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A CYLINDRICAL SHELL WITH AN ARBITRARILY ORIENTED CRACK

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ABSTRACT

In this paper the general problem of a shallow shell with constant curvatures is considered. It is assumed that the shell contains an arbitrarily oriented through crack and the material is specially orthotropic. The nonsymmetric problem is solved for arbitrary self-equilibrating crack surface tractions, which, added to an appropriate solution for an uncracked shell, would give the result for a cracked shell under most general loading conditions. The problem is reduced to a system of five singular integral equations in a set of unknown functions representing relative displacements and rotations on the crack surfaces. The stress state around the crack tip is asymptotically analyzed and it is shown that the results are identical to those obtained from the two-dimensional in-plane and anti-plane elasticity solutions. The numerical results are given for a cylindrical shell containing an arbitrarily oriented through crack. Some sample results showing the effect of the Poisson's ratio and the material orthotropy are also presented.

1. Introduction

Because of their potential applications to the strength and failure analysis of such structurally important elements as pressure vessels, pipes, and a great variety of aerospace and hydrospace components, in recent past the crack problems in shells have attracted considerable attention. Typical solutions obtained by using the classical shallow shell theory may be found, for example, in [1]-[4]. In a Mode I type of shell problem (that is, in a shell for which the geometry and the loading are symmetric with respect to the plane of the crack), particularly for membrane loading, the solution based on the classical theory seems to be adequate. However, in skewsymmetric or nonsymmetric problems, because of the Kirchhoff assumption regarding the

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transverse shear and the twisting moment, in the classical solution it is not possible to separate Mode II and Mode III (i.e., respectively in-plane and anti-plane shear) stress states around the crack tips. In this case a singularity of the form $r^{-\frac{1}{2}}$ in Mode II stress state automatically implies $r^{-3/2}$ singularity in Mode III. For flat plates such drawbacks of the classical theory was pointed out in [5] where it was found that the asymptotic results obtained from plate bending and two-dimensional elasticity could be brought in agreement provided one uses a sixth order plate theory (e.g., that of Reissner's [6]).

In the crack problems for shells even though the membrane and bending results are coupled, the asymptotic behavior of the membrane and bending stresses around the crack tips should be identical to those given by respectively the plane stress and plate bending solutions. This was shown to be the case for the classical shell results (see, for example, the review article [7]). Recent studies using a Reissner-type shell theory [8], [9] shows that similar agreement is also obtained between shell results and those given by the plane elasticity and a sixth order plate bending theory [10]-[13].

Because of the high likelihood of Mode I type fracture most of the previous studies of crack problems in shells were on the symmetrically loaded structures in which the crack is located in one of the principal planes of curvature. The advantage of this crack geometry is that one can always formulate the problem for one half of the shell only as a symmetric or an antisymmetric problem and reduce the number of unknowns. However, in such structural components as pipes and pipe elbows, if, in addition to internal pressure and bending the external loads include also torsion, then the most likely orientation of the crack initiation and propagation would be along a helix rather than a principal plane of curvature. In this case, the problem would have no symmetry and all five stress intensity factors associated with the five membrane, bending, and transverse shear resultants on the crack surfaces would be coupled. Consequently, the related mixed boundary value problem would reduce to a system of five pairs of dual integral equations or five singular integral equations.

In this paper we consider the simplest and yet, from a practical viewpoint, perhaps the most important such problem, namely a cylindrical shell containing a through crack along an arbitrary direction with respect to the axis of the cylinder. In formulating the problem it is assumed that the regular solution of the shell without the crack for the given applied loads is obtained and the problem is reduced to a perturbation problem in which the self-equilibrating crack surface tractions are the only external loads.

2. The Basic Equations

The problem under consideration is described in Fig. 1. As in [11]-[13] the material is assumed to be specially orthotropic, that is the elastic constants defined by

$$\varepsilon_{11} = \frac{1}{E_1}(\sigma_{11}-\nu_1\sigma_{22}) , \quad \varepsilon_{22} = \frac{1}{E_2}(\sigma_{22}-\nu_2\sigma_{11}) ,$$

$$\varepsilon_{12} = \sigma_{12}/2G_{12} , \quad \nu_1/E_1 = \nu_2/E_2 \tag{2.1}$$

satisfy the following factorization condition(*) [12]

$$G_{12} = \frac{\sqrt{E_1E_2}}{2(1+\sqrt{\nu_1\nu_2})} . \tag{2.2}$$

Defining the following "effective" material constants

$$E = \sqrt{E_1E_2} , \quad \nu = \sqrt{\nu_1\nu_2} , \quad B = \frac{5G}{6} , \quad G = \frac{E}{2(1+\nu)} , \quad c = (E_1/E_2)^{\frac{1}{4}}, \tag{2.3}$$

(*)The results given in [12] show that the effect of material orthotropy on the stress intensity factors can be quite significant. In practice the material may be orthotropic because it is either a composite laminate or a rolled sheet metal alloy. Orthotropic materials are also anisotropic with regard to their resistance to fracture and crack propagation. Hence, in a cylindrical shell if the axes of orthotropy do not coincide with the axial and circumferential directions, the solution of the general inclined crack problem becomes all the more important. The solution is also necessary to analyze the weld defects and cracks initiated in the weak cleavage plane of the rolled sheet in spirally welded pipes.

equations (2.1) may be written as

$$\varepsilon_{11} = \frac{1}{E}\left(\frac{\sigma_{11}}{c^2} - \nu\sigma_{22}\right) , \; \varepsilon_{22} = \frac{1}{E}\left(c^2\sigma_{22} - \nu\sigma_{11}\right) , \; \varepsilon_{12} = \frac{\sigma_{12}}{2G} . \qquad (2.4)$$

The derivation of the differential equations for a specially orthotropic shallow shell based on a transverse shear theory [8], [9] may be found in [11]-[13] and will not be repeated in this paper. Referring to Appendix A for notation and to [11]-[13] for details, in terms of a stress function ϕ and the z-component of the displacement w the problem may be formulated as follows:

$$\nabla^4\phi - \frac{1}{\lambda^2}\left(\lambda_1^2 \frac{\partial^2}{\partial y^2} - 2\lambda_{12}^2 \frac{\partial^2}{\partial x \partial y} + \lambda_2^2 \frac{\partial^2}{\partial x^2}\right) w = 0 , \qquad (2.5)$$

$$\nabla^4 w + \lambda^2(1-\kappa\nabla^2)\left(\lambda_1^2 \frac{\partial^2}{\partial y^2} - 2\lambda_{12}^2 \frac{\partial^2}{\partial x \partial y} + \lambda_2^2 \frac{\partial^2}{\partial x^2}\right)\phi$$
$$= \lambda^4(1-\kappa\nabla^2)\left(\frac{aq}{h}\right) , \qquad (2.6)$$

$$\kappa\nabla^2\psi - \psi - w = 0 , \qquad (2.7)$$

$$\frac{\kappa(1-\nu)}{2} \nabla^2\Omega - \Omega = 0 . \qquad (2.8)$$

The shell parameters λ_1, λ_2, λ_{12}, λ, and κ are defined in Appendix A, q(x,y) is the transverse loading, and the curvatures are given by

$$\frac{1}{R_1} = -\frac{\partial^2 Z}{\partial x_1^2} , \; \frac{1}{R_2} = -\frac{\partial^2 Z}{\partial x_2^2} , \; \frac{1}{R_{12}} = -\frac{\partial^2 Z}{\partial x_1 \partial x_2} , \qquad (2.9)$$

where $Z = Z(x_1, x_2)$ is the equation of the middle surface of the shell. The functions ψ and Ω are related to the components of the rotation vector by

$$\beta_x = \frac{\partial\psi}{\partial x} + \kappa \frac{1-\nu}{2} \frac{\partial\Omega}{\partial y} , \; \beta_y = \frac{\partial\psi}{\partial y} - \kappa \frac{1-\nu}{2} \frac{\partial\Omega}{\partial x} . \qquad (2.10)$$

The normalized membrane, moment, and transverse shear resultants are given by

$$N_{xx} = \frac{\partial^2\phi}{\partial y^2} \ , \ N_{yy} = \frac{\partial^2\phi}{\partial x^2} \ , \ N_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \ ; \tag{2.11}$$

$$M_{xx} = \frac{a}{h\lambda^4}\left(\frac{\partial\beta_x}{\partial x} + \nu\frac{\partial\beta_y}{\partial y}\right) \ , \ M_{yy} = \frac{a}{h\lambda^4}\left(\nu\frac{\partial\beta_x}{\partial x} + \frac{\partial\beta_y}{\partial y}\right) \ ,$$

$$M_{xy} = \frac{a}{h\lambda^4}\frac{1-\nu}{2}\left(\frac{\partial\beta_x}{\partial y} + \frac{\partial\beta_y}{\partial x}\right) \ ; \tag{2.12}$$

$$V_x = \frac{\partial w}{\partial x} + \beta_x \ , \ V_y = \frac{\partial w}{\partial y} + \beta_y \ . \tag{2.13}$$

3. General Solution of Differential Equations

Eliminating ϕ in (2.5) and (2.6) one obtains an eighth order differential equation for $w(x,y)$. If the solution of this differential equation is expressed as

$$w(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x,\alpha)e^{-iy\alpha}d\alpha \ , \ f(x,\alpha) = R(\alpha)e^{mx} \ , \tag{3.1}$$

the characteristic equation for m is found to be

$$\begin{aligned} D(m) = \ & m^8-(4\alpha^2+\kappa\lambda_2^4)m^6 - 4\kappa\lambda_{12}^2\lambda_2^2\alpha i m^5 \\ & + [6\alpha^4+\kappa(4\lambda_{12}^4+2\lambda_1^2\lambda_2^2+\lambda_2^4)\alpha^2+\lambda_2^4]m^4 \\ & + 4\lambda_{12}^2[\lambda_2^2+\kappa(\lambda_1^2+\lambda_2^2)\alpha^2]\alpha i m^3 \\ & - [4\alpha^4+\kappa\alpha^2(4\lambda_{12}^4+2\lambda_1^2\lambda_2^2+\lambda_1^4)+4\lambda_{12}^4+2\lambda_1^2\lambda_2^2]\alpha^2 m^2 \\ & - 4\lambda_1^2\lambda_{12}^2(1+\kappa\alpha^2)\alpha^3 i m + \alpha^4(\alpha^4+\kappa\lambda_1^4\alpha^2+\lambda_1^4) = 0 \ . \end{aligned} \tag{3.2}$$

In (3.2), by substituting m=is it may be seen that

$$D(is) = \sum_{0}^{8} a_k(\alpha)s^k = 0 \,, \tag{3.3}$$

where the coefficients a_k are real and, hence, the complex roots are in conjugate form. Since $Re(m_j) = Im(s_j)$, $(j=1,\ldots,8)$, by ordering the roots m_j of (3.2) properly it may be shown that they have the following property:

$$Re(m_{j+4}) = -Re(m_j) \;,\; Re(m_j) < 0 \;,\; (j=1,\ldots,4) \;. \tag{3.4}$$

Considering now the regularity conditions at $x=\mp\infty$ from (3.1) and (3.4) it follows that

$$f(x,\alpha) = \begin{cases} \sum_{1}^{4} R_j(\alpha)e^{m_j x} \;, & x > 0 \;, \\ \sum_{5}^{8} R_j(\alpha)e^{m_j x} \;, & x < 0 \;. \end{cases} \tag{3.5}$$

Similarly, if we let

$$\phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x,\alpha)e^{-iy\alpha}\, d\alpha, \tag{3.6}$$

from (2.5), (3.1) and (3.5) we obtain

$$g(x,\alpha) = \begin{cases} \sum_{1}^{4} R_j(\alpha)K_j(\alpha)e^{m_j x} \,, & x > 0 \;, \\ \sum_{5}^{8} R_j(\alpha)K_j(\alpha)e^{m_j x} \,, & x < 0 \;, \end{cases} \tag{3.7}$$

where

$$K_j(\alpha) = \frac{(\lambda_2^2-\lambda_1^2)\alpha^2 + \lambda_2^2 p_j + 2\lambda_{12}^2 \alpha m_j i}{\lambda^2\, p_j^2} \,,\; p_j = m_j^2 - \alpha^2 \;. \tag{3.8}$$

Also, by assuming that

$$\Omega(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x,\alpha)e^{-i\alpha y}\, d\alpha \ , \tag{3.9}$$

$$\psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta(x,\alpha)e^{-i\alpha y}\, d\alpha \ , \tag{3.10}$$

from (2.7), (2.8), (3.1) and (3.5) we find

$$h(x,\alpha) = \begin{cases} A_1(\alpha)e^{r_1 x} \ , & x > 0 \\ A_2(\alpha)e^{r_2 x} \ , & x < 0 \ , \end{cases} \tag{3.11}$$

$$r_1 = -r_2 = -[\alpha^2 + \frac{2}{\kappa(1-\nu)}]^{\frac{1}{2}} \ , \tag{3.12}$$

$$\theta(x,\alpha) = \begin{cases} \sum_1^4 \frac{R_j(\alpha)}{\kappa p_j - 1} e^{m_j x} \ , & x > 0 \\ \sum_5^8 \frac{R_j(\alpha)}{\kappa p_j - 1} e^{m_j x} \ , & x < 0 \ . \end{cases} \tag{3.13}$$

The expressions for the stress, moment, and transverse shear resultants may be obtained by substituting from the solution given above into (2.10)-(2.13). The results are given in Appendix B.

Since the problem has no symmetry, the preceding analysis would give its solution for the half regions x>0 and x<0 separately, and since each half would have five boundary conditions at x=0, ten unknown functions are needed to account for these conditions. Quite apart from the crack problem, $R_1,\ldots,R_8$, A_1, and A_2 are the ten unknowns which may be used to solve the two half shell problems. In the crack problem following are the continuity and boundary conditions which must be satisfied at x=0:

$$N_{xx}(+0,y) = N_{xx}(-0,y) \ , \ -\infty<y<\infty \ , \tag{3.14}$$

$$M_{xx}(+0,y) = M_{xx}(-0,y) \ , \ -\infty<y<\infty \ , \tag{3.15}$$

$$N_{xy}(+0,y) = N_{xy}(-0,y) \ , \quad -\infty<y<\infty \ , \tag{3.16}$$

$$M_{xy}(+0,y) = M_{xy}(-0,y) \ , \quad -\infty<y<\infty \ , \tag{3.17}$$

$$V_x(+0,y) = V_x(-0,y) \ , \quad -\infty<y<\infty \ ; \tag{3.18}$$

$$\left.\begin{aligned} &N_{xx}(+0,y) = F_1(y) \ , \ |y|<\sqrt{c} \ , \\ &u(+0,y) - u(-0,y) = 0 \ , \ |y|>\sqrt{c} \ , \end{aligned}\right\} \tag{3.19}$$

$$\left.\begin{aligned} &M_{xx}(+0,y) = F_2(y), \ |y|<\sqrt{c} \ , \\ &\beta_x(+0,y) - \beta_x(-0,y) = 0 \ , \ |y|>\sqrt{c} \ , \end{aligned}\right\} \tag{3.20}$$

$$\left.\begin{aligned} &N_{xy}(+0,y) = F_3(y) \ , \ |y|<\sqrt{c} \ , \\ &v(+0,y) - v(-0,y) = 0, \ |y|>\sqrt{c} \ , \end{aligned}\right\} \tag{3.21}$$

$$\left.\begin{aligned} &M_{xy}(+0,y) = F_4(y) \ , \ |y|<\sqrt{c} \ , \\ &\beta_y(+0,y) - \beta_y(-0,y) = 0 \ , \ |y|>\sqrt{c} \ , \end{aligned}\right\} \tag{3.22}$$

$$\left.\begin{aligned} &V_x(+0,y) = F_5(y) \ , \ |y|<\sqrt{c} \ , \\ &w(+0,y) - w(-0,y) = 0, \ |y|>\sqrt{c} \ , \end{aligned}\right\} \tag{3.23}$$

where $F_1,\ldots,F_5$ are the known crack surface loads in the perturbation problem under consideration.

From the expressions of the stress, moment, and the transverse shear resultants given in Appendix B, it may be seen that the homogeneous relations (3.14)-(3.18) can be used to eliminate five of the ten unknown functions $R_1,\ldots,R_8$, A_1, and A_2. The remaining five may then be obtained from the mixed boundary conditions (3.19)-(3.23). The problem may be reduced to a system of five dual integral equations by obtaining expressions for the displacements and rotations similar to that given in Appendix B and then by substituting into (3.19)-(3.13). However, in the problem under consideration this procedure would be extremely lengthy. A somewhat more convenient approach is the reduction of the mixed boundary conditions directly to a system of integral equations. From the nature of the mixed conditions it is clear that the integral equations will be singular. In order to avoid strong singularities in the resulting integral equations it is necessary that the new unknown functions be selected as the derivatives of the "displacement" quantities rather than the displacements and rotations. Of the "displacements" which appear in the mixed boundary conditions β_x, β_y, and w may readily be expressed in terms of R_j and A_k, $(j=1,\ldots,8;\ k=1,2)$ by using the solution given in this section. To find u and v we use the basic strain-displacement relations for the shallow shells, namely

$$\varepsilon_{ij} = \frac{1}{2}\left[u_{i,j} + u_{j,i} + Z_{,i}\, u_{3,j} + Z_{,j}\, u_{3,i}\right],\ (i,j=1,2)\ . \tag{3.24}$$

From (3.24) it can be shown that

$$\frac{\partial u_1}{\partial x_2^2} = 2\,\frac{\partial \varepsilon_{12}}{\partial x_2} - \frac{\partial \varepsilon_{22}}{\partial x_1} - \frac{\partial Z}{\partial x_1}\frac{\partial^2 u_3}{\partial x_2^2} - \frac{\partial^2 Z}{\partial x_2^2}\frac{\partial u_3}{\partial x_1}\ . \tag{3.25}$$

By substituting from (2.4), (2.9) and Appendix B into (3.25) we obtain

$$\frac{\partial u}{\partial y} = 2(1+\nu)N_{xy} - \int \frac{\partial N_{yy}}{\partial x}\,dy + \nu \int \frac{\partial N_{xx}}{\partial x}\,dy$$

$$+ \int \left[\left(\frac{\lambda_1}{\lambda}\right)^2 x + \left(\frac{\lambda_{12}}{\lambda}\right)^2 y\right] \frac{\partial^2 w}{\partial y^2}\,dy + \left(\frac{\lambda_2}{\lambda}\right)^2 \int \frac{\partial w}{\partial x}\,dy\ . \tag{3.26}$$

From (3.19)-(3.23) it is seen that the y-derivatives of the relative crack surface displacements and rotations are the natural choice for the new unknown functions. However, (3.26) suggests that for the in-plane displacements u and v such a choice would require very complicated analysis. In fact, in the present problem it is not feasible to express the functions

R_j and A_k in terms of the new unknowns in the desired form if they are selected as the derivatives of u^+-u^- and v^+-v^-. We thus define the new unknown functions as follows:

$$G_1(y) = \lim_{x\to+0} [\frac{\partial u}{\partial y} - (\frac{\lambda_{12}}{\lambda})^2 \int y \frac{\partial^2 w}{\partial y^2} dy]$$

$$- \lim_{x\to-0} [\frac{\partial u}{\partial y} - (\frac{\lambda_{12}}{\lambda})^2 \int y \frac{\partial^2 w}{\partial y^2} dy] , \quad (3.27)$$

$$G_2(y) = \lim_{x\to+0} \frac{\partial \beta_x}{\partial y} - \lim_{x\to-0} \frac{\partial \beta_x}{\partial y} , \quad (3.28)$$

$$G_3(y) = \lim_{x\to+0} [\frac{\partial v}{\partial y} - (\frac{\lambda_2}{\lambda})^2 y \frac{\partial w}{\partial y}] - \lim_{x\to-0} [\frac{\partial v}{\partial y} - (\frac{\lambda_2}{\lambda})^2 y \frac{\partial w}{\partial y}] , \quad (3.29)$$

$$G_4(y) = \lim_{x\to+0} \frac{\partial \beta_y}{\partial y} - \lim_{x\to-0} \frac{\partial \beta_y}{\partial y} , \quad (3.30)$$

$$G_5(y) = \lim_{x\to+0} \frac{\partial w}{\partial y} - \lim_{x\to-0} \frac{\partial w}{\partial y} . \quad (3.31)$$

By using now the solution given in this section and the results of Appendix B, the auxiliary functions $G_1,\ldots,G_5$ may be expressed in terms of $R_1,\ldots,R_8$, A_1, and A_2 as follows:

$$G_1(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{\alpha} [\sum_1^4 (\frac{\lambda_2^2}{\lambda^2} - K_j p_j) m_j R_j(\alpha)$$

$$- \sum_5^8 (\frac{\lambda_2^2}{\lambda^2} - K_j p_j) m_j R_j(\alpha)] e^{-i\alpha y} \, d\alpha , \quad (3.32)$$

$$G_2(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{-i\alpha[\sum_1^4 \frac{m_jR_j(\alpha)}{\kappa p_j-1} - \sum_5^8 \frac{m_jR_j(\alpha)}{\kappa p_j-1}]$$

$$- \frac{\alpha^2\kappa(1-\nu)}{2}[A_1(\alpha)-A_2(\alpha)]\}e^{-i\alpha y}\,d\alpha\ , \tag{3.33}$$

$$G_3(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\sum_1^4 K_jR_j[p_j+(1+\nu)\alpha^2] - \sum_5^8 K_jR_j[p_j+(1+\nu)\alpha^2]\}e^{-i\alpha y}\,d\alpha\ , \tag{3.34}$$

$$G_4(y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}[\alpha^2(\sum_1^4 \frac{R_j}{\kappa p_j-1} - \sum_5^8 \frac{R_j}{\kappa p_j-1})$$

$$- \frac{\kappa(1-\nu)}{2}\alpha\ i\ (r_1A_1-r_2A_2)]e^{-i\alpha y}\,d\alpha\ , \tag{3.35}$$

$$G_5(y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} i\alpha[\sum_1^4 R_j - \sum_5^8 R_j]e^{-i\alpha y}\,d\alpha\ . \tag{3.36}$$

Also, by substituting from Appendix B into (3.14)-(3.18) and inverting the Fourier integrals we find

$$\sum_1^4 K_jR_j - \sum_5^8 K_jR_j = 0\ , \tag{3.37}$$

$$\sum_1^4 \frac{p_j+(1-\nu)\ \alpha^2}{\kappa p_j-1}R_j - \sum_5^8 \frac{p_j+(1-\nu)\alpha^2}{\kappa p_j-1}R_j$$

$$- \frac{\kappa}{2}(1-\nu)^2\ i\,\alpha\,(r_1A_1-r_2A_2) = 0\ , \tag{3.38}$$

$$\sum_1^4 m_jK_jR_j - \sum_5^8 m_jK_jR_j = 0\ , \tag{3.39}$$

$$i\alpha \sum_1^4 \frac{m_j R_j}{\kappa p_j - 1} - i\alpha \sum_5^8 \frac{m_j R_j}{\kappa p_j - 1}$$

$$+ \frac{\kappa}{4}(1-\nu)(\alpha^2+r_1^2)(A_1-A_2) = 0 \quad , \tag{3.40}$$

$$\sum_1^4 \frac{m_j p_j R_j}{\kappa p_j - 1} - \sum_5^8 \frac{m_j p_j R_j}{\kappa p_j - 1} - \frac{i}{2}\alpha(1-\nu)(A_1-A_2) = 0 \ . \tag{3.41}$$

From (3.32)-(3.41) it then follows that

$$-\frac{i}{\alpha}\left(\sum_1^4 m_j p_j K_j R_j - \sum_5^8 m_j p_j K_j R_j\right) = q_1(\alpha) - \left(\frac{\lambda_2}{\lambda\alpha}\right)^2 q_2(\alpha) \ , \tag{3.42}$$

$$(A_1-A_2)/2 = q_2(\alpha) \ , \tag{3.43}$$

$$\sum_1^4 p_j K_j R_j - \sum_5^8 p_j K_j R_j = q_3(\alpha) \quad , \tag{3.44}$$

$$\sum_1^4 \frac{p_j R_j}{\kappa p_j - 1} - \sum_5^8 \frac{p_j R_j}{\kappa p_j - 1} = (1-\nu) q_4(\alpha) \ , \tag{3.45}$$

$$-i\alpha\left(\sum_1^4 R_j - \sum_5^8 R_j\right) = q_5(\alpha) \ , \tag{3.46}$$

where

$$q_j(\alpha) = \int_{-\sqrt{c}}^{\sqrt{c}} G_j(t) e^{i\alpha t}\, dt \ , \ (j=1,\ldots,5) \ . \tag{3.47}$$

In defining q_j by (3.47) it is assumed that $G_j(y) = 0$ for $|y| > \sqrt{c}$. From (3.37)-(3.46) the unknown functions $R_1,\ldots,R_8$, A_1, and A_2 may be obtained as follows:

$$R_j(\alpha) = \sum_{k=1}^{5} i\, B_{jk}(\alpha) q_k(\alpha) \ , \ (j=1,\ldots,8) \ , \tag{3.48}$$

$$A_j(\alpha) = \sum_{k=1}^{5} C_{jk}(\alpha) q_k(\alpha) \ , \ (j=1,2) \ ; \tag{3.49}$$

where B_{jk} and C_{jk} are known functions of α.

4. The Integral Equations

The relations to determine the functions $G_1,\ldots,G_5$ necessary to complete the solution of the problem are obtained by substituting from (3.47)-(3.49) and Appendix B into the mixed boundary conditions (3.19)-(3.23). From the definitions of $G_1,\ldots,G_5$ as given by (3.27)-(3.31) it is seen that they are related to the derivatives of the crack surface displacements and rotations. Thus, in addition to requiring that $G_j(y) = 0$ for $|y| > \sqrt{c}$, $(j=1,\ldots,5)$, further conditions must be imposed on these functions in order to insure the continuity of displacements and rotations in the shell for $x=0$, $|y| > \sqrt{c}$ (see (3.19)-(3.23)). That is, $G_1,\ldots,G_5$ must be such that

$$\int_{-\sqrt{c}}^{\sqrt{c}} \frac{\partial}{\partial y} [\omega_j(+0,y) - \omega_j(-0,y)]dy = 0 \quad , \tag{4.1}$$

where ω_j, $(j=1,\ldots,5)$ represents the displacements and rotations u, v, w, β_x, and β_y. From (3.27)-(3.31) the single-valuedness conditions of the form (4.1) may now be expressed as

$$\int_{-\sqrt{c}}^{\sqrt{c}} [G_1(t)+(\frac{\lambda_{12}}{\lambda})^2 tG_5(t)]dt - (\frac{\lambda_{12}}{\lambda})^2 \int_{-\sqrt{c}}^{\sqrt{c}} dt \int_{-\sqrt{c}}^{t} G_5(y)dy = 0 \ , \tag{4.2}$$

$$\int_{-\sqrt{c}}^{\sqrt{c}} G_2(t)dt = 0 \quad , \tag{4.3}$$

$$\int_{-\sqrt{c}}^{\sqrt{c}} [G_3(t) + (\frac{\lambda_2}{\lambda})^2 \ t \ G_5(t)]dt = 0 \quad , \tag{4.4}$$

$$\int_{-\sqrt{c}}^{\sqrt{c}} G_4(t)dt = 0 \quad , \tag{4.5}$$

$$\int_{-\sqrt{c}}^{\sqrt{c}} G_5(t)dt = 0 \quad . \tag{4.6}$$

With the requirements that $G_j(y)$ be zero for $|y| > \sqrt{c}$ and the conditions (4.2)-(4.6) be satisfied, the second part of the mixed boundary conditions (3.19)-(3.23) relating to the displacements and rotations has thus been taken care of. The first part of (3.19)-(3.23) relating to crack surface loading would then give the integral equations to determine $G_1,\ldots,G_5$ which may be expressed as

$$\lim_{x\to+0} -\frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha^2 \sum_1^4 K_j R_j e^{m_j x - i\alpha y}\, d\alpha = F_1(y),\ |y| < \sqrt{c}\ , \tag{4.7}$$

$$\lim_{x\to+0} \frac{1}{2\pi}\frac{a}{h\lambda^4}\int_{-\infty}^{\infty} [\sum_1^4 \frac{p_j+(1-\nu)\alpha^2}{\kappa p_j - 1} R_j\ e^{m_j x}$$

$$- \frac{\kappa}{2}(1-\nu)^2\ i\alpha A_1 r_1 e^{r_1 x}]e^{-i\alpha y} d\alpha = F_2(y),\ |y| < \sqrt{c}\ , \tag{4.8}$$

$$\lim_{x\to+0} \frac{i}{2\pi}\int_{-\infty}^{\infty} \alpha \sum_1^4 m_j K_j R_j e^{m_j x - i\alpha y} d\alpha = F_3(y),\ |y| < c\ , \tag{4.9}$$

$$\lim_{x\to+0} \frac{1}{2\pi}\frac{a(1-\nu)}{h\lambda^4}\int_{-\infty}^{\infty} [i\alpha \sum_1^4 \frac{m_j R_j}{\kappa p_j - 1}\ e^{m_j x}$$

$$+ \frac{\kappa}{2}(1-\nu)(\alpha^2+r_1^2)A_1 e^{r_1 x}]e^{-i\alpha y}\ d\alpha = -F_4(y),\ |y| < \sqrt{c}\ , \tag{4.10}$$

$$\lim_{x\to+0} \frac{1}{2\pi}\int_{-\infty}^{\infty} [\sum_1^4 \frac{\kappa p_j m_j R_j}{\kappa p_j - 1}\ e^{m_j x}$$

$$- \frac{\kappa}{2}(1-\nu)\alpha i A_1 e^{r_1 x}]e^{-i\alpha y} d\alpha = F_5(y),\ |y| < \sqrt{c}\ . \tag{4.11}$$

By substituting now from (3.47)-(3.49) into (4.7)-(4.11) and by changing the order of integrations we obtain a system of integral equations for $G_1,\ldots,G_5$ of the following form:

$$\lim_{x\to+0} \int_{-\sqrt{c}}^{\sqrt{c}} \sum_{1}^{5} G_j(t)dt \int_{-\infty}^{\infty} V_{kj}(x,\alpha)e^{i(t-y)}d\alpha = F_k(y),\ (k=1,\ldots,5),\ |y|<\sqrt{c} \tag{4.12}$$

where $V_{kj}(x,\alpha)$, $(k,j=1,\ldots,5)$ are known functions. The dependence of V_{kj} on α is primarily through $r_1(\alpha)$, $r_2(\alpha)$ and the roots $m_j(\alpha)$, $(j=1,\ldots,8)$ of the characteristic equation (3.2) and is therefore very complicated. However, the functions V_{kj} depend on x only through the exponential damping terms $\exp(m_jx)$, $(j=1,\ldots,4)$ and $\exp(r_1x)$ which simplifies the asymptotic analysis of the kernels in (4.12) quite considerably. To examine the singular behavior of the kernels given by the inner integrals in (4.12) the asymptotic analysis of the functions $V_{kj}(x,\alpha)$ for large values of $|\alpha|$ is needed. First, from (3.2) and (3.12) it can be shown that for large values of $|\alpha|$ the characteristic roots m_j and r_j have the following asymptotic values:

$$m_j(\alpha) = -|\alpha|(1 + \frac{p_j}{2\alpha^2} - \frac{p_j^2}{8\alpha^4} + \ldots)\ ,\ (j=1,\ldots,4)\ , \tag{4.13}$$

$$m_j(\alpha) = |\alpha|(1 + \frac{p_j}{2\alpha^2} - \frac{p_j^2}{8\alpha^4} + \ldots),\ (j = 5,\ldots,8)\ , \tag{4.14}$$

$$r_1(\alpha) = -|\alpha|(1 + \frac{1}{\kappa(1-\nu)\alpha^2} - \ldots)\ , \tag{4.15}$$

$$r_2(\alpha) = |\alpha|(1 + \frac{1}{\kappa(1-\nu)\alpha^2} - \ldots\ )\ , \tag{4.16}$$

where $p_j = m_j^2 - \alpha^2$. Then, observing that the coefficients B_{jk} and C_{jk} which appear in the expressions of R_j and A_j (see (3.48) and (3.49)) depend on α through m_j, $(j=1,\ldots,8)$, and r_k, $(k=1,2)$ only, the asymptotic expansion of $V_{kj}(x,\alpha)$ for large values of $|\alpha|$ may be obtained by using (4.13)-(4.14). Consider, for example, the integral equation (4.7). By using the asymptotic values found for $m_j(\alpha)$ and (3.37), (3.39), and (3.42) it can be shown that for large values of $|\alpha|$ we have

$$\sum_{1}^{4} K_jR_j(\alpha) \cong -\frac{\text{sign}(\alpha)}{4i\alpha^2}\, q_1(\alpha)\ . \tag{4.17}$$

By adding and subtracting the asymptotic value to and from the integrand, (4.7) may be written as

$$\lim_{x\to+0}\{-\frac{1}{2\pi}\int_{-\infty}^{\infty}[\alpha^2\sum_1^4 K_jR_je^{m_jx}+\frac{\text{sign}(\alpha)}{4i}q_1(\alpha)e^{-|\alpha|x}]e^{-i\alpha y}\,d\alpha$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\text{sign}(\alpha)}{4i}e^{-|\alpha|x}\,d\alpha\int_{-\sqrt{c}}^{\sqrt{c}}G_1(t)e^{i\alpha(t-y)}dt\}=F_1(y),\ |y|<\sqrt{c}\ . \tag{4.18}$$

By changing the order of integration, evaluating the resulting inner integral, and then going to limit, the second integral in (4.18) may be expressed as

$$\lim_{x\to+0}\frac{1}{2\pi}\int_{-\sqrt{c}}^{\sqrt{c}}G_1(t)dt\,\frac{1}{2}\int_{0}^{\infty}e^{-\alpha x}\sin\alpha(t-y)d\alpha$$

$$=\lim_{x\to+0}\frac{1}{4\pi}\int_{-\sqrt{c}}^{\sqrt{c}}\frac{(t-y)G_1(t)}{(t-y)^2+x^2}dt=\frac{1}{4\pi}\int_{-\sqrt{c}}^{\sqrt{c}}\frac{G_1(t)}{t-y}dt\ . \tag{4.19}$$

The first integral in (4.18) is uniformly convergent, and hence, the limit can be put under the integral sign. By substituting now from (3.47), (3.48) and (4.19) into (4.18) we obtain

$$\int_{-\sqrt{c}}^{\sqrt{c}}[\frac{G_1(t)}{t-y}+\sum_1^5 k_{1j}(y,t)G_j(t)]dt=4\pi F_1(y),\ |y|<\sqrt{c}\ , \tag{4.20}$$

where the kernels $k_{1j}(y,t)$ are known functions which are bounded for all values y and t in the closed interval $[-\sqrt{c},\sqrt{c}]$.

Similarly, the integral equations (4.8)-(4.11) can be reduced to

$$\int_{-\sqrt{c}}^{\sqrt{c}}[\frac{1-\nu^2}{\lambda^4}\frac{G_2(t)}{t-y}+\sum_1^5 k_{2j}(y,t)G_j(t)]dt=4\pi\frac{h}{a}F_2(y),\ |y|<\sqrt{c}\ , \tag{4.21}$$

$$\int_{-\sqrt{c}}^{\sqrt{c}}[\frac{G_3(t)}{t-y}+\sum_1^5 k_{3j}(y,t)G_j(t)]dt=4\pi F_3(y),\ |y|<\sqrt{c}\ , \tag{4.22}$$

$$\int_{-\sqrt{c}}^{\sqrt{c}} [\frac{1-\nu^2}{\lambda^4} \frac{G_4(t)}{t-y} + \sum_1^5 k_{4j}(y,t)G_j(t)]dt = 4\pi \frac{h}{a} F_4(y), \ |y|<\sqrt{c} , \tag{4.23}$$

$$\int_{-\sqrt{c}}^{\sqrt{c}} [\frac{G_5(t)}{t-y} + \sum_1^5 k_{5j}(y,t)G_j(t)]dt = 4\pi F_5(y), \ |y|<\sqrt{c} . \tag{4.24}$$

The expressions of the Fredholm kernels, $k_{ij}(y,t)$, $(i,j=1,\ldots,5)$, are given in Appendix C. The details of the analysis may be found in [14]. The system of singular integral equations (4.20)-(4.24) must be solved under the additional conditions (4.2)-(4.6). They may be solved in a straightforward manner by using the Gaussian integration technique (see, for example, [15]). The major work in this problem is the evaluation of the Fredholm kernels $k_{ij}(y,t)$, $(i,j=1,\ldots,5)$ which are given in terms of Fourier integrals. To improve the accuracy the asymptotic parts of all integrands are separated and the related integrals are evaluated in closed form. The details of this analysis may also be found in [14].

5. Asymptotic Stress Field Around the Crack Tips, Stress Intensity Factors

For the numerical solution of the system of singular integral equations (4.20)-(4.24) the interval $(-\sqrt{c}, \sqrt{c})$ is normalized by defining

$$\tau = t/\sqrt{c} , \ \eta = y/\sqrt{c} , \ \xi = x/\sqrt{c} ,$$

$$H_j(\tau) = G_j(\tau\sqrt{c}), (j = 1,\ldots,5) , -1<\tau<1 . \tag{5.1}$$

We now observe that the index of the system of singular integral equations is +1 and its solution is of the following form:

$$H_j(\tau) = h_j(\tau)/(1-\tau^2)^{\frac{1}{2}} , (-1<\tau<1) , (j=1,\ldots,5) , \tag{5.2}$$

where $h_1,\ldots,h_5$ are unknown bounded functions. The membrane, bending, and transverse shear resultants may be obtained by substituting from (5.1), (5.2), and (4.47)-(4.49) into the expressions given in Appendix B. The asymptotic behavior of the stress field around the crack tip could then be obtained by using the asymptotic expansions of m_j and r_k, ($j=1,\ldots,8$; $k=1,2$) given in (4.13)-(4.16) and the following asymptotic relation [16]

$$\int_{-1}^{1} \frac{h(\tau)}{\sqrt{1-\tau^2}} e^{i\alpha\tau} d\tau = \left(\frac{\pi}{2|\alpha|}\right)^{\frac{1}{2}} \{h(1)\exp[i(\alpha - \frac{\pi\alpha}{4|\alpha|})]$$

$$+ h(-1)\exp[-i(\alpha - \frac{\pi\alpha}{4|\alpha|})] + O(\frac{1}{|\alpha|})\} , \quad (|\alpha|\to\infty) , \tag{5.3}$$

giving

$$N_{xx} \cong \xi \frac{h_3(1)}{4\sqrt{2\pi}} \int_0^\infty \sqrt{\alpha}\, e^{-\alpha|\xi|} \cos[\alpha(1-\eta) - \frac{\pi}{4}]d\alpha$$

$$+ \frac{h_1(1)}{4\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{\alpha}} (1+\alpha|\xi|)e^{-\alpha|\xi|} \sin[\alpha(1-\eta) - \frac{\pi}{4}]d\alpha, \tag{5.4}$$

$$N_{yy} \cong \frac{h_3(1)}{2\sqrt{2\pi}} \int_0^\infty \frac{e^{-\alpha|\xi|}}{\sqrt{\alpha}} \cos[\alpha(1-\eta) - \frac{\pi}{4}]d\alpha$$

$$+ \frac{h_1(1)}{4\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{\alpha}} (1-\alpha|\xi|e^{-\alpha|\xi|} \sin[\alpha(1-\eta) - \frac{\pi}{4}]d\alpha , \tag{5.5}$$

$$N_{xy} \cong \frac{h_3(1)}{4\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{\alpha}} (1-\alpha|\xi|)e^{-\alpha|\xi|} \sin[\alpha(1-\eta) - \frac{\pi}{4}]d\alpha$$

$$+ \frac{h_1(1)}{4\sqrt{2\pi}} \int_0^\infty \xi \sqrt{\alpha}\, e^{-\alpha|\xi|} \cos[\alpha(1-\eta) - \frac{\pi}{4}]d\alpha \quad , \tag{5.6}$$

$$M_{xx} = \frac{h}{12a}\frac{\xi h_4(1)}{4\sqrt{2\pi}}\int_0^\infty \sqrt{\alpha}\, e^{-\alpha|\xi|} \cos[\alpha(1-\eta) - \tfrac{\pi}{4}]d\alpha$$

$$+ \frac{h}{12a}\frac{h_2(1)}{4\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{\alpha}}(1+\alpha|\xi|)e^{-\alpha|\xi|} \sin[\alpha(1-\eta) - \tfrac{\pi}{4}]d\alpha, \qquad (5.7)$$

$$M_{yy} = \frac{h}{12a}\frac{h_4(1)}{2\sqrt{2\pi}}\int_0^\infty \frac{e^{-\alpha|\xi|}}{\sqrt{\alpha}} \cos[\alpha(1-\eta) - \tfrac{\pi}{4}]d\alpha$$

$$+ \frac{h}{12a}\frac{h_2(1)}{4\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{\alpha}}(1-\alpha|\xi|)e^{-\alpha|\xi|} \sin[\alpha(1-\eta) - \tfrac{\pi}{4}]d\alpha, \qquad (5.8)$$

$$M_{xy} \cong \frac{1}{12}\frac{h}{a}\frac{h_4(1)}{4\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{\alpha}}(1-\alpha|\xi|)e^{-\alpha|\xi|}\sin[\alpha(1-\eta) - \tfrac{\pi}{4}]d\alpha$$

$$+ \frac{1}{12}\frac{h}{a}\frac{h_2(1)}{4\sqrt{2\pi}}\int_0^\infty \sqrt{\alpha}|\xi|e^{-\alpha|\xi|}\cos[\alpha(1-\eta) - \tfrac{\pi}{4}]d\alpha \quad , \qquad (5.9)$$

$$V_x \cong \frac{h_5(1)}{2\sqrt{2\pi}}\int_0^\infty \frac{e^{-\alpha|\xi|}}{\sqrt{\alpha}} \sin[\alpha(1-\eta) - \tfrac{\pi}{4}]d\alpha \quad , \qquad (5.10)$$

$$V_y \cong \frac{h_5(1)}{2\sqrt{2\pi}}\int_0^\infty \frac{e^{-\alpha|\xi|}}{\sqrt{\alpha}} \cos[\alpha(1-\eta) - \tfrac{\pi}{4}]d\alpha \quad . \qquad (5.11)$$

Defining the new coordinates r, θ in η, ξ plane by

$$\xi = r\sin\theta \ , \ \eta-1 = r\cos\theta \ , \qquad (5.12)$$

and using the relation [16]

$$\int_0^\infty z^{\mu-1} e^{-sz} \left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(rz)dz = \frac{\Gamma(\mu)}{(s^2+r^2)^{\mu/2}} \left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\} (\mu \tan^{-1} \tfrac{r}{s})$$

$$(s>0 \ , \ \mu>0) \ , \qquad (5.13)$$

equations (5.4)-(5.11) can be reduced to the following form

$$N_{xx} \cong -\frac{h_3(1)}{4\sqrt{2r}}[-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}] - \frac{h_1(1)}{4\sqrt{2r}}[\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{4}\cos\frac{5\theta}{2}] , \tag{5.14}$$

$$N_{yy} \cong -\frac{h_3(1)}{4\sqrt{2r}}[-\frac{7}{4}\sin\frac{\theta}{2}-\frac{1}{4}\sin\frac{5\theta}{2}] - \frac{h_1(1)}{4\sqrt{2r}}[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}], \tag{5.15}$$

$$N_{xy} \cong -\frac{h_3(1)}{4\sqrt{2r}}[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}] - \frac{h_1(1)}{4\sqrt{2r}}[-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}], \tag{5.16}$$

$$M_{xx} \cong -\frac{h_4(1)}{4\sqrt{2r}}\frac{h}{12a}[-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}] - \frac{h_2(1)}{4\sqrt{2r}}\frac{h}{12a}[\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{4}\cos\frac{5\theta}{2}], \tag{5.17}$$

$$M_{yy} \cong -\frac{h_4(1)}{4\sqrt{2r}}\frac{h}{12a}[-\frac{7}{4}\sin\frac{\theta}{2}-\frac{1}{4}\sin\frac{5\theta}{2}] - \frac{h_2(1)}{4\sqrt{2r}}\frac{h}{12a}[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}], \tag{5.18}$$

$$M_{xy} \cong -\frac{h_4(1)}{4\sqrt{2r}}\frac{h}{12a}[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}] - \frac{h_2(1)}{4\sqrt{2r}}\frac{h}{12a}[-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}], \tag{5.19}$$

$$V_x \cong -\frac{h_5(1)}{2\sqrt{2r}}\cos\frac{\theta}{2} , \tag{5.20}$$

$$V_y \cong \frac{h_5(1)}{2\sqrt{2r}}\sin\frac{\theta}{2} . \tag{5.21}$$

By observing that the membrane and bending components of the stresses are given by (see Appendix A)

$$\sigma_{ij}^{\ m} = N_{ij}, \sigma_{ij}^{\ b} = \frac{12az}{h} M_{ij} \quad (i,j=x,y) \tag{5.22}$$

from equations (5.14)-(5.19) for the leading terms of the combined in-plane stresses $\sigma_{ij} = \sigma_{ij}^{\ m} + \sigma_{ij}^{\ b}$, $(i,j=x,y)$ one obtains

$$\sigma_{xx} \cong -\frac{h_3(1)+zh_4(1)}{4\sqrt{2r}}[-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}]$$

$$-\frac{h_1(1)+zh_2(1)}{4\sqrt{2r}}[\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{4}\cos\frac{5\theta}{2}] , \quad (5.23)$$

$$\sigma_{yy} \cong -\frac{h_3(1)+zh_4(1)}{4\sqrt{2r}}[-\frac{7}{4}\sin\frac{\theta}{2}-\frac{1}{4}\sin\frac{5\theta}{2}]$$

$$-\frac{h_1(1)+zh_2(1)}{4\sqrt{2r}}[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}] , \quad (5.24)$$

$$\sigma_{xy} \cong -\frac{h_3(1)+zh_4(1)}{4\sqrt{2r}}[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}]$$

$$-\frac{h_1(1)+zh_2(1)}{4\sqrt{2r}}[-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}] . \quad (5.25)$$

Similarly, for the transverse shear stresses from

$$\sigma_{iz} = \frac{3}{2} V_i[1 - (\frac{az}{h/2})^2] , (i = x,y) , \quad (5.26)$$

we obtain

$$\sigma_{xz} \cong -\frac{3}{2}\frac{h_5(1)}{2\sqrt{2r}}\cos\frac{\theta}{2}[1 - (\frac{az}{h/2})^2] , \quad (5.27)$$

$$\sigma_{yz} \cong -\frac{3}{2}\frac{h_5(1)}{2\sqrt{2r}}\sin\frac{\theta}{2}[1 - (\frac{az}{h/2})^2] . \quad (5.28)$$

Note that for the isotropic materials c=1 and the asymptotic stress fields (5.23)-(5.25) and (5.27)-(5.28) found from the shell solution are identical to those given by respectively the in-plane and the anti-plane elasticity solution of a two-dimensional crack problem. If we now define the Modes I, II, and III stress intensity factors (for a crack along $x_1=0$, $-a<x_2<a$) by

$$k_j(x_3) = \lim_{x_2 \to a} \sqrt{2(x_2-a)}\, \sigma_{1j}(0,x_2,x_3) \ , \ (j=1,2,3) \ , \tag{5.29}$$

from (5.23)-(5.25) and (5.27) and Appendix A we obtain

$$k_1(x_3) = -\frac{cE}{4}\sqrt{a}\,[h_1(1) + \frac{x_3}{a} h_2(1)] \ , \tag{5.30}$$

$$k_2(x_3) = -\frac{E\sqrt{a}}{4}[h_3(1) + \frac{x_3}{a} h_4(1)] \ , \tag{5.31}$$

$$k_3(x_3) = -\frac{3}{4} B\sqrt{a}\sqrt{c}\, h_5(1)[1 - (\frac{x_3}{h/2})^2] \ . \tag{5.32}$$

6. The Results and Discussion

The main interest in this study is in evaluating the stress intensity factors in shells for various crack geometries and loading conditions. For each crack geometry the problem is solved by assuming only one of the five possible crack surface loadings to be nonzero at a time. For a general loading the result may then be obtained by superposition. From (5.30) and (5.31) it is seen that the in-plane stress intensity factors k_1 and k_2 have a "membrane" and a "bending" component, and h_1 and h_3 are related to the membrane and h_2 and h_4 are related to the bending stresses. For simplicity, the related stress intensity factors are defined separately. The calculated results are normalized with respect to a standard stress intensity factor $\sigma_j\sqrt{a}$ where σ_j stands for any one of the following five nominal ("membrane", "bending", in-plane "shear", "twisting", and "transverse shear") stresses:

$$\sigma_m = N_{11}/h \ , \ \sigma_b = 6M_{11}/h^2 \ , \ \sigma_s = N_{12}/h \ ,$$

$$\sigma_t = 6M_{12}/h^2 \ , \ \sigma_v = (3/2)V_1/h \ , \tag{6.1 a-e}$$

where crack lies in x_2x_3 plane and N_{11}, M_{11}, N_{12}, M_{12}, and V_1 are (a measure or amplitude of) the crack surface tractions.

The normalized stress intensity factors are then defined and calculated in terms of $h_i(1)$, $(i=1,\ldots,5)$ as follows:

$$k_{mj} = \frac{k_1(0)}{\sigma_j\sqrt{a}} = -\frac{cE}{4\sigma_j} h_1(1) \ , \tag{6.2}$$

$$k_{bj} = \frac{k_1(h/2)-k_1(0)}{\sigma_j\sqrt{a}} = -\frac{cE}{4\sigma_j}\frac{h}{2a} h_2(1) \ , \tag{6.3}$$

$$k_{sj} = \frac{k_2(0)}{\sigma_j\sqrt{a}} = -\frac{E}{4\sigma_j} h_3(1) \ , \tag{6.4}$$

$$k_{tj} = \frac{k_2(h/2)-k_2(0)}{\sigma_j\sqrt{a}} = -\frac{E}{4\sigma_j}\frac{h}{2a} h_4(1) \ , \tag{6.5}$$

$$k_{vj} = \frac{k_3(0)}{\sigma_j\sqrt{a}} = -\frac{3}{4}\frac{B}{\sigma_j}\sqrt{c}\, h_5(1) \ , \ (j=m,b,s,t,v) \ , \tag{6.6}$$

where for each individual loading σ_j is given by (6.1). In the case of uniform crack surface loads N_{11}, M_{11}, N_{12}, M_{12}, and V_1, referring to (3.19)-(3.23), Appendix A, and (6.1) the input functions of the system of integral equations (4.20)-(4.24) are given by

$$F_1(y) = \frac{\sigma_m}{cE} \ , \ F_2(y) = \frac{\sigma_b}{6cE} \ , \ F_3(y) = \frac{\sigma_s}{E} \ , \ F_4(y) = \frac{\sigma_t}{6E} \ ,$$

$$F_5(y) = \frac{2}{3}\frac{\sigma_v}{B\sqrt{c}} \ . \tag{6.7}$$

Even though the formulation given in this paper is valid for any shell with constant curvatures $1/R_1$, $1/R_2$, and $1/R_{12}$, the results are obtained for the practical problem of a cylindrical shell containing an arbitrarily oriented crack only (Fig. 1). The crack is assumed to be in a plane defined by the angle β shown in Fig. 1. For the shallow cylindrical shell the curvatures referred to x_1, x_2 axes shown in the figure and defined by (2.9)

Table 1. Stress intensity factor ratios in an isotropic cylindrical shell containing an inclined crack under uniform membrane loading N_{11}, $\nu = 0.3$, $\beta = 45°$.

	h/R \ a/h	1	2	3	5	10
k_{mm}	1/5	1.097	1.302	1.544	2.030	3.486
	1/10	1.049	1.167	1.321	1.665	2.516
	1/15	1.033	1.116	1.230	1.501	2.199
	1/25	1.020	1.072	1.148	1.341	1.886
	1/50	1.010	1.037	1.079	1.194	1.563
	1/100	1.005	1.019	1.041	1.106	1.337
	1/200	1.002	1.010	1.021	1.056	1.192
k_{bm}	1/5	0.084	0.122	0.100	-0.069	-0.761
	1/10	0.058	0.108	0.126	0.079	-0.299
	1/15	0.046	0.093	0.121	0.118	-0.125
	1/25	0.032	0.073	0.104	0.132	0.023
	1/50	0.020	0.049	0.076	0.117	0.120
	1/100	0.012	0.031	0.051	0.089	0.139
	1/200	0.007	0.019	0.033	0.062	0.120
k_{sm}	1/5	-0.036	-0.108	-0.190	-0.333	-0.517
	1/10	-0.018	-0.060	-0.113	-0.227	-0.424
	1/15	-0.012	-0.041	-0.081	-0.173	-0.365
	1/25	-0.007	-0.025	-0.052	-0.119	-0.289
	1/50	-0.004	-0.013	-0.028	-0.068	-0.192
	1/100	-0.002	-0.007	-0.014	-0.037	-0.117
	1/200	-0.001	-0.003	-0.007	-0.019	-0.067
k_{tm}	1/5	0.012	-0.029	-0.119	-0.432	-4.232
	1/10	0.010	-0.008	-0.053	-0.219	-1.853
	1/15	0.008	-0.002	-0.031	-0.144	-1.244
	1/25	0.006	0.002	-0.015	-0.082	-0.757
	1/50	0.004	0.003	-0.004	-0.036	-0.379
	1/100	0.003	0.003	0.000	-0.015	-0.186
	1/200	0.002	0.002	0.001	-0.006	-0.090
k_{vm}	1/5	-0.051	-0.139	-0.261	-0.609	-2.630
	1/10	-0.026	-0.070	-0.131	-0.302	-1.117
	1/15	-0.018	-0.047	-0.088	-0.201	-0.736
	1/25	-0.011	-0.029	-0.053	-0.121	-0.441
	1/50	-0.005	-0.015	-0.028	-0.062	-0.221
	1/100	-0.003	-0.008	-0.014	-0.032	-0.111
	1/200	-0.001	-0.004	-0.008	-0.017	-0.056

may be expressed as

$$\frac{1}{R_1} = \frac{\sin^2\beta}{R}, \quad \frac{1}{R_2} = \frac{\cos^2\beta}{R}, \quad \frac{1}{R_{12}} = -\frac{\sin\beta\cos\beta}{R}. \tag{6.8}$$

Some numerical results obtained for an isotropic cylinder are shown in Figures 2-11. Figures 2-6 show the primary stress intensity factor ratios k_{mm}, k_{bb}, k_{ss}, k_{tt}, and k_{vv} for a cylinder having a crack inclinced 45° with respect to the axis. The unusual results here are those found for k_{tt} and k_{vv}. Under a twisting moment M_{12} uniformly distributed along the crack, the Mode II stress intensity factor ratio k_{tt} appears to be nearly independent of the shell curvature 1/R but highly dependent on a/h. Fig. 6 shows that the monotonic variation of the stress intensity factor ratios with a/h and h/R observed in Figures 2-5 and in previous shell solutions is not valid for k_{vv}. This seems to be the case for all values of β varying from zero to ninety degrees.

The effect of β on the primary stress intensity ratios k_{mm}, k_{bb}, k_{ss}, k_{tt}, and k_{vv} is shown in Figures 7-11. Extensive results giving all stress intensity ratios k_{ij} $(i,j=m,b,s,t,v)$ for $\beta=0$, 15°, 30°, 45°, 60°, 75°, 90° and for varying h/R and a/h may be found in [14]. Table 1 shows some sample results regarding the secondary stress intensity ratios in a cylinder with a 45° crack under torsion (i.e., N_{11} = constant and all other crack surface tractions zero).

The stress intensity factors given in Figures 2-11 and in Table 1 are obtained for the Poisson's ratio $\nu = 0.3$. Some sample results showing the effect of ν on the stress intensity factors are given in Table 2. It is seen that this effect is not really significant.

It should be noted that the Poisson's ratio ν in isotropic shells and $\nu = \sqrt{\nu_1\nu_2}$ and the stiffness ratio $c = (E_1/E_2)^{1/4}$ in specially orthotropic shells appear in the expressions of the kernels of the integral equations. Thus, to investigate the effect of the material orthotropy on the stress intensity factors both ν and c must be varied. However, as seen from Table 2 the influence of ν is rather insignificant. Therefore, to study the effect of the material orthotropy it may be sufficient to vary c only.

Table 2. The effect of Poisson's ratio on the stress intensity factor ratios in an isotropic cylindrical shell containing an inclined crack, $\beta=45°$, $a/h=2$, $h/R=1/10$.

k \ ν	0.0	0.1	0.2	0.3	0.4	0.5
k_{mm}	1.166	1.167	1.167	1.167	1.166	1.164
k_{bm}	0.063	0.077	0.092	0.108	0.124	0.140
k_{sm}	-0.058	-0.059	-0.059	-0.060	-0.060	-0.059
k_{tm}	-0.009	-0.008	-0.008	-0.008	-0.008	-0.008
k_{vm}	-0.075	-0.073	-0.072	-0.070	-0.069	-0.067
k_{mb}	0.018	0.023	0.028	0.034	0.039	0.045
k_{bb}	0.605	0.617	0.626	0.632	0.634	0.631
k_{sb}	-0.016	-0.018	-0.019	-0.021	-0.024	-0.026
k_{tb}	-0.005	-0.005	-0.006	-0.006	-0.006	-0.006
k_{vb}	0.004	0.003	0.003	0.003	0.003	0.003
k_{ms}	-0.058	-0.059	-0.059	-0.060	-0.060	-0.060
k_{bs}	-0.054	-0.059	-0.064	-0.069	-0.074	-0.080
k_{ss}	1.059	1.059	1.059	1.059	1.058	1.057
k_{ts}	0.007	0.007	0.008	0.008	0.009	0.010
k_{vs}	0.133	0.131	0.129	0.128	0.126	0.124
k_{mt}	0.005	0.005	0.005	0.004	0.004	0.003
k_{bt}	-0.005	-0.005	-0.006	-0.006	-0.006	-0.006
k_{st}	-0.007	-0.007	-0.007	-0.006	-0.006	-0.005
k_{tt}	0.309	0.325	0.339	0.353	0.366	0.379
k_{vt}	-0.095	-0.094	-0.093	-0.091	-0.090	-0.088
k_{mv}	-0.223	-0.244	-0.266	-0.287	-0.308	-0.330
k_{bv}	-0.004	0.001	0.007	0.014	0.022	0.032
k_{sv}	-0.174	-0.187	-0.200	-0.213	-0.226	-0.238
k_{tv}	1.138	1.166	1.191	1.213	1.233	1.250
k_{vv}	2.304	2.287	2.272	2.258	2.244	2.231

Table 3. The effect of material orthotropy on the stress intensity factor ratios in a cylindrical shell containing an inclined crack; β=45, a/h=3, h/r=1/10.

k \ E_1/E_2	0.037	1.000	26.667
k_{mm}	1.127	1.321	1.984
k_{bm}	0.078	0.126	0.125
k_{sm}	-0.056	-0.113	-0.181
k_{tm}	-0.010	-0.053	-0.179
k_{vm}	-0.074	-0.131	-0.263
k_{mb}	0.024	0.044	0.050
k_{bb}	0.569	0.567	0.534
k_{sb}	-0.011	-0.026	-0.028
k_{tb}	-0.004	-0.005	-0.002
k_{vb}	0.004	0.007	0.012
k_{ms}	-0.057	-0.115	-0.189
k_{bs}	-0.031	-0.073	-0.079
k_{ss}	1.082	1.111	1.205
k_{ts}	0.019	0.068	0.179
k_{vs}	0.238	0.228	0.331
k_{mt}	0.005	0.004	0.003
k_{bt}	-0.004	-0.006	-0.005
k_{st}	-0.010	-0.006	-0.005
k_{tt}	0.314	0.273	0.189
k_{vt}	-0.095	-0.093	-0.087
k_{mv}	-0.166	-0.577	-1.100
k_{bv}	-0.009	0.030	0.090
k_{sv}	-0.277	-0.491	-0.872
k_{tv}	1.033	1.888	2.724
k_{vv}	2.089	2.671	3.573

For a strongly orthotropic material (graphite-epoxy composite) this effect is shown in Table 3. The axes of material orthotropy are along 45° directions with respect to the cylinder axis and the crack is located along one or the other axis of orthotropy. The Poisson's ratio is $\nu = \sqrt{\nu_1\nu_2} = 0.037$ for the orthotropic shells and $\nu = 0.3$ for the isotropic results included for the purpose of comparison. The table shows that the effect of material orthotropy on the stress intensity factors could be very significant.

The quantity which is of some interest in certain fracture studies is the rate of internally released or externally added energy per unit fracture area created as a result of crack propagation. If U is the work of the external loads, V the total strain energy, and A the fracture surface, then in a quasistatic problem the rate of energy available for fracture would be d(U-V)/dA. For elastic problems this energy rate is known to be the same for "fixed grip" and "fixed load" conditions. It can therefore be calculated as the crack closure energy under fixed grip conditions. Under these conditions, dU = 0 and for a crack going from $x_2=a$ to $x_2=a+da$, dV may be expressed as

$$dV = -\int_{-h/2}^{h/2}\int_{0}^{da} \frac{1}{2}\sum_{j=1}^{3} \sigma_{1j}(0,x_2,x_3)[u_j(+0,x_2-da,x_3) - u_j(-0,x_2-da,x_3)]dx_2dx_3 \tag{6.9}$$

where the minus sign is due to the fact that during the "release" of the crack surfaces in $a<x_2<a+da$, $-h/2<x_3<h/2$ the tractions and displacements are in opposite directions (consequently, the total strain energy of the shell decrease). For small values of da we now observe that

$$\sigma_{1j}(0,x_2,x_3) = \frac{k_j(x_3)}{\sqrt{2(x_2-a)}}\ , \ (j=1,2,3) \tag{6.10}$$

$$u_j(+0,x_2-da,x_3)-u_j(-0,x_2-da,x_3) = \frac{4k_j(x_3)}{E}\sqrt{2(a+da-x_2)},\ (j=1,2) \tag{6.11}$$

$$u_3(+0,x_2-da,x_3)-u_3(-0,x_2-da,x_3) = \frac{k_3(x_3)}{G}\sqrt{2(a+da-x_2)}\ , \tag{6.12}$$

where k_1, k_2, and k_3 are the Modes I, II, and III stress intensity factors around the crack border x_2=a.

Referring to the definitions of the stress intensity ratios k_{ij} (i,j=m,b,s,t,v), given by (6.2)-(6.6) we can define the "membrane", "bending", "shear", "twisting" and the "transverse shear" components of the stress intensity factors at the crack tip x_2=a as follows:

$$k_i = \sum_j k_{ij}\sigma_j\sqrt{a} \ , \ (i,j = m,b,s,t,v) \ . \tag{6.13}$$

From (5.30)-(5.32), (6.2)-(6.6), and (6.13) the stress intensity factors may then be expressed as

$$k_1(x_3) = k_m + k_b \left(\frac{x_3}{h/2}\right) \ , \tag{6.14}$$

$$k_2(x_3) = k_s + k_t \left(\frac{x_3}{h/2}\right) \ , \tag{6.15}$$

$$k_3(x_3) = k_v\left[1 - \left(\frac{x_3}{h/2}\right)^2\right] \ . \tag{6.16}$$

By substituting from (6.10)-(6.12) and (6.14)-(6.16) into (6.9) we obtain

$$dV = -\frac{\pi}{E}\left[k_m^{\ 2} + \frac{k_b^{\ 2}}{3} + k_s^{\ 2} + \frac{k_t^{\ 2}}{3} + \frac{4(1+\nu)}{15} k_v^{\ 2}\right]hda. \tag{6.17}$$

Observing that hda = dA, for the rate of externally added or internally released energy (at one crack tip x_2=a, per unit shell thickness, per unit crack extension in the plane of the original crack) we find

$$\frac{d}{dA}(U-V) = \frac{\pi}{E}\left[k_m^{\ 2} + \frac{k_b^{\ 2}}{3} + k_s^{\ 2} + \frac{k_t^{\ 2}}{3} + \frac{4(1+\nu)}{15} k_v^{\ 2}\right] \ . \tag{6.18}$$

Finally it is again worthwhile to remember that all shell theories are, to varying degrees, approximations of the three dimensional elasticity. Therefore, even if the "shallowness" assumption is satisfied, the theory used in this paper and the results given are only approximate. Strictly speaking, the crack problems considered in plates and shells are three-dimensional elasticity problems. Such problems in their simplest form do not seem to be as yet analytically tractable. However, from a structural viewpoint, the shell solutions can be useful in the sense that the "plane stress" crack solutions are, that is, the results should be interpreted and used in a certain thickness-average sense. Since the shell theories are quite numerous, there is always the question as to what theory to use in the crack problem. Clearly there is no unique answer for this question. However, one could try to establish some guidelines and set certain minimum requirements. In crack problems the most important information (from an application viewpoint) is imbedded in the asymptotic solution of the problem around the crack tips. The first requirement then is that the asymptotic results found from the shell solution must be compatible with that of the in-plane and anti-plane elasticity solutions of the crack problem. This means that the stresses around the crack tips must have the standard square root singularity and their angular distribution must be identical to that given by the related two-dimensional elasticity solutions.

In crack problems since one is interested in the behavior of the solution very near the crack tip, it is natural to assume that all local length parameters would have some influence on the results which are of interest. In a general shallow shell there are five local length parameters, namely three radii of curvature, R_1, R_2, R_{12}, the crack length 2a, and the thickness h. A particular shell theory to be suitable for crack problems should therefore contain four dimensionless (independent) length parameters.

Again, since it is desired that the shell theory give a reasonably accurate solution near the crack tip, it would be necessary that the theory should accommodate all the stress boundary conditions on the crack surfaces separately.

Reissner's transverse shear theory, which has been used in this paper, seems to be the simplest theory which satisfy all these requirements. Aside from a certain degree of confidence one may have in its results,an advantage of such a compatible theory, is that it makes it possible to carry out calculations such as that of energy release rate (see (6.18)) routinely. This, of course, is primarily due to the fact that the asymptotic results (5.23)-(5.25) and (5.27)-(5.28) are identical to that of the corresponding elasticity solutions. However, since a higher order shell theory does not necessarily imply higher accuracy in (certain calculated) results, there are still unresolved questions. Are the results of the crack problems obtained from the Reissner's shell theory, for example, more reliable than that given by the classical shell theory? For the stress intensity factors we think the answer is yes. The reason for this is largely the fact that the classical theory satisfies none of the requirements listed above. Could one improve the solution further by considering "higher order" theories which may take into account additional features of deformations and stresses (such as, for example, the stretch in thickness direction)? Even if one can solve such problems with the same degree of numerical accuracy as the problems based on simpler shell theories, it would be difficult to know which solution is more reliable. In our view, therefore, it would be very difficult to justify the use of a more complex theory than Reissner's in solving the crack problem in shells unless one has a demonstrable reason for it.

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Appendix A

Dimensionless and normalized quantities used in the analysis,

$$x = x_1/a\sqrt{c}\ ,\ y = x_2\sqrt{c}/a\ ,\ z = x_3/a\ ; \tag{A.1}$$

$$u = u_1\sqrt{c}/a\ ,\ v = u_2/a\sqrt{c}\ ,\ w = u_3/a\ ; \tag{A.2}$$

$$\beta_x = \beta_1\sqrt{c}\ ,\ \beta_y = \beta_2/\sqrt{c}\ ; \tag{A.3}$$

$$\phi(x,y) = F(x_1,x_2)/Eha^2; \tag{A.4}$$

$$\sigma_{xx} = \frac{\sigma_{11}}{Ec}\ ,\ \sigma_{yy} = \frac{c\sigma_{22}}{E}\ ,\ \sigma_{xy} = \frac{\sigma_{12}}{E}\ ,\ \sigma_{xz} = \frac{\sigma_{13}}{B\sqrt{c}}\ ,\ \sigma_{yz} = \frac{\sqrt{c}\,\sigma_{23}}{B}\ ; \tag{A.5}$$

$$N_{xx} = N_{11}/Ehc\ ,\ N_{yy} = cN_{22}/Eh\ ,\ N_{xy} = N_{12}/Eh\ ; \tag{A.6}$$

$$M_{xx} = M_{11}/Ech^2\ ,\ M_{yy} = cM_{22}/Eh^2\ ,\ M_{xy} = M_{12}/Eh^2\ ; \tag{A.7}$$

$$V_x = V_1/Bh\sqrt{c}\ ,\ V_y = V_2\sqrt{c}/Bh\ ; \tag{A.8}$$

$$\lambda_1^4 = 12(1-\nu^2)\,\frac{c^2a^4}{h^2R_1^2}\ ,\ \lambda_2^4 = 12(1-\nu^2)\,\frac{a^4}{c^2h^2R_2^2}\ ,$$

$$\lambda_{12}^4 = 12(1-\nu^2)\,\frac{a^4}{h^2R_{12}^2}\ ,\ \lambda^4 = 12(1-\nu^2)\,\frac{a^2}{h^2}\ ,\ \kappa = E/B\lambda^4\ ; \tag{A.9}$$

$$E = \sqrt{E_1/E_2}\ ,\ \nu = \sqrt{\nu_1\nu_2}\ ,\ (\nu_1/E_1 = \nu_2/E_2)\ ,\ c^4 = \frac{E_1}{E_2}\ ,\ B = \frac{5E}{12(1+\nu)}\ . \tag{A.10}$$

Appendix B

The stress, moment, and transverse shear resultants.

$$N_{xx}(x,y) = \begin{cases} -\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \alpha^2 \sum_{j=1}^{4} K_j R_j(\alpha) e^{m_j x} e^{-i\alpha y} d\alpha & x > 0, \\[2ex] -\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \alpha^2 \sum_{j=5}^{8} K_j R_j(\alpha) e^{m_j x} e^{-i\alpha y} d\alpha & x < 0, \end{cases} \tag{B.1}$$

$$M_{xx}(x,y) = \begin{cases} \dfrac{1}{2\pi}\dfrac{a}{h\lambda^4}\displaystyle\int_{-\infty}^{\infty} \sum_{j=1}^{4} \dfrac{(1-\nu)\alpha^2+p_j}{\kappa p_{j-1}} R_j(\alpha) e^{m_j x} e^{-i\alpha y} d\alpha \\[2ex] -\dfrac{1}{2\pi}\dfrac{a}{h\lambda^4}\dfrac{\kappa(1-\nu)^2}{2}\displaystyle\int_{-\infty}^{\infty} i A_1(\alpha)\alpha r_1 e^{r_1 x} e^{-i\alpha y} d\alpha & x>0, \\[2ex] \dfrac{1}{2\pi}\dfrac{a}{h\lambda^4}\displaystyle\int_{-\infty}^{\infty} \sum_{j=5}^{8} \dfrac{(1-\nu)\alpha^2+p_j}{\kappa p_j-1} R_j(\alpha) e^{m_j x} e^{-i\alpha y} d\alpha \\[2ex] -\dfrac{1}{2\pi}\dfrac{a}{h\lambda^4}\dfrac{\kappa(1-\nu)^2}{2}\displaystyle\int_{-\infty}^{\infty} i\, A_2(\alpha)\alpha r_2 e^{r_2 x} e^{-i\alpha y} d\alpha \\ & x<0\ . \end{cases} \tag{B.2}$$

$$N_{xy}(x,y) = \begin{cases} \dfrac{i}{2\pi}\displaystyle\int_{-\infty}^{\infty} \alpha \sum_{j=1}^{4} R_j(\alpha) K_j m_j e^{m_j x} e^{-i\alpha y} d\alpha & x>0\ , \\[2ex] \dfrac{i}{2\pi}\displaystyle\int_{-\infty}^{\infty} \alpha \sum_{j=5}^{8} R_j(\alpha) K_j m_j e^{m_j x} e^{-i\alpha y} d\alpha & x\ 0. \end{cases} \tag{B.3}$$

$$M_{xy}(x,y) = \begin{cases} -\frac{1}{2\pi}\frac{(1-\nu)a}{h\lambda^4}\int_{-\infty}^{\infty} i\alpha \sum_{j=1}^{4} \frac{m_j R_j(\alpha)}{\kappa p_j - 1} e^{m_j x} e^{-i\alpha y} d\alpha \\ -\frac{1}{2\pi}\frac{a\kappa}{h\lambda^4}\frac{(1-\nu)^2}{4}\int_{-\infty}^{\infty}(\alpha^2+r_1{}^2)A_1(\alpha)e^{r_1 x}e^{-i\alpha y}d\alpha, \\ \qquad x>0, \\ -\frac{1}{2\pi}\frac{(1-\nu)a}{h\lambda^4}\int_{-\infty}^{\infty} i\alpha \sum_{j=5}^{8} \frac{m_j R_j(\alpha)}{\kappa p_j - 1} e^{m_j x} e^{-i\alpha y} d\alpha \\ -\frac{1}{2\pi}\frac{a\kappa}{h\lambda^4}\frac{(1-\nu)^2}{4}\int_{-\infty}^{\infty}(\alpha^2+r_2{}^2)A_2(\alpha)e^{r_2 x}e^{-i\alpha y}d\alpha, \\ \qquad x<0 . \end{cases} \quad (B.4)$$

$$V_x(x,y) = \begin{cases} \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{j=1}^{4} \frac{\kappa p_j R_j(\alpha) m_j}{\kappa p_j - 1} e^{m_j x} e^{-i\alpha y} d\alpha \\ -\frac{1}{2\pi}\frac{\kappa(1-\nu)}{2}\int_{-\infty}^{\infty} iA_1(\alpha)\alpha\, e^{r_1 x}e^{-i\alpha y}d\alpha, \; x>0, \\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{j=5}^{8} \frac{\kappa p_j R_j(\alpha) m_j}{\kappa p_j - 1} e^{m_j x} e^{-i\alpha y} d\alpha \\ -\frac{1}{2\pi}\frac{\kappa(1-\nu)}{2}\int_{\infty}^{\infty} iA_2(\alpha)\alpha\, e^{r_2 x}e^{-i\alpha y}d\alpha, x<0 . \end{cases} \quad (B.5)$$

$$N_{yy}(x,y) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \sum_{j=1}^{4} R_j K_j m_j^2 e^{m_j x} e^{-i\alpha y}\, d\alpha, & x>0, \\[2ex] \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \sum_{j=5}^{8} R_j K_j m_j^2 e^{m_j x} e^{-i\alpha y}\, d\alpha, & x<0 . \end{cases} \tag{B.6}$$

$$M_{yy}(x,y) = \begin{cases} \dfrac{1}{2\pi}\dfrac{a}{h\lambda^4}\displaystyle\int_{-\infty}^{\infty} \sum_{j=1}^{4} \frac{\nu p_j-(1-\nu)\alpha^2}{\kappa p_j-1} R_j e^{m_j x} e^{-i\alpha y}\, d\alpha \\[2ex] + \dfrac{1}{2\pi}\dfrac{a}{h\lambda^4}\dfrac{\kappa(1-\nu)^2}{2}\displaystyle\int_{\infty}^{\infty} i A_1(\alpha)\alpha r_1 e^{r_1 x} e^{-i\alpha y}\, d\alpha, & x>0, \\[4ex] \dfrac{1}{2\pi}\dfrac{a}{h\lambda^4}\displaystyle\int_{-\infty}^{\infty} \sum_{j=5}^{8} \frac{\nu p_j-(1-\nu)\alpha^2}{\kappa p_j-1} R_j e^{m_j x} e^{-i\alpha y}\, d\alpha \\[2ex] + \dfrac{1}{2\pi}\dfrac{a}{h\lambda^4}\dfrac{\kappa(1-\nu)^2}{2}\displaystyle\int_{-\infty}^{\infty} i A_2(\alpha)\alpha r_2 e^{r_2 x} e^{-i\alpha y}\, d\alpha, & x<0. \end{cases} \tag{B.7}$$

$$V_y(x,y) = \begin{cases} -\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} i\alpha \sum_{j=1}^{4} \frac{\kappa p_j}{\kappa p_j-1} R_j e^{m_j x} e^{-i\alpha y}\, d\alpha \\[2ex] -\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\kappa(1-\nu)}{2} A_1(\alpha) r_1 e^{r_1 x} e^{-i\alpha y}\, d\alpha, & x>0 , \\[2ex] -\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} i\alpha \sum_{j=5}^{8} \frac{\kappa p_j}{\kappa p_j-1} R_j e^{m_j x} e^{-i\alpha y}\, d\alpha \\[2ex] -\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\kappa(1-\nu)}{2} A_2(\alpha) r_2 e^{r_2 x} e^{-i\alpha y}\, d\alpha, & x<0 . \end{cases} \tag{B.8}$$

APPENDIX C

The Kernels of Integral Equations:

$$k_{11}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = \int_0^\infty [2\alpha^2\mathrm{Re}(\sum_{j=1}^{8} K_j B_{j1})-1]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$+ 2\int_0^\infty \alpha^2\mathrm{Im}\,(\sum_{j=1}^{8} K_j B_{j1})\cos\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.1)$$

$$k_{12}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = 2\int_0^\infty \alpha^2\mathrm{Re}[\sum_{j=1}^{8} K_j B_{j2}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$+ 2\int_0^\infty \alpha^2\mathrm{Im}[\sum_{j=1}^{8} K_j B_{j2}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.2)$$

$$k_{13}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = 2\int_0^\infty \alpha^2\mathrm{Re}[\sum_{j=1}^{8} K_j B_{j3}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$+ 2\int_0^\infty \alpha^2\mathrm{Im}[\sum_{j=1}^{8} K_j B_{j3}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.3)$$

$$k_{14}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = 2\int_0^\infty \alpha^2\mathrm{Re}[\sum_{j=1}^{8} K_j B_{j4}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$+ 2\int_0^\infty \alpha^2\mathrm{Im}[\sum_{j=1}^{8} K_j B_{j4}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.4)$$

$$k_{15}(\sqrt{c}\ \tau,\sqrt{c}\ \eta) = 2\int_0^\infty \alpha^2 \mathrm{Re}[\sum_{j=1}^{8} K_j B_{j5}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$+ 2\int_0^\infty \alpha^2 \mathrm{Im}[\sum_{j=1}^{8} K_j B_{j5}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.5)$$

$$k_{21}(\sqrt{c}\ \tau,\sqrt{c}\ \eta) = -\frac{2}{\lambda^4}\int_0^\infty \mathrm{Re}[\sum_{j=1}^{8} \frac{p_j+(1-\nu)\alpha^2}{\kappa p_j-1} B_{j1}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$-\frac{2}{\lambda^4}\int_0^\infty \mathrm{Im}[\sum_{j=1}^{8} \frac{p_j+(1-\nu)\alpha^2}{\kappa p_j-1} B_{j1}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.6)$$

$$k_{22}(\sqrt{c}\ \tau,\sqrt{c}\ \eta) = \int_0^\infty \{-\frac{2}{\lambda^4}\mathrm{Re}[\sum_{j=1}^{8} \frac{p_j+(1-\nu)\alpha^2}{\kappa p_j-1} B_{j2}] + \frac{2k(1-\nu)^2\alpha r_1}{\lambda^4} - \frac{1-\nu^2}{\lambda^4}\}$$

$$\times\ \sin\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$-\frac{2}{\lambda^4}\int_0^\infty \mathrm{Im}[\sum_{j=1}^{8} \frac{p_j+(1-\nu)\alpha^2}{\kappa p_j-1} B_{j2}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.7)$$

$$k_{23}(\sqrt{c}\ \tau,\sqrt{c}\ \eta) = -\frac{2}{\lambda^4}\int_0^\infty \mathrm{Re}[\sum_{j=1}^{8} \frac{p_j+(1-\nu)\alpha^2}{\kappa p_j-1} B_{j3}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$-\frac{2}{\lambda^4}\int_0^\infty \mathrm{Im}[\sum_{j=1}^{8} \frac{p_j+(1-\nu)\alpha^2}{\kappa p_j-1} B_{j3}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.8)$$

$$k_{24}(\sqrt{c}\ \tau,\sqrt{c}\ \eta) = -\frac{2}{\lambda^4}\int_0^\infty \mathrm{Re}[\sum_{j=1}^{8} \frac{p_j+(1-\nu)\alpha^2}{\kappa p_j-1} B_{j4}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$-\frac{2}{\lambda^4}\int_0^\infty \mathrm{Im}[\sum_{j=1}^{8} \frac{p_j+(1-\nu)\alpha^2}{\kappa p_j-1} B_{j4}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.9)$$

$$k_{25}(\sqrt{c}\ \tau,\sqrt{c}\ \eta) = -\frac{2}{\lambda^4}\int_0^\infty \mathrm{Re}[\sum_{j=1}^{8} \frac{p_j+(1-\nu)\alpha^2}{\kappa p_j-1} B_{j5}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$-\frac{2}{\lambda^4}\int_0^\infty \mathrm{Im}[\sum_{j=1}^{8} \frac{p_j+(1-\nu)\alpha^2}{\kappa p_j-1} B_{j5}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha, \qquad (C.10)$$

$$k_{31}(\sqrt{c}\ \tau,\sqrt{c}\ \eta) = -2\int_0^\infty \alpha\ \mathrm{Re}[\sum_{j=1}^{8} K_j m_j B_{j1}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$+ 2\int_0^\infty \alpha\ \mathrm{Im}[\sum_{j=1}^{8} K_j m_j B_{j1}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.11)$$

$$k_{32}(\sqrt{c}\ \tau,\sqrt{c}\ \eta) = -2\int_0^\infty \alpha\ \mathrm{Re}[\sum_{1}^{8} K_j m_j B_{j2}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$+ 2\int_0^\infty \alpha\ \mathrm{Im}[\sum_{1}^{8} K_j m_j B_{j2}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha, \qquad (C.12)$$

$$k_{33}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = -2\int_0^\infty \alpha\, \mathrm{Re}[\sum_{j=1}^{8} K_j m_j B_{j3}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$+\int_0^\infty \{2\alpha\,\mathrm{Im}[\sum_{j=1}^{8} K_j m_j B_{j3}]-1\}\sin\alpha\sqrt{c}(\tau-\eta)d\alpha, \qquad (C.13)$$

$$k_{34}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = -2\int_0^\infty \alpha\, \mathrm{Re}[\sum_{j=1}^{8} K_j m_j B_{j4}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$+ 2\int_0^\infty \alpha\, \mathrm{Im}[\sum_{j=1}^{8} K_j m_j B_{j4}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha, \qquad (C.14)$$

$$k_{35}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = -2\int_0^\infty \alpha\, \mathrm{Re}[\sum_{j=1}^{8} K_j m_j B_{j5}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$+ 2\int_0^\infty \alpha\, \mathrm{Im}[\sum_{j=1}^{8} K_j m_j B_{j5}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha, \qquad (C.15)$$

$$k_{41}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = \frac{2(1-\nu)}{\lambda^4}\int_0^\infty \alpha\, \mathrm{Re}[\sum_{j=1}^{8} \frac{m_j}{\kappa p_j-1} B_{j1}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$-\frac{2(1-\nu)}{\lambda^4}\int_0^\infty \alpha\, \mathrm{Im}[\sum_{j=1}^{8} \frac{m_j}{p_j-1} B_{j1}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha,$$

$$(C.16)$$

$$k_{42}(\sqrt{c}\ \tau,\sqrt{c}\ \eta) = \frac{2(1-\nu)}{\lambda^4}\int_o^\infty \alpha\ Re[\sum_{j=1}^{8}\frac{m_j}{\kappa p_j - 1}B_{j2}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$- \frac{2(1-\nu)}{\lambda^4}\int_o^\infty \alpha\ Im[\sum_{j=1}^{8}\frac{m_j}{\kappa p_j - 1}B_{j2}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.17)$$

$$k_{43}(\sqrt{c}\ \tau,\sqrt{c}\ \eta) = \frac{2(1-)}{\lambda^4}\int_o^\infty \alpha\ Re[\sum_{j=1}^{8}\frac{m_j}{\kappa p_j - 1}B_{j3}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$- \frac{2(1-\nu)}{\lambda^4}\int_o^\infty \alpha\ Im[\sum_{j=1}^{8}\frac{m_j}{\kappa p_j - 1}B_{j3}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.18)$$

$$k_{44}(\sqrt{c}\ \tau,\sqrt{c}\ \eta) = \frac{2(1-\nu)}{\lambda^4}\int_o^\infty \alpha\ Re[\sum_{j=1}^{8}\frac{m_j}{\kappa p_j - 1}B_{j4}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$- \int_o^\infty \{\frac{2(1-\nu)}{\lambda^4}\alpha\ Im[\sum_{j=1}^{8}\frac{m_j}{\kappa p_j - 1}B_{j4}] + \frac{1-\nu^2}{\lambda^4} + \frac{1+\alpha^2\kappa(1-\nu)}{\alpha r_1\lambda^4}(\alpha^2+r_1{}^2)$$

$$x(1-\nu)\}\sin\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.19)$$

$$k_{45}(\sqrt{c}\ \tau,\sqrt{c}\ \eta) = \frac{2(1-\nu)}{\lambda^4}\int_o^\infty \alpha\ Re[\sum_{j=1}^{8}\frac{m_j}{\kappa p_j - 1}B_{j5}]\cos\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$- \frac{2(1-\nu)}{\lambda^4}\int_o^\infty \alpha\ Im[\sum_{j=1}^{8}\frac{m_j}{\kappa p_j - 1}B_{j5}]\sin\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$+ \frac{(1-\nu)}{\lambda^4}\int_o^\infty \frac{\alpha^2+r_1{}^2}{r_1}\cos\alpha\sqrt{c}(\tau-\eta)d\alpha\ , \qquad (C.20)$$

$$k_{51}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = -\int_0^\infty \mathrm{Re}\Big[\sum_{j=1}^{8} \frac{\kappa p_j m_j}{\kappa p_j - 1} B_{j1}\Big] \sin\alpha\sqrt{c}(\tau-\eta)\,d\alpha$$

$$- \int_0^\infty \mathrm{Im}\Big[\sum_{j=1}^{8} \frac{\kappa p_j m_j}{\kappa p_j - 1} B_{j1}\Big] \cos\alpha\sqrt{c}(\tau-\eta)\,d\alpha\,, \qquad (C.21)$$

$$k_{52}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = -\int_0^\infty \mathrm{Re}\Big[\sum_{j=1}^{8} \frac{\kappa p_j m_j}{\kappa p_j - 1} B_{j2}\Big] \sin\alpha\sqrt{c}(\tau-\eta)\,d\alpha$$

$$-\int_0^\infty \mathrm{Im}\Big[\sum_{j=1}^{8} \frac{\kappa p_j m_j}{\kappa p_j - 1} B_{j2}\Big] \cos\alpha\sqrt{c}(\tau-\eta)\,d\alpha \qquad (C.22)$$

$$k_{53}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = -\int_0^\infty \mathrm{Re}\Big[\sum_{j=1}^{8} \frac{\kappa p_j m_j}{\kappa p_j - 1} B_{j3}\Big] \sin\alpha\sqrt{c}(\tau-\eta)\,d\alpha$$

$$-\int_0^\infty \mathrm{Im}\Big[\sum_{j=1}^{8} \frac{\kappa p_j m_j}{\kappa p_j - 1} B_{j3}\Big] \cos\alpha\sqrt{c}(\tau-\eta)\,d\alpha\,, \qquad (C.23)$$

$$k_{54}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = -\int_0^\infty \mathrm{Re}\Big[\sum_{j=1}^{8} \frac{\kappa p_j m_j}{\kappa p_j - 1} B_{j4}\Big] \sin\alpha\sqrt{c}(\tau-\eta)\,d\alpha$$

$$-\int_0^\infty \mathrm{Im}\Big[\sum_{j=1}^{8} \frac{\kappa p_j m_j}{\kappa p_j - 1} B_{j4}\Big] \cos\alpha\sqrt{c}(\tau-\eta)\,d\alpha$$

$$-\int_0^\infty \frac{1+\alpha^2\kappa(1-\nu)}{r_1} \cos\alpha\sqrt{c}(\tau-\eta)\,d\alpha\,, \qquad (C.24)$$

$$k_{55}(\sqrt{c}\,\tau,\sqrt{c}\,\eta) = -\int_0^\infty \{Re[\sum_{j=1}^{8} \frac{\kappa p_j m_j}{\kappa p_j - 1} B_{j5}] + (\frac{\alpha}{r1} + 1)\} \sin\alpha\sqrt{c}(\tau-\eta)d\alpha$$

$$-\int_0^\infty I m[\sum_{j=1}^{8} \frac{\kappa p_j m_j}{\kappa p_j - 1} B_{j5}] \cos\alpha\sqrt{c}(\tau-\eta)d\alpha \,, \qquad (C.25)$$

In the expressions given above $B_{jk}(\alpha)$, $(j=1,\ldots,8;\ k=1,\ldots,5)$ are the coefficients given in (3.48) which are obtained by solving the linear algebraic equations (3.37)-(3.46) for the unknowns $R_1,\ldots,R_8$, A_1, and A_2.

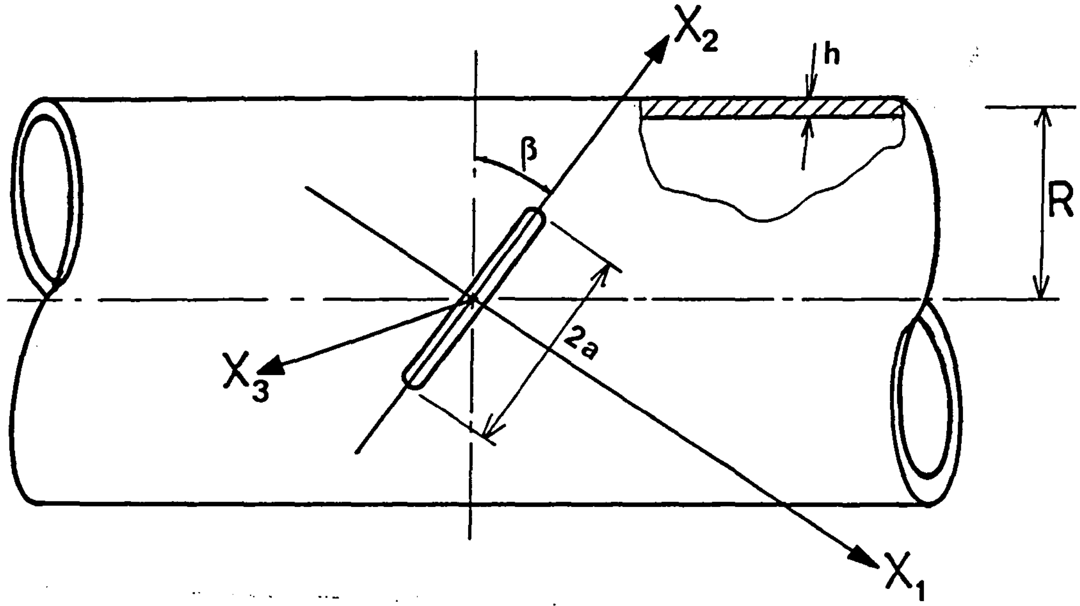

Figure 1. Geometry of a cylindrical shell containing an inclined crack.

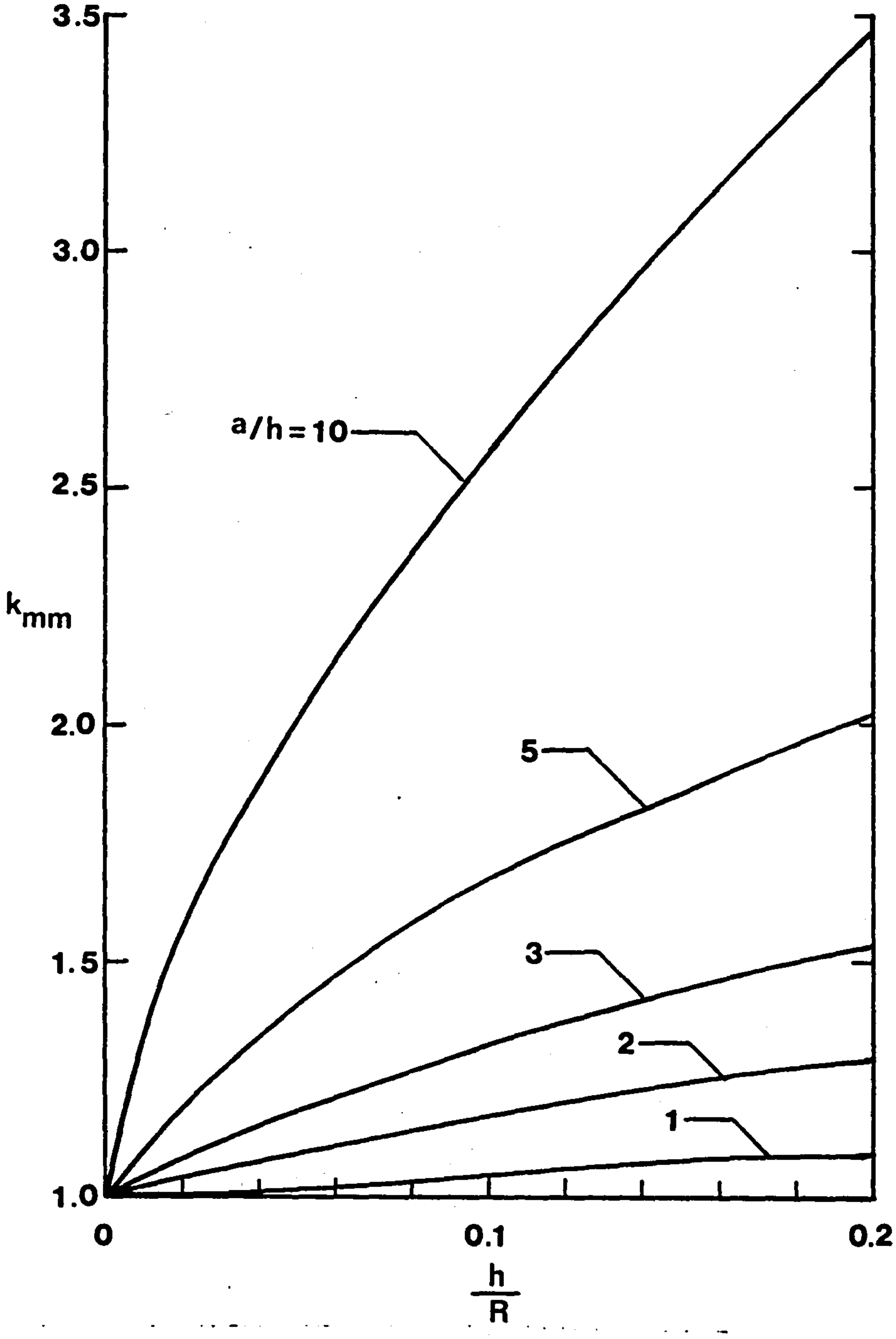

Figure 2. Stress intensity factor ratio k_{mm} in an isotropic cylindrical shell containing an inclined crack under uniform membrane loading N_{11}; β=45°, ν=0.3.

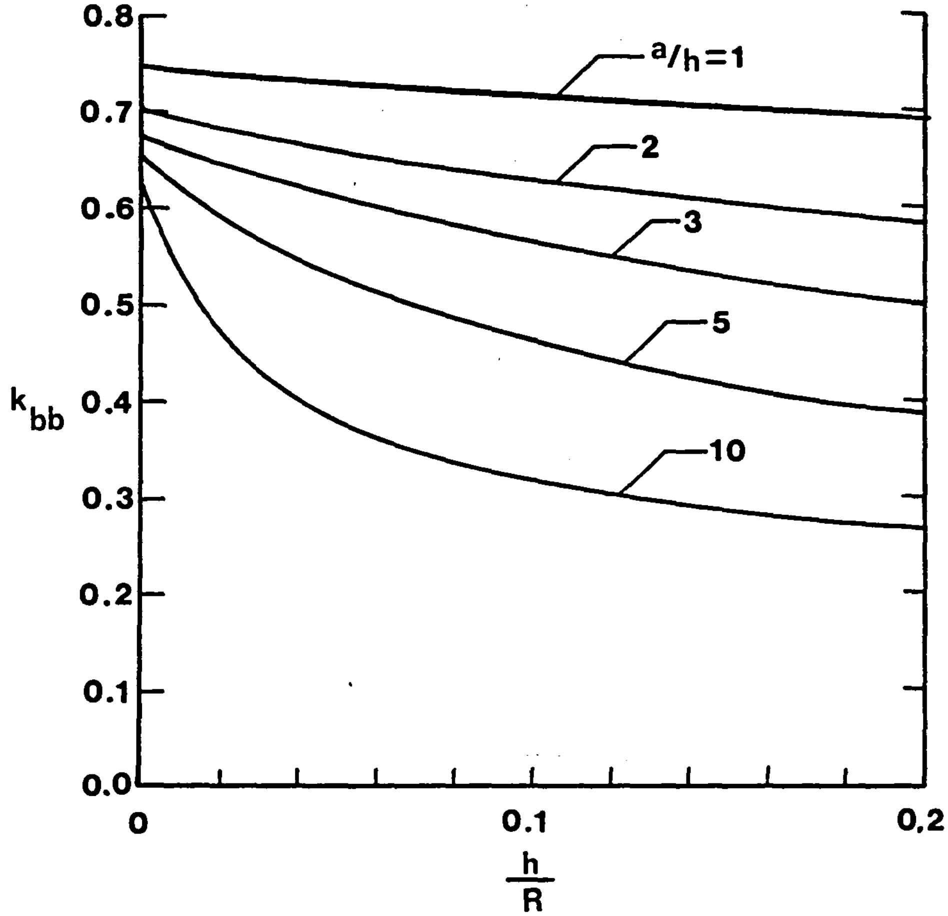

Figure 3. Stress intensity factor ratio k_{bb} in an isotropic cylindrical shell containing an inclined crack under uniform bending moment M_{11}; $\beta=45°$, $\nu=0.3$.

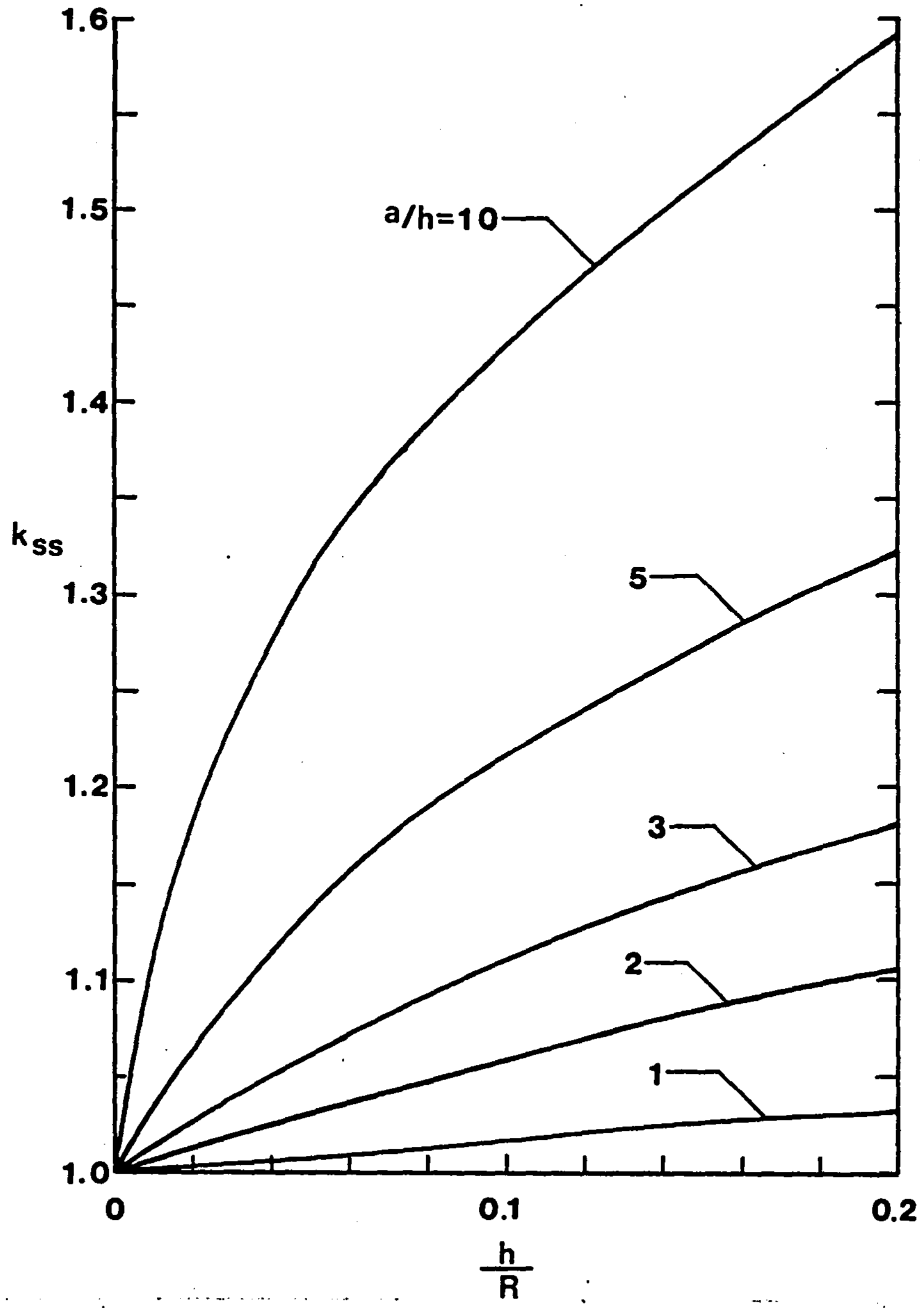

Figure 4. Stress intensity factor ratio k_{ss} in an isotropic cylindrical shell containing an inclined crack under uniform in-plane shear loading N_{12}; β=45°, ν=0.3.

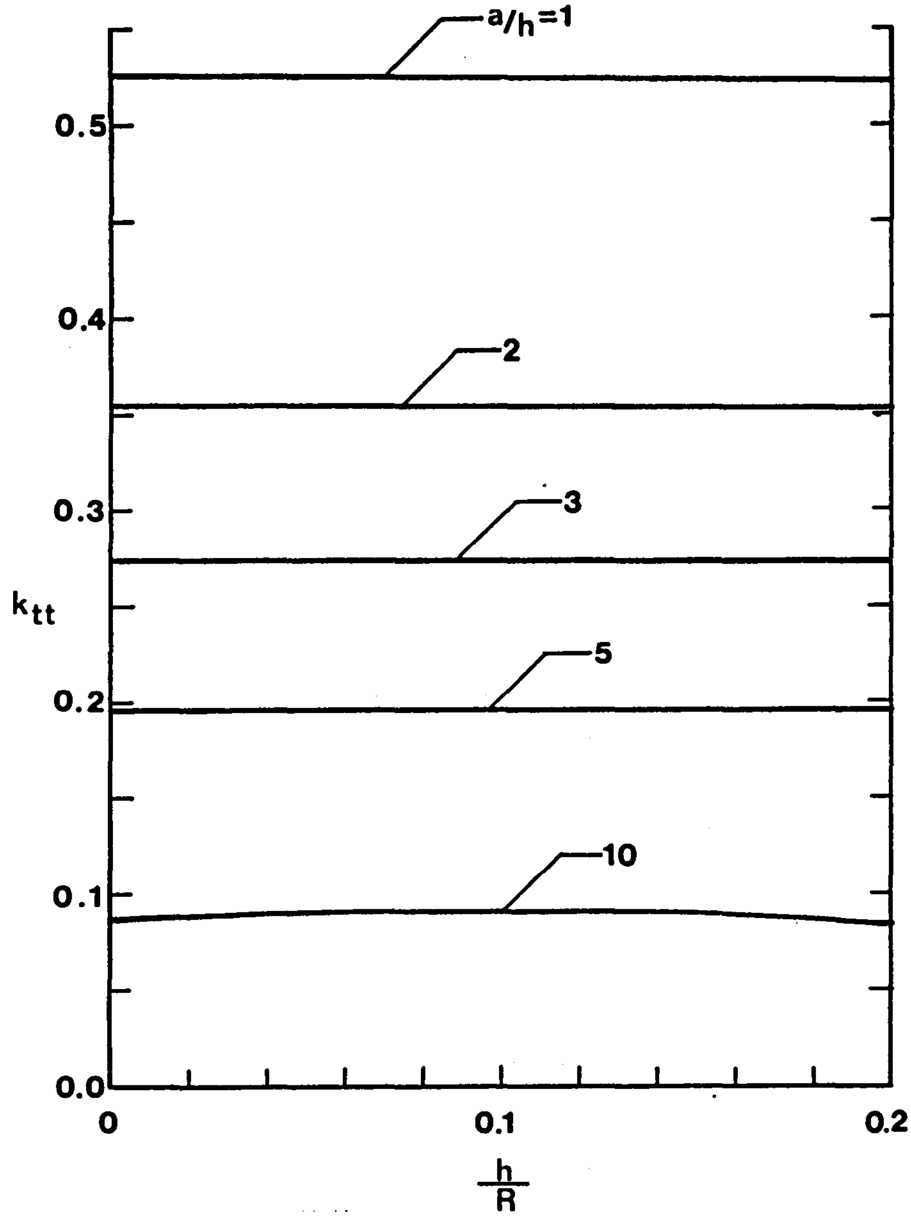

Figure 5. Stress intensity factor ratio k_{tt} in an isotropic cylindrical shell containing an inclined crack under uniform twisting moment M_{12}; β=45°, ν=0.3.

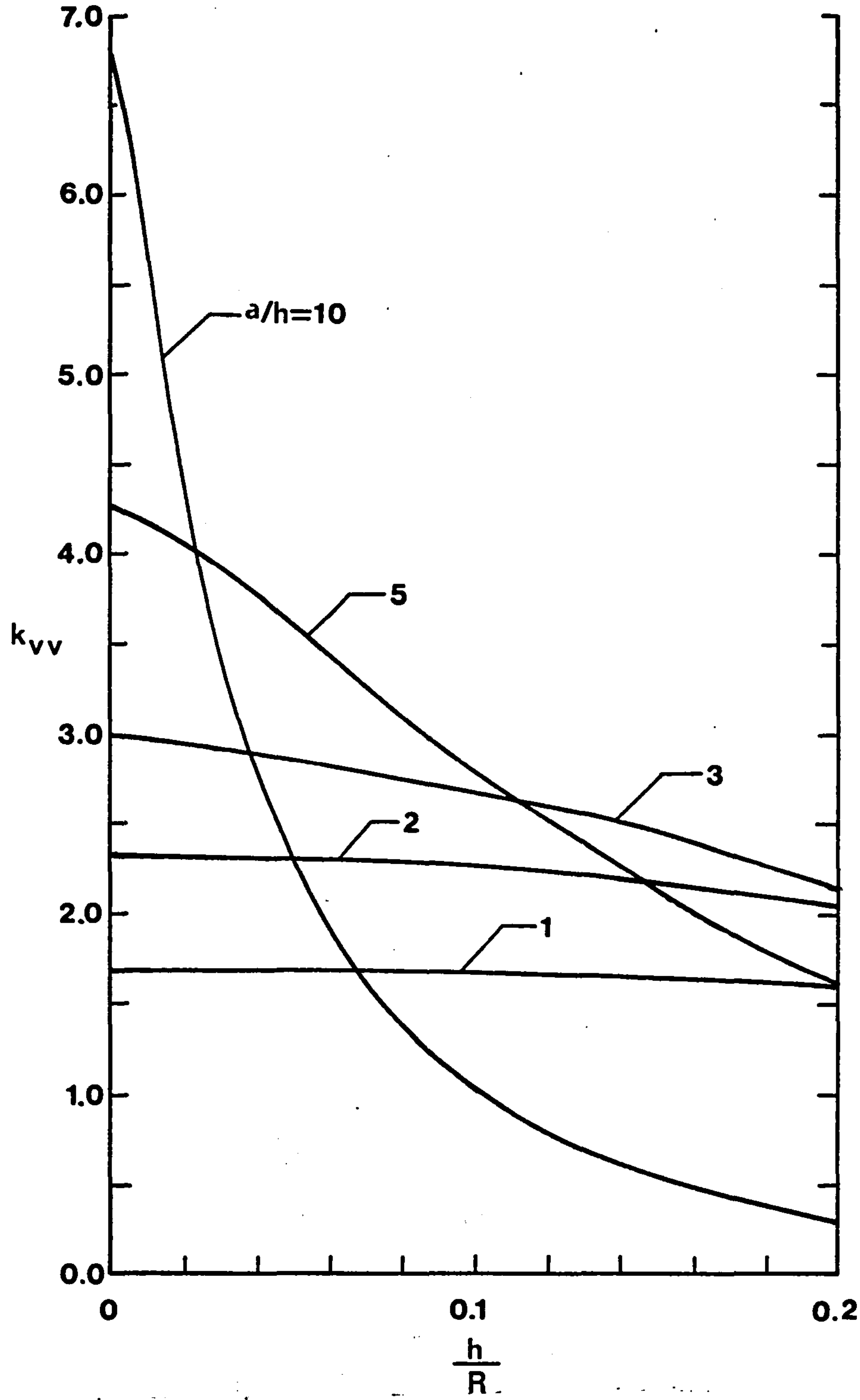

Figure 6. Stress intensity factor ratio k_{vv} in an isotropic cylindrical shell containing an inclined crack under uniform transverse shear loading V_1; β=45°, ν=0.3.

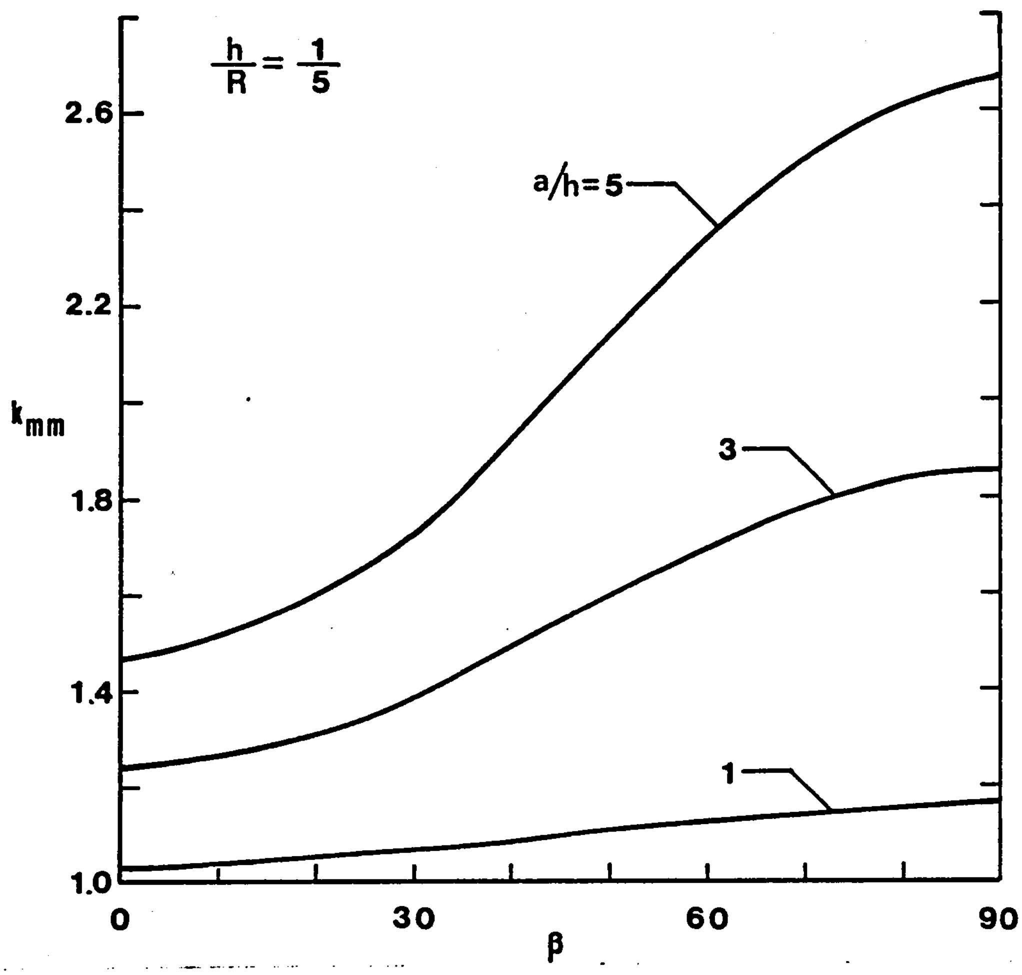

Figure 7. Stress intensity factor ratio k_{mm} in an isotropic cylindrical shell containing an inclined crack under uniform membrane loading N_{11}; ν=0.3, h/R=1/5.

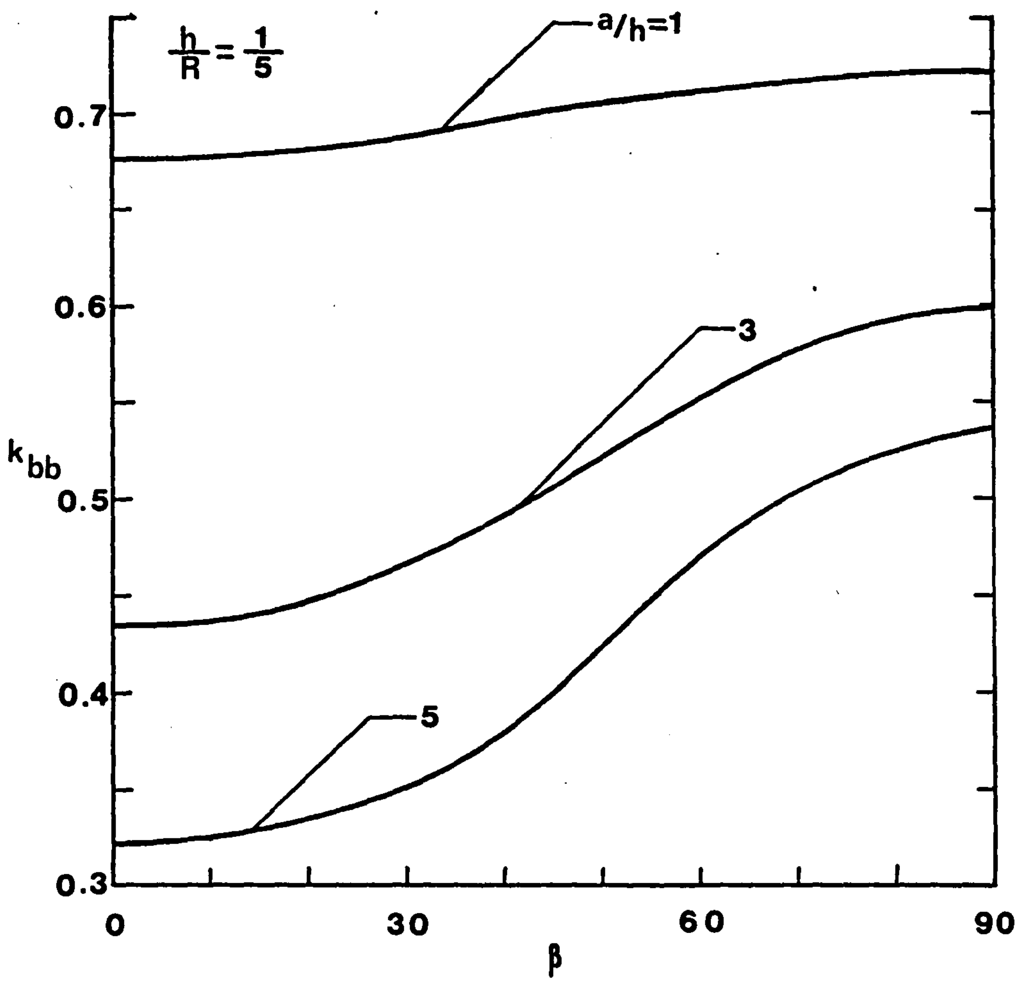

Figure 8. Stress intensity factor ratio k_{bb} in an isotropic cylindrical shell containing an inclined crack under uniform bending moment M_{11}; ν=0.3, h/R=1/5.

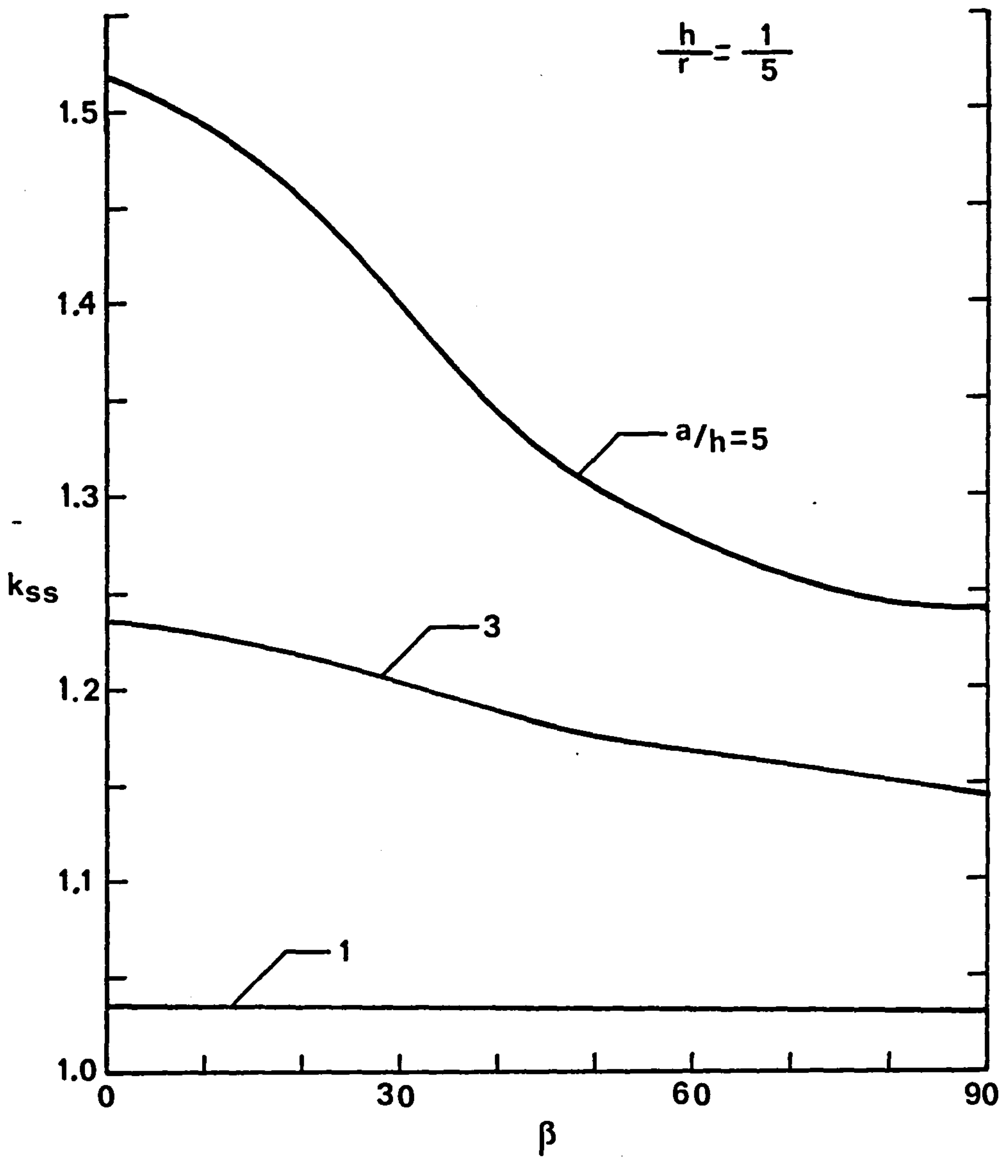

Figure 9. Stress intensity factor ratio k_{ss} in an isotropic cylindrical shell containing an inclined crack under uniform in-plane shear loading N_{12}; ν=0.3, h/R=1/5.

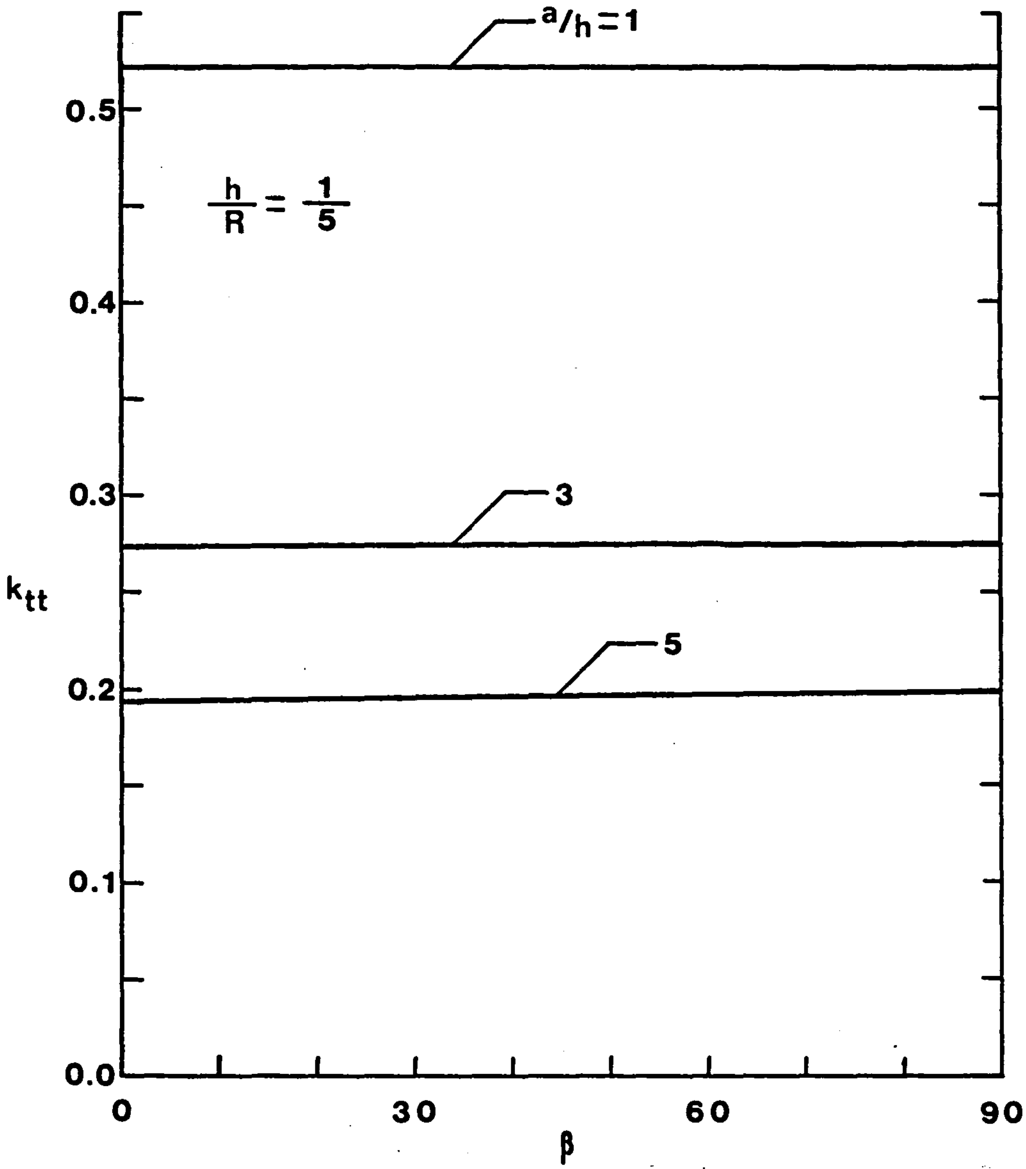

Figure 10. Stress intensity factor ratio k_{tt} in an isotropic cylindrical shell containing an inclined crack under uniform twisting moment M_{12}; ν=0.3, h/R=1/5.

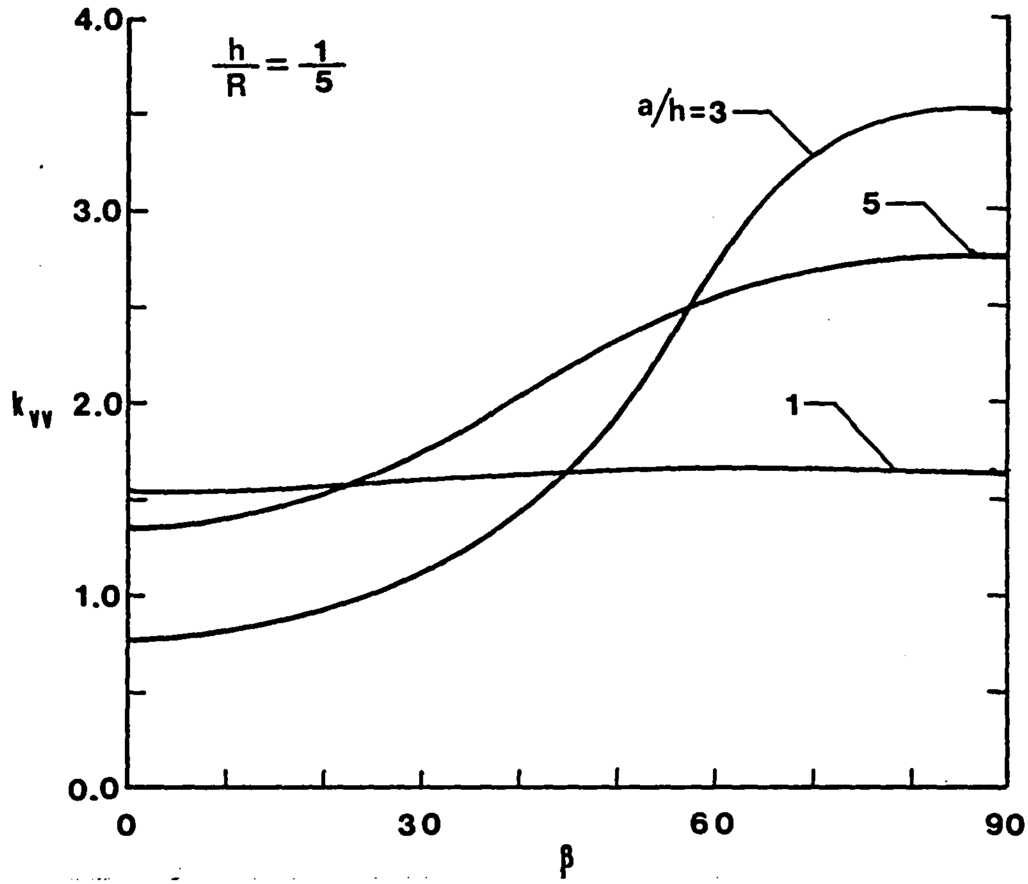

Figure 11. Stress intensity factor ratio k_{yy} in an isotropic cylindrical shell containing an inclined crack under uniform transverse shear loading V_1; $\nu=0.3$, $h/R=1/5$.

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15. Supplementary Notes

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16. Abstract

In this paper the general problem of a shallow shell with constant curvatures is considered. It is assumed that the shell contains an arbitrarily oriented through crack and the material is specially orthotropic. The nonsymmetric problem is solved for arbitrary self-equilibrating crack surface tractions, which, added to an appropriate solution for an uncracked shell, would give the result for a cracked shell under most general loading conditions. The problem is reduced to a system of five singular integral equations in a set of unknown functions representing relative displacements and rotations on the crack surfaces. The stress state around the crack tip is asymptotically analyzed and it is shown that the results are identical to those obtained from the two-dimensional in-plane and anti-plane elasticity solutions. The numerical results are given for a cylindrical shell containing an arbitrarily oriented through crack. Some sample results showing the effect of the Poisson's ratio and the material orthotropy are also presented.

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