(NASA-CR-169862) A POISSON PROCESS APPROXIMATION FOR GENERALIZED K-S CONFIDENCE REGIONS (George Washington Univ.) 43 p HC A03/MF A01 CSCL 12A

N83-18379

Unclas
G3/65 02742

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Serial T-470
19 November 1982

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Research partially supported by
National Aeronautics and Space Administration
Grant NAG 1-179

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Abstract
of
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One-sided confidence regions for continuous cumulative distribution functions are constructed using empirical cumulative distribution functions and the generalized Kolmogorov-Smirnov distance. The band width of such regions becomes narrower in the right or left tail of the distribution. To avoid tedious computation of confidence levels and critical values, an approximation based on the Poisson process is introduced. This approximation provides a conservative confidence region; moreover, the approximation error decreases monotonically to 0 as sample size increases. Critical values necessary for implementation are given. Applications are made to the areas of risk analysis, investment modeling, reliability assessment, and analysis of fault-tolerant systems.

Key words and Phrases: One-sided confidence regions for continuous cdf, Empirical cdf, Generalized K-S statistics, Poisson process, Risk analysis (assessment), Investment modeling, Reliability assessment, Fault-tolerant systems.

Research Partially Supported by
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1. INTRODUCTION AND SUMMARY

For constructing confidence regions for a continuous cumulative distribution function (cdf) $F(\cdot)$, based upon the empirical cdf $F_n(\cdot)$ of sample size n, the Kolmogorov-Smirnov (K-S) distances have been widely applied. One problem in applying the K-S distances is that the constructed region has a constant band width for a given sample size n and significance level $(1-\alpha)$.

It is well known that by the definition of the empirical cdf, for each x, $nF_n(x)$ is a binomial $(n,F(x))$ random variable. Therefore the usual binomial confidence interval for $F(x)$, x fixed, can be obtained. This confidence interval is valid at the single point x only and not for all x simultaneously.

The goal of this paper is to give a compromise between these two extremes. For the point x in either tail one can construct a one-sided confidence region, either upper or lower, with a narrower band in the neighborhood of x ; that is, one has more confidence in the one-tail probabilities at the expense of less confidence in the central and other tail probabilities. Numerou statistics may be used to construct such a confidence region, and these are discussed in Section 2.2.

In this paper the confidence region has one of the following desirable forms:

$$F(x) \leq (\delta/n) + \gamma F_n(x) \ , \ \forall\, x \ , \ \gamma > 1, \ \delta > 0 \qquad (1.1)$$

$$F(x) \leq (\delta/n) + \gamma F_n(x) \ , \ \forall\, x \ , \ 0 < \gamma < 1, \ \delta > 0 \qquad (1.2)$$

$$F(x) \geq -(\delta/n) + \gamma F_n(x), \ \forall\, x \ , \ \gamma > 1, \ \delta > 0 \qquad (1.3)$$

$$F(x) \geq -(\delta/n) + \gamma F_n(x), \ \forall\, x \ , \ 0 < \gamma < 1, \ \delta > 0 \qquad (1.4)$$

The distribution of the generalized K-S statistics may be used to obtain the significance level $(1-\alpha)$ of these desired confidence regions. Although the closed form of α in terms of (γ,δ,n) is available we have shown that, due to computational difficulties and, moreover, the need for extensive tables with three entries, a meaningful upper bound on the value of α can easily and quickly be computed based on the Poisson process. With this Poisson process approximation, a conservative confidence region of the desired shape is obtained. Moreover, it is shown that the error committed by this approximation becomes monotonically smaller as the sample size grows larger. In the following sections we provide the relevant background leading to the use of the generalized K-S distances, describe the difficulties involved in implementing such distances, and prescribe the Poisson process approximation to overcome these difficulties. Some areas of approximation are identified, and tables and graphs provided, along with examples of how they are used.

2. CONFIDENCE REGION WITH NONCONSTANT WIDTH

Let $F(\cdot)$ be a cumulative distribution function, continuous on R^1. The ordered sample from this distribution function will be denoted by $X_{1,n},\ldots,X_{n,n}$, and the related empirical cdf by $F_n(\cdot)$. Let D denote a general "distance" between the two distribution functions $F(\cdot)$, $F_n(\cdot)$ (we use "distance" in a nonmathematical sense, essentially different from the mathematical conception of "norm"). Then $D(F,F_n)$ is said to be distribution free in the family of continuous F if and only if

$$P[D(F,F_n) \leqslant d] = P[D(F_n(F^{-1}),U) \leqslant d] , \qquad d \in R^1 \tag{2.1}$$

where $U(\cdot)$ denotes the cdf of the uniform [0,1] random variable. In the following subsections we explain how some distribution-free distances are used to construct a confidence region over $F(\cdot)$ based on $F_n(\cdot)$. Most of the distances we used in our study are those which, under a simple null hypothesis on the form of $F(\cdot)$, F continuous, become the usual statistics widely discussed and used in the goodness-of-fit literature.

2.1 The Generalized K-S Distances

The generalized K-S distances are defined to be [see, for example, Dempster (1959), Dwass (1959), or Pyke (1959)]

$$D_n^-(\gamma) = \sup_x \,[\gamma F(x) - F_n(x)] \tag{2.2}$$

$$D^+(\gamma) = \sup_x \,[F_n(x) - \gamma F(x)] \tag{2.3}$$

Arsham (1982) tabulated the right tail distribution of these distances for some values of n and γ. Using $D_n^-(\gamma)$ one can construct a confidence region of the form $F(x) < \delta/n + \gamma F_n(x)$ having a narrower band over either the left ($\gamma > 1$) or the right ($0 < \gamma < 1$) tail. Similarly, a confidence region of the form $F(x) > \gamma F_n(x) - \delta/n$ can be obtained by utilizing the distribution of $D_n^+(\gamma)$. The confidence region becomes narrower over either the left ($0 < \gamma < 1$) or the right ($\gamma > 1$) tail. In the following we illustrate how such confidence regions can be constructed.

Suppose one is interested in constructing a lower confidence region that narrows over the right tail, $F(x) \geq \gamma F_n(x) - \delta/n$, $\forall x$, $\gamma > 1$, $\delta > 0$. The significance level can be obtained by noting that

$$P\left[D_n^+\left(\frac{1}{\gamma}\right) \leq \frac{\delta}{n\gamma}\right] = P\left[F(x) \geq \gamma F_n(x) - \delta/n,\ \forall x\right] = 1 - \alpha,\ \gamma > 1,\ \delta > 0$$

By the standard distribution-free argument, this probability can be written as

$$P\left[D_n^-\left(\frac{1}{\gamma}\right) \leq \delta/n\gamma\right] = P\left[U_n(x) \leq \frac{1}{\gamma}x + \delta/n\gamma,\ 0 \leq x \leq 1\right] \doteqdot P_n(\gamma,\delta)$$

where $U_n(\cdot)$ is the empirical cdf of the uniform [0,1] random variate and $\alpha = 1 - P_n(\gamma,\delta)$ can be interpreted as a crossing probability. Specifically,

$$1 - P_n(\gamma,\delta) = P\left[U_n(\cdot) \text{ crosses } Y(x) = \frac{1}{\gamma}x + \delta/n\gamma\right] \tag{2.4}$$

The closed formula of $P_n(\gamma,\delta)$ in terms of (n,δ,γ) is given by Dwass (1959) and by Durbin (1973):

$$P_n(\gamma,\delta) = 1 - \frac{n^2-n\gamma+\delta}{n^{2n}} \sum_{j=[1+(\delta/n\gamma)]}^{n} \binom{n}{j} (n\gamma j - \delta)^j (n^2 - n\gamma j + \delta)^{n-j-1} \tag{2.5}$$

for $n^2(\gamma - 1) \leq \delta \leq n^2$ and $\gamma > 1$ where the notation [z] stands for the largest integer $\leq z$. In later sections we return to the generalized K-S confidence region and establish an approximation to it based on the Poisson process. In the last section of this paper we have graphically displayed some confidence regions of the forms $F(x) \geq \gamma F_n(x) - \delta/n$ and $F(x) \leq \gamma F_n(x) + \delta/n$, $\forall\, x,\ \gamma > 1,\ \delta > 0$.

2.2 Other Nonconstant Width Confidence Regions

K-S Distance with a Particular Weight Function. This distance is defined by Anderson-Darling (1952) as

$$K_{n,w} = \sup_{x:0<F(x)<1} \left| F_n(x) - F(x) \right| \cdot W[F(x)] \tag{2.6}$$

where $W[\cdot]$ is a nonnegative weight function. When a suitable weight function is chosen, many distribution-free distances are reduced to $K_{n,w}$. For example, $W(y) = 1$ leads to the two-sided K-S distance. With the weight function $W[y] = [y(1-y)]^{-1/2}$ this distance can provide a two-sided confidence region discussed in Doksum (1977). Consider the following normalized version of $K_{n,w}$:

$$D_{n,w} = \sqrt{n}\, K_{n,w} = \sqrt{n} \sup_{x:0<F(x)<1} \frac{\left| F_n(x) - F(x) \right|}{\sqrt{F(x)(1-F(x))}} \tag{2.7}$$

The two-sided confidence region using $D_{n,w}$ can be obtained by noting that

$$P[D_{n,w} < d(\alpha,n,w)] = 1 - \alpha .$$

Following Doksum (1977), this can be written as:

$$P\{(1+a)F^2(x) - [2F_n(x) + a] \cdot F(x) + F_n^2(x) \leq 0,\ \forall\, x\} = 1 - \alpha$$

where $a = (d^2(n,\alpha,w))/n$, or equivalently

$$P\left[\frac{2F_n(x) + a + \sqrt{\Delta(F_n(x))}}{2(1+a)} \geq F(x) \geq \frac{2F_n(x) + a - \sqrt{\Delta(F_n(x))}}{2(1+a)}, \forall x\right] = 1 - \alpha \tag{2.8}$$

where $\Delta[F_n(x)] = -4aF_n^2(x) + 4aF_n(x) + a^2$.

Noé (1972) obtained a truncated power series which approximates α for a given n and d . The inversion of d in terms of α for the statistic in (2.7) is

$$d^{-2} = 2^{-1}\alpha - 2^{-3}(3 - 5n^{-1})\alpha^2 - 2^{-5}(14 - 132n^{-1} + 114n^{-2})\alpha^{-3}$$
$$- 2^{-7}(151 - 4035n^{-1} + 12981n^{-2} - 9105n^{-3})\alpha^{-4} . \tag{2.9}$$

Neither the general term nor a general truncation error bound is known. In practice, to construct the confidence region by using $D_{n,w}$ one chooses a level of significance $1 - \alpha$, then by means of the truncated series (2.9) determines the corresponding value of $a = d^2/n$. Thus one obtains the two jagged shaped bounds c_1 and c_2 whose equations are given in the probabilistic equation (2.8). Figure 1 shows a realization of a sample of size 20 from the uniform distribution [0,1] with its bounds for 95% confidence. In Figure 2 the sample size is set to be n = 1000. A comparison of these two figures shows that for a larger sample size, both bounds "come in" at both tails. The limitation of using this distance as a solution to our problem is that one can obtain only an approximated confidence region. Moreover, Canner (1975) has noted that this distance is very sensitive to first and last order statistics; this implies that the confidence interval is very narrow in the tails at the expense of the center of the distribution.

A Modified K-S Confidence Region Based on Censoring. When a K-S confidence region is constructed using the truncated or censored data,

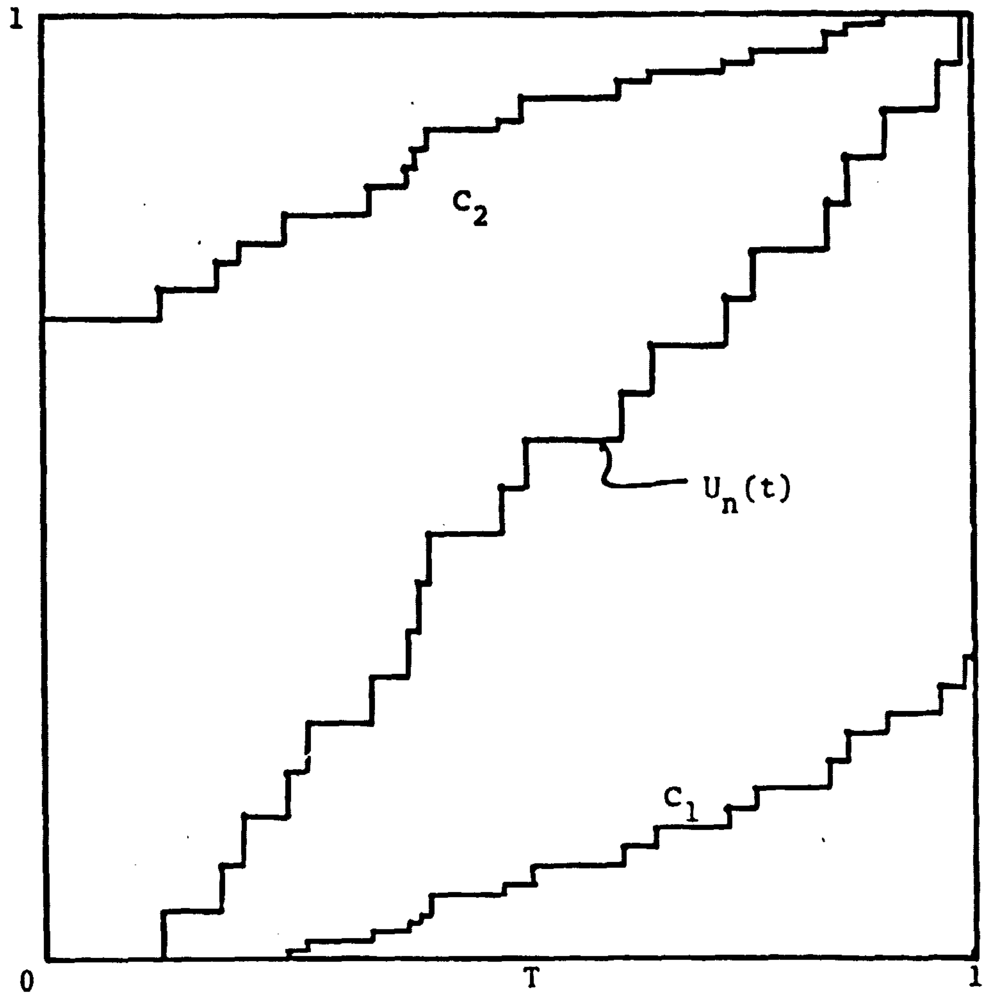

Figure 1.--A realization of sample size $n = 20$ from $U[0,1]$ together with the 95% confidence region based on $D_{n,w}$.

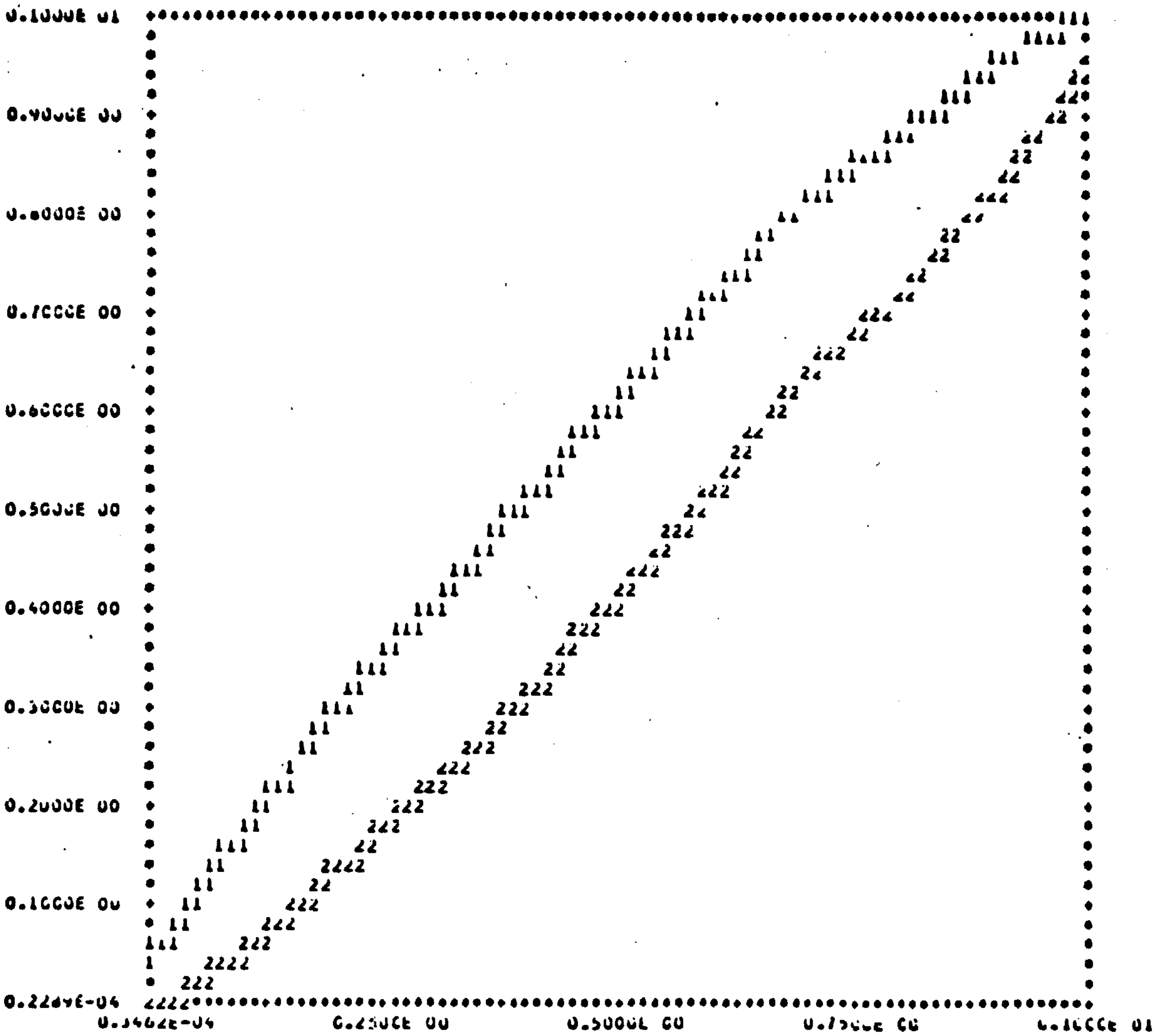

Figure 2.--A 95% confidence region based on a sample of size n = 1000 from U[0,1] based on $D_{n,w}$ distance.

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the actual region would be a fixed width band over the right tail, since in this case it is not required to remain within the band beyond the fth failure in an "f out of n censored" plan. Consider the following distance given in Barr and Davidson (1973):

$$T_{f,n} = \sup_{x:0 \leq F(x) \leq (f/n)} |F(x) - F_n(x)| \qquad (2.11)$$

The two-sided confidence region using $T_{f,n}$ can be obtained by noting that

$$P[T_{f,n} \leq d(f,n,\alpha)] = 1 - \alpha$$

where the critical values $d(f,n,\alpha)$ are tabulated for some value of n [Barr and Davidson (1973)]. Later Koziol and Byar (1975) provided the asymptotic critical values as n approaches infinity. A good approximation formula for significance points is given by Dufour and Maag (1978) when sample size exceeds 25. Figure 3 shows a typical confidence region using a sample from a uniform distribution based upon the distance defined by (2.11).

Manija Confidence Region. Manija (1949) introduced the following distance:

$$d_n^-(a,b) = \sup_{x \in S} [F_n(x) - F(x)] , \qquad a < b$$

where

$$S = \{x \mid F(x) \leq a\} \cup \{x \mid F(x) \geq b\} .$$

By the general distribution-free argument, the distribution of $d_n^-(a,b)$ is independent of $F(\cdot)$, F continuous over the set S . A lower confidence region using this distance can be obtained by noting that

$$P[U(x) \geq U_n(x) - z(a,b,n) \text{ for all } x \in S] = 1 - \alpha$$

based upon a uniform empirical cdf path. A typical lower confidence

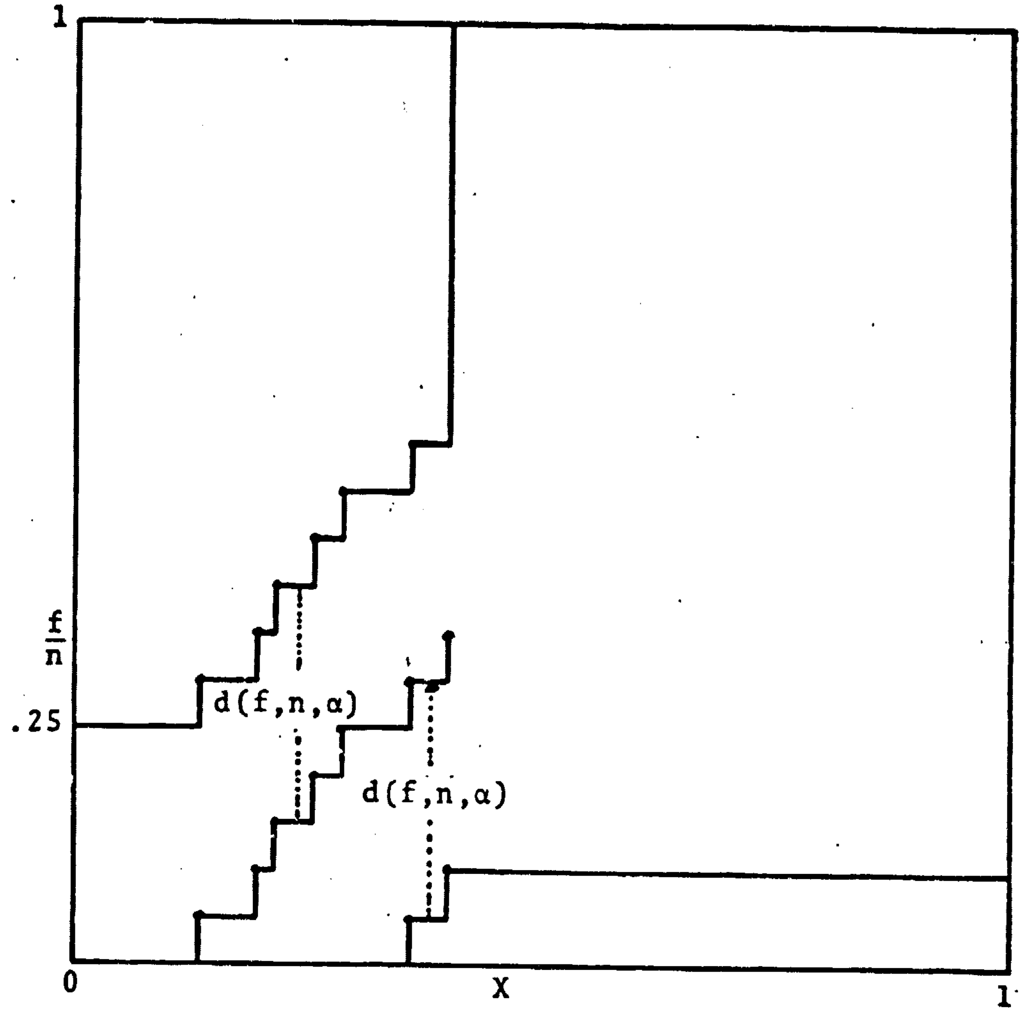

Figure 3.--A modified K-S confidence region based upon censoring data of U[0,1] .

region based on this distance using a sample from $U[0,1]$ is shown in Figure 4. The limitation of applying this distance in construction of a confidence region is that it provides an approximation region only. The asymptotic distribution of $d_n^-(a,b)$ is available, and is given in Sahler (1968).

Tang Confidence Region. Tang (1962) developed a distance based upon the ratio between the empirical and hypothetical cdf's. This distance is a special case of the Renyi (1953) distance. The Tang distance is defined for $0 < b \leq n$ as

$$r_n(b) = \sup_{x:0<F(x)\leq(b/n)} \frac{F_n(x)}{F(x)}$$

By the usual distribution-free argument, $r_n(b)$ has distribution independent of $F(\cdot)$, if $F(\cdot)$ is continuous over the set

$$S = \{x \mid 0 < F(x) \leq b/n \leq 1\}$$

A one-sided confidence region using $r_n(b)$ can be obtained by noting that

$$P\left[F(x) \geq \frac{F_n(x)}{d(\alpha,n,b)} \text{ for all } x \in S \right] = 1 - \alpha$$

Figure 5 shows a typical confidence region using a uniform empirical cdf based on the $r_n(b)$ distance. The distribution of $r_n(b)$ in closed form is available but it is not easy to implement.

3. THEORY OF POISSON APPROXIMATION TO GENERALIZED K-S PROBABILITIES

For generalized K-S confidence regions it is necessary to calculate the crossing probabilities $1 - P_n(\gamma,\delta)$ from equation (2.5). For a given confidence level $(1 - \alpha)$ it is necessary to find solutions of

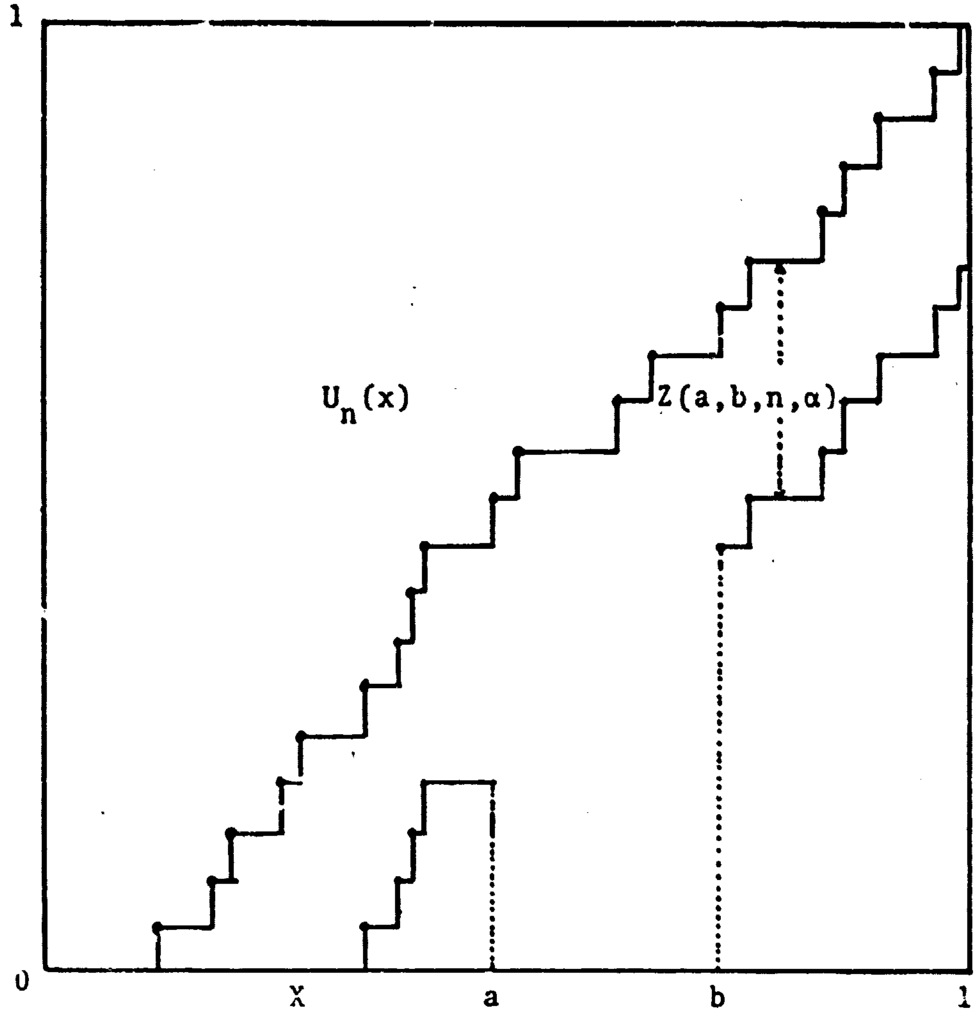

Figure 4.--A typical lower confidence region based on Manija distance using sample size n from U[0,1] .

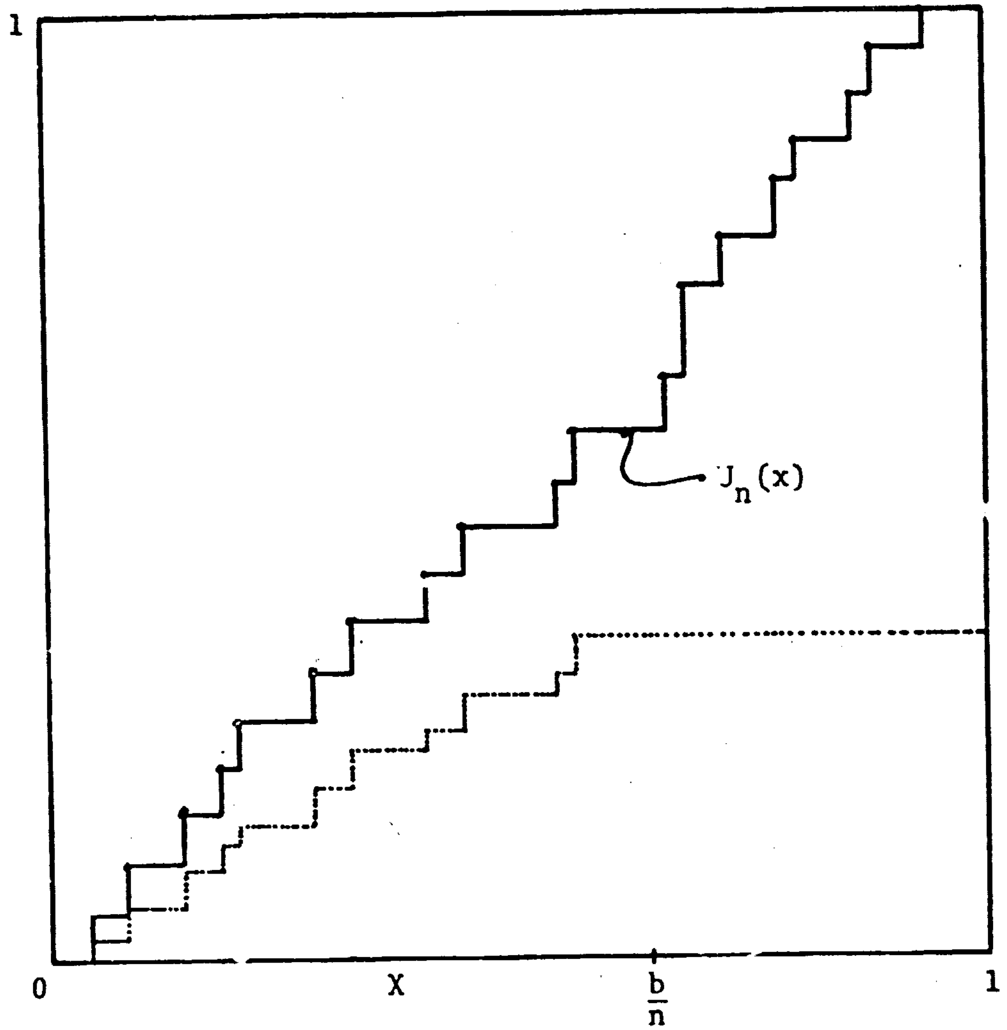

Figure 5.--A typical confidence region using a Uniform [0,1] empirical cdf, based on Tang distance.

the equation $\alpha = P_n(\gamma,\delta)$. To avoid these computational difficulties and, moreover, to avoid generating very extensive tables with the three parameters n , γ , and δ , we have developed a conservative bound on $P_n(\gamma,\delta)$ which is quite accurate and easy to compute based on the Poisson process. The theory for this approximation is presented in the four theorems of this section.

Let $\{X(t),\ 0 \leq t\}$ be a homogeneous Poisson process with unit rate. Let $\{U_n(t),\ 0 \leq t \leq 1\}$ be the empirical cdf of a sample of n $U[0,1]$ random variables.

Theorem 1 For $0 < \gamma < 1$ and $\delta > 0$, $P\left(nU_n(t/n) \leq t/\gamma + \delta/\gamma,\ 0 \leq t \leq n\right)$ decreases monotonically as n increases. Furthermore,

$$\lim_{n\to\infty} P\left(nU_n(t/n) \leq t/\gamma + \delta/\gamma,\ 0 \leq t \leq n\right)$$

$$= P\left(X(t) \leq t/\gamma + \delta/\gamma,\ 0 \leq t\right) \tag{3.1}$$

$$= (1-\gamma) \sum_{j=[\delta/\gamma]+1}^{\infty} \frac{(\delta j-\delta)^j}{j!} \exp(\delta - \gamma j)$$

Proof Dwass (1974) shows that, for $c > 1$ and $d \geq 0$,

$$P\left(nU_n(t/n) \leq ct + d,\ 0 \leq t \leq n\right) = P\left(\sum_{i=1}^{N_n} U_i \leq d\right) \tag{3.2}$$

where U_i , $i = 1,2,\ldots,n$, are i.i.d. $U[0,1]$ random variables and N_n is an independent random variable with

$$P(N_n \geq k) = \left(\frac{1}{nc}\right)^k \frac{n!}{(n-k)!}\ , \qquad k = 0,1,2,\ldots \tag{3.3}$$

and

$$P\left(X(t) \leq ct + d,\ 0 \leq t\right) = P\left(\sum_{i=1}^{M} U_i \leq d\right) \tag{3.4}$$

where M is an independent random variable with

$$P(M \geq k) = \left(\frac{1}{c}\right)^k , \qquad k = 0,1,2,\ldots . \tag{3.5}$$

From (3.3) it follows that

$$N_n \overset{st}{\leq} N_{n+1} . \tag{3.6}$$

It follows from (3.3) and (3.5) that

$$\lim_{n\to\infty} P(N_n \geq k) = P(M \geq k) , \qquad k = 0,1,\ldots . \tag{3.7}$$

Define $f_d(m) = P\left(\sum_{i=1}^{m} U_i > d\right)$; f_d is a monotonically nondecreasing function such that

$$P\left(\sum_{i=1}^{M} U_i > d\right) = \sum_{m=0}^{\infty} P\left(\sum_{i=1}^{M} U_i > d \mid M = m\right) P(M = m)$$

$$= \sum_{m=0}^{\infty} f_d(m)\ P(M = m) \tag{3.8}$$

$$= E\left(f_d(M)\right) .$$

It follows from (3.6) and the increasing nature of f_d that $f_d(N_n) \overset{st}{\leq} f_d(N_{n+1})$, which implies that $E(f_d(N_n) \leq E(f_d(N_{n+1}))$ which, together with (3.2) and (3.8), proves the monotonicity in the statement of the theorem. It follows from the discreteness of the random variables and (3.7) that $\lim_{n\to\infty} f_d(N_n) = f_d(M)$ in distribution, from which $\lim_{n\to\infty} E(f_d(N_n)) = E(f_d(M))$ follows by dominated convergence; this, together with (3.2), (3.4), and (3.8), proves the limiting result in (3.1). Finally, the second equality in (3.1) is given by Pyke (1959).//

Theorem 1 provides the necessary theory regarding crossing of upper lines, $y(t) = (1/\gamma)t + (\delta/n\gamma)$, and Theorems 2, 3, and 4 deal with the lower lines, $y(t) = (1/\gamma)t - (\delta/n\gamma)$.

<u>Theorem 2</u> For $\gamma > 1$ and $\delta > 0$, $P\big(nU_n(t/n) > t/\gamma - \delta/\gamma,\ 0 \leq t \leq n\big)$ is monotone nonincreasing in n.

<u>Proof</u> Define $c = 1/\gamma$, $d = \delta/\gamma$, and $V_n(t) = nU_n(t/n)$, $0 \leq t \leq n$. Let $P_n(c,d) = P\big(V_n(t) > ct - d,\ 0 \leq t \leq n\big)$; then, for $0 < c < 1$ and $d > 0$, we must verify

$$P_m(c,d) \geq P_n(c,d)\ , \qquad m < n\ . \tag{3.9}$$

The proof is based on induction. We first verify (3.9) for $m = 1$: Letting $S_{n,j} = \min\big(t\colon V_n(t) = j\big)$, $P_1(c,d) = P(S_{1,1} < d/c) = d/c$ if $d/c < 1$, and 1 otherwise. For $n > 1$, if $d/c < 1$, $P_n(c,d) \leq P(S_{n,1} < d/c) = 1 - P(S_{n,1} \geq d/c) = 1 - (1 - d/c)^n < d/c$. This verifies (3.9) for $m = 1$.

We now make the inductive hypothesis that (3.9) holds for $m < k$. To complete the proof it suffices to show that this implies that (3.9) is true for $m = k$. This inductive step of the proof uses, for fixed k and n, dependent versions of $\{V_k(t),\ 0 \leq t \leq k\}$ and $\{V_n(t),\ 0 \leq t \leq n\}$ defined on the same probability space.

The process $\{V_m(t),\ 0 \leq t \leq m\}$ is a pure birth process with initial distribution $P\big(V_m(0) = 0\big) = 1$ and transition rate function $\lambda_m(i,j;t) = (m-i)/(m-t)$ if $j = i + 1$, and 0 otherwise, $i \neq j$, $0 \leq t \leq m$. Let $D_k = \min\big(t > 0\colon V_k(t) = t\big)$. We shall define a modified version of V_k which jumps to ∞ when it crosses the diagonal:

$$V_k'(t) = \begin{cases} V_k(t)\ , & 0 \leq t < D_k \\ \infty\ , & D_k \leq t \leq k \end{cases}$$

This process has transition function $\lambda_k'(i,j;t) = (k-i)/(k-t)$ if $j = i + 1$, $i + 1 < t \leq k$ or if $j = \infty$, $t < i + 1$, and $= 0$ otherwise.

Note that $\lambda_k'(t) \geq \lambda_n'(t)$, $0 \leq t \leq k$, if $k < n$. This implies that $\{V_k'(t) , 0 \leq t \leq k\} \overset{st}{\geq} \{V_n'(t) , 0 \leq t \leq k\}$ and furthermore that V_k' and V_n' can be defined on the same probability space so that $P\big(V_k'(t) \geq V_n'(t), 0 \leq t \leq k\big) = 1$ [see Kamae, Krengel, and O'Brien (1977) and references therein]. From this we get the joint distribution of D_k and $V_n(D_k)$. Using these distributions we define $\{V_k''(t), D_k \leq t \leq k\}$ and $\{V_n''(t), D_k \leq t \leq n\}$: Given $\{D_k = s\}$, V_k'' is a birth process with initial state $V_k''(s) = j$, where $j = \min(i: i \geq s)$ and transition rate function $\lambda_k''(i,j;t) = \lambda_k(i,j;t)$, $s \leq t \leq k$. Given $\{D_k = s, V_n(D_k) = i\}$, V_n'' is a birth process with initial state $V_n''(s) = i$ and transition rate function $\lambda_n''(i,j;t) = \lambda_n(i,j;t)$, $s \leq t \leq n$. Furthermore, $\{\big(V_k'(t), V_n'(t)\big), 0 \leq t < D_k\}$, $\{V_k''(t), D_k \leq t \leq k\}$, and $\{V_n''(t), D_k \leq t \leq n\}$ are conditionally independent given D_k , $V_n'(D_k)$, and $V_n''(D_k)$. Let $V_k(t) = V_k'(t)$, $0 \leq t < D_k$, and $V_k''(t)$, $D_k \leq t \leq k$. Let $V_n(t) = V_n'(t)$, $0 \leq t < D_k$, and $V_n''(t)$, $D_k \leq t \leq n$. These dependent processes, V_k and V_n , will be used in the induction step of the proof.

(For the sake of completeness, we give an explicit construction which can be shown to yield the above (V_k, V_n) : Let $X_{k,i}$, $0 \leq i \leq k-1$, and $X_{n,i}$, $0 \leq i \leq n-1$ be $k + n$ i.i.d. uniform [0,1] random variables defined on the same probability space. Let $G_{m,i}(y|s) = 1 - \exp\left(- \int_s^{s+y} \lambda_m(i,i+1;t)dt\right)$ be the cdf of the holding time of V_m in state i given the passage to i occurs at s, $m = k$, $0 \leq i \leq k-1$, and $m = n$, $0 \leq i \leq n-1$. We first construct $\{V_k(t), 0 \leq t \leq k\}$. Let $S_{k,0} = 0$, $Y_{k,0} = G_{k,0}^{-1}(X_{k,0} \mid S_{k,0})$, $S_{k,1} = Y_{k,0}$, ..., $Y_{k,i} = G_{k,i}^{-1}(X_{k,i} \mid S_{k,i})$, $S_{k,i+1} = S_{k,i} + Y_{k,i}$, ... , $S_{k,k} = S_{k,k-1} + Y_{k,k-1}$.

Let $V_k(t) = \max(i: S_{k,i} \leq t)$, $0 \leq t \leq k$. This defines V_k and also $D_k = \min(t > 0: V_k(t) = t)$. Now let us consider $\{V_n(t), 0 \leq t \leq n\}$. On the interval $[0,D_k]$, V_n and V_k must be ordered with probability one. Let $Y_{n,i} = G^{-1}_{n,i}(X_{k,i} \mid S_{n,i})$, if $S_{n,i+1} = S_{n,i} + Y_{n,i} \leq D_k$, $0 \leq i < k$. Let $j = \min(i: S_{n,i} + G^{-1}_{n,i}(X_{k,i} \mid S_{n,i}) > D_k)$; define $Y_{n,j} = G^{-1}_{n,j}(X_{n,j} \mid D_k)$ and $S_{n,j+1} = D_k + Y_{n,j}$. Then for $j < \ell \leq n-1$ let $Y_{n,\ell} = G^{-1}_{n,\ell}(X_{n,\ell} \mid S_{n,\ell})$ and $S_{n,\ell+1} = S_{n,\ell} + Y_{n,\ell}$. As before, let $V_n(t) = \max(i: S_{n,i} \leq t)$. It can be shown that the processes V_k and V_n have the properties claimed in the preceding paragraph by appealing to standard construction techniques such as are found in Heyman and Sobel (1982, Ch. 4) and comparison techniques such as those found in Kamae, *et al.* (1977) and Stoyan (1977).)

Now consider the dependent processes $\{V_k(t), 0 \leq t \leq k\}$ and $\{V_n(t), 0 \leq t \leq n\}$ constructed above. Letting $G(s,i) = P\{D_k \leq s, V_n(D_k) \leq i\}$,

$$P_n(c,d) = \int P\big(V_n(t) > ct - d, 0 \leq t \leq n \mid D_k = s, V_n(D_k) = i\big)dG(s,i)$$

$$= \int P\big(V_n(t) > ct - d, 0 \leq t \leq s \mid D_k = s, V_n(D_k) = i\big) \cdot P\big(V_n(t) > ct - d, s \leq t \leq n \mid D_k = s, V_n(D_k) = i\big)dG(s,i) \tag{3.10}$$

and

$$P_k(c,d) = \int P\big(V_k(t) > ct - d, 0 \leq t \leq s \mid D_k = s, V_n(D_k) = i\big) \cdot P\big(V_k(t) > ct - d, s \leq t \leq k \mid D_k = s, V_n(D_k) = i\big)\, dG(s,i) . \tag{3.11}$$

The inequality $P\big(V_k(t) > ct - d, 0 \leq t \leq s \mid D_k = s, V_n(D_k) = i\big) \geq P\big(V_n(t) > ct - d, 0 \leq t \leq s \mid D_k = s, V_n(D_k) = i\big)$ follows from the ordering between V'_k and V'_n . Thus to prove (3.9) for $m = k$ from (3.10) and (3.11), it suffices to verify

$$\begin{aligned} &P\big(V_k(t) > ct - d,\ s \leq t \leq k \mid D_k = s,\ V_n(D_k) = i\big) \\ &\geq P\big(V_n(t) > ct - d,\ s \leq t \leq n \mid D_k = s,\ V_n(D_k) = i\big) . \end{aligned} \tag{3.12}$$

However, the left-hand side of (3.12) is independent of i and the right-hand side achieves its largest value for $i = j-1$, where $j = \min(\ell\colon \ell \geq s)$, thus it suffices to verify

$$\begin{aligned} &P\big(V_k(t) > ct - d,\ s \leq t \leq k \mid D_k = s,\ V_k(s) = j\big) \\ &\geq P\big(V_n(t) > ct - d,\ s \leq t \leq n \mid D_k = s,\ V_n(s) = j - 1\big) \end{aligned} \tag{3.13}$$

The probability expressions in (3.13) are equivalent to those involving processes with a fewer number of transitions: The right-hand side may be evaluated by labeling $(s,j-1)$ as the origin and recognizing that in the remaining interval of length $n - s$, the process is equivalent to counting $n - j + 1$ order statistics. With the appropriate scaling this gives

$$\begin{aligned} &P\big(V_n(t) > ct - d,\ s \leq t \leq n \mid D_k = s,\ V_n(s) = j-1\big) \\ &= P\left(V_{n-j+1}(t) > y_1(t) = \frac{c(n-s)}{n-j+1}\, t + cs-d-j+1,\ 0 \leq t \leq n-j+1\right) \end{aligned} \tag{3.14}$$

Similarly, for the left-hand side of (3.13),

$$\begin{aligned} &P\big(V_k(t) > ct - d,\ s \leq t \leq k \mid D_k = s,\ V_k(s) = j\big) \\ &= P\left(V_{k-j}(t) > y_2(t) = \frac{c(k-s)}{k-j}\, t + cs-d-j,\ 0 \leq t \leq k-j\right) \\ &\geq P\big(V_{k-j}(t) > y_1(t),\ 0 \leq t \leq k-j\big) \end{aligned} \tag{3.15}$$

the last inequality following from $y_1(t) \geq y_2(t)$, $0 \leq t \leq k-j$. If $y_1(0) \geq 0$, then (3.14) equals zero and (3.13) follows trivially. If $y_1(0) > 0$, we use the facts that the slope of $y_1(\cdot)$ is less than 1,

$m = k-j < k$ and $n-j+1 > k-j$, to invoke the inductive hypothesis, proving (3.13). This completes the proof of Theorem 2. //

Theorem 3 For $\gamma > 1$ and $\delta > 0$,

$$P\left(nU_n(t/n) > t/\gamma - \delta/\gamma,\ 0 \leq t \leq n\right) \geq P\left(X(t) > t/\gamma - \delta/\gamma,\ 0 \leq t\right) \tag{3.16}$$

Proof The proof of Theorem 3 parallels that of Theorem 2: Define $c = 1/\gamma$, $d = \delta/\gamma$, and $U_n(t) = nU_n(t/n)$, $0 \leq t \leq n$. Let $P_n(c,d) = P\left(V_n(t) > ct - d,\ 0 \leq t \leq n\right)$, and let $P_\infty(c,d) = P\left(X(t) > ct - d,\ 0 \leq t\right)$. Note that $P_1(c,d) \geq P_\infty(c,d)$ follows from the fact that a uniform [0,1] random variable is stochastically less than an exponential random variable with mean 1. Next, make the inductive hypothesis that $P_n(c,d) \geq P_\infty(c,d)$, $n < k$. Consider the process V_k , letting $D_k = \min\left(t > 0: V_k(t) = t\right)$, and define the process $V_k'(t) = V_k(t)$, for $0 \leq t < D_k$, $= \infty$, for $D_k \leq t \leq k$. Note that the transition function of V_k' is greater than the transition function of the Poisson process X ; thus, it is possible to construct dependent versions of V_k' and X such that $P\left(V_k'(t) \geq X(t),\ 0 \leq t \leq k\right) = 1$ and from this get a construction of V_k and X such that $P\left(V_k(t) \geq X(t),\ 0 \leq t \leq D_k\right) = 1$. Using analogs of (3.10) and (3.11) we obtain

$$P\left(V_k(t) > ct - d,\ 0 \leq t \leq s \mid D_k = s,\ X(D_k) = i\right)$$

$$\geq P\left(X(t) > ct - d,\ 0 \leq t \leq s \mid D_k = s,\ X(D_k) = i\right)$$

from the construction on $[0,D_k)$ and note that it suffices to demonstrate

$$P\left(V_k(t) > ct - d,\ s \leq t \leq k \mid D_k = s,\ X(D_k) = i\right)$$
$$\geq P\left(X(t) > ct - d,\ s \leq t \mid D_k = s,\ X(D_k) = i\right) \tag{3.17}$$

to complete the proof. The right-hand side of (3.17) is less than or equal to

$$P\big(X(t) > ct - d,\ s \leq t \mid X(s) = j - 1\big)$$
$$= P\big(X(t) > y_3(t) = ct + cs - d - j + 1,\ 0 \leq t\big) \tag{3.18}$$

where $j = \min(\ell\colon \ell \geq s)$. The left-hand side of (3.17) equals

$$P\big(V_k(t) > ct - d,\ s \leq t \leq k \mid D_k = s,\ V_k(s) = j\big)$$
$$= P\Big(V_{k-j}(t) > y_2(t) = \frac{c(k-s)}{k-j}\,t + cs - d - j,\ 0 \leq t \leq k-j\Big) \tag{3.19}$$
$$\geq P\big(V_{k-j}(t) > y_3(t),\ 0 \leq t \leq k-j\big)\ .$$

Equations (3.18) and (3.19) combined with the inductive hypothesis verify (3.17) completing the proof of Theorem 3. //

Theorem 4 For $\gamma > 1$, $\delta > 0$,

$$\lim_{n\to\infty} P\big(nU_n(t/n) > t/\gamma - \delta/\gamma,\ 0 \leq t \leq n\big)$$
$$= P\big(X(t) > t/\gamma - \delta/\gamma,\ 0 \leq t\big) \tag{3.20}$$
$$= \exp(-\delta z/\gamma)\ ,$$

where z is the nonnegative root of the equation

$$\gamma(1 - e^{-z}) = z \tag{3.21}$$

Proof Given $\varepsilon > 0$, let k_ε be an integer such that

$$P\big(X(t) > t/\gamma - \delta/\gamma,\ 0 \leq t \leq \gamma k_\varepsilon + \delta\big)$$
$$\leq P\big(X(t) > t/\gamma - \delta/\gamma,\ 0 \leq t\big) + \varepsilon \tag{3.22}$$

This follows from Pyke (1959, Theorem 2, equation 9). Define $U_{i,n} = \min\big(t\colon nU_n(t/n) = i\big)$, $i = 1,2,\ldots,n$, and $X_i = \min\big(t\colon X(t) = i\big)$, $i = 1,2,\ldots,n$. It follows from Miller (1976) that the joint distributions of $\{U_{i,n},\ i = 1,2,\ldots,\min(n,k_\varepsilon)\}$ converges to that of

$\{X_i,\ i = 1,2,\ldots,k_\varepsilon\}$. This implies

$$\lim_{n\to\infty} P\big(nU_n(t/n) > t/\gamma - \delta/\gamma,\ 0 \leq t \leq n\big)$$

$$\leq \lim_{n\to\infty} P\big(nU_n(t/n) > t/\gamma - \delta/\gamma,\ 0 \leq t \leq \min(n,\gamma k_\varepsilon + \delta)\big) \quad (3.23)$$

$$= P\big(X(t/n) > t/\gamma - \delta/\gamma,\ 0 \leq t \leq \gamma k_\varepsilon + \delta\big) .$$

Equations (3.22) and (3.23) imply that

$$\lim_{n\to\infty} P\big(nU_n(t/n) > t/\gamma - \delta/\gamma,\ 0 \leq t \leq n\big)$$
$$\leq P\big(X(t/n) > t/\gamma - \delta/\gamma,\ 0 \leq t\big) + \varepsilon . \quad (3.24)$$

Theorem 3 and (3.24) verify the limit in (3.20). The second equality on (3.20) is given by Pyke (1959). //

4. IMPLEMENTATION AND SOME NUMERICAL RESULTS

In the following we provide some aspect of our findings related to Theorems 2, 3, and 4 in more detail. The goal is to construct confidence regions of the form $F(\cdot) \leq \gamma F_n(\cdot) + \delta/n$, $\gamma > 1$. Some numerical results are provided, together with some examples of how these results are used. In the construction of an upper confidence region, one is interested in at least one of the following problems.

(i) Given the desired shape of an upper confidence region of the form $F(\cdot) \leq \gamma F_n(\cdot) + \delta/n$, that is, given (δ,γ,n) , what is the significance level α of such a confidence region?

(ii) Given (n,γ,α) , find a δ such that the corresponding upper confidence region has $100(1-\alpha)\%$ confidence.

(iii) What value of γ can ensure that the upper confidence region of the form above has $100(1-\alpha)\%$ confidence?

(iv) What is the smallest sample size n necessary to ensure that $\theta + \gamma F_n(x)$, given (θ,γ) , is an upper confidence region with $100(1-\alpha)\%$ confidence, where $\theta = \delta/n$?

The "exact" solutions to all these problems can be found using formula (2.5). Due to the computational difficulties in implementing such a formula and, moreover, the need to construct several extensive tables for each problem, we use Theorem 3, which shows that a conservative approximate solution for all these problems is possible. Moreover, by Theorems 2 and 4 the error committed by this approximation goes to zero monotonically as the sample size grows larger. Table I shows the numerical results for some values of δ , γ , and n as computed by formula (2.5). The last column of this table provides the Poisson approximation computed by formula (3.20) of Theorem 4. We notice that these "exact" values of α converge to their Poisson approximations as n , the sample size, increases as expected. The curves of Figure 6 are derived from additional computation by formulas from Theorem 4. Specifically, for a given δ and n find a γ such that the upper confidence region has at least $100(1-\alpha)\%$ confidence. By Theorem 4 one can approximate α as

$$\alpha = \exp\left(-\frac{\delta \cdot z}{\gamma}\right)$$

where z is the nonnegative root of the equation (3.21). Thus after some manipulation, one obtains

$$\gamma = \frac{\delta}{\log(\alpha)} \log\left[\left(\delta + \log(\alpha)\right)/\delta\right] . \qquad (4.1)$$

The above results are used in the next section, where we provide some real world applications. Similar results can be developed by

TABLE I

The Significance Level α as a Function of (n,γ,δ) for the Confidence Region $F(\cdot) \leq \gamma F_n(\cdot) + \delta/n$. Values of α Either Read Directly or Interpolated for Additional Value of n. The Last Column Shows the Asymptotic Results by the Poisson Process.

γ	δ	n = 20	40	60	80	100	150	200	250	300	400	500	∞
6.00	1.613	.1861	.1931	.1954	.1965	.1972	.1982	.1986	.1989	.1991	.1993	.1994	.200
	1.902	.1357	.1428	.1452	.1464	.1471	.1481	.1486	.1489	.1490	.1493	.1494	.150
	2.308	.0861	.0930	.0953	.0965	.0973	.0981	.0986	.0989	.0991	.0993	.0994	.100
	2.597	.0620	.0684	.0706	.0717	.0724	.0732	.0737	.0739	.0741	.0743	.0745	.075
	3.003	.0386	.0442	.0461	.0471	.0477	.0484	.0488	.0491	.0492	.0494	.0495	.050
	3.227	.0296	.0347	.0364	.0373	.0378	.0386	.0389	.0391	.0393	.0395	.0396	.040
	3.515	.0210	.0253	.0268	.0276	.0281	.0287	.0290	.0292	.0294	.0295	.0296	.030
	3.922	.0127	.0162	.0174	.0180	.0184	.0189	.0192	.0194	.0195	.0196	.0197	.020
	4.210	.0089	.0117	.0128	.0133	.0136	.0141	.0143	.0145	.0145	.0147	.0147	.015
	4.617	.0053	.0074	.0082	.0087	.0089	.0093	.0095	.0096	.0096	.0097	.0098	.010
4.00	1.642	.1833	.1916	.1944	.1958	.1965	.1970	.1978	.1982	.1985	.1989	.1991	.200
	1.935	.1330	.1414	.1442	.1457	.1465	.1470	.1477	.1482	.1485	.1489	.1491	.150
	2.349	.0838	.0918	.0945	.0958	.0967	.0971	.0978	.0982	.0986	.0989	.0991	.100
	2.643	.0600	.0673	.0698	.0711	.0719	.0723	.0730	.0734	.0736	.0740	.0742	.075
	3.056	.0371	.0433	.0454	.0466	.0472	.0476	.0482	.0486	.0488	.0491	.0493	.050
	3.284	.0283	.0339	.0358	.0369	.0375	.0378	.0384	.0387	.0389	.0392	.0393	.040

Table I--*continued*

γ	δ	n = 20	40	60	80	100	150	200	250	300	400	500	∞
4.00	3.577	.0198	.0246	.0263	.0272	.0278	.0281	.0286	.0288	.0290	.0293	.0294	.030
	3.991	.0119	.0156	.0170	.0177	.0182	.0184	.0188	.0191	.0192	.0194	.0195	.020
	4.285	.0082	.0113	.0124	.0131	.0134	.0137	.0140	.0142	.0143	.0145	.0146	.015
	4.698	.0048	.0071	.0080	.0085	.0088	.0089	.0092	.0094	.0095	.0096	.0097	.010
3.00	1.711	.1786	.1891	.1927	.1945	.1956	.1977	.1982	.1986	.1989	.1992	.1993	.200
	2.017	.1284	.1389	.1426	.1444	.1455	.1477	.1483	.1486	.1488	.1491	.1493	.150
	2.448	.0796	.0895	.0929	.0946	.0957	.0978	.0983	.0987	.0989	.0992	.0993	.100
	2.754	.0564	.0652	.0684	.0700	.0710	.0729	.0734	.0737	.0739	.0742	.0744	.075
	3.185	.0342	.0416	.0442	.0456	.0465	.0482	.0486	.0489	.0491	.0493	.0494	.050
	3.423	.0258	.0323	.0348	.0360	.0368	.0383	.0387	.0390	.0391	.0394	.0395	.040
	3.728	.0178	.0233	.0254	.0265	.0272	.0285	.0289	.0291	.0292	.0294	.0295	.030
	4.160	.0104	.0146	.0163	.0172	.0177	.0188	.0191	.0193	.0194	.0195	.0196	.020
	4.465	.0070	.0105	.0118	.0126	.0130	.0139	.0142	.0144	.0145	.0146	.0147	.015
	4.897	.0040	.0065	.0075	.0081	.0084	.0092	.0094	.0095	.0096	.0097	.0097	.010
2.50	1.803	.1731	.1862	.1907	.1930	.1944	.1962	.1972	.1977	.1981	.1986	.1989	.200
	2.125	.1231	.1361	.1406	.1429	.1443	.1462	.1471	.1477	.1481	.1486	.1488	.150
	2.580	.0751	.0868	.0911	.0932	.0946	.0963	.0973	.0978	.0982	.0986	.0989	.100
	2.902	.0523	.0628	.0667	.0687	.0699	.0716	.0724	.0729	.0733	.0737	.0740	.075

Table I--*continued*

γ	δ	n = 20	40	60	80	100	150	200	250	300	400	500	∞
2.50	3.356	.0309	.0396	.0428	.0445	.0456	.0470	.0477	.0482	.0485	.0489	.0491	.050
	3.606	.0230	.0305	.0335	.0350	.0360	.0373	.0379	.0383	.0386	.0390	.0392	.040
	3.928	.0155	.0218	.0243	.0256	.0265	.0276	.0282	.0285	.0288	.0291	.0293	.030
	4.383	.0088	.0135	.0154	.0165	.0171	.0180	.0185	.0188	.0190	.0192	.0194	.020
	4.705	.0058	.0095	.0111	.0120	0126	.0133	.0137	.0140	.0141	.0144	.0145	.015
	5.159	.0032	.0058	.0070	.0077	.0081	.0087	.0090	.0092	.0093	.0095	.0096	.010
2.00	2.020	.1607	.1794	.1861	.1895	.1915	.1943	.1957	.1966	.1971	.1979	.1983	.200
	2.381	.1113	.1294	.1360	.1394	.1414	.1442	.1457	.1465	.1471	.1478	.1483	.150
	2.890	.0648	.0808	.0868	.0899	.0919	.0945	.0959	.0967	.0972	.0979	.0983	.100
	3.251	.0435	.0574	.0628	.0657	.0675	.0699	.0711	.0719	.0724	.0730	.0734	.075
	3.760	.0242	.0351	.0395	.0419	.0435	.0455	.0466	.0473	.0477	.0483	.0486	.050
	4.040	.0173	.0266	.0305	.0327	.0341	.0359	.0369	.0375	.0379	.0384	.0387	.040
	4.401	.0111	.0185	.0218	.0236	.0248	.0264	.0273	.0278	.0282	.028	.0289	.030
	4.910	.0058	.0110	.0135	.0149	.0158	.0171	.0178	.0182	.0185	.0189	.0191	.020
	5.271	.0036	.0075	.0095	.0107	.0115	.0125	.0131	.0135	.0137	.0140	.0142	.015
	5.779	.0018	.0044	.0058	.0067	.0072	.0081	.0085	.0088	.0090	.0092	.0094	.101
1.75	2.258	.1470	.1717	.1807	.1853	.18[illegible]	.1921	.1940	.1952	.1960	.1970	.1976	.200
	2.662	.0985	.1218	.1306	.1352	.1381	.1419	.1439	.1451	.1459	.1469	.1475	.150

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Table I--*continued*

γ	δ	n = 20	40	60	80	100	150	200	250	300	400	500	∞
1.75	3.231	.0542	.0740	.0819	.0861	.0887	.0923	.0942	.0953	.0961	.0971	.0976	.100
	3.634	.0347	.0347	.0514	.0622	.0646	.0679	.0696	.0706	.0714	.0723	.0728	.075
	4.203	.0179	.0303	.0359	.0390	.0410	.0438	.0453	.0462	.0468	.0476	.0481	.050
	4.519	.0122	.0225	.0273	.0301	.0318	.0344	.0357	.0365	.0371	.0378	.0383	.040
	4.920	.0074	.0152	.0191	.0214	.0229	.0251	.0262	.0270	.0274	.0281	.0284	.030
	5.489	.0035	.0086	.0114	.0132	.0143	.0160	.0169	.0175	.0179	.0184	.0187	.020
	5.893	.0020	.0057	.0079	.0093	.0102	.0116	.0124	.0129	.0132	.0136	.0139	.015
	6.362	.0009	.0031	.0046	.0056	.0063	.0074	.0080	.0083	.0086	.0089	.0091	.010
1.50	2.762	.1177	.1538	.1679	.1754	.1801	[illegible]	.1898	.1918	.1931	.1948	.1959	.200
	3.255	.0725	.1047	.1181	.1254	.1300	.1364	.1397	.1417	.1430	.1447	.1458	.150
	3.951	.0346	.0593	.0707	.0771	.0813	.0871	.0902	.0921	.0933	.0950	.0959	.100
	4.444	.0197	.0389	.0485	.0541	.0578	.0631	.0659	.0676	.0688	.0703	.0712	.075
	5.140	.0084	.0209	.0281	.0325	.0354	.0397	.0421	.0436	.0446	.0459	.0467	.050
	5.523	.0051	.0146	.0206	.0243	.0269	.0307	.0328	.0343	.0351	.0362	.0370	.040
	6.017	.0026	.0091	.0137	.0167	.0188	.0220	.0238	.0249	.0257	.0267	.0273	.030
	6.712	.0009	.0046	.0075	.0097	.0112	.0136	.0150	.0159	.0165	.0173	.0173	.020
	7.206	.0004	.0027	.0049	.0065	.0077	.0096	.0108	.0115	.0120	.0127	.0131	.015
	7.902	.0001	.0013	.0026	.0037	.0045	.0059	.0067	.0073	.0077	.0082	.0085	.010

Table I--*continued*

γ	δ	n = 20	40	60	80	100	150	200	250	300	400	500	∞
1.25	4.334	.0423	.0928	.1200	.1364	.1473	.1631	.1717	.1770	.1806	.1853	.1882	.200
	5.108	.0175	.0521	.0743	.0886	.0985	.1134	.1216	.1268	.1304	.1351	.1379	.150
	6.200	.0042	.0213	.0359	.0464	.0546	.0665	.0736	.0783	.0816	.0858	.0885	.100
	6.975	.0013	.0106	.0206	.0285	.0347	.0449	.0510	.0551	.0580	.0619	.0643	.075
	8.067	.0002	.0037	.0089	.0138	.0179	.0252	.0300	.0332	.0356	.0387	.0408	.050
	8.668	.0001	.0020	.0055	.0091	.0122	.0182	.0222	.0250	.0270	.0298	.0316	.040
	9.442	.0000	.0008	.0029	.0052	.0074	.0118	.0149	.0172	.0189	.0212	.0227	.030
	10.534	.0000	.0002	.0011	.0023	.0035	.0063	.0084	.0100	.0112	.0130	.0142	.020
	11.309	.0000	.0001	.0005	.0012	.0020	.0040	.0055	.0068	.0077	.0091	.0101	.015
	12.400	.0000	.0000	.0002	.0005	.0009	.0020	.0030	.0038	.0045	.0055	.0062	.010

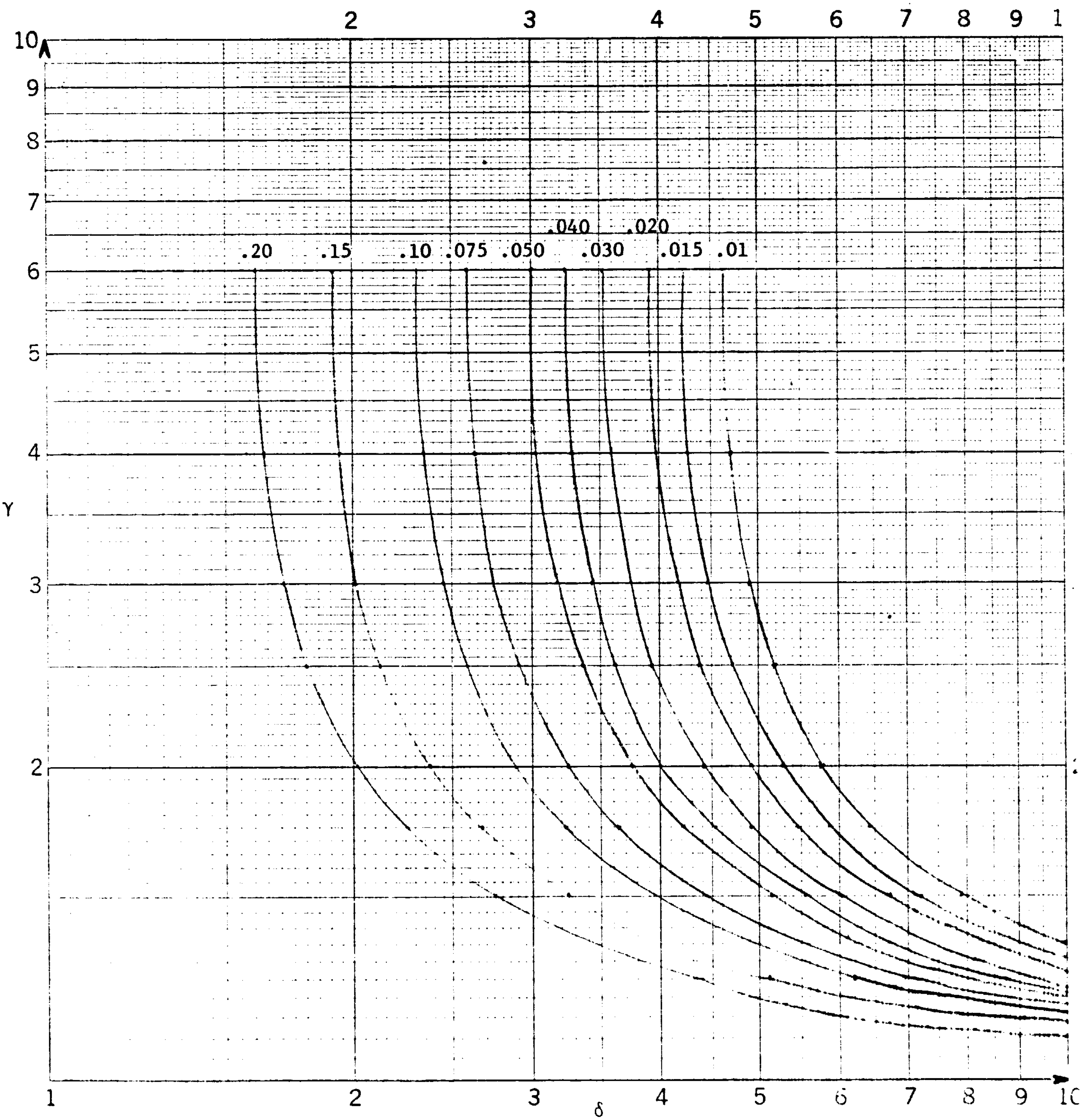

Figure 6. Poisson approximation to the significance level α, for the confidence region of form $F(\cdot) \leqslant \gamma F_n(\cdot) + \delta/n$.

implementing Theorem 1, but since we have no compelling application for these cases, we have not pursued their implementation.

5. SOME APPLICATIONS

Since the idea of constructing a confidence region over the cdf, based on the generalized K-S distance, is new, we present some areas of application where this idea is useful.

5.1 Application in Risk Analysis

Gross, Miller, and Soland (1980) studied and gave details of confidence region construction of a risk profile defined as $R(t) = 1 - F(t)$. In the following we take one of their examples and apply our findings. Their data base is a typical simulated risk profile based on a sample of 500 observations. It is desired to construct a confidence region $R(t) < 2R_n(t) + d$ with $1-\alpha = 95\%$ confidence. They utilized the formula given in (2.5) and obtained the "exact" value $d = .0075$. Although their approach is straightforward, it was necessary to write a large and tedious program to find the d value. The desired region is equivalent to the upper confidence region $2F_n(t) + d$. Using the result of Theorem 4 one obtains the following relation from formula (4.1):

$$\frac{\delta+\log(\alpha)}{\delta} = \exp\left(\frac{\gamma\log(\alpha)}{\delta}\right)$$

where $\log(\alpha)$ is the natural logarithm of α. This relatively easy equation can be solved by numerical methods. We employed the method of binary search and obtained $\delta = 3.760$ and therefore the conservative value for $d = \frac{3.760}{500} = .00752$. In fact, from Figure 6 one can easily obtain an accurate enough solution. Figure 7 shows a typical simulated risk profile with its confidence region.

5.2 Application in Investment Modeling

The Investment Department of the World Bank developed a trading strategy for U.S. Treasury Notes. The strategy aims to maximize the rate of return from its investment in Treasury Notes. The basic idea behind the strategy is based on trend-following. Turning points in the movement of prices or yields can be identified as generating "buy" and "sell" signals. "Buy" signals imply that Treasury Notes be bought for all cash proceeds, and "sell" signals imply that all Treasury Notes held be sold and all cash proceeds invested immediately in Federal funds until the next "buy" signal. Federal funds represent money that banks hold and which can be lent to other banks to fulfill their reserve requirements. The interest rate that banks pay when they borrow Federal funds is called the Federal funds rate; these loans usually are made on an overnight basis.

When the "buy" and "sell" signals are generated from the trend-following strategy, the rate of return is calculated on a quarterly basis. They are then compared with some "neutral" strategy--for example, the rate of return in pure Federal funds investment strategy; that is, investing all money in Federal funds daily, on an overnight basis.

The differential rates of return, or the difference between the rates of return from the trend-following strategy and the rates of return from the Federal funds strategy, are calculated on a quarterly

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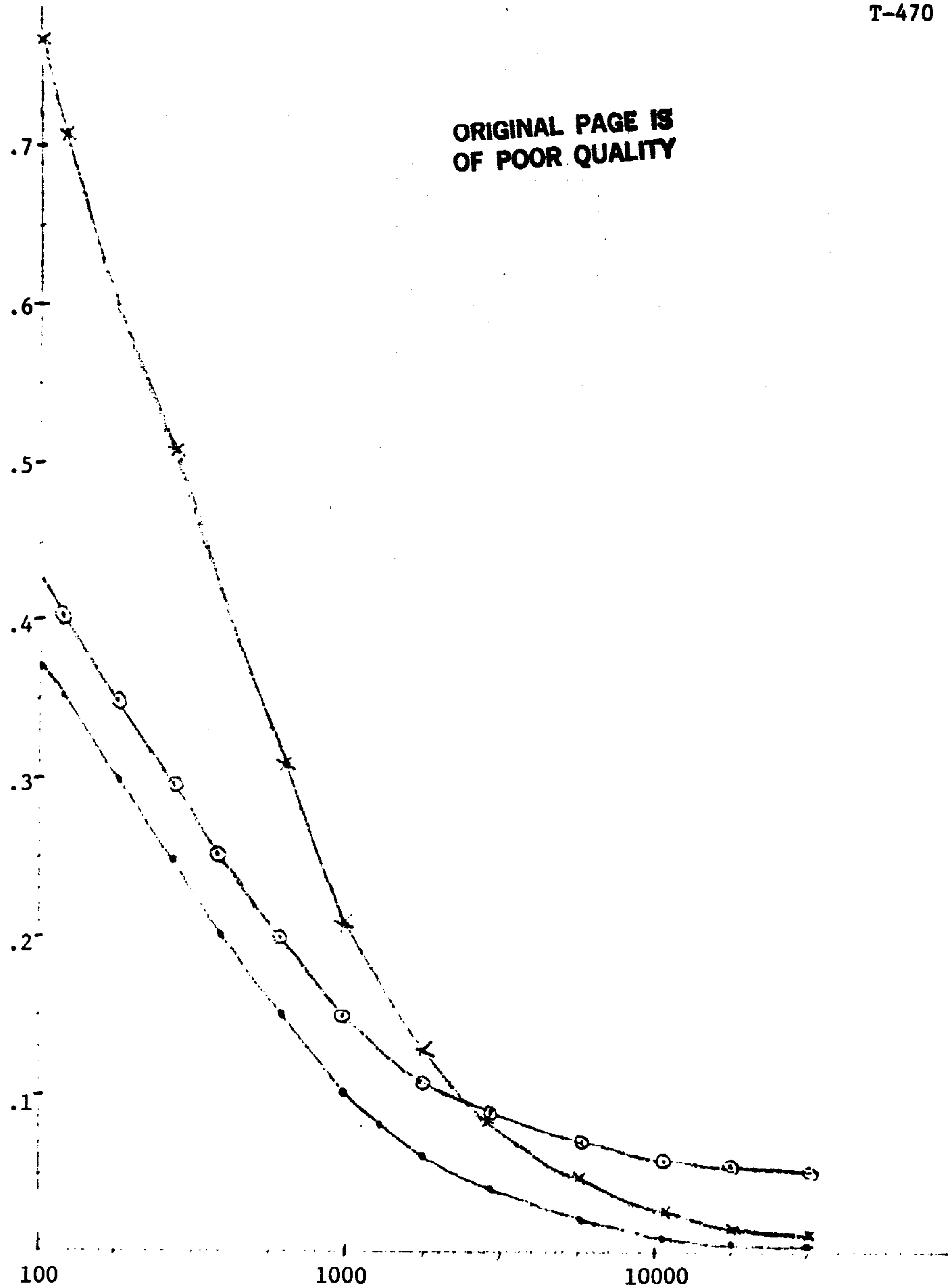

Figure 7. Simulated risk profile and confidence regions: risk profile R_n (—•—•—) based on 500 simulated observations; K-S 95% upper bound $R_n + .056$ (—⊙—⊙—), generalized K-S 95% upper bound $2R_n + .0075$ (—X—X—) approximated by the Poisson process.

basis. Some measures of performance are calculated. For example, if the differential rate of return is positive, the trading strategy is superior in that quarter; if it is negative, the Federal funds strategy is superior. The measures of performance chart how much better or worse the trading strategy performs than the Federal funds investment strategy in the long run. The World Bank defines "reward" as the expected value of positive differential return, and "risk" as the expected value of negative differential return. Let

$$r_d = \text{differential rate of return;}$$

then

$$\text{Reward} = E[\text{Max}(0, r_d)]$$

$$\text{Risk} = E[\text{Min}(0, r_d)] \ .$$

Using the daily historical prices and yields from June 1974 through December 1981, Table II can be obtained, where the quarterly rates of return during this period are presented.

The World Bank is interested in constructing an upper confidence region for the cdf of the differential rate of return of the following form:

$$F(r_d) \leqslant \gamma F_n(r_d) + \theta \ , \qquad \gamma > 1, \ \theta > 0$$

with a 95% confidence. This can easily be done as follows. Let us, for a given $\theta = .17$, construct an upper confidence region of the form (1.1) with $\alpha \leqslant .05$. With $\delta = 30(.17) = 5.14$ and using (4.1) we obtain,

$$\gamma = \frac{\delta}{\log(\alpha)} \log[(\delta + \log(\alpha))/\delta]$$

$$\gamma = 1.5 \ .$$

The same result can be obtained directly from Figure 6. Thus,

$$1.5\ F_n(r_d) + .17\ , \qquad -\infty < r_d < +\infty$$

is an upper bound for the cdf of the differential rate of return with at least 95% confidence, with the characteristic of having more confidence in risk taking circumstances. Figures 8 and 9 show the empirical cdf obtained from Table II together with the upper confidence region following two-year and five-year Treasury notes, respectively.

5.3 Reliability Estimation

Suppose an item has a lifetime distribution $F(t) = P(L \leqslant t)$, $t \geqslant 0$. In some contexts, such as the analysis of a pro-rated warranty, it is desirable to have more accurate estimates in the left tail of the distribution. This leads to a confidence interval of the form:

$$F(t) \leqslant \gamma F_n(t) + \delta\ , \qquad t \geqslant 0$$

with $\gamma > 1$.

5.4 Recovery Times in Fault-tolerant Systems

Critical systems must often meet very high reliability requirements. This high reliability is achieved by incorporating fault tolerance into the system. [A typical application is flight-critical avionics computers for aircraft, Hopkins, *et al.* (1978), and Wensley, *et al.* (1978).] When a fault occurs in such a system the system must detect it and take appropriate remedial action, reconfiguring itself so that the offending component no longer has potential for contributing to system failure. The length of time needed to achieve detection and reconfiguration has a very strong influence on system reliability; thus

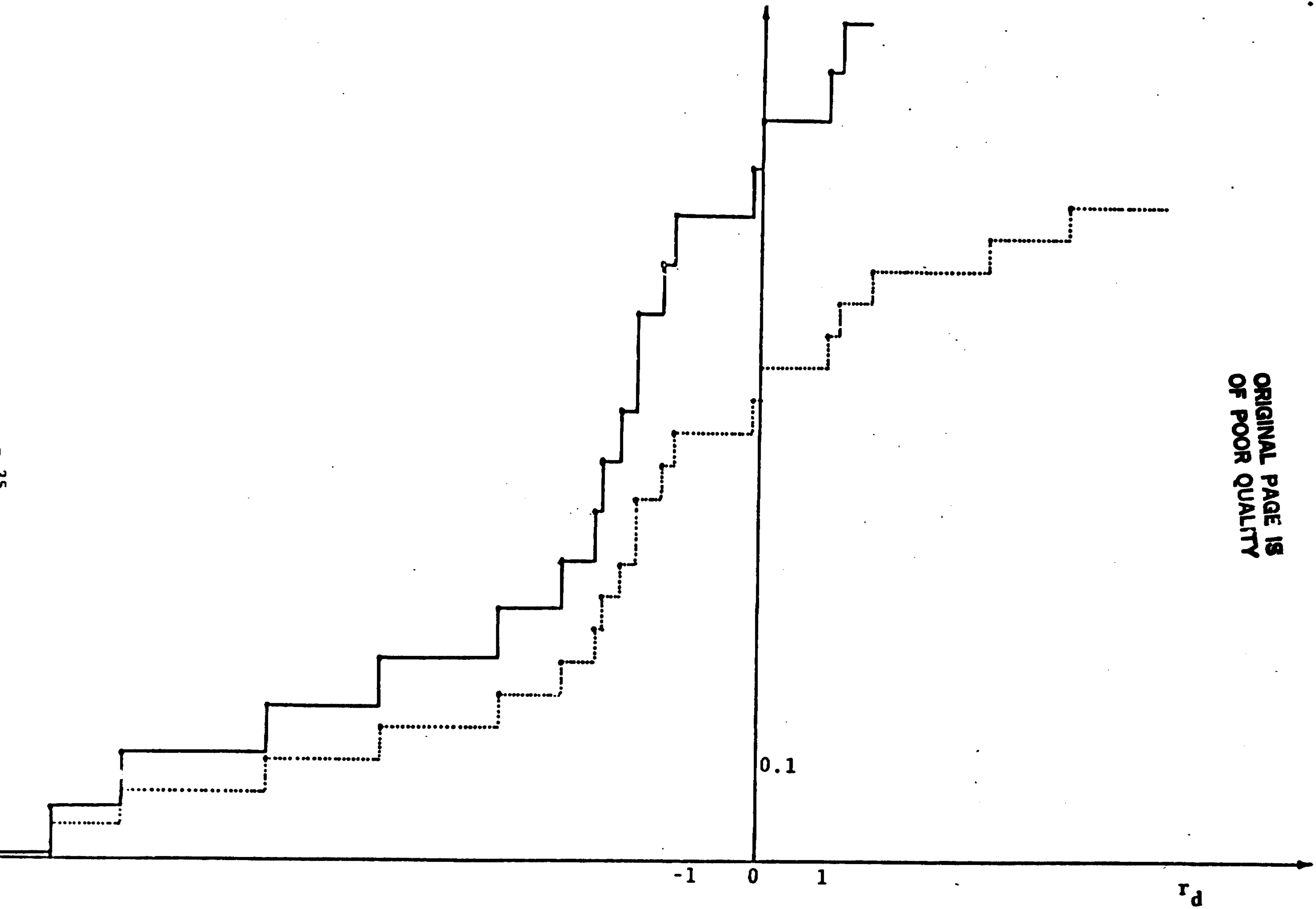

Figure 8.--An upper confidence region for cdf of differential rate of return following five-year notes trend.

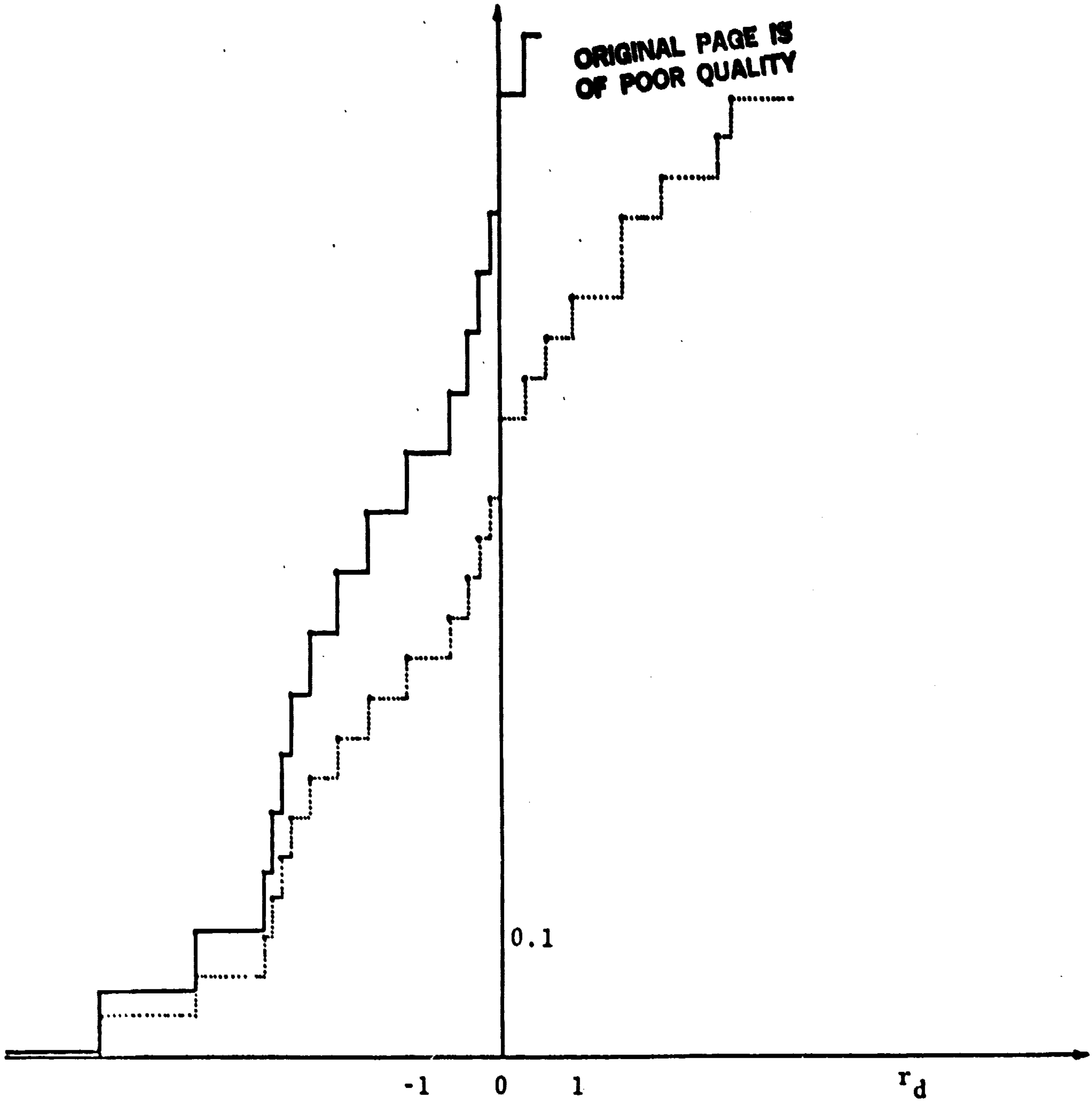

Figure 9.--An upper confidence region for cdf of differential rate of return following two-year notes trend.

TABLE II

QUARTERLY RATES OF RETURN*

Year	Quarter of the Year	Trend-following		Federal Funds	Differential of Rate of Return	
		5-Year Notes	2-Year Notes		5-Year Notes	2-Year Notes
1974	3	6.1	5.6	7.4	-1.3	-1.8
	4	21.9	13.4	9.4	12.5	4.0
1975	1	9.7	9.1	6.3	3.4	2.8
	2	10.2	8.5	5.5	4.7	3.0
	3	2.4	3.5	6.3	-3.9	-2.8
	4	17.8	14.6	5.4	12.4	9.2
1976	1	5.9	4.0	4.9	1.0	2.1
	2	6.9	7.3	5.2	1.7	2.1
	3	14.1	10.3	5.3	8.8	5.0
	4	15.6	10.3	4.9	9.7	5.4
1977	1	2.2	4.4	4.7	-2.5	-0.3
	2	12.0	6.8	5.2	6.8	1.6
	3	3.4	4.5	5.8	-2.4	-1.3
	4	4.7	6.1	6.5	-1.8	-0.4
1978	1	4.7	6.1	6.8	-2.1	-0.7
	2	5.9	6.5	7.4	-1.5	0.9
	3	9.8	8.4	8.1	1.7	0.3
	4	7.7	6.9	9.5	-1.8	-2.6
1979	1	10.2	10.1	10.1	0.1	0.0
	2	18.3	16.1	10.2	8.1	5.9

Table II--*continued*

Year	Quarter of the Year	Trend-following		Federal Funds	Differential of Rate of Return	
		5-Year Notes	2-Year Notes		5-Year Notes	2-Year Notes
1979	3	8.0	8.8	11.0	-3.0	-2.2
	4	19.6	13.6	13.4	6.2	0.2
1980	1	7.7	11.1	15.1	-7.4	-4.0
	2	57.5	43.5	12.7	44.8	30.8
	3	9.8	9.8	9.8	0.0	0.0
	4	6.2	12.9	15.8	-9.6	-2.9
1981	1	6.1	11.3	16.5	-10.4	-5.2
	2	18.8	18.2	17.6	1.2	0.6
	3	11.8	14.4	17.5	-5.7	-3.1
	4	30.3	23.7	13.5	16.8	10.2

*Data obtained from the Investment Department of the World Bank, Washington, D.C.

it is important to accurately estimate the recovery (or coverage) time distribution $C(t) = P(T_R \leq t)$, $t \geq 0$, from data which may be obtained from bench tests, simulations, or actual operation. Since long recovery times pose a much greater threat than shorter ones, a confidence interval should take the form

$$C(t) \geq \gamma C_n(t) - \delta , \qquad t \geq 0 ,$$

where $C_n(\cdot)$ is the empirical cdf and $\gamma > 1$.

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