V_{\parallel}^2 \left(1+\gamma^2\right) = V^2 , \tag{34}$$

where V is the velocity magnitude, which is strictly constant. Equations (11) and (13) are the x and z momentum conservation equations and equation (34) expresses conservation of energy for the particles interacting only with the magnetic field. The y momentum equation is not included because it has already constrained any new uniform fields to lie in the x-z plane.

Consider a uniform field solution characterized by the parameters θ_o, $V_{\parallel o}$, γ_o. When n_o and B_z are also specified, the constants in equations (11), (13), and (34) are determined. Are there then other sets (θ, V_{par}, γ), representing other uniform field solutions, which also satisfy equations (11), (13), and (34) with the constants determined from the <u>initial</u> uniform field solution? To find such sets, the sums which appear in equations (11) and (13) can be evaluated using equations (31) - (33). Inserting these latter expressions into equations (11) and

(13), and evaluating the constants P_x and P_z from the initial parameter set results in the equations:

$$M m_0 V_{\parallel_0} \cos\theta_0 V_\parallel \sin\theta \left(1-\tfrac{1}{2}\gamma^2\right) - \frac{B_z B_x}{4\pi}$$
$$= M m_0 V_{\parallel_0}^2 \cos\theta_0 \sin\theta_0 \left(1-\tfrac{1}{2}\gamma_0^2\right) - \frac{B_z B_{x_0}}{4\pi}$$

$$M m_0 V_{\parallel_0} \cos\theta_0 V_\parallel \cos\theta \left(1+\tfrac{1}{2}\gamma\beta^2\right) + \frac{B^2}{8\pi}$$
$$= M m_0 V_{\parallel_0}^2 \cos^2\theta_0 \left(1+\tfrac{1}{2}\gamma_0^2\beta_0^2\right) + \frac{B_0^2}{8\pi} .$$

Using equation (34) to eliminate $V_\parallel$, there result two equations in the two unknowns γ, θ (recall that $\beta = \tan\theta$):

$$M m_0 V^2 \cos\theta_0 \left(1-\tfrac{1}{2}\gamma_0^2\right) \sin\theta \left(1-\gamma^2\right) - \frac{\beta B_z^2}{4\pi}$$
$$= M m_0 V^2 \cos\theta_0 \sin\theta_0 \left(1-\tfrac{3}{2}\gamma_0^2\right) - \frac{\beta_0 B_z^2}{4\pi} \qquad (35)$$

$$M m_0 V^2 \cos\theta_0 \left(1-\tfrac{1}{2}\gamma_0^2\right) \cos\theta \left(1+\tfrac{1}{2}\gamma^2\beta^2-\tfrac{1}{2}\gamma^2\right) + \frac{\beta^2 B_z^2}{8\pi}$$
$$= M m_0 V^2 \cos^2\theta_0 \left(1+\tfrac{1}{2}\gamma_0^2\beta_0^2-\gamma_0^2\right) + \frac{\beta_0^2 B_z^2}{8\pi} . \qquad (36)$$

These equations can be made dimensionless by defining a suitable ratio of kinetic to magnetic energy densities. Normally this would be the Alfvén number, given by:

$$\mathcal{M}_A^2 = \frac{4\pi n M V_\parallel^2}{B^2} = \frac{4\pi n M V^2}{B^2}\left(1-\gamma^2\right), \tag{37}$$

but here it appears that a modified Alfvén number, given by:

$$\mathcal{M}^2 = \frac{4\pi n M V^2\left(1 - \frac{3}{2}\gamma^2\right)}{B^2} \tag{38}$$

plays a more fundamental role. When this number is close to unity the definition (38) can be shown to be a special case of Hudson's (1970) modified Alfvén number appropriate to anisotropic plasmas, and under these conditions will be referred to as the Hudson number, m_H. The plasma under consideration is indeed anisotropic, since the pressure, as determined by the quasirandom perpendicular velocity components of the N beams, is entirely perpendicular to the magnetic field.

Equation (35) can be solved for γ^2, and by using the modified Alfvén number the two equations become:

$$\gamma^2 = \gamma_0^2 + 1 + \frac{1}{m_0^2}\left(\frac{\sin\theta_0}{\sin\theta} - \frac{\cos\theta_0}{\cos\theta}\right) - \frac{\sin\theta_0}{\sin\theta} \tag{39}$$

$$\frac{\cos\theta}{\cos\theta_0}\left\{1 + \gamma_0^2 + \frac{1}{2}\gamma^2\left(\beta^2 - 1\right)\right\} + \frac{\beta^2}{2m_0^2} = 1 + \frac{1}{2}\gamma_0^2\left(\beta_0^2 + 1\right) + \frac{\beta_0^2}{2m_0^2}. \tag{40}$$

Equation (39) can be inserted into equation (40) to give a single equation for θ. This can be solved numerically to reveal two classes of solutions, one in which the field undergoes a near reversal and the other in which the field angle changes only slightly. There is, of course, also the trivial solution $\theta = \theta_0$, which corresponds to the soliton solutions of Saffman and others.

More understanding of the solutions can be obtained by approaching equations (39) and (40) analytically. Because the one beam theory admits an exact reversal ($\theta = -\theta_0$) when $m_A^2 = 1$ (m_H and m_A are indistinguishable with one beam, since $\gamma = 0$), it is reasonable to look for solutions with m_H^2 close to one and which nearly reverse the field. Let $\epsilon = m_H^2 - 1$ and $\theta = -(\theta_o + \Delta)$.

Putting these forms into equations (39) and (40)., and performing considerable algebra while retaining only first order terms in ϵ and Δ results in the near reversal solution:

$$\theta = -\left\{ \theta_0 + \frac{2\epsilon\left(1-\beta_0^2\right)}{\beta_0\left(1+\beta_0^2\right)} \right\} \tag{41}$$

$$\gamma^2 = \gamma_0^2 + 4\epsilon \sin^2\theta_0 \ . \tag{42}$$

Thus the extra degree of freedom afforded by nonzero temperature has allowed field rotating solutions to satisfy the momentum conservation integrals for values of the Hudson number other than unity. The solutions are shock-like in that the temperature rises across the wave, as indicated by equation (42). With the initial Hudson number given by $m_{Ho}^2 = 1 + \epsilon$, the Hudson number in the reversed field region can be evaluated from equations (38), (41), and (42), and is found to be $m_H^2 = 1 - \epsilon$. Thus the Hudson number goes from above unity to below. However, the Alfven number, given in this case by $m_A^2 = m_H^2 + \frac{1}{2}\gamma^2$, remains above unity. It is unclear how to classify these possibilities; they might describe intermediate waves but if so their evolutionary character is unclear, since it is not known whether intermediate waves in anisotropic plasma are evolutionary (Hudson 1970).

The second class of solution can be analyzed when the Alfven number is close to $\sec^2\theta$. In this case numerical work suggests only a small field rotation, so a solution of the form $\theta = \theta_o + \Delta$ is assumed. The analysis is complicated by the proximity of the trivial root $\theta = \theta_o$, but eventually yields the solution, good to terms of order γ^2:

$$\theta = \theta_0 + \frac{2}{3}\left(\frac{\epsilon}{\beta_0} - \gamma^2\beta_0\right) \tag{43}$$

$$\gamma^2 = \gamma_0^2 , \tag{44}$$

where now $\epsilon = m_o^2/\sec^2\theta_o - 1$. Equation (44) shows that this situation is possible even in the one beam case where γ is always zero. Evaluation of the modified Alfvén number in the rotated field region gives

$$m^2 = \sec^2\theta_0\left(1 + \frac{1}{3}\epsilon + \frac{2}{3}\gamma_0^2\beta_0^2\right). \tag{45}$$

It is useful to compare the new Hudson number with the secant of the new field angle. Introducing equation (43) into equation (45) gives the result

$$m^2 - (1+\beta^2) = (1+\beta_0^2)(2\gamma_0^2\beta_0^2 - \epsilon). \tag{46}$$

Thus a Hudson number initially less than $\sec^2\theta_0$ (ie, $\epsilon < 0$) becomes greater than $\sec^2\theta$ in the rotated field region. In the analysis leading to both rotated field situations, it is assumed that

$$\gamma_0^2 \ll \beta \ll 1/\gamma_0^2 .$$

5.0 ASYMPTOTIC SOLUTIONS

In Saffman's cold plasma theory it is shown that solitary waves can exist only when the Alfven number, as defined in the present paper, satisfies the condition

$$1 < \mathcal{M}_A^2 < \sec^2\theta \tag{47}$$

(Saffman 1961). Small perturbations of uniform field solutions outside this range are incapable of growth. Both field rotating possibilities in the multibeam warm plasma theory take the modified Alfvén number from a region of permissible Alfvén numbers in Saffman's theory to a region where asymptotic growth is not possible. However, the Alfvén number may remain within the permissible region. Saffman's results cannot be applied to the multibeam theory because the nonzero temperature and associated pressure anisotropy creates a distinction between the Alfvén and modified Alfvén numbers. It is necessary to perform an asymptotic analysis of the multibeam theory to see whether the waves admitted by the momentum integrals can actually exist.

It is convenient to make the equations dimensionless through the following definitions:

$$\underset{\sim}{U}_k = \frac{\underset{\sim}{V}_k}{V_{\parallel 0}}$$

$$\underset{\sim}{b} = \frac{\underset{\sim}{B}}{B_0}$$

$$\zeta = \frac{e B_0}{M c V_{\parallel 0}} z$$

$$\mathcal{M}_A^2 = \frac{4\pi n_0 M V_{\parallel 0}^2}{B_0^2}$$

$$\eta_k = \frac{n_k}{n_{0k}} .$$

Here the subscript $_0$ refers to conditions in a uniform field region. The definition of b gives immediately $b_z = \cos\theta$ where θ is the initial angle between $\underline{B}$ and the z axis. With these definitions, equations (11), (12), and (6) become:

$$\sum \eta_{0k} U_{z0k} U_{xk} - \frac{b_x \cos\theta}{\mathcal{M}_A^2} = \rho_x \tag{48}$$

$$\sum \eta_{0k} U_{z0k} U_{yk} - \frac{b_y \cos\theta}{m_A^2} = 0 \tag{49}$$

$$\frac{d\,\underset{\sim}{U}_k}{d\rho} = \frac{1}{U_{zk}}\left(\underset{\sim}{U}_k \times \underset{\sim}{b}\right), \tag{50}$$

where ρ_x is a constant and the analogous y constant has been set to 0 so that the field lies initially in the x-z plane. Let $\eta = \eta_{0k}U_{z0k}$ and require this quantity to be independent of k, so that initially the beams differ only in phase. Then equations (48) and (49) can be solved for the magnetic field components to give:

$$b_x = \sin\theta + \frac{m_A^2}{\cos\theta}\,\eta \sum \left(U_{xj} - U_{xj0}\right) \tag{51}$$

$$b_y = \frac{m_A^2}{\cos\theta}\,\eta \sum \left(U_{yj} - U_{yj0}\right). \tag{52}$$

Introducing these forms into the x and y components of equation (50) gives

$$\frac{dU_{xk}}{d\rho} = \frac{U_{yk}}{U_{zk}}\cos\theta - \frac{m_A^2}{\cos\theta}\eta\sum\left(U_{yj} - U_{yj_0}\right) \tag{53}$$

$$\frac{dU_{yk}}{d\rho} = \sin\theta + \frac{m_A^2}{\cos\theta}\eta\sum\left(U_{xj} - U_{xj_0}\right) - \frac{U_{xk}}{U_{zk}}\cos\theta . \tag{54}$$

To see whether growth from the uniform field solution is possible, let $\underline{u}_k = \underline{v}_k + \underline{w}_k$, where $\underline{v}_k$ is a uniform field solution. To first order in the $\underline{w}_k$'s, equations (53) and (54) become

$$\frac{dW_{xk}}{d\rho} = \frac{\cos\theta}{U_{zk}}\left(W_{yk} - \frac{U_{yk}}{U_{zk}}W_{zk}\right) - \frac{m_A^2}{\cos\theta}\eta\sum W_{yj} , \tag{55}$$

$$\frac{dW_{yk}}{d\rho} = \frac{m_A^2}{\cos\theta}\eta\sum W_{xk} - \frac{\cos\theta}{U_{zk}}\left(W_{xk} - \frac{W_{zk}U_{xk}}{U_{zk}}\right) . \tag{56}$$

Conservation of energy requires that $u_{xk}^2 + u_{yk}^2 + u_{zk}^2$ remain constant, so that

$$U_{xk} W_{xk} + U_{yk} W_{yk} + U_{zk} W_{zk} = 0 . \tag{57}$$

Assuming γ to be small, dimensionless forms of the uniform field solution (equations (28) - (30)) can be used for the $\underline{v}_k$'s. Introducing these into equations (55) - (57) and using equation (57) to eliminate w_{zk} yields, after much algebra, the 2N equations

$$\frac{dw_{xk}}{d\rho} = w_{yk} - \frac{m_A^2}{N}\sum w_{yj} + \gamma\beta\left(w_{yk}\cos\alpha_k - \frac{\sin\alpha_k}{\cos\theta} w_{xk}\right) + \gamma^2\left[\left(\beta^2\cos 2\alpha_k + (1+\beta^2)\sin^2\alpha_k\right)w_{yk} - \frac{\sin 2\alpha_k}{2\cos\theta}(1+4\beta^2)w_{xk}\right] \tag{58}$$

$$\frac{dw_{yk}}{d\rho} = \frac{m_A^2}{N}\sum w_{xj} - (1+\beta^2)w_{xk} + \gamma\beta\left[\frac{\sin\alpha_k\, w_{yk}}{\cos\theta} - 3\cos\alpha_k(1+\beta^2)w_{xk}\right] + \gamma^2\left[\frac{\sin 2\alpha_k}{2\cos\theta}(1+4\beta^2)w_{yk} - (1+\beta^2)\left\{3\beta^2\cos 2\alpha_k + (1+3\beta^2)\cos^2\alpha_k\right\}w_{xk}\right] . \tag{59}$$

Here $\alpha_k = \zeta/\cos\theta - \rho_k$, where $\zeta/\cos\theta = \omega t$. These linear homogeneous equations admit solutions of the form

$$\underset{\sim}{W}_k = \underset{\sim}{g}_k e^{a\zeta} , \tag{60}$$

where the exponential represents either a growth or oscillation common to all beams, and the g_k's are individual and presumably small oscillations at the larmor frequency. When equation (60) is introduced into equations (58) and (59) there result

$$\begin{aligned} \frac{d g_{xk}}{d\zeta} &= g_{yk} - a g_{xk} - \frac{m_A^2}{N} \sum g_{yj} + \gamma\beta \left(g_{yk} \cos\alpha_k - \frac{\sin\alpha_k}{\cos\theta} g_{xk} \right) \\ &+ \gamma^2 \left[\left\{ \beta^2 \cos 2\alpha_k + (1+\beta^2)\left(\frac{1-\cos 2\alpha_k}{2} \right) \right\} g_{yk} - \frac{\sin 2\alpha_k}{2\cos\theta} (1+4\beta^2) g_{xk} \right] \end{aligned} \tag{61}$$

$$\begin{aligned} \frac{d g_{yk}}{d\zeta} &= \frac{m_A^2}{N} \sum g_{xj} - (1+\beta^2) g_{xk} - a g_{yk} \\ &+ \gamma\beta \left[\frac{\sin\alpha_k \, g_{yk}}{\cos\theta} - 3\cos\alpha_k (1+\beta^2) g_{xk} \right] \\ &+ \gamma^2 \left[\frac{\sin 2\alpha_k}{2\cos\theta} (1+4\beta^2) g_{yk} - (1+\beta^2) \left\{ 3\beta^2 \cos 2\alpha_k + (1+3\beta^2)\left(\frac{1+\cos 2\alpha_k}{2} \right) \right\} g_{x} \right. \end{aligned} \tag{62}$$

The periodic wiggles represented by the q_k's can be described by a fourier series solution of the form

$$q_k = B_{k0} + \sum_{n=1}^{\infty} \left(A_n \sin n\alpha_k + B_n \cos n\alpha_k \right). \tag{63}$$

The growth or oscillatory nature of the overall solution can be determined by introducing equation (63) into equations (61) and (62), looking only at the $\int$ independent terms, and summing over k, which yields

$$m_A^2 B_{Y_0} = B_{Y_0} \left\{ 1 + \frac{1}{2} \gamma^2 (1+\beta^2) \right\} - a B_{X_0} \tag{64}$$

$$m_A^2 B_{X_0} = a B_{Y_0} + (1+\beta^2) B_{X_0} \left\{ 1 + \gamma^2 \left(1 + \frac{3\beta^2}{2} \right) \right\} , \tag{65}$$

(65)

where $\underline{B}_o = \sum_k \underline{B}_{ko}$. One of the B's in these equations is an arbitrary scale factor, so that these are two equations in the two unknowns a and the ratio of the B's. Multiplying the first equation by B_{yo}, the second by B_{xo} and adding gives

$$\left[m_A^2 - \left\{1 + \tfrac{1}{2}\gamma^2\left(1+\beta^2\right)\right\}\right] B_{y_0}^2 + \left[m_A^2 - \left\{\left(1+\beta^2\right)\left(1+\gamma^2\left(1+\tfrac{3}{2}\beta^2\right)\right)\right\}\right] B_{x_0}^2 = 0 . \tag{66}$$

Solving equation (64) for B_{xo} and introducing the result into equation (66) gives

$$B_{y_0}^2 \left[m_A^2 - \left\{1 + \tfrac{1}{2}\gamma^2\left(1+\beta^2\right)\right\}\right]\left[a^2 + \left[m_A^2 - \left\{1 + \tfrac{1}{2}\gamma^2\left(1+\beta^2\right)\right\}\right]\left[m_A^2 - \left[\left\{\left(1+\beta^2\right)\left(1+\gamma^2\left(1+\tfrac{3}{2}\beta^2\right)\right)\right\}\right]\right]\right] = 0 , \tag{67}$$

The root $B_{yo}^2 = 0$ is of no interest here because, even if it does permit growth, the average values of the field perturbations remain zero so that such growth would not permit solutions with different field orientations to be connected. The second root, given by

$$m_A^2 - \left\{ 1 + \frac{1}{2}\gamma^2(1+\beta^2) \right\} = 0 \tag{68}$$

requires that the Alfvén number be close to but above unity. In this case the modified Alfvén number is the Hudson number and is given by $m_H^2 = m_A^2 \left(1 - \frac{1}{2}\gamma^2\right)$, so that the Hudson number must remain above one, thus excluding the field rotating solution of equations (41) and (42). The third root of equation (67) occurs when

$$a^2 + \left[m_A^2 - \left\{ 1 + \frac{1}{2}\gamma^2(1+\beta^2) \right\} \right] \left[m_A^2 - (1+\beta^2)\left\{ 1 + \gamma^2\left(1 + \frac{3\beta^2}{2}\right) \right\} \right] = 0 \tag{69}$$

Growing solutions require $a^2 > 0$, so that

$$\left[m_A^2 - \left\{1 + \frac{1}{2}\gamma^2(1+\beta^2)\right\}\right]\left[m_A^2 - (1+\beta^2)\left\{1+\gamma^2\left(1+\frac{3\beta^2}{2}\right)\right\}\right] < 0 \;. \tag{70}$$

When $\gamma = 0$ this collapses to Saffman's condition $1 < m_A^2 < (1+\beta^2)$ (note that $1 + \beta^2 = \sec^2\theta$). For the field reversing solutions of equations (41) and (42), m remains close to unity, so that the second term in equation (70) is less than zero. Then a growing solution requires $m_A^2 - (1 + \frac{1}{2}\gamma^2(1 + \beta^2)) > 0$, which requires $m_H^2 > 1 + \frac{1}{2}\gamma^2\beta^2$. Thus the uniform field solution in the rotated field region cannot grow, and the field rotating solutions admitted by the momentum conservation integrals do not exist.

The second class of solutions, described by equations (43) - (46), involves an Alfven number of order $\sec^2\theta = 1 + \beta^2$. Thus the first term in equation (70) is greater than zero, so that a growing solution requires

$$m_A^2 - (1+\beta^2) < \gamma^2(1+\beta^2)\left(1+\frac{3}{2}\beta^2\right) \;. \tag{71}$$

This condition must hold on either side of a nonuniform field --that is, for the values of m_A^2, β, and γ found on the side in question. To examine the situation in the rotated region, equation (46) may be written in terms of m_A^2 as

$$m_A^2 - (1+\beta^2) = (1+\beta^2)\left(\frac{1}{2}\gamma^2 - \epsilon + 2\gamma^2\beta^2\right), \tag{72}$$

where use has been made of the fact that $\gamma^2 = \gamma_o^2$ and that β and β_o are interchangeable to terms of order γ^2 when they appear multiplied by γ^2. Using equation (72), the condition (71) becomes

$$\epsilon > \frac{1}{2}\gamma^2\beta_o^2. \tag{73}$$

A second constraint is placed on ϵ when the initial uniform region is taken into account. The inequality (71) holds here as well, while the definition $\epsilon = m_o^2/(1+\beta_o^2) - 1$ leads to the expression:

$$M_{A_0}^2 - (1+\beta_0^2) = (1+\beta_0^2)\left(\frac{1}{2}\gamma_0^2 + \epsilon\right). \tag{74}$$

Using equation (74) in inequality (71) leads to the condition

$$\epsilon < \frac{3}{2}\gamma_0^2\beta_0^2. \tag{75}$$

Comparison of inequalities (73) and (75) shows that growing solutions are possible in both the initial and rotated field regions provided

$$\frac{1}{2}\gamma_0^2\beta_0^2 < \epsilon < \frac{3}{2}\gamma_0^2\beta_0^2. \tag{76}$$

6.0 NUMERICAL WORK

Inequality (76) shows that in warm plasma ($\gamma^2 > 0$) growing solutions are possible for both uniform field solutions admitted by the momentum conservation equations. However, the conditions of momentum conservation and asymptotic growth do not guaranty that two different uniform field solutions corresponding to the same initial conditions are actually connected. In the warm plasma case the existence of such a connecting solution can be determined only by numerical integration, since the analytic integrals of the system are not sufficient to specify B_x as a function of B_y as is possible with cold plasma.

Extensive numerical integration of the equations for systems of three to fifteen beams has confirmed the limitations on growth suggested by the asymptotic analysis. The resulting B_x, B_y hodograms resemble those of Saffman even in the extended modified Alfven number range given by inequality (76). Although the parameters γ_o, β_o, and m^2_o have been varied extensively within appropriate ranges, no solutions have been found which take one uniform field solution into another with different field angle. Instead, true soliton solutions occur which are qualitatively no different from those of Saffman and which take the initial uniform field solution back to itself. At the midpoint of the soliton, the y component of the magnetic field vanishes but its derivative is always a maximum. Furthermore, the x component at this point does not have the appropriate rotated value given by equation (43). Thus the warm plasma theory appears incapable of producing the desired solutions.

CONCLUSION

The inclusion of thermal effects through use of a multibeam distribution function enriches and complicates the description of large amplitude disturbances. The extra degree of freedom afforded by the inclusion of the temperature-like variable γ^2 allows two classes of field rotating solutions to satisfy the momentum conservation integrals of the system. However, only those solutions associated with small changes in field angle appear capable of asymptotic growth on both sides of the disturbance, and even these have not been found after extensive numerical investigations. It is unclear whether the apparent nonexistence of field reversing disturbances is a peculiarity of the particular and somewhat artificial model used or rather a reflection of the impossibility of such disturbances in collisionless plasma.

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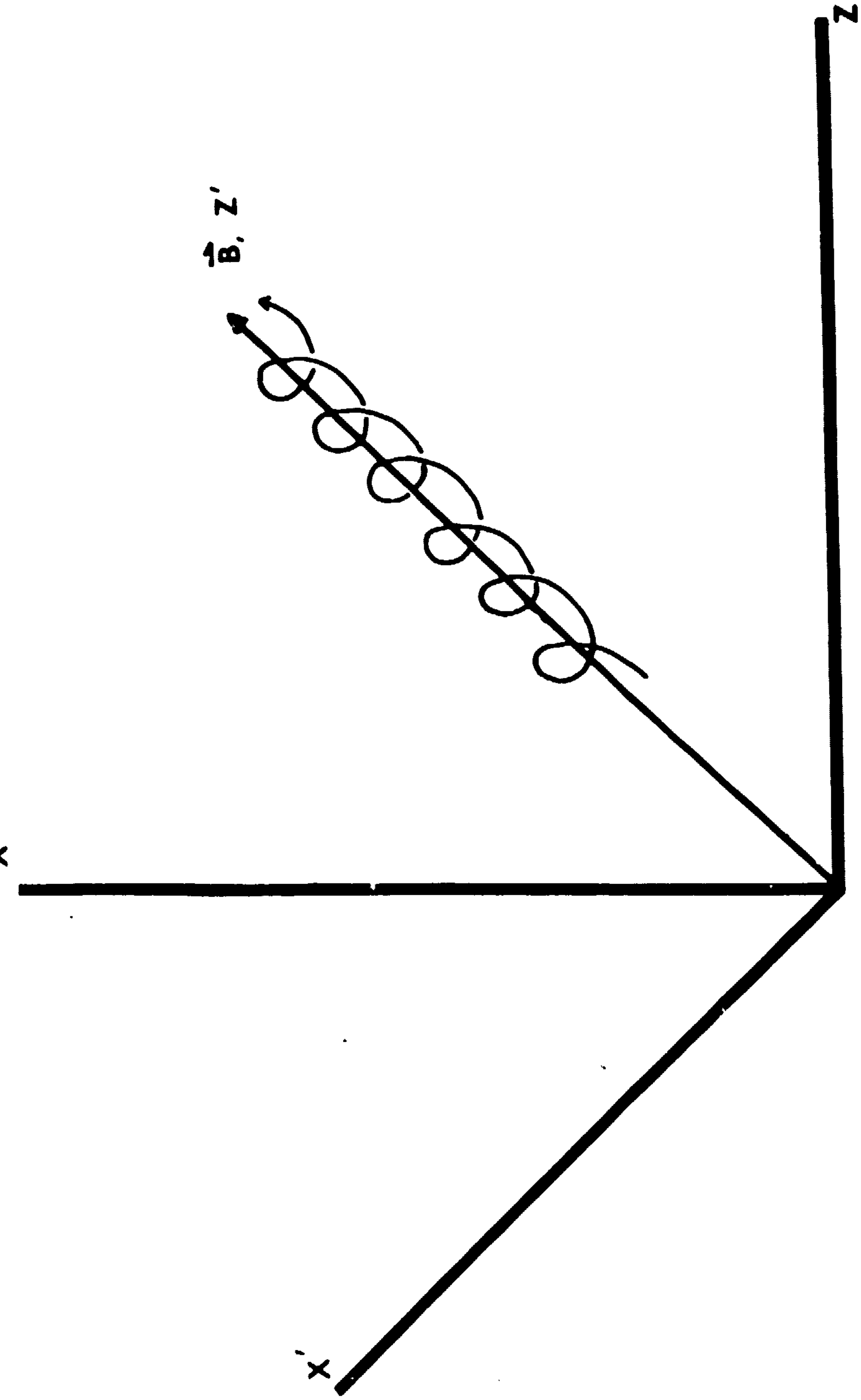
Figure Caption

Figure 1: A particle in a helical trajectory about the magnetic field. All space is assumed filled with such particles, whose orbits are in phase along the z axis, not along the field direction (z' axis).


Z
B, z'
x
x'