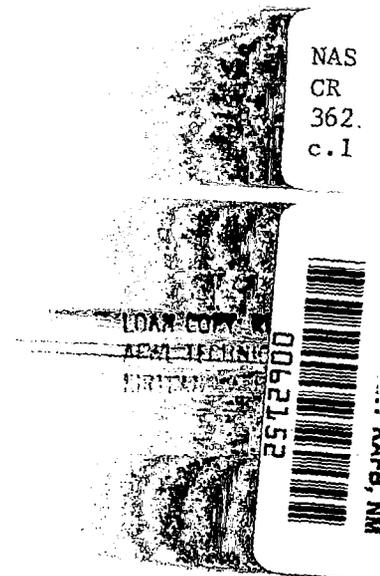


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Phenomenological and Mechanics  
Aspects of Nondestructive  
Evaluation and Characterization  
by Sound and Ultrasound of  
Material and Fracture Properties



Li-Sheng W. Fu

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**Phenomenological and Mechanics  
Aspects of Nondestructive  
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Material and Fracture Properties**

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## INTRODUCTION

The recent thrust in the study of elastic wave scattering has been highly motivated by its applications in various fields such as seismic explorations, nondestructive testing, material property evaluation and dynamic stress concentration. An excellent account of history and fundamentals of elastic wave motions is given in [1-2]<sup>1</sup> and a comprehensive discussion of applications in nondestructive evaluation from a theoretical viewpoint can be found in [3-4].

The scattering of a single ellipsoidal scatterer in an infinite medium is of fundamental importance and is attacked by the method of matched asymptotic expansions [5], the direct volume integral formulation [6], the surface integral formulations [7], and also, recently, the polarization approach [8].

It appears that Mal and Knopoff [6] were first in presenting a direct volume integral formulation where they gave the scattered displacements in terms of volume integrals involving the displacements and strains inside the scatterer. Not knowing these fields, they used the solution when no scatterer was present to obtain an approximate solution for a perfect sphere. The same approach was later taken by Gubernatis [9] for an ellipsoidal inhomogeneity. These solutions are appropriate at longwave scattering,  $ka < 1$ , where  $k$  is wave number and "a" is a typical geometric dimension.

Using the equivalent inclusion method, Eshelby [10,11,12] studied the static elastic fields "inside" and "outside" an ellipsoidal inclusion

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1 Number in brackets designate References at the end of paper.

or inhomogeneity embedded in an infinite isotropic elastic medium under applied tension. All through this paper, an "inhomogeneity" is referred to as a region of different elastic moduli and density compared with its surrounding matrix and an "inclusion" is referred to as a region with the same elastic moduli and density as its surrounding matrix but include in it a distribution of eigenstrains. Eigenstrains are strains that are not derived from mechanical loading. As examples we note that thermal strains and also the swelling strains due to the presence of moisture are special types of eigenstrains.

The method of equivalent inclusion is a method where the inhomogeneity is replaced by an inclusion such that solutions for the two problems are exactly the same. The basic concept is sketched in Figure 1. Wheeler and Mura [13] first developed but did not apply a complete eigenstrain formulation to the dynamic case. The purpose of this study is to extend the method of equivalent inclusion to fully develop the equivalence conditions and to present a method for complete determination of the eigenstrains and/or their derivatives as appropriate.

The equivalence conditions and the solutions to the scattering of an inhomogeneity in terms of eigenstrains and/or their derivatives are first developed. Agreement with other approaches is then easily seen. The formulation is general and both the inhomogeneity and the host medium can be anisotropic. The scattering of an ellipsoidal inhomogeneity in a linear elastic isotropic whole space subjected to plane time-harmonic wave is studied and the differential and total cross-sections for a uniformly distributed eigenstrain are detailed and shown.

Mr. H.S. Jing and Mr. Y. Paul Hsu assisted in the computer programming and graph plotting. Numerical work was conducted by using the AMDAHL 470 computing system and plotting was done on the MINC System of the Department of Engineering Mechanics, Ohio State University.

## EQUIVALENCE CONDITIONS

In this section, we develop the equivalence conditions by requiring that the displacement and stress fields obtained in the inclusion problem be identical to those for the inhomogeneity problem, Fig 1(a) and 1(b).

### The Inhomogeneity Problem (Problem I)

Consider the problem of a single inhomogeneity occupying the region  $\Omega$  in the infinitely extended region  $D-\Omega$  subjected to applied incident wave field  $u_j^{(i)}$ , Figure 1. Let the elastic moduli and the mass density be denoted by  $C'_{ijkl}$  and  $\rho'$  for the inhomogeneity, and by  $C_{ijkl}$  and  $\rho$  for the host medium, respectively.

The governing equations for the displacement field are:

$$C_{jkrs} u_{r,sk} + \Delta C_{jkrs} u_{r,sk} = \rho \ddot{u}_j + \Delta \rho \ddot{u}_j \quad \text{in } D \quad (1)$$

in which we used

$$\sigma_{jk} = C_{jkrs} u_{r,s} + \Delta C_{jkrs} u_{r,s} \quad (2)$$

$$\Delta \rho = \begin{cases} 0 & \text{in } D-\Omega \\ \rho' - \rho & \text{in } \Omega \end{cases} \quad (3)$$

$$\Delta C_{jkrs} = \begin{cases} 0 & \text{in } D-\Omega \\ C'_{jkrs} - C_{jkrs} & \text{in } \Omega \end{cases} \quad (4)$$

Let the superscripts (i) and (m) denote fields associated with the incident wave and the mis-match in mass density and elastic moduli. It is clear that

$$u_j = u_j^{(i)} + u_j^{(m)} \quad (5)$$

as in the absence of mis-match,  $u_j^{(m)}$  disappears and the total field is identical to the incident field.

The boundary conditions are that (1) the displacements and tractions at the intersection of the regions  $\Omega$  and  $D-\Omega$  must be continuous, and (2) the characteristics of out-going wave field and that the stresses die out at infinity must be observed.

#### The Inclusion Problem (Problem II)

Consider next an infinite elastic solid of homogeneous moduli  $C_{ijkl}$  and density  $\rho$  with distributed eigenstrains, denoted by  $\epsilon_{ij}^*$ , in a region  $\Omega$ , such that  $\Omega$  is identical in shape and size to that in Problem I, and

$$\epsilon_{ij}^* = \begin{cases} 0 & \text{in } D-\Omega \\ \epsilon_{ij}^* & \text{in } \Omega \end{cases} \quad (6)$$

The total strain field is

$$\epsilon_{rs} = (u_{r,s} + u_{s,r})/2 = \epsilon_{rs}^e + \epsilon_{rs}^* \quad (7)$$

where

$$\sigma_{jk} = C_{jkrs} \epsilon_{rs}^e \quad (8)$$

Using Equations (6,7) in the equations of motion, we easily obtain the governing equations for the total displacement field as follows:

$$C_{jkrs} u_{r,sk} = \rho \ddot{u}_j + C_{jkrs} \epsilon_{rs,k}^* \quad \text{in } D \quad (9)$$

It is clear from Equation (9) that

$$u_j = u_j^{(i)} + u_j^* \quad (10)$$

where  $u_j^*$  is the displacement field due to the presence of  $\epsilon_{ij}^*$  and it disappears when  $\epsilon_{ij}^*$  vanish. The only boundary conditions are those regular conditions at infinity and the radiation condition.

### Equivalence Conditions

For a complete equivalence between Problem I and Problem II, we require that the displacement and stress fields in the two problems be identical. Hence, for equivalence in stress field, we require, from Equation (2) and Equations (7,8),

$$\left( C_{jkrs} u_{r,s} + \Delta C_{jkrs} u_{r,s} \right)_I = \left( C_{jkrs} (u_{r,s} - \epsilon_{rs}^*) \right)_{II} \quad (11)$$

For equivalence in displacement fields we require that the Equations (1) and (9) be identical, hence

$$\left( \Delta \rho \ddot{u}_j - \Delta C_{jkrs} u_{r,sk} \right)_I = \left( C_{jkrs} \epsilon_{rs,k}^* \right)_{II} \quad (12)$$

It is clear that Equations (11,12) are automatically satisfied in the region  $D-\Omega$  by observing the definitions given in Equations (3,4,6). It is convenient to split the RHS of Equations (12) into two parts such that

$$C_{jkrs} \epsilon_{rs,k}^{*(1)} = -\Delta C_{jkrs} u_{r,sk} \quad (13a)$$

$$C_{jkrs} \epsilon_{rs,k}^{*(2)} = \Delta \rho \ddot{u}_j \quad (13b)$$

Employing Equations (11-13) and Equations (5,10) we obtain the equivalence conditions as:

$$\Delta C_{jkrs} u_{r,s}^{(m)}(\bar{r}) + C_{jkrs} \varepsilon_{rs}^{*(1)}(\bar{r}) = -\Delta C_{jkrs} u_{r,s}^{(i)}(\bar{r}) , \quad \text{in } \Omega \quad (14a)$$

$$\Delta \rho \ddot{u}_j^{(m)}(\bar{r}) + C_{jkrs} \varepsilon_{rs,k}^{*(2)}(\bar{r}) = -\Delta \rho \ddot{u}_j^{(i)}(\bar{r}) \quad \text{in } \Omega \quad (14b)$$

These conditions can be used to determine the eigenstrain distribution that is necessary for the equivalence of Problems I and II provided that we can write  $u_r^{(m)}$  in terms of the eigenstrains. One such method is given in [15]. It is of interest to note that Equations (14a) are identical in form as the equivalence conditions in the static case and that only the j<sup>th</sup> components of  $C_{jkrs} \varepsilon_{rs,k}^{*(2)}$  are needed for determining  $u_j^{(m)}$ . Further discussion on the determination of the eigenstrains will follow in the next section.

## THE SCATTERING OF AN INHOMOGENEITY

Instead of finding the solution to the physical problem stated we seek the solution to the equivalent inclusion problem. The governing equations are Equations (6,7,8,9). Let the incident wave field be plane time-harmonic then the time harmonic displacement and eigenstrain fields can be written as, for example,

$$u_j^{(m)}(\vec{r}, t) = u_j^{(m)}(\vec{r}) \exp(-i\omega t)$$

$$\epsilon_{rs}^*(\vec{r}, t) = \epsilon_{rs}^*(\vec{r}) \exp(-i\omega t)$$

where  $\omega$  is the frequency of the incident wave field and  $i \cdot i = -1$ . Using the dynamic version of the Betti-Rayleigh reciprocal theorem and suppressing the time dependence we obtain the displacement field as:

$$u_m(\vec{r}') = u_m^{(i)}(\vec{r}) - \iiint_V C_{jkrs} g_{jm}(\vec{r}, \vec{r}') \epsilon_{rs,k}^*(\vec{r}) dV \quad (15)$$

or, upon employing Gauss' theorem,

$$u_m(\vec{r}') = u_m^{(i)}(\vec{r}') + \iiint_V C_{jkrs} g_{jm,k}(\vec{r}, \vec{r}') \epsilon_{rs}^*(\vec{r}) dV \quad (16)$$

where  $g_{jm}(\vec{r}, \vec{r}')$  is the spatial part of the solution to the associated Green's function problem for Equation (9). Note that the use of the Green's function preserves the characteristics of an outward propagating wave and satisfies the boundary condition on stress at infinity. Since there are two types of eigenstrains, we write the displacements as

$$\begin{aligned}
u_m(\bar{r}') &= u_m^{(i)}(\bar{r}') - \iiint_{\Omega} g_{jm}(\bar{r}, \bar{r}') C_{jkrs} \epsilon_{rs,k}^{*(2)}(\bar{r}) dV \\
&- \iiint_{\Omega} g_{jm,k'}(\bar{r}, \bar{r}') C_{jkrs} \epsilon_{rs}^{*(1)}(\bar{r}) dV \quad (17)
\end{aligned}$$

where Equations (6,15,16) are used. It is clear that only the j<sup>th</sup> component of  $C_{jkrs} \epsilon_{rs,k}^{*(2)}$  are needed to determine the displacements. We can therefore view this as a vector quantity, say  $\pi_j^*$ , where

$$\pi_j^* = C_{jkrs} \epsilon_{rs,k}^{*(2)} \quad (18)$$

Substituting Equation (18) in Equation (17) we obtain the solution form given as Equation (2.19) in Reference [8], p. 291, for the time-harmonic case. If the Equations (14) are substituted in Equation (17), we find the solution form Equation (12) in Reference [6], p. 379, or Equation (2.25) in Reference [14], p. 2806.

The stress field that is arisen from the presence of mis-match or equivalently by the presence of eigenstrains can be obtained by using Hooke's law and Equations (5, 10, 17) as:

$$\begin{aligned}
\sigma_{pq}^{(m)}(r') &= -\frac{1}{2} C_{pqmn} \iiint_{\Omega} [(g_{jm,n'}(\bar{r}, \bar{r}') + g_{jn,m'}(\bar{r}, \bar{r}')) \pi_j^*(\bar{r}) dV \\
&- \frac{1}{2} C_{pqmn} \iiint_{\Omega} (g_{jm,k'n'} + g_{jn,k'm'}) C_{jkrs} \epsilon_{rs}^*(\bar{r}) dV \quad (19)
\end{aligned}$$

Quantities of interest such as the differential cross section  $dP(\omega)/d\Omega$  defined as: [16,14]

$$\frac{dP(\omega)}{d\Omega} = \lim_{r \rightarrow \infty} \frac{\langle r^2 \ell_i \sigma_{ij}^s \dot{u}_i^s \rangle}{\langle I^0 \rangle} \quad (20)$$

can be obtained in terms of the eigenstrains via Equations (17-20), where  $I^0$  is the incident power,

$$I^0 = n_i \sigma_{ij}^{(i)} \dot{u}_j^{(i)},$$

$\langle f(t) \rangle$  denotes time averaging of a function

$$\langle f(t) \rangle = \frac{1}{T} \int_0^T f(t) dt,$$

and  $\ell_i, n_i$  are direction cosines for  $\vec{r}$  and  $\vec{n}$ , respectively.

The differential  $d\Omega$  is the differential element of a solid angle. The total cross section is simply

$$p(\omega) = \int_{4\pi} \frac{dP(\omega)}{d\Omega} \cdot d\Omega \quad (21)$$

In what follows we give examples for the scattering of an isotropic smooth inhomogeneity in a linear elastic isotropic infinite medium subjected to plane time harmonic incident wave field.

LINEAR ELASTIC ISOTROPIC MEDIUM

For such a medium the spatial part of the Green's function is

$$g_{jm}(\vec{r}-\vec{r}') = \frac{1}{4\pi\rho\omega^2} \{ \beta^2 \delta_{jm} (\exp i\beta R)/R + [(\exp i\beta R)/R - (\exp i\alpha R)/R],_{jm} \} \quad (22)$$

where

$$R = |\vec{r}-\vec{r}'|, \quad \alpha^2 = \frac{\omega^2}{v_L^2} = \frac{\rho\omega^2}{\lambda+2\mu}, \quad \beta^2 = \frac{\omega^2}{v_T^2} = \frac{\rho\omega^2}{\mu}$$

and  $\lambda$ ,  $\mu$ ,  $v_L$ ,  $v_T$  are the Lamé's constants, longitudinal wave speed, transverse wave speed, respectively. Before we substitute Equation (22) in Equation (17), we expand the unknown quantities associated with the eigenstrains in form of a polynomial as [15,17]:

$$\pi_j^*(\vec{r}) = A_j + A_{jk} x_k + A_{jkl} x_k x_l + \dots \quad (23)$$

$$\epsilon_{ij}^{*(1)}(\vec{r}) = B_{ij} + B_{ijk} x_k + B_{ijkl} x_k x_l + \dots \quad (24)$$

where  $A_j$ ,  $A_{jk}$ , ...,  $B_{ij}$ ,  $B_{ijk}$ , ... are constants. Substituting Equations (22,23,24), with  $\pi_j^*$  defined by Equation (18), we obtain

$$u_m^{(m)}(\vec{r}) = u_m(\vec{r}) - u_m^{(i)}(\vec{r}) = f_{mj}(\vec{r}) A_j + f_{mjk}(\vec{r}) A_{jk} + \dots + F_{mij}(\vec{r}) B_{ij} + F_{mijk}(\vec{r}) B_{ijk} + \dots \quad (25a)$$

where

$$4\pi\rho\omega^2 f_{mj}(\vec{r}) = -\beta^2 \phi \delta_{mj} + \psi_{,mj} - \phi_{,mj} \quad (25b)$$

$$4\pi\rho\omega^2 f_{mjk}(\vec{r}) = -\beta^2 \phi_k \delta_{mj} + \psi_{k,mj} - \phi_{k,mj} \quad (25c)$$

...

$$4\pi\rho\omega^2 F_{mij}(\vec{r}) = -[\lambda\alpha^2 \psi_{,m} \delta_{ij} + 2\mu\beta^2 \phi_{,i} \delta_{mj} - 2\mu\psi_{,mij} + 2\mu\phi_{,mij}] \quad (25d)$$

$$4\pi\rho\omega^2 F_{mijk}(\vec{r}) = -[\lambda\alpha^2 \psi_{k,m} \delta_{ij} + 2\mu\beta^2 \phi_{k,i} \delta_{mj} - 2\mu\psi_{k,mij} + 2\mu\phi_{k,mij}] \quad (25e)$$

...

and

$$\phi(\vec{r}) = \iiint_{\Omega} (\exp i\beta R)/R dV' \quad (26a)$$

$$\phi_k(\vec{r}) = \iiint_{\Omega} x_k (\exp i\beta R)/R dV' \quad (26b)$$

...

$$\phi_{k\ell\dots s}(\vec{r}) = \iiint_{\Omega} x_k x_{\ell} \dots x_s (\exp i\beta R)/R dV' \quad (26c)$$

$$\psi(\vec{r}) = \iiint_{\Omega} (\exp i\alpha R)/R dV' \quad (26d)$$

$$\psi_k(\vec{r}) = \iiint_{\Omega} x_k (\exp i\alpha R)/R dV' \quad (26e)$$

...

$$\psi_{k\ell ms}(\bar{r}) = \iiint_{\Omega} x_k x_\ell \dots x_s (\exp i\alpha R)/R dV' \quad (26f)$$

The  $\phi$ - and  $\psi$ -integrals given in Equations (26) are the volume integrals associated with the inhomogeneous Helmholtz equation. They can be carried out for an ellipsoidal region by expanding  $(\exp ikR)/R$  in Taylor series expansions with respect to  $\bar{r}'$ , for  $r > r'$  and with respect to  $\bar{r}$  for  $r < r'$ . Here  $k$  can be either  $\alpha$  or  $\beta$ . Details are given in Reference [18]. This type of expansion for the integrand is particularly useful in determining the coefficients of a "polynomial" distribution of  $\pi_j^*$  and  $A_{ij}^*$ .

To determine the coefficients  $A_j, A_{jk}, \dots, B_{jk}, B_{jkl}, \dots$ , we substitute Equations (18,23,24,25) in Equations (14) and note that we are dealing with time-harmonic displacements. Since the  $\pi_j^*$  and  $\epsilon_{ij}^{*(1)}$  are given in terms of polynomials we expand the  $u_j^{(m)}(\bar{r}), u_{r,s}^{(m)}(\bar{r})$  and  $u_{r,s}^{(i)}(\bar{r})$  in Taylor series expansions with respect to the coordinate origin, by matching the coefficients of terms to the same power of  $x_i^0, x_i, x_i x_j, \dots$ , we obtain a set of infinite number of algebraic equations for  $A_j, A_{jk}, \dots, B_{jk}, B_{jkl}, \dots$ . The scattering cross sections are given in terms of these coefficients by way of Equations (17,19,20).

EXAMPLES:

Uniformly Distributed Egenstrains

Let the region  $\Omega$  be an ellipsoidal region of  $2a_1$ ,  $2a_2$  and  $2a_3$  along the  $x$ ,  $y$ , and  $z$ -axis, respectively, Figure 2. We first expand the integrals and their derivatives in Taylor series for  $r < r'$  and obtain the Taylor series for  $u_j^{(m)}(\bar{r})$  from Equations (25). Substituting this series for  $u_j^{(m)}(\bar{r})$  and its derivatives in Equations (14) with Equations (23,24) and the Taylor series for  $u_j^{(i)}(\bar{r})$ , we obtain the governing algebraic equations for  $A_j$ ,  $A_{jk}$ , ...,  $B_{ij}$ ,  $N_{ijk}$ , ... by comparing the order in the power series i.e.  $x_i^0$ ,  $x_i$ ,  $x_i x_j$ , etc. To save space these equations are not shown here. Once these coefficients are determined, we go back to Equations (25) and find the scattered displacement field which is  $u_j^m(\bar{r})$  when  $r \rightarrow \infty$ . By using formulas given in [18] the scattered displacement field is given in terms of a triple sum.

In what follows we consider the case for a plane time harmonic wave propagating in the  $+z$ -direction, i.e.

$$u_j^{(i)}(\bar{r}, t) = u_0 \exp i(\alpha z - \omega t) \quad (27)$$

For a given  $(ka)$  enough terms must be taken in determining the coefficients  $A$ 's and  $B$ 's in Equations (23,24) from

$$\begin{bmatrix} f \end{bmatrix} \begin{Bmatrix} A \end{Bmatrix} + \begin{bmatrix} F \end{bmatrix} \begin{Bmatrix} B \end{Bmatrix} = \begin{Bmatrix} H \end{Bmatrix}$$

$m \times m$     $m \times 1$              $m \times m$     $m \times 1$              $m \times 1$

$$\begin{bmatrix} d \end{bmatrix} \begin{Bmatrix} A \end{Bmatrix} + \begin{bmatrix} D \end{bmatrix} \begin{Bmatrix} B \end{Bmatrix} = \begin{Bmatrix} E \end{Bmatrix}$$

$m \times m$     $m \times 1$              $m \times m$     $m \times 1$              $m \times 1$

where  $[f]$ ,  $[F]$  are defined by the Taylor expansion of the  $f$ - and  $F$ - functions

at  $r = 0$ , and  $[d]$ ,  $[D]$  are the average of the symmetric part of  $[f]$ ,  $[F]$ . The RHS are obtained from the Taylor series expansion for  $u_j^{(i)}$  and  $\epsilon_{ij}^{(i)}$ , respectively.

Using Equations (12,17) in [18] and Equation (25), we find that, for the lowest order of  $\epsilon_{ij}^{*(1)}$  and  $\pi_j^*$  or  $C_{jkrs} \epsilon_{rs}^{*(2)}$ , i.e. keeping only the constant terms in Equations (23,24), at a distance far away from the ellipsoid:

$$\begin{aligned}
 u_m^{(s)}(\bar{r}, t) &= u_m^{(m)}(\bar{r}, t) \Big|_{r \rightarrow \infty} \\
 &= (4\pi\rho\omega^2)^{-1} \sum_{n=0}^{\infty} \sum_{\ell=0}^n \sum_{k=0}^{n-\ell} \left[ \frac{(-1)^n 4\pi a_1 a_2 a_3 a_1^\ell a_2^k a_3^{n-\ell-k} \ell! k! (n-\ell-k)!}{(2n+3)(2n+1)n! (\ell/2)!(k/2)!(n-\ell-k)/2!} \right] \cdot \\
 &\cdot \left\{ \frac{\exp i\alpha r}{r} \left[ -\alpha^2 (i\alpha)^n \ell_m \ell_j A_j - \lambda i\alpha^3 (i\alpha)^n \ell_m \delta_{kj} B_{kj} \right. \right. \\
 &\quad \left. \left. - 2\mu i\alpha^3 (i\alpha)^n \ell_m \ell_k \ell_j B_{kj} \right] \right. \\
 &\quad \left. + \frac{\exp i\beta r}{r} \left[ \beta^2 (\ell_m \ell_j - \delta_{mj}) A_j - 2\mu i\beta^3 \ell_k \delta_{mj} B_{kj} \right. \right. \\
 &\quad \left. \left. + 2\mu i\beta^3 \ell_m \ell_k \ell_j B_{kj} \right] (i\beta)^n \right\} \exp(-i\omega t) \tag{28}
 \end{aligned}$$

where the repeated subscripts must be summed from 1 to 3. Note that  $B_{kj}$  is non-dimensional and is homogeneous in  $(\alpha u_0)$  and  $A_j$  is of dimension  $(\Delta\rho \cdot \omega^2 \cdot u_0)$ . The first term in the expression, i.e.  $n = 0$ , gives the following:

$$\frac{u_m^{(s)}(\vec{r})}{(\alpha a_1)^3 u_0} = \frac{\exp i\alpha r}{\alpha r} G_m^O(\theta, \phi) + \frac{\exp i\beta r}{\beta r} H_m^O(\theta, \phi) \quad (29a)$$

where  $(r, \theta, \phi)$  are spherical coordinates and

$$G_m^O(\theta, \phi) = - (a_2 a_3 / 3 a_1 a_1) [\ell_m \ell_j A_j^*(\Delta\rho/\rho) + (1 - \alpha^2/\beta^2) \ell_m B_{jj}^* + 2 \ell_m \ell_k \ell_j B_{kj}^*] \quad (29b)$$

$$H_m^O(\theta, \phi) = (a_2 a_3 / 3 a_1 a_1) [(\beta/\alpha)^3 (\ell_m \ell_j - \delta_{mj}) A_j^*(\Delta\rho/\rho) - 2(\beta/\alpha)^4 \ell_k B_{km}^* + 2(\beta/\alpha)^4 \ell_m \ell_k \ell_j B_{kj}^*] \quad (29c)$$

in which

$$A_j^* = A_j / (\Delta\rho \omega^2 u_0) \quad (29d)$$

$$B_{kj}^* = -B_{kj} / (i\alpha u_0) \quad (29e)$$

By using Equations (20,27,28) we obtain the differential scattering cross section, after manipulation, as

$$\frac{dP(\omega)}{d\Omega} = \sigma^P(\theta, \phi) + (\alpha/\beta) \sigma^S(\theta, \phi) \quad (30)$$

where

$$\alpha^2 \sigma^P(\theta, \phi) = (\alpha a_1)^6 G_m^O(\theta, \phi) \bar{G}_m^O(\theta, \phi) (\vec{C}_n \cdot \bar{\vec{C}}_{n'})$$

$$\beta^2 \sigma^S(\theta, \phi) = (\alpha a_1)^6 \cdot H_m^O(\theta, \phi) H_m^O(\theta, \phi) (D_n \cdot \bar{D}_n)$$

$$C_n = \sum_{n=0}^{\infty} \sum_{\ell=0}^n \sum_{k=0}^{n-\ell} \frac{(-i)^n (\alpha a_1)^n (a_2/a_1)^\ell (a_3/a_1)^{n-\ell-k} \ell_1^\ell \ell_2^k \ell_3^{n-\ell-k}}{(2n+3)(2n+1)n! (\ell/2)! (k/2)! (k/2)! (n-\ell-k)/2!}$$

$$D_n = \sum_{n=0}^{\infty} \sum_{\ell=0}^n \sum_{k=0}^{n-\ell} \frac{(-i)^n (\beta a_1)^n (a_2/a_1)^{n-\ell-k} \ell_1^\ell \ell_2^k \ell_3^{n-\ell-k}}{(2n+3)(2n+1)n! (\ell/2)! (k/2)! (n-\ell-k)/2!}$$

The super bar here denotes complex conjugate. The total differential cross-section can be easily obtained as

$$\begin{aligned} P(\omega)/a_1^2 &= (\alpha a_1)^4 \int_{\Sigma} G_m^O(\theta, \phi) \bar{G}_m^O(\theta, \phi) C_n \bar{C}_n \, d\Omega \\ &= (\alpha/\beta)^3 (\alpha a_1)^4 \int_{\Sigma} H_m^O \bar{H}_m^O D_n \bar{D}_n \, d\Omega \end{aligned} \quad (31)$$

## Discussion

An eigenstrain approach to the scattering of a single ellipsoidal inhomogeneity is studied in detail. A complete formulation of the equivalence between the inhomogeneity problem and the inclusion problem is given while the case  $\Delta\rho=0$  was given in NASA contractor report #3445 [15]. It is shown that this approach is identical to other approaches such as the direct volume integral formulation, Refs. [6,8,9].

The eigenstrains are expanded as a geometric series and the coefficients are determined by the equivalence conditions derived here and also Ref. [19]. The scattered displacements and stresses are given in an analytic series form in terms of these coefficients. The scattering of an ellipsoidal inhomogeneity in an isotropic elastic medium under plane time-harmonic incident wave is worked out as an example. The eigenstrains are assumed to be uniformly distributed.

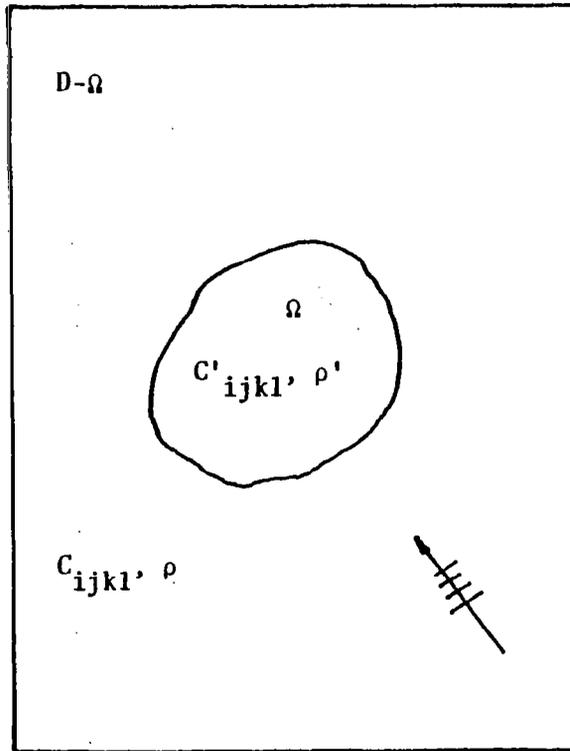
The advantages in using the approach taken is at least three-fold: (i) the radiation condition for out-going waves and the continuity conditions at the interface between the inhomogeneity and the matrix are automatically satisfied, (ii) the solution for special inhomogeneity geometric shape, such as sphere, cylinder, plate, disk, prolate and oblate spheroids, can be easily obtained by setting the appropriate ratios between  $a_1, a_2, a_3$  in evaluating the volume integrals [18], (iii) with the assistance of asymptotic expansion method the solution can be easily modified to obtain solutions appropriate for different ranges of wavelengths that are of interest.

Numerical results and graphical plots for an inhomogeneity of spheroidal geometry, Fig. 2, are given in Figs. 3-9, for the case of uniformly distributed eigenstrains. A comparison of the results obtained with "exact" solution can only be done for simple cases. Such is done for the cases of a planar or a spherical geometry. The comparison is given in a subsequent report. Finally, it should be mentioned that a generalization of this work for investigating the attenuation and velocity factors is being planned.

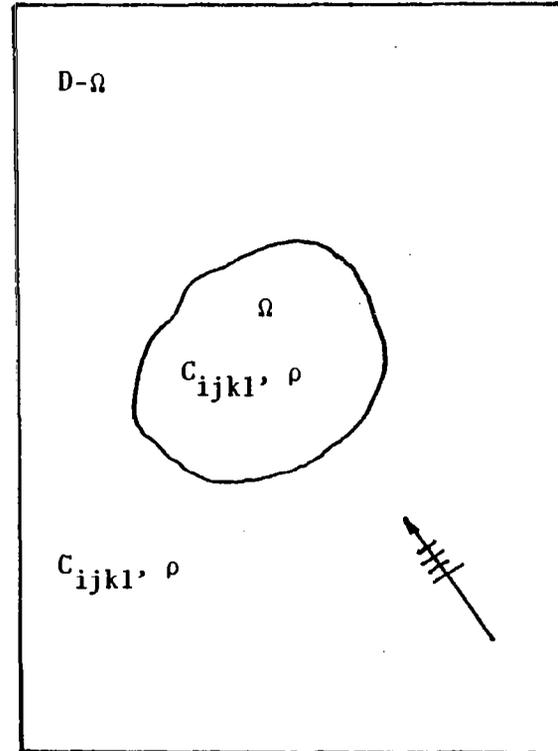
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(a)



(b)

Fig. 1 (a) The inhomogeneity problem,  
(b) The inclusion problem.

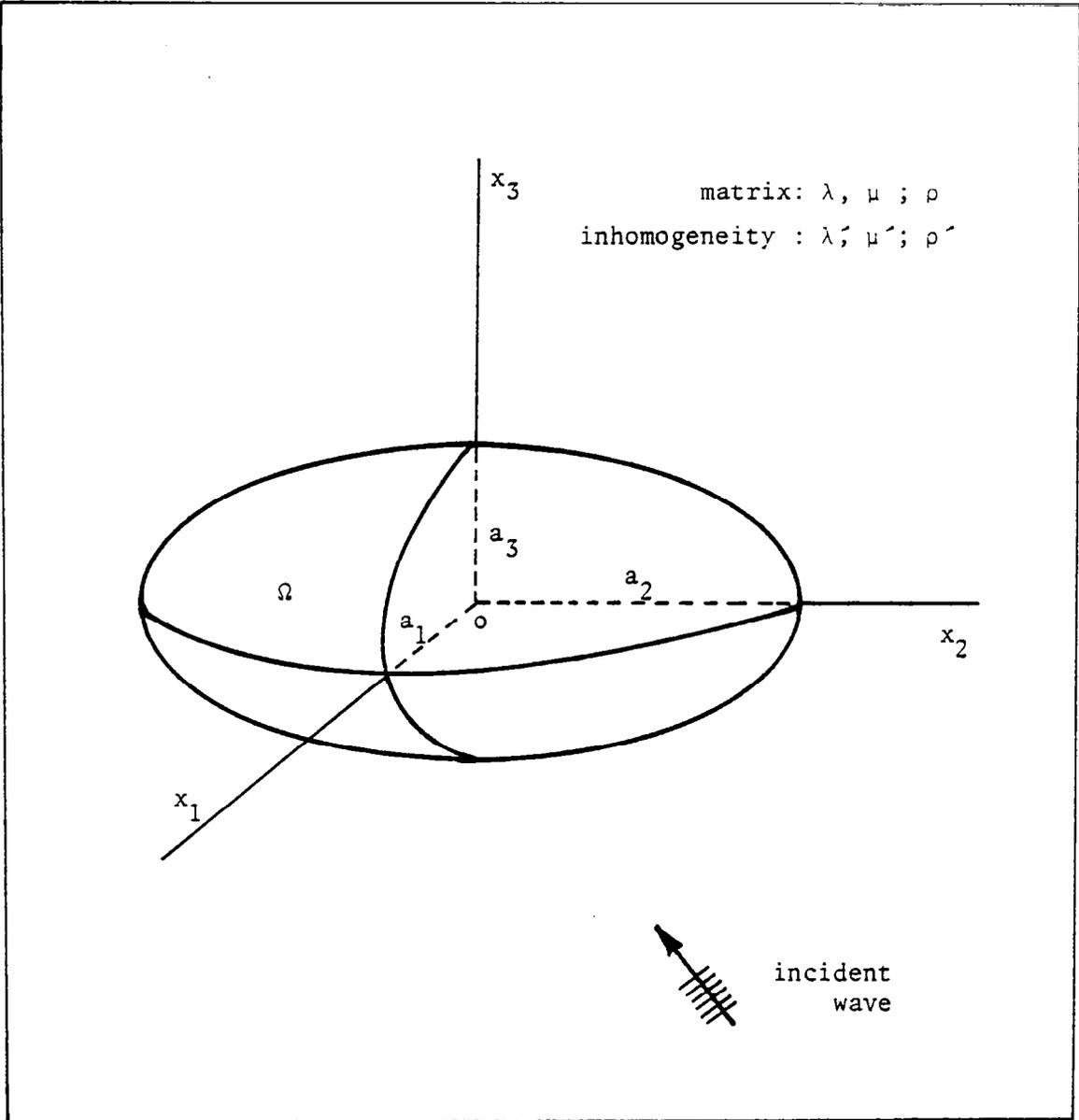


Fig. 2 An ellipsoidal inhomogeneity under incident wave

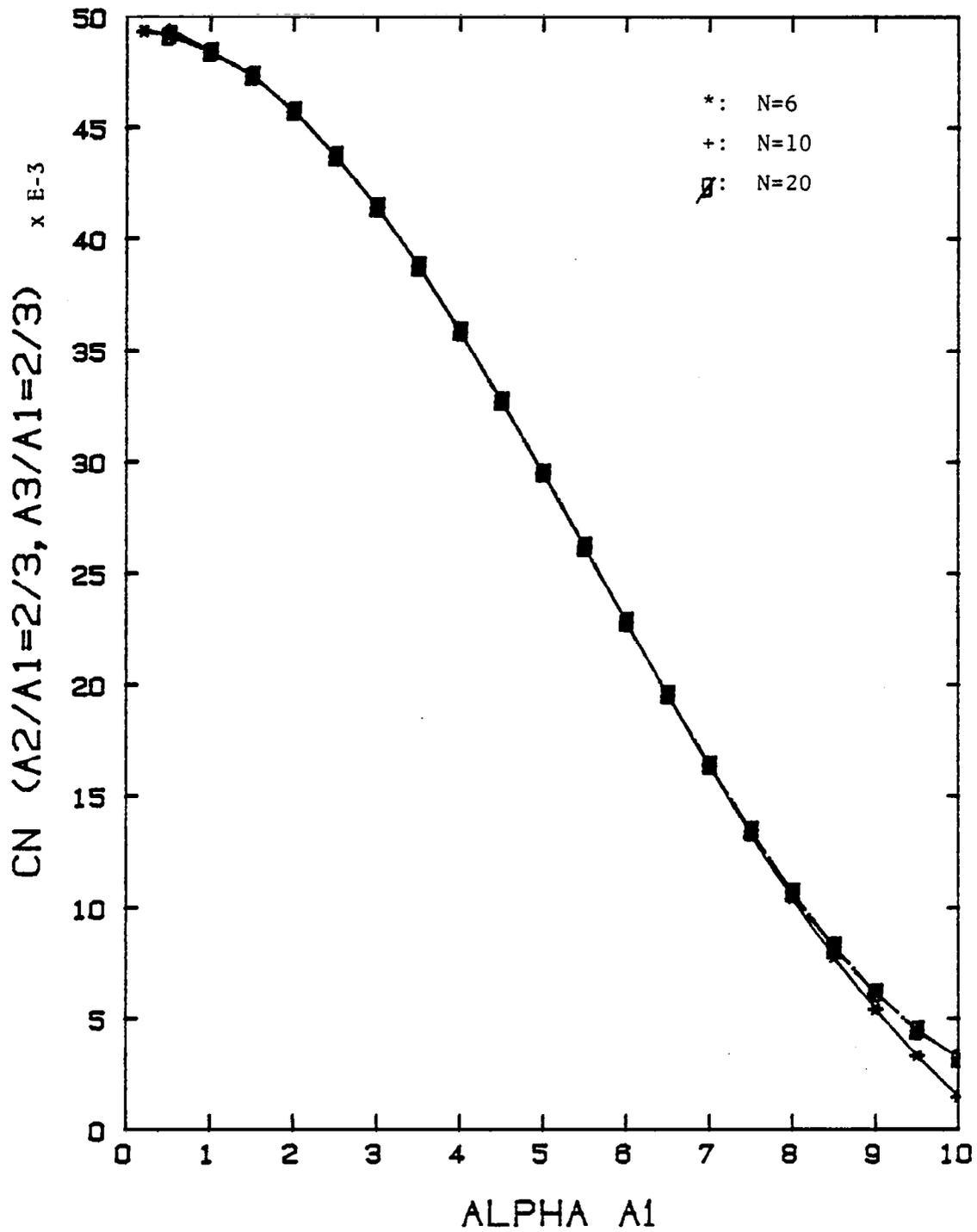


Fig. 7 Convergence of  $C_n$  as a function of  $\alpha_1$ :  $a_2/a_1 = 2/3$ ,  $a_3/a_1 = 2/3$ .

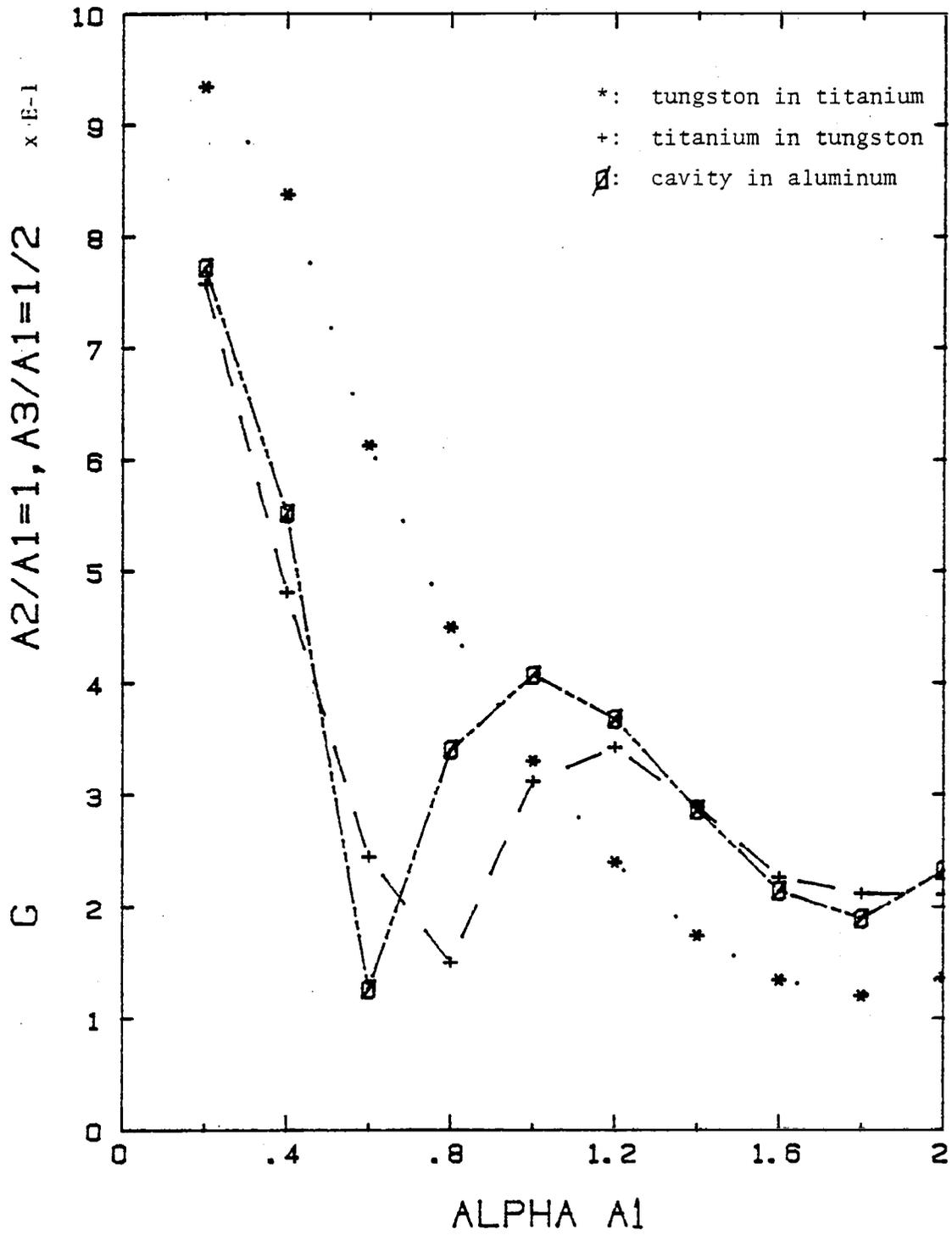


Fig. 6  $G(\theta, \phi)$  vs  $\alpha a_1$ :  $a_2/a_1 = 1$ ,  $a_3/a_1 = 1/2$ ,  $\phi = 0$ ,  $\theta = 0$ ,  
 $G \cdot G = G_m \cdot \bar{G}_m$ .

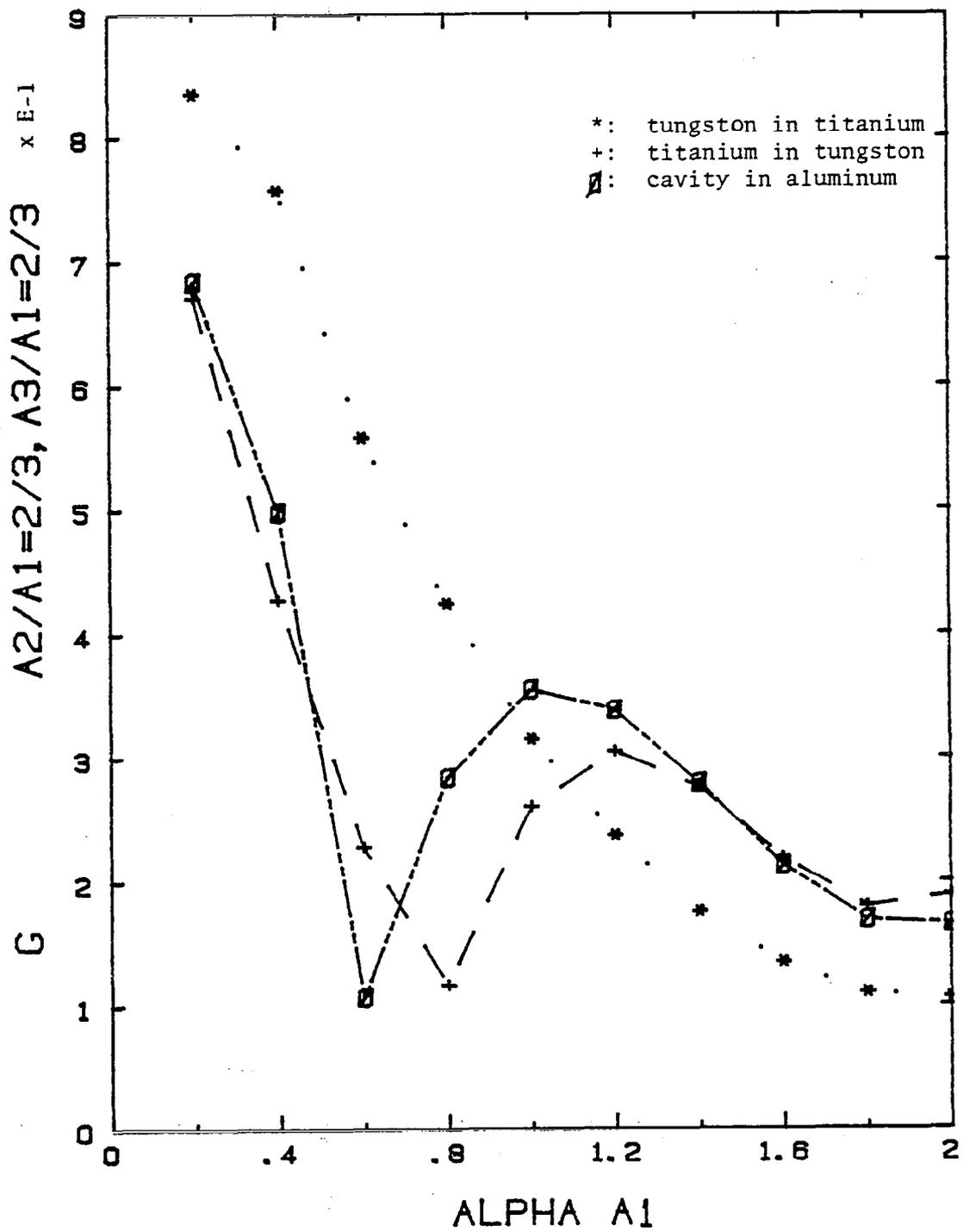


Fig. 5  $G(\theta, \phi)$  vs  $\alpha a_1$ :  $a_2/a_1 = 2/3$ ,  $a_3/a_1 = 2/3$ ,  $\phi = 0$ ,  $\theta = 0$

$$G \cdot G = G_m \cdot \bar{G}_m$$

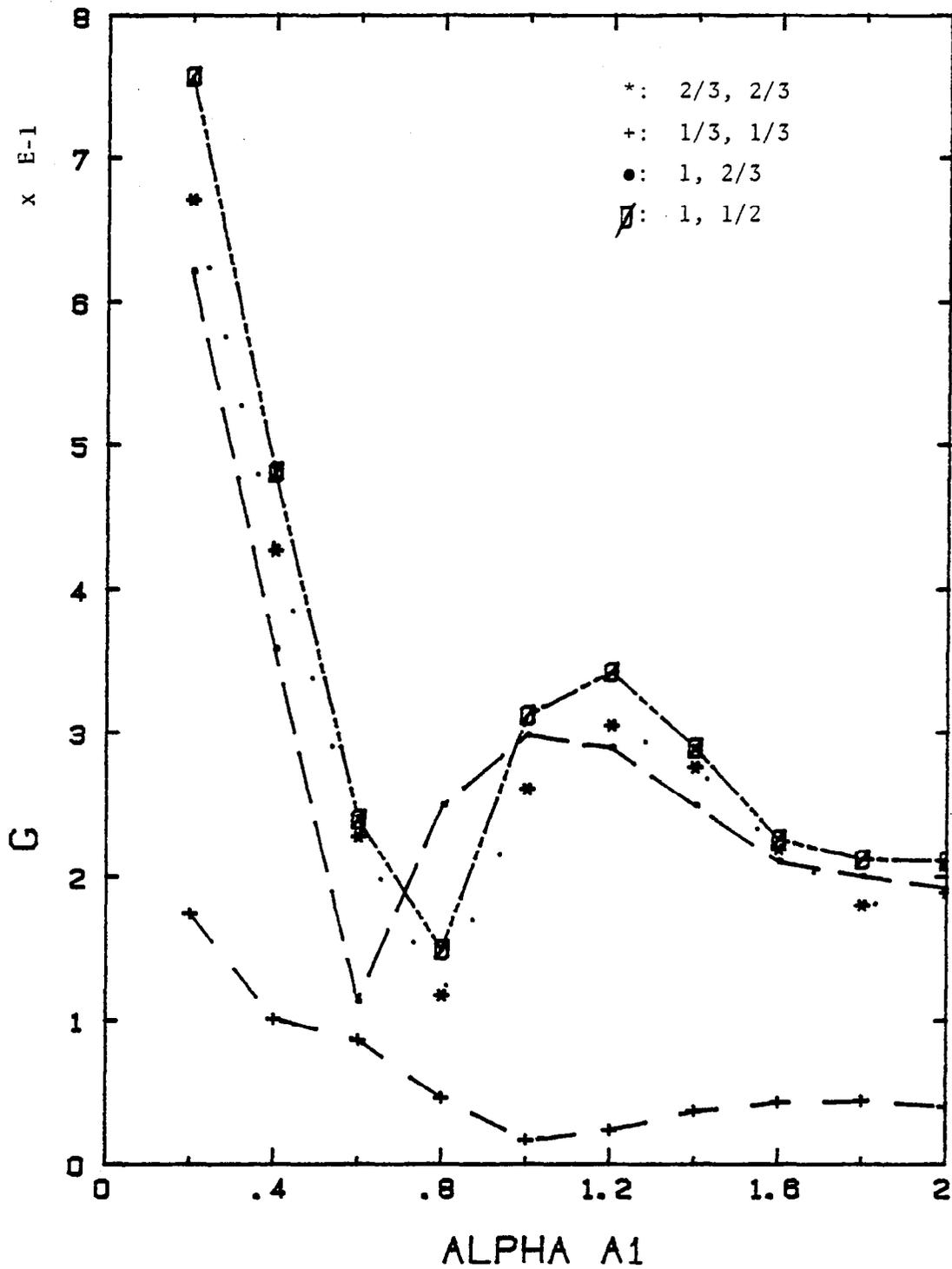


Fig. 4  $G(\theta, \phi)$  vs.  $\alpha a_1$  for different ratio of  $a_3/a_1$  and  $a_2/a_1$ : titanium in tungsten,  $\phi = 0$ ,  $\theta = 0$ ,  $G \cdot G = G_m \cdot \bar{G}_m$ .

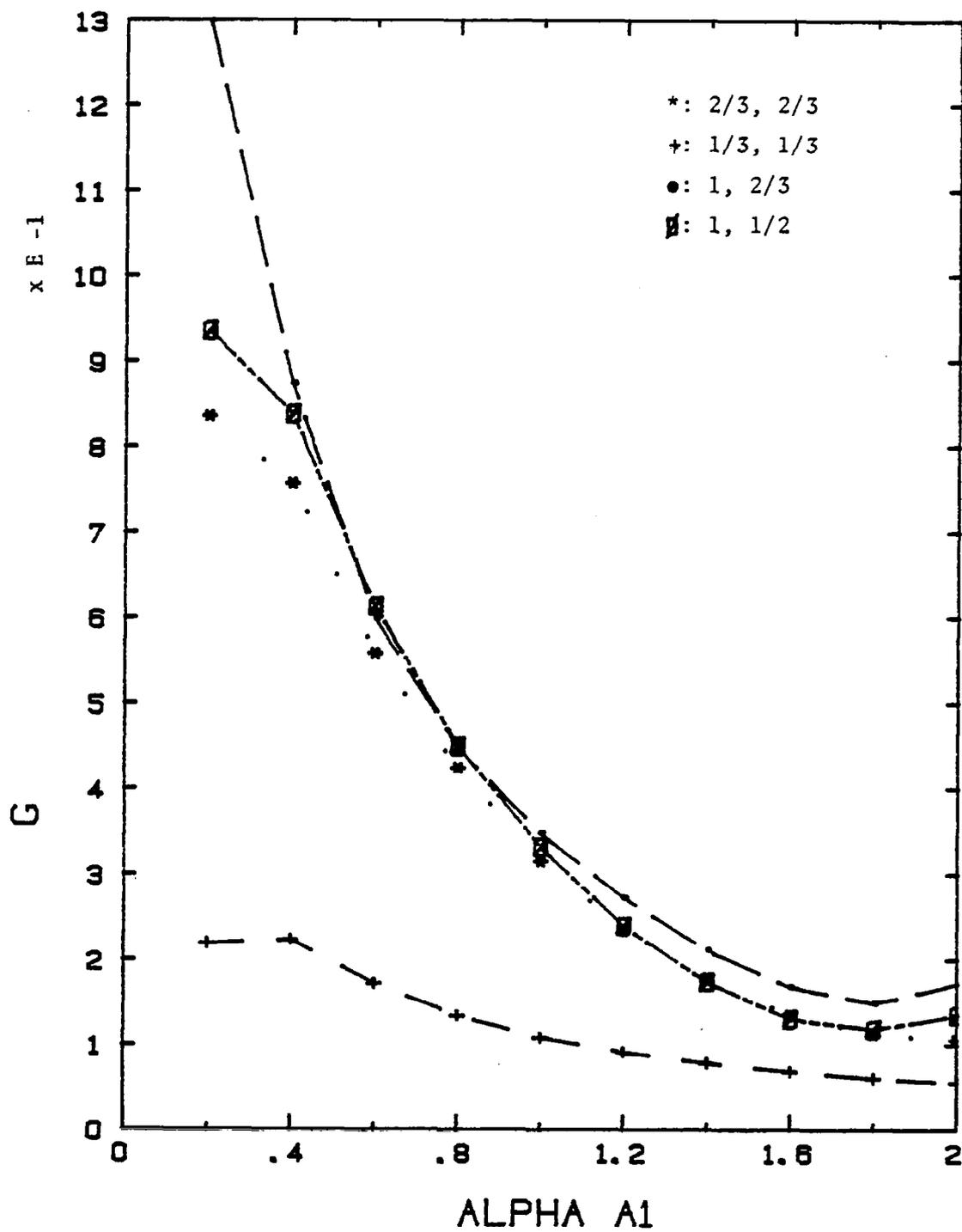


Fig. 3  $G(\theta, \phi)$  vs.  $\alpha a_1$  for different ratios of  $a_2/a_1$  and  $a_3/a_1$ : tungsten in titanium,  $\phi = 0$ ,  $\theta = 0$ ,  $G:G = G_m \bar{G}_m$ .

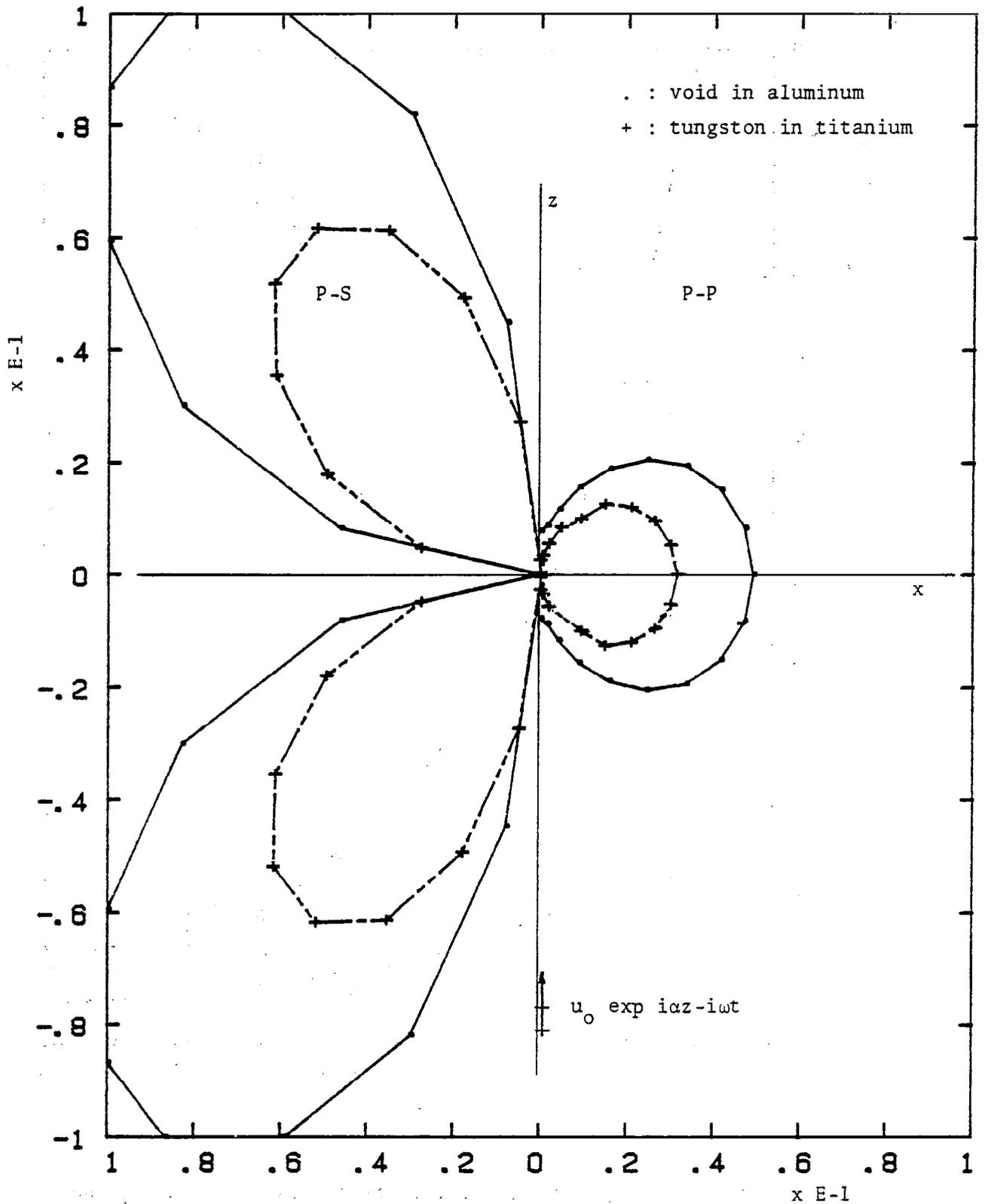


Fig. 8 Polar plots for the scattering amplitudes for the case  $\alpha a_1 = 2.0$ ;  $\phi = 0$ ,  $a_2/a_1 = a_3/a_1 = 2/3$ .

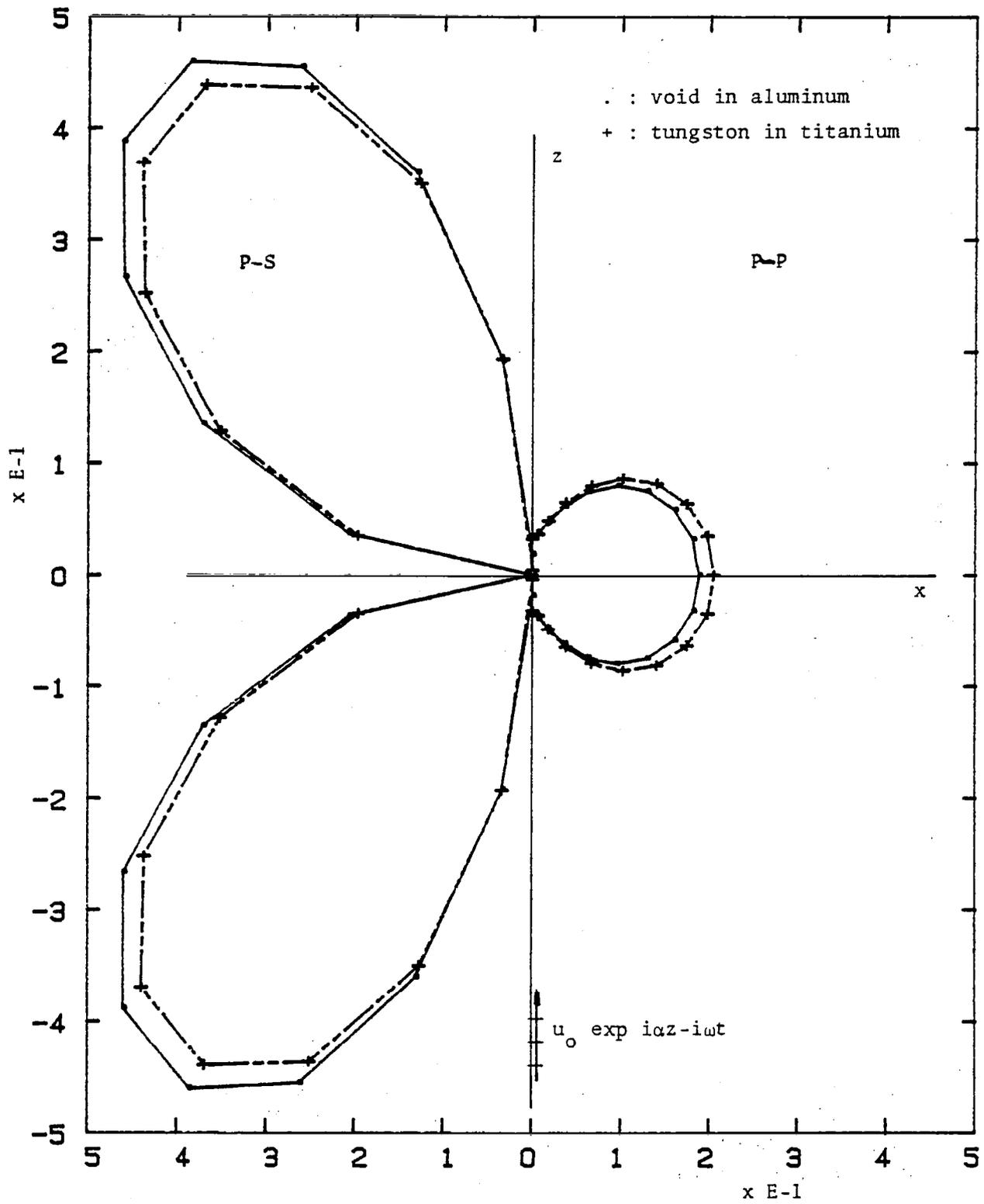


Fig. 9 Polar plots for the scattering amplitudes for the case  $\alpha a_1 = 2.0$ ,  $\phi = 0$ ,  $a_2/a_1 = a_3/a_1 = 1/3$ .

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16. Abstract  Advances have been made on a broad front in nondestructive testing (NDT) in terms of measurement methods, instrumentation, automation and computer-assisted signal acquisition and processing while recent developments in fracture mechanics and elastic wave theory have enabled the understanding of many physical phenomena in a mathematical context. The purpose of this review is to bring together the available literature in the material and fracture characterization by NDT, and the related mathematical methods in mechanics that provide fundamental underlying principles for its interpretation and evaluation. Information on the energy release mechanism of defects and the interaction of microstructures within the material is basic in the formulation of the mechanics problems that supply guidance for nondestructive evaluation (NDE).					
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