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Summary

T. J. R. Hughes and W. K. Liu described a new type of implicit explicit algorithm for the solution of transient problems in structural dynamics. The method involved dividing the finite elements into implicit and explicit groups while automatically satisfying the interface conditions. It is the purpose of this report to apply this algorithm to the solution of the linear, transient, two-dimensional heat equation subject to an initial condition derived from the solution of a steady state problem over an L-shaped region made up of a good conductor and an insulating material.

Using the IIT/PRIME computer with virtual memory, a FORTRAN computer program code was developed to make accuracy, stability, and cost comparisons among the fully explicit Euler, the Hughes-Liu, and the fully implicit Crank-Nicholson algorithms. This report illustrates the Hughes-Liu claim that the explicit group governs the stability of the entire region while maintaining the unconditional stability of the implicit group.

Introduction

Transient heat flow in solids can be described mathematically in terms of a parabolic differential equation. For certain very simple boundary conditions, this equation may be solved analytically, but when complex problems are considered, this is usually not possible. In that case the transient heat transfer problem may be descretized by finite elements and a weak solution is obtained at the nodes of the mesh. The weak solution is adequate provided it converges.

The truncation error, (the difference between the exact solution of the transient problem, and the exact solution of the discretized problem) is due to the finite distance between mesh points, the approximation of the time derivatives by finite differences, and the choice of the integration scheme. To find conditions under which the truncation error goes to zero, is the problem of convergence [1]. By a well-known result, stability and consistency imply convergence, so that stability can be a computationally crucial question in such problems.

There are two major types of algorithms for the solution of transient structural heat transfer problems: implicit and explicit. In implicit algorithms the nodal temperatures at time $t + \Delta t$ are expressed as functions of temperatures as time t and $t + \Delta t$, while in explicit algorithms they are expressed solely as functions of temperatures at time t. Explicit algorithms require much less computation per time step, but the time step size is limited (often severely) by the stability consideration. These stability limitations are most severe for good conducting materials. [2].

The present work considers the solutions of the transient heat transfer problem in an L-shaped region, one part of which consists of a very good conductor, and the other part is made up of insulation. In this type of situation researchers have begun to consider the possibility of solving the problem by a mixed implicit-explicit algorithm. Belytschko, and Mullen [3] have an implicit method that works on the following idea of strong coupling: The explicit group is integrated first, and the information then moves between adjacent nodes during a particular time step. This then is used as a boundary condition to integrate the implicit group of elements, and information in the implicit group travels across the entire mesh. In order to accomplish this, it is necessary to partition the element into implicit, explicit, and interface groups, while partitioning the nodes into implicit and explicit groups [3].

Hughes and Liu [4,5] have a simpler implicit-explicit algorithm where the elements are subdivided into implicit and explicit groups, and no provision needs to be made for interface elements since these are automatically taken care of during the assembly process. It is the purpose of this report to apply the Hughes-Liu algorithm to the heat equation, and to make appropriate analysis of both stability and accuracy. The performance of the Hughes-Liu algorithm is then compared with the implicit Crank-Nicolson and the explicit Euler methods. These three methods are also compared from the point of view of cost in terms of the CPU time (all calculations were performed on the IIT prime 400 computer).

Problem Description

Heat flow in solids in the absence of internal heat generation is described by the equation:

$$\nabla \cdot (k \nabla u) = c u_t \qquad (1)$$

k = thermal conductivity $\left(\frac{\text{Watt}}{\text{m - K}}\right)$

c = specific heat capacity $\left(\frac{\text{Joule}}{\text{m}^3\text{ - K}}\right)$

μ = temperature (K)

t = subscript on u denoting differentiation
with respect to time in seconds

Equation (1) is solved herein by the finite element method over the L-shaped region composed of two materials. The geometry and boundary conditions are indicated in Fig. 1.

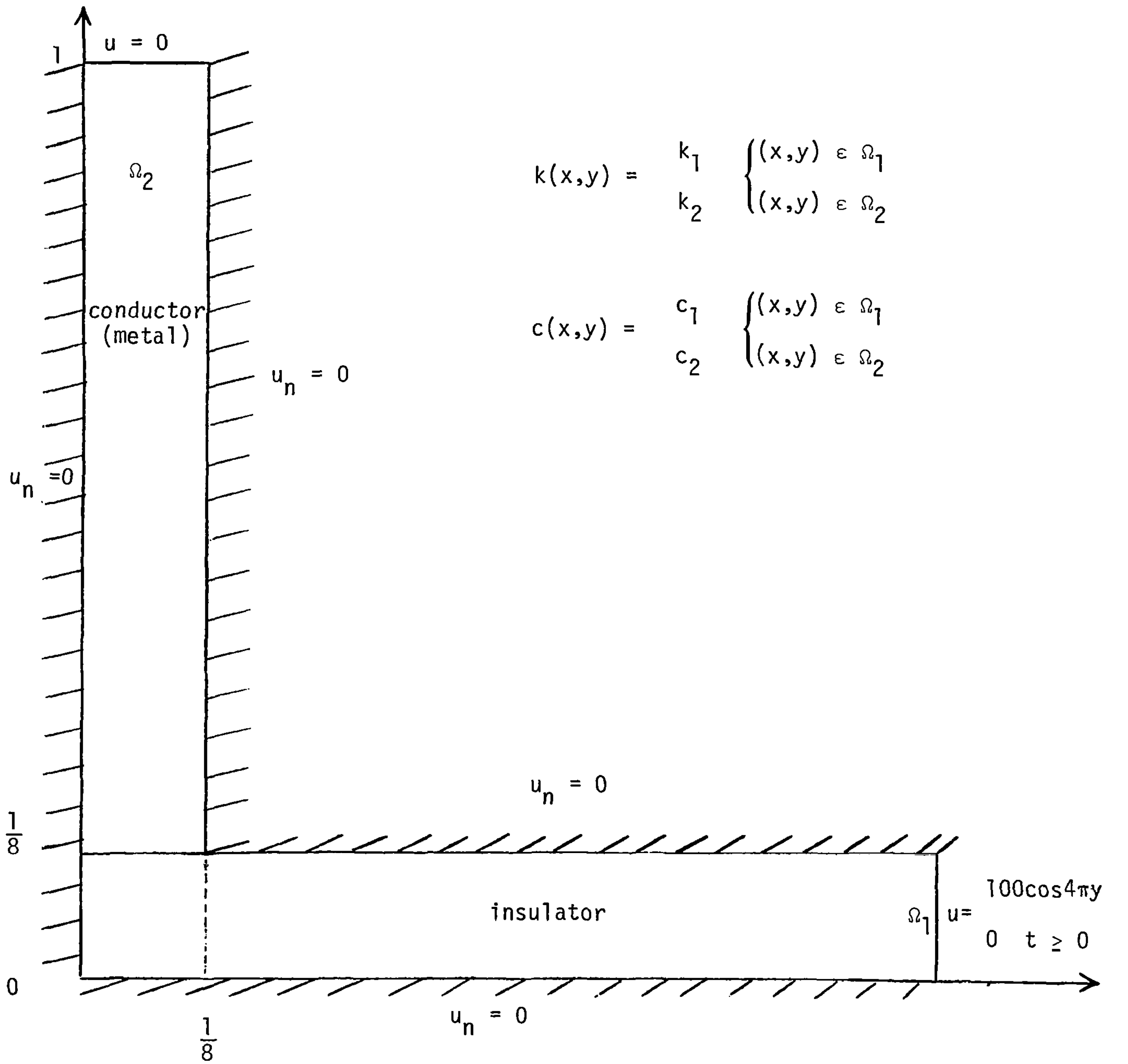


Figure 1

The L-shaped Region

The spatial discretization of Figure (1) is accomplished by a "crude" mesh of twelve elements (twenty one nodes) of dimensions (1/16) by (1/16), and (1/16) by (7/16), and by a "fine" mesh of two hundred and forty elements (three hundred and five nodes) of dimension (1/32) by (1/32). At the element level, both k and c are constant, so that at each point of the region, equation (1) may be simplified when expressed in terms of the diffusion coefficient α given by

$$\alpha = \frac{k}{c} \ \frac{m^2}{sec} \tag{2}$$

so that

$$\alpha\nabla^2 u = u_t \tag{3}$$

In the present work the initial condition:

$$u(0) = U \tag{4}$$

is obtained by solving the steady state problem:

$$\alpha\nabla^2 u = 0 \tag{5}$$

with the same boundary conditions. By means of the Galerkin Weighted Residual method, for example, equation (3) is reduced to the equivalent matrix equation:

$$C\dot{u} + Ku = 0 \tag{6}$$

where:

C = global heat capacity matrix
K = global thermal conductivity matrix
u = vector of unknown nodal temperatures

and the dot denotes time differentiation. Also the steady state problem (5) is reduced to the matrix equation:

$$Ku = 0 \tag{7}$$

The solution of Eq. 7 subject to the boundary condition in Fig. 1 gives the initial condition used in Eq. 4.

Explicit and Implicit Algorithms

The following notation is used in the solution algorithm:

$$d = \dot{u} \tag{8}$$

$$d_n \cong d(t_n) \tag{9}$$

$$u_n \cong u(t_n) \tag{10}$$

In equations (9) and (10) d_n and u_n are the numerical approximations to the exact value at time

$$t_n = n\Delta t \tag{11}$$

where

t_n = n^{th} timestep

Δt = time increment

The explicit algorithm used is the Euler method which consists of the first two terms of the Taylor series. Thus it is a first order method and is given by:

$$u_{n+1} = u_n + \Delta t d_n \tag{12}$$

The stability of this method may be characterized in terms of the Fourier number which is defined as the ratio of the rate of heat conduction to the rate of heat storage. In physical terms it gives an indication as to how fast the temperature of the material changes due to a heat input [7]. If Δx, and Δy are the minimum element lengths in the model of the region along the x, and y directions, respectively, then the Fourier number, F_o, is defined by:

$$F_o = a\Delta t \tag{13}$$

where

$$a = \frac{\alpha}{(\Delta x)^2 + (\Delta y)^2} \tag{14}$$

In the present problem we are dealing with an inhomogeneous material region consisting of a good conductor and an insulator. We have found that the size of Δt

is limited by the higher diffusion ratio of the good conductor. Furthermore, Δt must be chosen small enough to meet the Courant stability criterion:

$$F_o \leq \tfrac{1}{4} \qquad (15)$$

The implicit algorithm used is the Crank-Nicholson method. It is a second order method given by:

$$u_{n+1} = u_n + \frac{\Delta t}{2} (d_n + d_{n+1}) \qquad (16)$$

which is unconditionally stable. In the numerical experiments the solution obtained with the implicit method with the time step of the explicit method is considered to be standard (baseline) for accuracy considerations.

The Hughes-Liu Algorithm

The elements of the mesh are divided into implicit and explicit groups. The Hughes-Liu algorithm replaces equation (6) by:

$$Cd_{n+1} + K^I u_{n+1} + K^E \tilde{u}_{n+1} = 0 \qquad (17)$$

where

I indicates implicit group

E indicates explicit group

The predictor value $\tilde{u}$ for u is

$$\tilde{u}_{n+1} = u_n + \tfrac{1}{2}\Delta t d_n \qquad (18)$$

which is just the central difference formula for d_n. The corrector equation is

$$u_{n+1} = \tilde{u}_{n+1} + \tfrac{1}{2}\Delta t d_{n+1} \qquad (19)$$

Reduction of Hughes-Liu Algorithm to a Single Equation

The predictor equation (18) is solved for d_n and the corrector equation (19) is evaluated at the n^{th} step. Then with d_n eliminated from equation (19) the following predictor-corrector relation is obtained:

$$u_n = \tfrac{1}{2}(\tilde{u}_n + \tilde{u}_{n+1}) \qquad (20)$$

With equation (20) the equivalent split operator problem (17), (18), and (19) is reduced to the single equation:

$$u_{n+1} = G(\Delta t)u_n \qquad (21)$$

where

$$G(\Delta t) = (I - \tfrac{1}{2}\Delta tA^I)^{-1}(I + \tfrac{1}{2}\Delta tA^T + \Delta tA^E) \qquad (22)$$

with

$$A^I = - C^{-1}K^I \qquad (23)$$

$$A^E = - C^{-1}K^E \qquad (24)$$

For the fully explicit case ($K^I = 0$, $K^E = K$) equation (25) simplifies to:

$$G(\Delta t) = I + \Delta tA \qquad (26)$$

where

$$A = A^E \qquad (27)$$

Thus the fully explicit equation (26) is the first order Euler method. For the fully implicit case ($K^I = K$, $K^E = 0$) equation (22) simplifies to:

$$G(\Delta t) = (I - \tfrac{1}{2}\Delta tA)^{-1}(I + \tfrac{1}{2}\Delta tA) \qquad (28)$$

where

$$A = A^I \qquad (29)$$

and this is the second order Crank-Nicholson method.

Stability Analysis for Hughes-Liu Algorithm

Equation (21) may be put into the form:

$$u_{n+1} = (I - \Delta tB)u_n \qquad (30)$$

(See Appendix for a one-dimensional example). Equation (30) represents a contraction map if for the largest eigenvalue of B we have:

$$|1 - \lambda\Delta t| \leq 1 \qquad (31)$$

Using an energy type norm of the discrete solution, Hughes and Liu show for a dynamics problem that the largest eigenvalue of B is the same as if only the explicit region was analyzed [4]. For heat conduction this is demonstrated in the Appendix for the one-dimensional case. Because the explicit region is chosen to be the insulator, the stability circle for the Hughes-Liu method is considerably larger than the unit circle centered at -1 which is the stability circle for the explicit Euler method. This is shown in Figure (2). The fully implicit Crank-Nicholson method is unconditionally stable, so it has a stability region consisting of the entire left half of the complex plane. (See Figure (3).

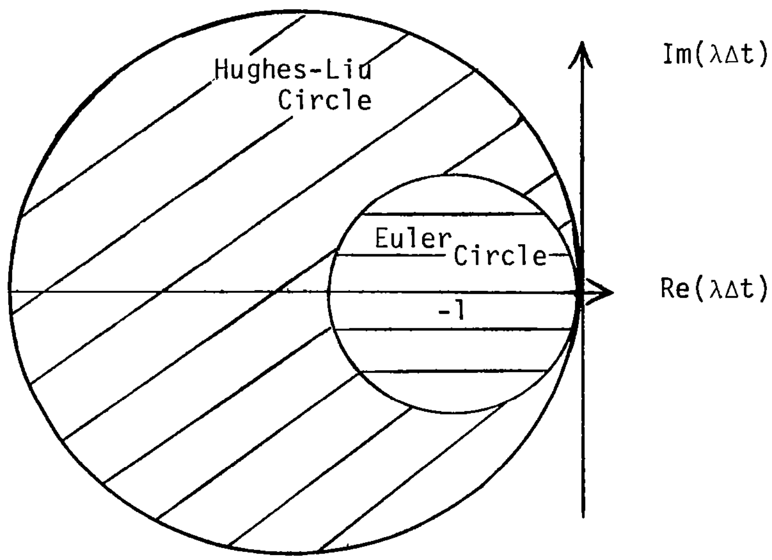


Figure (2)

Stability regions for the Euler and Hughes-Liu methods

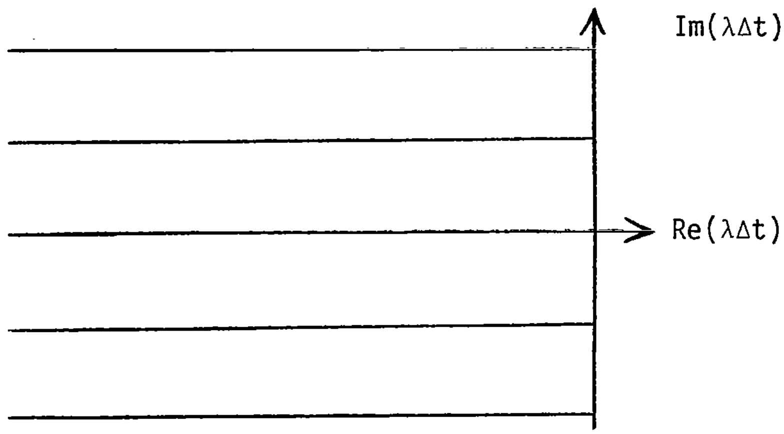


Figure (3)

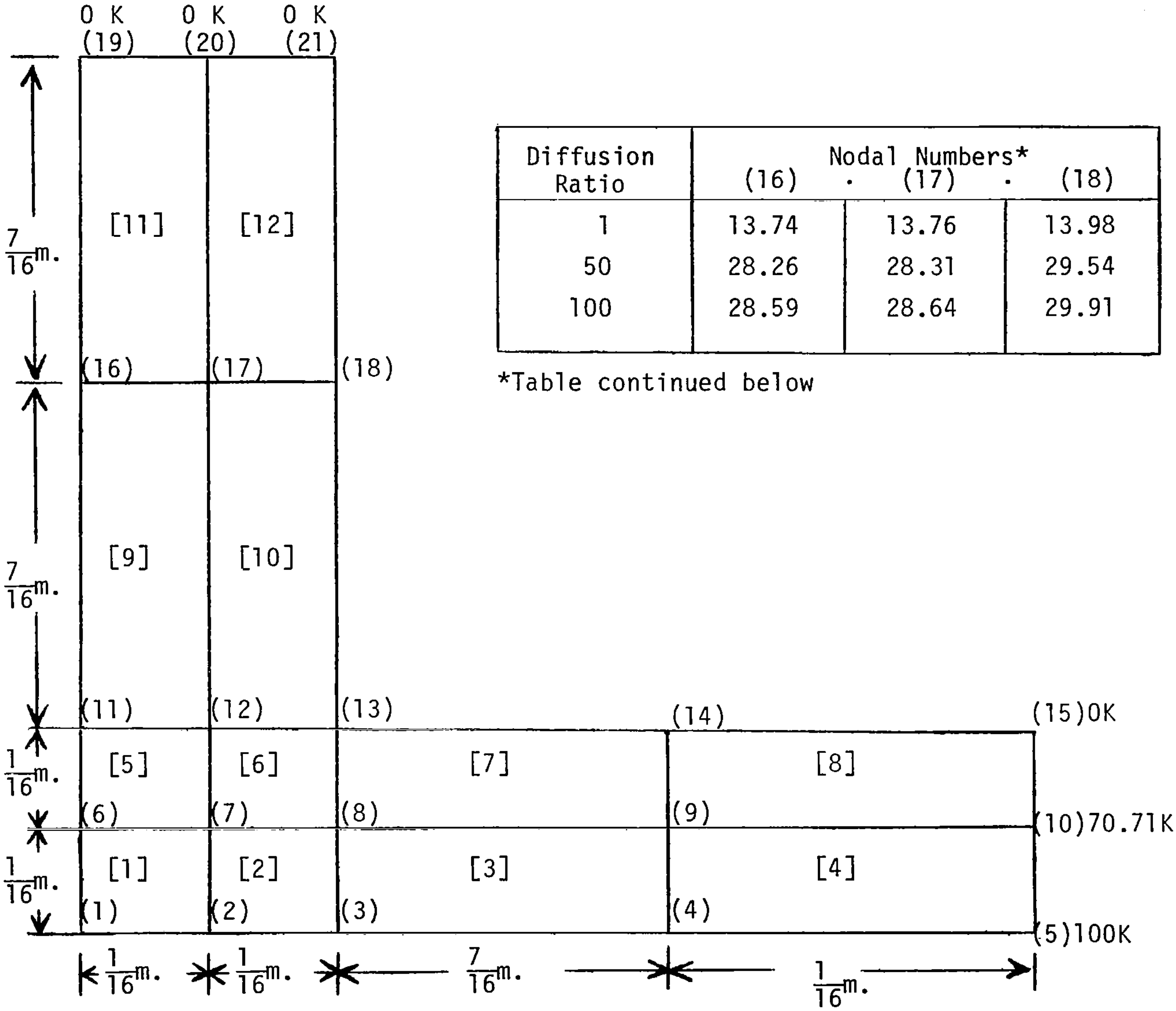
Stability region for the Crank-Nicholson method

Steady State Solution

Figure (4) shows the coarse mesh discretization of the L-shaped region and the steady state distribution of the initial temperature which is input to the transient problem. Material properties are given in Table 1 and the value of the Fourier constant (Eq. 14) in Table 2. The temperature distribution is tabulated for diffusion ratios of one, fifty, and one hundred. While the normal derivative is zero along the edge between nodes 14 and 15, it has the approximate value of -1.1 (K/m) along the edge between nodes 15 and 10, resulting in a sharp discontinuity at node 15. Careful examination of Figure (8.1) shows the while the temperature along the essential boundary nodes (5-10-15) decreases 100K → 70.71K → 0K, the temperature along the opposite end of elements 4 and 8 along nodes (4-9-14) increases 36.52K → 43.72K → 62.23K. This phenomenon of "reflection" is the direct result of the normal derivative discontinuity at the corner node 15 and it has been resolved by mesh refinement. In particular, with a finer mesh, where elements 4 and 8 of the coarse mesh have been replaced by the 56 elements, at the position corresponding to nodes (4-9-14) the temperature became constant at 47.70K.

STEADY STATE DISTRIBUTION FOR COARSE MESH:

Figure 4



Diffusion Ratio	Nodal Numbers* (16)	(17)	(18)
1	13.74	13.76	13.98
50	28.26	28.31	29.54
100	28.59	28.64	29.91

*Table continued below

(1)	(6)	(11)	(2)	(7)	(12)	(3)	(8)	(13)	(4)	(9)	(14)
30.86	29.99	27.93	32.04	30.71	27.81	36.30	33.87	26.92	36.52	43.72	62.23
59.21	59.05	58.76	59.85	58.96	58.40	62.76	60.39	53.29	49.76	56.94	75.49
59.84	59.20	59.46	60.46	59.59	59.12	63.34	60.96	53.84	50.04	57.22	75.77

Experiment 1: Accuracy of Fully Explicit vs. Hughes-Liu Algorithm

The purpose of this experiment was to compare the accuracy of the Hughes-Liu method and fully explicit method for the same step size. The fully implicit method which is of order two (compared to order one for the fully explicit) is taken as the standard.

For a fixed time step the conductor has a Fourier number two orders of magnitude larger than that of the insulation. For this reason implicit integration is required over the conductor while explicit integration over the insulation seems sufficient. The time step must be chosen small enough to meet Courant's stability criterion ($F_0 \leq \frac{1}{4}$) for the conducting material. In particular, in case of the fine mesh, the conducting material has a = 3.2 x 10^{-2} and for Δt = 7.81 seconds the value of F_0 is $\frac{1}{4}$ as shown in Table 1.

As shown in Table 3, after 77 steps, (elapsed time is 602 seconds), the maximum error of 0.21K occurred at nodes 26, 92, and 258 of the fine mesh for the fully explicit method relative to the standard. This is a negligible error occurring over the conductor near its essential boundary which illustrates that the Courant condition is sufficient for stability of explicit time integration. The Hughes-Liu method showed the same accuracy as the standard but had the advantage in CPU time: 983 sec. for the Hughes-Liu method vs. 1327 sec. for the fully implicit.

While the fully implicit method is unconditionally stable convergence is not guaranteed. In a run with Δt = 600 sec., negative temperature values appeared near the essential boundary of the conductor when calculated for the first and only step. The reason for this is the presence of large truncation error (the Fourier number is 19.2).

Experiment 2: Determination of Maximum Time Step

The purpose of this experiment was to determine the maximum time step for each method to achieve accuracy within one degree of the temperature. The baseline for this coincided with the standard at the time step of 7.8 sec., and the final time was 78 sec. The material properties for the insulator were varied to achieve various diffusion ratios by fixing heat capacity at 9.4×10^4 and adjusting thermal conductivity as shown in Table 4. Using the standard, Table 5 shows that varying the diffusion ratio has very little effect on the baseline temperature, or on the variation from the baseline by the three methods.

The data presented in Table 5 is at nodes 31, 64, 97, 130, and 163, located 1/16 meters from the essential boundary of the conductor. These nodes were selected because the maximum temperature variation within the one degree limit had been observed there. It can be seen in Table 5 that the fully explicit method is within one degree of baseline using a time step of 7.1 sec. This corresponds to $F_o = 0.2273$ just 9.09% below the Courant value. In case of the fully implicit method the corresponding time step is 26.04* sec., which is 3.33 times larger than the fully explicit time step. In the next two experiments with larger material properties this ratio was increased to 9.00. The Hughes-Liu method was compared to the baseline only at the diffusion ratio of fifty. It was found to be identical to the fylly implicit results shown in Table 5. The CPU times for the three methods are shown in Table 6. It is seen that the Hughes-Liu method is the least expensive of the three methods.

Experiment 3: Effect of the Diffusion Ratio

In this experiment, the thermal conductivity for the good conductor was increased from its value in Experiment 2 to 1500 (Watt/m-K)), the time step was correspondingly reduced to 0.78 sec., and the thermal conductivity was increased by a factor of ten over those listed in Table 4. The data is presented at the nodes 32, 65, 131, 98, and 164, located 1/32 meter from the essential boundary of the

* Note that the choice of a time step is constrained by the requirement that the final time is an integer multiple of the time step.

conductor, slightly closer than in Experiment 2, because there is where the maximum variation has shifted as the result of the above changes while remaining below one-degree of baseline. Table 7 shows that the effect of changing the diffusion ratio is negligible. It is seen that baseline temperatures are identical at each node. In the fully implicit case of Table 7, a difference of 0.01 K shows up at nodes 65 and 131 as compared to Table 5 with nodes 31, 64, and 130.

The fully explicit method is within one degree of baseline at t = 0.79 sec., corresponding to F_o = 0.2525, which is just 1.01% above the Courant value of 0.25. The fully implicit method remains within one degree of baseline at the same nodes when the time step equals 7 sec. The ratio between the fully implicit and fully explicit time steps is 9.0.

The Hughes-Liu method was compared to the baseline only at the diffusion ratio of fifty, and the temperature values were found to be identical to the values obtained in the fully implicit case. The CPU times for the three methods are shown in Table 8 and show that while the CPU time for the fully implicit and Hughes-Liu method doubled, the CPU time for the fully explicit method increased by a factor of 7.65, making it much more expensive than either of the other two methods. Since the Hughes-Liu method is as accurate as the fully implicit method, the Hughes-Liu is the superior method for the present problem. The increased efficiency of the implicit method and the Hughes-Liu method over the explicit method is due to the increased diffusivity. Because of the higher value of the diffusivity the temperatures decay faster from the initial state to the final state. As a result, a larger share of the temperature history is in a region where there is little temperature variation as compared to the previous experiment.

Experiment 4: Effect on Implicit Algorithm of Variable Time Steps

The purpose of this experiment was to find the effect of a variable time step on the implicit method. Because there is applied heat loading, accuracy requirements dictate small time steps at the beginning of the temp history. Later, as the temperatures become more uniform, the time step may be increased. For the implicit and Hughes-Liu methods the penalty for variable time step is the need to refactorize the Jacobian. We thus used the crude mesh in order to minimize the cost of refactorization. Table 9 summarizes the results obtained for this experiment. It shows the effect on accuracy of using six integrating steps of variable size. Node 14 was selected because that is where the maximum temperature variation from the baseline was observed. It is seen from Table 9 that by taking small time steps initially, the deviation of -10.66 K at fifty sec. is drastically reduced to -5.15 K at 53 sec. and temperature values more accurate. However, the three refactorizations required to achieve this increased accuracy costs more in terms of the CPU time. More specifically, while a single factorization run takes 3 sec., two additional refactorizations raise the CPU time to 5 sec.

The number of steps to reach 300 sec. was increased from six to eight. The results are given in Table 10. Table 10 shows that by taking several small time steps of three seconds initially, the huge variation of 9.30°K at 37.5 sec. is reduced to -4.42 K at 40.5 sec., -2.35 K at 43.5 sec., and -1.32 K at 46.5 sec. While the one factorization run costs 3 sec. the (2-4-2) run costs 5 sec., and the (1-6-1) and (3-2-3) runs cost 6 sec. The effect of taking eight instead of six steps was that the initial oscillations are slightly smaller. This is simply due to the fact that in this case the maximum step size is 37.5 sec. as opposed to the 50 sec. step size in Table 9. Tables 9 & 10 both show that for parabolic problems the initial oscillations do not affect the final temperature because they die out sooner.

Conclusions

This report describes the application of the Hughes-Liu algorithm to the solution of linear, transient heat equations, subject to an initial condition. The technique was applied to an L-shaped region made up of a good conductor and an insulating material. The report has illustrated that the explicit time step determined by the largest eigenvalue of the explicit region governs the stability of the entire region while unconditional stability is achieved for the implicit region. Numerical experiments have verified that this stable behavior leads to substantial computational savings in the two dimensional problem.

Numerical examples have illustrated that when the diffusivity of the good conductor is high enough to warrant the use of a fully implicit integrator, while the poor conductor can be stably treated by explicit integration, the Hughes-Liu algorithm can lead to substantial computational savings. Caution should be observed in interpreting these findings because of the simple, rapidly decaying character of the test problem. Nevertheless, the implicit/explicit integration methods show promise as tools in the numerical solution of transient thermal analysis problems.

APPENDIX

One Dimensional Stability Analysis for the Time Integration Methods

If x_1, x_2 are sampling points; w_1, w_2 are weights for two-point Gaussian quadrature rule over the closed interval from zero to one, then

$$\int_0^1 F(s)ds \cong w_1 F(x_1) + w_2 F(x_2) \qquad \text{(A1)}$$

If, further, we require that we have equal weights w, and symmetrically distributed sampling points x, and 1 - x, then

$$\int_0^1 F(s)ds \cong w[F(x) + F(1 - x)] \qquad \text{(A2)}$$

Being a two-point formula, the quadrature must be exact for polynomials of degree three, which leads to

$$\int_0^1 F(s)ds \cong \tfrac{1}{2}[F(\tfrac{1}{2} - \tfrac{\sqrt{3}}{6}) + F(\tfrac{1}{2} + \tfrac{\sqrt{3}}{6})] \qquad \text{(A3)}$$

Finally, by moving the sampling points to the endpoints of the interval, we obtain the "lumped" quadrature rule [6]

$$\int_0^1 F(s)ds \cong \tfrac{1}{2}[F(0) + F(1)] \qquad \text{(A4)}$$

The special notation used is listed below:

k = Thermal conductivity for conductor

σ_1 = Insulator to conductor conductivity ratio ($0 < \sigma_1 \le 1$)

c = Heat capacity for conductor

σ_2= Insulator to conductor capacity ratio ($0 < \sigma_2 \leq 1$)

h = Element length

x = Spatial coordinate

t = Elapsed time

u = Temperature

I,E= Superscripts used to identify Implicitland Explicit groups

Special Notation

The mathematical formulation:

$$k(x) = \begin{cases} k & 0 \leq x \leq h \\ \sigma_1 k & h \leq x \leq 2h \end{cases} \quad \text{(A6)}$$

$$c(x) = \begin{cases} c & 0 \leq x \leq h \\ \sigma_2 c & h \leq x \leq 2h \end{cases} \quad \text{(A7)}$$

$$k(x)u_{xx} = c(x)u_t \qquad 0 \leq x \leq 2h, \ t \geq 0 \quad \text{(A8)}$$

Subject to:

Initial Condition

$$u(x,0) = U(x) \quad 0 \leq x \leq 2h \quad \text{(A9)}$$

BIC Conditions:

(i) Essential: $u(2h,t) = 0 \quad t \geq 0$ (A10)

(ii) Natural: $u_x(0,t) = 0 \quad t \geq 0$ (A11)

(iii) Compatibility: $u^I(h,t) = u^E(h,t) \ t \geq 0$ (A12)

(iv) Interface: $u_x^I(h,t) = \sigma_1 u_x^E(h,t) \ t \geq 0$ (A13)

Solution Space: Hilbertian Sobolev Manifold

(i) Trial function: $H_0^1 = \{u \in L^2[0,2h]:u, u_x \in BIC\}$ (A14)

(ii) Test function: $H_0^{1*} = \{v \in L^2[0,2h]:v,v_x \in BIC\}$ (A15)

$= H_0^1$ for Galerkin formulation

Korn Inequality: $\alpha||u||_1 \le a(u,u)^{\frac{1}{2}} \le \beta||u||_1$ (A16)

where α, β are constants

$$||u||_1^2 \equiv \int_0^{2h} (u^2 + u_x^2)dx \qquad \text{(A17)}$$

and $a(u,v) \equiv \int_0^{2h} k(x)u_{xx}vdx$

$$= - \int_0^{h} v_x^I u_x^I dx - \int_h^{2h} v_x^E u_x^E dx \qquad \text{(A18)}$$

induces the energy norm equivalent to $||\cdot||_1$.

If

$$\hat{f}(v) \equiv \int_0^{2h} c(x)u_t vdx \qquad \text{(A19)}$$

then the Galerkin method consists of solving for the residual

$$R \equiv a(u,v) - \hat{f}(v) = 0 \qquad \text{(A20)}$$

Using isoparametric transformation between global and local coordinates given by

$$x = x(s) = hs + x_1 \qquad 0 \le s \le 1 \qquad \text{(A21)}$$

and assuming linear variation of temperature function u, u and x may be written in terms of shape functions

$$\begin{aligned} \phi_1(s) &= 1 - s \\ \phi_2(s) &= s \end{aligned} \qquad \text{(A22)}$$

so that

$$a(u^e, v^e) = - v^T K^e u \qquad \text{(A23)}$$

$$\hat{f}(v^e) = v^T C^e u \qquad \text{(A24)}$$

where

$$K^e = \frac{K^e}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{A25}$$

$$C^e = hc^e \begin{bmatrix} \int_0^1 (1-s)^2 ds & \int_0^1 s(1-s)ds \\ \int_0^1 s(1-s)ds & \int_0^1 s^2 ds \end{bmatrix} \tag{A26}$$

Applying integration formula (A4) equation (A26) becomes

$$C^e = \frac{hc^e}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{A27}$$

Then separating out the formulas for the implicit and explicit groups, we obtain:

$$K^I = \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{A28}$$

$$K^E = \frac{\sigma_1 k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{A29}$$

$$C^I = \frac{hc}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{A30}$$

$$C^E = \frac{\sigma_2 hc}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{A31}$$

which leads to the assembled matrices:

$$K = \frac{k}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+\sigma_1 & -\sigma_1 \\ 0 & -\sigma_1 & \sigma_1 \end{bmatrix} \tag{A32}$$

$$C = \frac{hc}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\sigma_2 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} \tag{A33}$$

The essential boundary condition $u_3^E = 0$ forces the elimination of the 3rd row and 3rd column from K, and C resulting in:

$$K_b = \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1+\sigma_1 \end{bmatrix} \quad \text{(A34)}$$

$$C_b = \frac{hc}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1+\sigma_2 \end{bmatrix} \quad \text{(A35)}$$

and K_b can be split into implicit and explicit parts:

$$K_b^I = \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{(A36)}$$

$$K_b^E = \frac{k\sigma_1}{h}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(A37)}$$

whose eigenvalues are given by:

$$\lambda_1^E = 0 \quad \text{(A38)}$$

$$\lambda_2^E = -2\left(\frac{\sigma_1}{1+\sigma_2}\right)\frac{k}{h^2c} \quad \text{(A39)}$$

$$\lambda_1^I = 0 \quad \text{(A40)}$$

$$\lambda_2^I = -2\left(\frac{2+\sigma_2}{1+\sigma_2}\right)\frac{k}{h^2c} \quad \text{(A41)}$$

Employing simultaneous diagonalization on the Hughes-Liu equation, it may be put into the form

$$u_{n+1} = (I - B\Delta t)u_n \quad \text{(A42)}$$

and this same form may be achieved in both partially and fully explicit cases where:

$$B = \begin{cases} \begin{bmatrix} q^E & -\dfrac{q^E}{\sqrt{1+\sigma_2}} \\ \dfrac{-\sqrt{1+\sigma_2}}{1+\sigma_2+q^I\Delta t}\, q^E & \dfrac{2q^I + q^E}{1+\sigma_2+q^I\Delta t} \end{bmatrix} & \text{In Hughes-Liu Equation} \\ \begin{bmatrix} -\lambda_1^E & 0 \\ 0 & -\lambda_2^E \end{bmatrix} & \text{In partially Explicit Case } (K_b^I = 0) \\ \dfrac{2k}{h^2 c} \begin{bmatrix} 1 & -1 \\ \dfrac{-1}{1+\sigma_2} & \dfrac{1+\sigma_1}{1+\sigma_2} \end{bmatrix} & \text{In Fully Explicit Case } (K_b^I = 0,\ K_b^E = K_b) \end{cases} \tag{A43}$$

With corresponding eigenvalues:

$$\lambda_1 = \begin{cases} r - \sqrt{r^2 - s} & \text{In Hughes-Liu Case} \\ -\lambda_1^E & \text{In Partially Explicit Case} \\ 2[r^* - \sqrt{(r^*)^2 - s^*}\,]\dfrac{k}{h^2 c} & \text{In Fully Explicit Case} \end{cases}$$

$$\lambda_2 = \begin{cases} r + \sqrt{r^2 - s} & \text{In Hughes-Liu Case} \\ -\lambda_2^E & \text{In Partially Explicit Case} \\ 2[r^* + \sqrt{(r^*)^2 - s^*}\,]\dfrac{k}{h^2 c} & \text{In Fully Explicit Case} \end{cases} \tag{A45}$$

Where:

$$q^I = (2 + \sigma_2) \frac{k}{h^2 c} \tag{A46}$$

$$q^E = \frac{2\sigma_1}{2 + \sigma_2} \frac{k}{h^2 c} \tag{A47}$$

$$r = \frac{q^I q^E \Delta t + (2 + \sigma_2) q^E + 2q^I}{2(q^I \Delta t + 1 + \sigma_2)} \tag{A48}$$

$$s = \frac{2q^I q^E}{q^I \Delta t + 1 + \sigma_2} \tag{A49}$$

$$r^* = \tfrac{1}{2}\left(1 + \frac{1 + \sigma_1}{1 + \sigma_2}\right) \tag{A50}$$

$$s^* = \frac{\sigma_1}{1 + \sigma_2} \tag{A51}$$

The stability condition corresponding to equation (A42) is given by

$$|1 - \Delta t \lambda| \leq 1 \tag{A52}$$

and since in all cases

$$\lambda_1 < \lambda_2 \tag{A53}$$

is equivalent to

$$0 \leq \Delta t \leq \frac{2}{\lambda_2} \tag{A54}$$

so that

$$0 \leq \Delta t \leq \begin{cases} \dfrac{1 + \sigma_2}{\sigma_1} \dfrac{h^2 c}{k} & \text{in Hughes-Liu Case} \\[2ex] \dfrac{1 + \sigma_2}{\sigma_1} \dfrac{h^2 c}{k} & \text{In partially Explicit Case} \\[2ex] \dfrac{2 + \sigma_1 + \sigma_2 - \sqrt{(\sigma_1 - \sigma_2)^2 + 4(1 + \sigma_2)}}{2\sigma_1} \dfrac{h^2 c}{k} & \\ \text{In Fully Explicit Case} & \end{cases} \tag{A55}$$

This illustrates the energy theorem of Hughes and Liu [4] which states that whenever the partially explicit case is stable over the explicit region, the Hughes-Liu method will be stable over the entire inhomogeneous region.

Finally, in the fully implicit case ($K_b^E = 0$, $K_b^I = K_b$) we obtain the equation

$$(I + \frac{\Delta t}{2} B)u_{n+1} = (I - \frac{\Delta t}{2} B)u_n \tag{A56}$$

where B is the matrix obtained in the fully explicit case.

This shows that the stability criterion is

$$\left| \frac{1 - \frac{\Delta t}{2} \lambda_2}{1 + \frac{\Delta t}{2} \lambda_2} \right| \leq 1 \tag{A57}$$

so that $\Delta t \geq 0$ unconditionally.

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Table 1

Material Properties of L-shape Configuration

Material	Thermal Conductivity	Heat Capacity	Diffusion Coefficient	Fourier Number, F_o, Eq. 13
Conductor	150	2.4×10^{6}	6.25×10^{-5}	0.25
Insulation	0.06	9.4×10^{4}	6.38×10^{-7}	0.00255

Table 2

Value of Fourier Constant a (Eq. 14)

Material	Crude Mesh	Fine Mesh
Conductor	8×10^{-3}	3.2×10^{-2}
Insulation	8.17×10^{-5}	3.27×10^{-4}

Table 3

Comparison of Accuracy of the
Fully Implicit, Fully Explicit, and Hughes-Liu Methods at Selected Nodes

Time Step = 7.8 sec.

Number of Steps = 77

Duration - 602 sec.

Node Number	Implicit	Explicit	Error	Hughes Liu	Error
1	63	63	0.01	63	0.00
2	63	63	0.00	63	0.00
3	63	63	0.01	63	0.09
26	36.13	36	0.21	36.13	0.00
92	36.13	36	0.21	36.13	0.00
258	20.19	20.19	0.00	20.19	0.00
CPU TIME	1327 sec.	507 sec.		983 sec.	TOTAL 2939 sec.

Table 4

Insulator Thermal Conductivity for Various Diffusion Ratios:

Diffusion Ratio	Thermal Conductivity
1	5.875
50	0.1175
100	0.05875

Table 5

Temperature Variation from Baseline for

Various Diffusion Ratios (Δt chosen to achieve errors below 1K)

Node	Diffusion Ratio	Baseline Temperature(K) $\Delta t = 7.8$ sec.	Temperature Variation from Baseline	
			Fully Implicit* $\Delta t = 26.0$ sec.	Fully Explicit $\Delta t = 7.1$ sec.
31	1	29.71	0.78	-0.74
	50	29.87	0.78	-0.73
	100	29.87	0.79	-0.73
64	1	29.67	0.71	-0.71
	50	29.84	0.70	-0.72
	100	29.84	0.71	-0.72
97	1	29.58	0.48	-0.67
	50	29.75	0.48	-0.68
	100	29.75	0.48	-0.67
130	1	29.50	0.22	-0.63
	50	29.66	0.23	-0.63
	100	29.67	0.22	-0.63
163	1	29.46	0.09	-0.61
	50	29.63	0.09	-0.62
	100	29.63	0.09	-0.62

*The Hughes-Liu method produced identical results for diffusion ratio of 50

Table 6

CPU Times for various algorithms diffusion ratio = 50
(Experiment 2)

Integration Method	CPU TIME
Fully Implicit	127 sec.
Fully Explicit	33 sec.
Hughes-Liu	77 sec.

Table 7

Temperature Variation from Baseline for Various Diffusion Ratios (Experiment 3 increased conductivity, Δt chosen to achieve errors below 1K)

Node	Diffusion Ratio	Baseline Temperature (K) Δt = 7.81 sec.	Temperature Variation from Baseline	
			Fully Implicit* Δt = 7.1 sec.	Fully Explicit Δt = 0.79 sec.
32	1	4.92	-0.59	-0.10
	50	5.01	-0.60	-0.11
	100	5.01	-0.60	-0.11
65	1	4.92	-0.55	-0.11
	50	5.01	-0.56	-0.12
	100	5.01	-0.55	-0.12
98	1	4.92	-0.43	-0.11
	50	5.01	-0.44	-0.12
	100	5.01	-0.44	-0.11
131	1	4.92	-0.20	-0.10
	50	5.01	-0.21	-0.11
	100	5.01	-0.20	-0.11
163	1	4.92	-0.14	-0.11
	50	5.01	-0.15	-0.12
	100	5.01	-0.15	-0.12

*The Hughes-Liu method produced identical results for diffusion ratio of 50

Table 8

CPU Times for Experiment 3 at diffusion ratio = 50

Integration Method	CPU TIME
Fully Implicit	245 sec.
Fully Explicit	635 sec.
Hughes-Liu	158 sec.

Table 9

Effects of variable time step on error in temperature at Node 14 (six time steps, implicit method)

Elapsed Time sec.	Baseline at t=1 sec.	6 steps at Δt=50 sec.	1 steps at Δt=3 sec. 4 steps at Δt=50 sec. 1 steps at Δt=50 sec.	2 steps at Δt=3 sec. 2 steps at Δt=50 sec. 2 steps at Δt=97 sec.
3	67.89		-0.27	-0.27
6	63.67			-0.19
50	57.89	-10.66		
53	51.45		-5.15	
56	51.03			-2.78
100	45.56	7.25		
103	45.23		3.24	
106	44.90			1.64
150	40.66	- 5.15		
153	40.40		-2.30	
200	36.73	3.52		
203	36.51		1.42	-1.56
250	33.47	- 2.55		
300	30.69	1.87	-1.33	0.97

Table 10

Effect of Variable time steps on temperature error (node 14)

Elapsed Time in Sec.	Δt=.05 sec.	8 steps Δt=37.5 sec.	1 step Δt=3 sec. 6 steps Δt=37.5 sec. 1 step Δt=72 sec.	2 steps Δt=3 sec. 4 steps Δt=37.5 sec. 2 steps Δt=72 sec.	3 steps Δt=3 sec. 2 steps Δt=37.5 sec. 3 steps Δt=72 sec.
3.0	67.91		-0.29	-0.29	-0.29
6.0	63.68			-0.20	-0.20
9.0	61.20				-0.14
37.5	53.79	-9.30			
40.5	53.31		-4.42		
43.5	52.85			-2.35	
46.5	52.40				-1.32
75.0	48.50	5.42			
78.0	48.13		2.45		
81.0	47.76			1.22	
84.0	47.40				0.64
112.5	44.22	-3.60			
115.5	43.91		-1.52		
118.5	43.61			-0.77	
150.0	40.66	2.26			
153.0	40.40		0.83		
156.0	40.14			0.36	0.61
187.5	37.64	-1.60			
190.5	37.41		-0.59		
225.0	35.03	1.03			
228.0	34.83		0.32	-0.38	0.26
300.0	30.69	0.52	-0.33	0.18	-0.35

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16. Abstract

This report describes the application of the Hughes-Liu mixed implicit/explicit algorithm to the solution of linear, transient heat equations, subject to an initial condition. The technique was applied to an L-shaped region made up of a good conductor and an insulating material. The report illustrates that the explicit time step determined by the largest eigenvalue of the explicit region governs the stability of the entire region while unconditional stability is achieved for the implicit region. Numerical experiments verify that this stable behavior leads to substantial computational savings in the two-dimensional problem.

Numerical examples illustrate that when the diffusivity of the good conductor is high enough to warrant the use of a fully implicit integrator, while the poor conductor can be stably treated by explicit integration, the Hughes-Liu algorithm can lead to substantial computational savings. While caution should be observed in interpreting these findings because of the simple, rapidly decaying character of our test problem, implicit/explicit integration methods show promise as tools in the numerical solution of transient thermal problems.

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