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"DESIGN OF HELICOPTER ROTOR BLADES FOR OPTIMUM DYNAMIC CHARACTERISTICS"

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1. INTRODUCTION

1.1 Overview of Research Project

The design of helicopter rotor blades involves not only considerations of strength, survivability, fatigue, and cost, but also requires that blade natural frequencies be significantly separated from the fundamental aerodynamic forcing frequencies (e.g. Ref. 1). A proper placement of blade frequencies is a difficult task for several reasons. First, there are many forcing frequencies (at all integer-multiples of the rotor RPM) which occur at rather closely-spaced intervals. For example, 5/rev and 6/rev are less than 20% apart. Second, the rotor RPM may vary over a significant range throughout the flight envelope, thus reducing even further the area of acceptable natural frequencies. Third, the natural modes of the rotor blade are often coupled because of pitch angle, blade twist, offset between the mass center and elastic axis, and large aerodynamic damping. These couplings complicate the calculation of natural frequencies. In fact, the dependence on pitch angle makes frequencies a function of loading condition, since loading affects collective pitch. Fourth, the centrifugal stiffness often dominates the lower modes, making it difficult to alter frequencies by simple changes in stiffness.

In the early stages of the development of the helicopter, it was believed that helicopter vibrations could be reduced (and even eliminated) by the correct choice of structural coupling and mass stiffness distributions. However, it is easy to imagine how difficult it is to find just the proper parameters such that the desired natural frequencies can be obtained. The difficulties in placement of natural frequencies have led, in many cases, to preliminary designs which ignore frequency placement. Then, after the structure is "finalized" (either on paper or in a prototype blade), the frequencies are calculated (or measured) and final adjustments made. Reference (2) describes the development of the XH-17 helicopter in which a 300-lb weight was added to each blade in order to change the spanwise and chordwise mass distribution and thereby move the first flapwise frequency away from 3/rev. However, these types of alterations are detrimental to blade weight, aircraft development time, and blade cost. In addition, corrections usually are not satisfactory; and the helicopter is often left with a noticeable vibration problem.

The state-of-the-art in helicopter technology is now to the point, however, that it should be possible to correctly place rotor frequencies during preliminary design stages. There are several reasons for this. First, helicopter rotor blades for both main rotors and tail rotors are now being fabricated from composite materials (Refs. 3 and 4). This implies that the designer can choose, with certain restrictions, the exact EI distribution desired. Furthermore, the lightness of composite blades for the main rotor usually necessitates the addition of weight to give sufficient autorotational blade inertia. Thus, there is a considerable amount of freedom as to how this weight may be distributed. Second, the methods of structural optimization and parameter identification are now refined to the point where they can be efficiently applied to the blade structure. Some elementary techniques have already been used for the design of rotor fuselages (Ref. 5). It follows that the time is right for the use of structural optimization in helicopter blade design. Some work on this is already under development, and, although not published, some companies are already experimenting with the optimum way to add weight to an existing blade in order to improve vibrations.

The purpose of the work discussed here is to investigate the possibilities (as well as the limitations) of tailoring blade mass and stiffness distributions to give an optimum blade design in terms of weight, inertia, and dynamic characteristics.

The major objectives of the work are:

1) To determine to what extent changes in mass or stiffness distribution can be used to place rotor frequencies at desired locations.
2) To establish theoretical limits to the amount of frequency shift.
3) To formulate realistic constraints on blade properties based on weight, mass moment of inertia, size, strength, and stability.
4) To determine to what extent the hub loads can be minimized by proper choice of EI distribution.
5) To determine if the design for minimum hub loads can be approximated by a design for a given set of natural frequencies.
6) To determine to what extent aerodynamic couplings might affect the optimum blade design.
7) To determine the relative effectiveness of mass and stiffness distribution on the optimization procedure.
8) To determine to what extent an existing blade could be optimized with minimal changes in blade structure.
9) To develop several "optimum profiles" for rotor blades operating under various standard conditions.

The work is to focus on configurations that are simple enough to yield clear, fundamental insights into the structural mechanisms but which are sufficiently complex to result in a realistic result for an optimum rotor blade.

1.2 Overview of Current Report

This second semi-annual report serves two purposes: 1) it informs our sponsors and other interested parties what we have accomplished during the last six months, and 2) it serves as an archive of data to which we expect to refer frequently in coming months, as our research proceeds. As a result of the latter, archival purpose, the report contains much information which is not necessarily new, but which needs to be recorded in an orderly fashion so that it may be easily accessed in the future.

The first section of the report details our experience with the CONMIN optimization program applied to the problem of finding the optimal design of a vibrating cantilever beam (see Fig. 1). This section gives the results of parameter changes, the results obtained with various constraint forumulations, and the effect of allowing lumped weights at discrete points. Other aspects that are studied include the autorotational constraint and the effect of rotation. The principal conclusion of these investigations is summarized as follows: CONMIN works reasonably well for all problems we have considered so far.

The second part of the report discusses some numerical aspects of the problem. One important aspect is the effect of the number of finite elements on both the frequency calculation and the sequence of optimal designs. Also of importance is the effect of errors in the eigenvalue analysis as well as the sensitivity of the frequency to small changes in blade dimensions (manufacturing tolerances). The conclusion to be drawn from these numerical studies is that if reasonable care is taken, numerical difficulties are not significant for the problems we have considered thus far.

2. EXPERIMENTS WITH THE CONMIN PROGRAM AND PARAMETERS

The CONMIN (Ref. 6) program was employed to minimize the weight of the cantilever beam, described in the introduction. Before extensive optimization problems are solved, it is necessary to experiment with the program parameters and determine the best values for the particular class of problems at hand. The problem chosen for the numerical experiments involved minimization of weight subject to two frequency constraints and to side constraints on the thickness. The non-rotating cantilever beam undergoing flapping vibrations was examined. Specifically, the following questions were to be answered:

1) Do the gradients produced by analytical techniques match those obtained by finite differences?
2) Is there any difference between results obtained with constraints on the frequency in Hz. and those obtained with constraints on the square of the circular natural frequency in $(rad/sec)^2$? Can scaling of the constraint function improve convergence?
3) Under what conditions do the starting values for thickness influence the convergence properties of the problem? Can an original, infeasible design (i.e., one which violates one or both of the frequency constraints) be expected to converge to a feasible, optimal design?

The determination of the so-called "optimal design" in CONMIN is influenced by several important parameters. Changes in the parameters can change the final answer and can influence the speed of convergence. The most important parameters are ITRM, DELFUN, DABFUN. Convergence is defined whenever ITRM consecutive iterations are encountered such that the values of DELFUN or DABFUN (or both) are less than the stated values input to the program. The parameters and default values are defined as follows:

ITRM: Default Value = 3. Number of consecutive iterations to indicate convergence by relative or absolute changes (DELFUN or DABFUN respectively).

DELFUN: Default Value = 0.0001. Minimum relative change in the objective function to indicate convergence. DELFUN = ABS (1.0 - OBJ(J-1)/OBJ(J)), where the objective functions for the current, J^{th}, iterate and the previous, $J-1^{st}$, iterate are tested.

DABFUN: Default Value = 0.001 times the initial objective function value. Minimum absolute change in the objective function to indicate convergence. DABFUN = ABS (OBJ(J) - OBJ(J-1)).

Note that a practical criterion for convergence is employed by the program. Thus, slight differences in answers can be expected if problems are started from different initial points, or if different values of ITRM, DELFUN, and DABFUN are employed.

The parameter CT is used to define whether or not a constraint is active. The exact satisfaction of the J^{th} constraint, $G(J) \equiv 0$ is numerically unusable. Rather, a band, CT in magnitude, on each side of the exact zero is employed. The default value and formal definition of this parameter is:

CT: Default Value = -0.1. The J^{th} constraint, G(J), is considered to be active if $CT \leq G(J) \leq ABS(CT)$. The value of CT is sequentially reduced in magnitude during the optimization process.

The parameter THETA is called the "Push Off Factor", and is used by the programmed Method of Feasible Directions (Ref. 7) to go from one feasible design to another feasible, improved design. The default value is 1.0. Larger values of THETA are appropriate for highly non-linear constraint functions. Lower values are appropriate as the constraints approach linear functions.

The parameter PHI controls how quickly an infeasible design will be moved in the direction of the feasible region. The default value = 5.0. Values of PHI above 5 should be employed if a feasible solution cannot be obtained.

Finally, the value of ITMAX defines the maximum number of interations in the optimization process. If a solution cannot be obtained in ITMAX iterations, the program is terminated. The default value is 10.

Herein, experiments with the parameters ITRM, DELFUN, DABFUN, CT, THETA, PHI and ITMAX were performed to enable definition of the numerical values that best fit the class of problem at hand. There are numerous other parameters within CONMIN, but in this study the default values for these other parameters were considered to be adequate.

2.1 The Basic Problem: Two Frequency Constraints

The problem to be considered is the non-rotating cantilever shown in Figure 1. Each 24 inch long element has a different thickness of flange. Letting t_1 be the thickness of the flange nearest the fixed end, element thicknesses are numbered in order, such that the tip end has a thickness denoted by t_{10}. Corresponding values of moment of inertia of area are denoted by I_1 through I_{10}. A material density and modulus of elasticity of 0.1 lbf/cu.in. and 10 x 10^6 psi are used.

It can be shown that the area, $A_i^{(in^2)}$, weight, $W_i^{(lbf)}$, and moment of inertia, $I_i^{(in^4)}$ are the following functions of the thickness, $t_i^{(in)}$:

$$A_i = 0.50 + 7.6\, t_i \tag{1}$$

$$W_i = 1.20 + 18.24\, t_i \tag{2}$$

$$I_i = \frac{25}{96} + \frac{t_i}{24} \{285 - 228\, t_i + 60.8\, t_i^2\} \tag{3}$$

The thicknesses are constrained to lie within the range 0.05 inches to 1.25 inches (the value for which the cross section would be a solid rectangle).

A uniform cantilever with a thickness of 0.10 inches for each flange would have the following first two frequencies:

$$f_1 = 1.98 \text{ Hz}$$

$$f_2 = 12.4 \text{ Hz}$$

To ensure that a feasible starting solution could be obtained with at least one chosen thickness, the frequencies are constrained to be within $\pm$ 0.2 Hz of the frequencies 1.98 and 12.4. Thus, $1.8 \leq f_1 \leq 2.2$ and $12.2 \leq f_2 \leq 12.6$ Hz.

The mathematical programming problem becomes:

$$\text{Min.}W = \text{Min} \sum_{i=1}^{10} W_i = 12 + 18.24 \sum_{i=1}^{10} t_i$$

$$\text{S.T.} \quad 1.8 \text{ Hz} \leq f_1 \leq 2.2 \text{ Hz}$$

$$12.2 \text{ Hz} \leq f_2 \leq 12.6 \text{ Hz}$$

$$\text{and,} \quad 0.05 \leq t_i \leq 1.25$$

$$i = 1,2,\ldots 10 \qquad (4)$$

The above problem has ten thickness decision variables. To preserve generality for later work with more complex sections, it was decided to use Moment-of-Inertia decision variables. The problem then becomes:

$$\text{Min.}Z = \sum_{i=1}^{10} Q(I_i)$$

$$\text{S.T.} \quad G_1 = 1.8 - f_1 \leq 0$$

$$G_2 = f_1 - 2.2 \leq 0$$

$$G_3 = 12.2 - f_2 \leq 0$$

$$G_4 = f_2 - 12.6 \leq 0$$

$$\text{and,} \quad 0.83073 \leq I_i \leq 5.20833$$

$$i = 1,2\ldots 10 \qquad (5)$$

In the above formulation, the i^{th} thickness has been written as a non-linear function, Q, of the i^{th} moment of inertia

$$t_i = Q(I_i) \tag{6}$$

The problem is formulated in terms of ten moment of inertia decision variables, I_1 through I_{10}, four frequency constraint functions, and twenty side constraints. The objective function and frequency constraints are non-linear functions of the decision variables.

2.1a Analytical Gradients vs Finite Differences

The CONMIN program has the option to compute gradients of the objective function and constraints via Finite Differences. If possible, however, it is more efficient for the user to provide analytically-derived gradients. In Appendix I, the method used to obtain gradients is given in detail. By using both methods on the same problem, it is possible to provide checks on the derivation and programming of the analytical gradient method.

Tables 1A and 1B present the results from using the two options. An original design with constant thickness of 0.05 inches leads to the designs in Table 1A, whereas a starting thickness of 0.10 inches is the basis for Table 1B. It is noted that the finite difference method leads to essentially the same results as obtained with analytical gradients.

The four designs all converge to a common design defined by:

1) A thickness of element 1 of aobut 0.086 inches - 0.091 inches
2) The thickness of all other elements are governed by the lower bound constraint (0.05 inches)
3) The second natural frequency is governed by the lower bound constraint (f_2 = 12.2 Hz)
4) The first natural frequency is not actively constrained, and ranges from 1.94 to 2.00 Hz.

The sensitivity of the first (lowest) frequency to small changes in thickness will be considered later, along with the accuracy of the single precision routine used for eigenvalue extraction.

It is instructive to make a further check on the solutions by running an optimization problem with only one decision variable (I_1) and one frequency constraint ($f_2 \geq 12.2$ Hz). All other thicknesses are held constant at 0.05 inches. The solution is started with $t_1 = 0.10$ inches, and in 6 iterations converges to the results that follow:

$I_1(t_1)$:1.256 inches^4(0.0902 inches)

f_1:2.01 Hz

f_2:12.21 Hz

Weight:21.85 lbf

These results can be considered to be the values toward which all four runs should approach. Within the limits of numerical accuracy, all four cases do converge to the values obtained in the calibration run.

Hereafter, all analyses are based on the use of analytical gradients.

2.1b Various Forms of the Frequency Constraint Function

The constraint relations, described earlier, have been

$$G_1 = 1.8 - f_1 \leq 0$$

$$G_2 = f_1 - 2.2 \leq 0$$

$$G_3 = 12.2 - f_2 \leq 0$$

$$G_4 = f_2 - 12.6 = \leq 0$$

where, f_1, f_2 are the natural frequencies in Hz. (7)

The authors of CONMIN recommend use of constraint functions that are mutually of the same order of magnitude, and that property is satisfied by the formulations in Eqn. 7.

It is instructive to see whether the use of eigenvalue constraints alter the convergence of the problem.

Let λ_i = i^{th} eigenvalue (8)

$= \omega_i^2$ (the square of the circular natural frequency)

If λ_i is extracted in units of $(rad/sec)^2$, the alternate form of the constraints becomes

$$G_1^* = (1.8(2\pi))^2 - \lambda_1$$

$$G_2^* = \lambda_1 - (2.2(2\pi))^2$$

$$G_3^* = (12.2(2\pi))^2 - \lambda_2$$

$$G_4^* = \lambda_2 - (12.6(2\pi))^2$$

where, λ_1, λ_2 are the eigenvalues in $(rad/sec)^2$ (7a)

The G_i^* constraints in Eqn. 7a are no longer expected to be of the same order of magnitude and convergence difficulties may result.

To test the convergence properties, the two forms of the constraint function are used on a problem with initial thickness of 0.05 inches. As seen in Table 2, the formulation with eigenvalue constraints does converge, but the optimal solution is not as good as that obtained by using frequency constraints .

When the eigenvalue constraints are used on a problem starting with $t = 0.10$ inches, a very poor optimal solution is obtained. A good optimal solution could be obtained by scaling the objective functions by 10 and 100.

$$G_1^{**} = G_1^*/10$$

$$G_2^{**} = G_2^*/10$$

$$G_3^{**} = G_3^*/100$$

$$G_4^{**} = G_4^*/100 \qquad (7b)$$

Alternately, G_i^* constraints could lead to a good optimum if the constraint thickness, CT, were changed from -0.1 (the default value) to -800.0.

When the eigenvalue constraints are used on a problem starting with t = 1.25 inches, no feasible design is found. Attempts to move toward a feasible design by changing the parameter PHI from 5 to 50 to 150 are unsuccessful.

For all further studies, frequency constraints of the form shown in Eqn. 7 are employed. It is to be noted that if troubles are encountered, a more efficient form of Equation 7 can be employed to ensure objective function values of the same order of magnitude.

$$1 - (f_i/f_{iL}) \leq 0$$

$$\text{and } (f_i/f_{iU}) - 1 \leq 0 \qquad (9)$$

where, f_i = i^{th} frequency, in Hz

f_{iL} = lower bound on i^{th} frequency, in Hz

f_{iU} = upper bound on i^{th} frequency, in Hz

2.1c Influence of Initial Design on Convergence and Optimal Design

Consider the three following initial designs:

Case 1 - constant thickness of 0.05 inches

Case 2 - Constant thickness of 0.10 inches

Case 3 - Constant thickness of 1.10 inches

In Case 1, the initial design violates the lower bound on f_2, while in Case 3, the lower bound on both frequencies is violated. Only in Case 2 does the initial design result in a feasible initial solution.

The information for the three runs is presented in Table 3. In all cases, feasible solutions were readily obtained, and eventually, the optimal solution was obtained. If the most difficult case (Case 3) had been required to meet more rigorous convergence criteria, a few more iterations would have resulted in an improved optimal design.

Certainly, initial designs which are feasible and close to optimal are ideal. But it is possible to start with designs which are not feasible, and which are far from optimal. Unfortunately, such conclusions are problem dependent. For more severe frequency constraints, or for added problem constraints, it may be necessary to start with feasible or close-to-feasible designs in order to optimize.

2.1d Parameters ITMAX, ITRM, DELFUN, DABFUN

All data collected are based on the following values of the CONMIN convergence control parameters.

ITMAX: Maximum number of iterations (default value = 10)
Values used: 40, 80

ITRM: Consecutive iterations for convergence (default value = 3)
Values used: 3, 5, 8

DELFUN: Relative change parameter (default value = 0.0001)
Values used: 0.0001, 0.00005

DABFUN: Absolute change parameter (default value = 0.001 times the initial objective function)
Values used: 0.011 (default value for t = 1.10 inches)
0.001 (default value for t = 0.10 inches)
0.0005 (default value for t = 0.05 inches).

In general, the default values for ITRM, DELFUN, and DABFUN lead to good convergence properties. In about half of the runs, the objective function does not change, or just barely changes, during the last ITRM iterations. The other half have convergence governed by the DABFUN parameter.

When larger thicknesses are used, higher initial objective function results, and the default value for DABFUN can be larger than desired.

To test the adequacy of the default values, Case 2 in Table 3 may be rerun with tighter controls. The results, iteration by iteration, are shown in Table 4. The tighter controlled run essentially doubles the number of iterations, and halves DELFUN and DABFUN. Yet, the final results are essentially identical.

The recommended parameters for such runs are:

ITMAX = 40

ITRM = 3 (default value), or 5

DELFUN = 0.0001 (default value)

DABFUN = default value (0.05 inches $\leq$ t $\leq$ 0.25 inches)

= 0.0025 (t > 0.25 inches).

The parameters stated may not be applicable either for larger problems, or for more severely constrained problems. The parameter will be critically examined at several stages of the study.

2.1e Parameters THETA, PHI

Higher values of THETA (The Push Off Factor) are recommended for highly non-linear constraint functions. The default value ($\theta = 1$), 10, 100, and 700 were tried on a specific initial design (t = 0.10 inches I = 1.355 inches^4). The results are shown in Table 5. Clearly, changing the parameter from the default value does not improve the rapidity of the convergence or the quality of the answer.

In section 2.1b, experiments with PHI were reported. Changing PHI from 5 (the default value) to 50 and 150 did not enable an initially infeasible design to be moved into the feasible design space.

Further designs are hereafter based on default values for THETA and PHI.

2.2 Placement of Frequencies (More Severe Constraints)

The constraints defined by Eqn. 7 can be generalized as follows:

$$G_1 = f_{1L} - f_1 \leq 0$$

$$G_2 = f_1 - f_{1U} \leq 0$$

$$G_3 = f_{2L} - f_2 \leq 0$$

$$G_4 = f_2 - f_{2U} \leq 0 \qquad (10)$$

where, the lower and upper bounds on the i^{th} frequency, in Hz, is given by f_{iL} and f_{iU} respectively.

Three cases are examined, as follows:

Case 4: Cases 1, 2, 3 were constrained by a band of ± 0.2 Hz around 2.0 Hz and 12.4 Hz. The band is now narrowed to ± 0.1 Hz.

Case 5: The first two frequencies are separated, such that a ± 0.2 Hz band is defined around 1.7 Hz and 13.0 Hz.

Case 6: The first two frequencies are brought closer together, such that a ± 0.2 Hz band is defined around 2.3 Hz and 11.8 Hz.

The results of the optimizations are shown in Table 6. Thus, within reason, it is possible to re-proportion initial designs (feasible or infeasible) such that frequencies are placed where desired during an optimization of weight.

2.2a Further Studies of Convergence Parameters

The recommended ITMAX, ITRM, DELFUN, and DABFUN parameters (30, 3, 0.0001, 0.0025) are used in Case 4. The convergence of the objective function to 3 significant figures seems to be incomplete in the 3rd figure.

For Case 5, ITRM is changed from 3 to 5 and a surprisingly substantial reduction in objective function is obtained. Five iterations appear to be appropriate to maintain objective functions of 3 significant figures. For Case 6, another check (not shown in Table 6) was made by tightening the parameters to 40, 6, 0.00005, 0.0001. After 40 iterations, the convergence criteria had not been satisfied, but probably would be in another few cycles. The objective function was only changed from 0.593 to 0.590, but elements 2 and 3, originally 0.0703 and 0.0886 inches, were appreciably changed to 0.0832 and 0.0791 inches respectively. The objective function is quite flat near convergence, and small changes in objective function can be accompanied by appreciable changes in structural configuration. The accuracy of eigenvalue extraction, herein done by a standard library routine in single precision, obviously has an effect on the defined final configuration. A numerical study of eigenvalue calculations is presented in section 5.

3. OPTIMAL DESIGNS FOR A NON-ROTATING CANTILEVER

With the experience gained by studying two frequency constraints, it is possible to intelligently formulate the more difficult problems that follow:

1) Added Constraints - A third frequency
2) Added Design Variables - Non-structural lumped mass
3) Added Constraint - The auto-rotation constraint (i.e., minimum mass moment of inertia about an axis normal to the beam, and located at the root).

3.1 Optimization with Three Frequency Constraints

In addition to the constraints shown in Eqn. 10, two more must be added:

$$G_5 = f_{3L} - f_3 \leq 0$$

$$G_6 = f_3 - f_{3U} \leq 0$$

where, the lower and upper bounds on the 3rd frequency, in Hz, is given by f_{3L} and f_{3U} respectively (11)

Three cases are examined, as reported in Tables 7A and 7B:

Case 7: A $\pm$ 0.2 Hz constraint band is placed around the first three frequencies of the initial design.

Case 8: The spread between the desired values of f_1 and f_2 is narrowed, and the desired value of f_3 is decreased by 1.0 Hz from that defined in Case 7.

Case 9: The spread between the desired values of f_1 and f_2 is increased, and the desired value of f_3 is raised by 1.0 Hz from that defined in Case 7.

The more difficult optimizations (Cases 8 and 9) are repeated with different starting points, and (in Case 9) with different convergence criteria. Convergence to an estimated 99% accurate value of objective function is accomplished for runs using ITMAX = 80, ITRM = 5, DELFUN = 0.0001, and DABFUN = 0.0005. Designs initiated with various constant thickness values converge to similar optimal designs. For example, Cases 9B and 9C both result in minimum thickness for elements 2, 3, 4, 5, 9, and 10. Despite identical objective functions, individual thicknesses for elements 6, 7, and 8 in Cases 9B and 9C are far from identical. Again it is noted that near optimum there can be appreciable changes in structural configuration with very minimal effect on the objective function.

3.2 Addition of Lumped Weights

The introduction of lumped weights (assumed to have mass, but not to contribute to the moment of inertia of area, I, used in defining the stiffness matrix) increases the number of decision variables in the optimization problem. For one such weight at the center of each element, there are now twenty decision variables:

I_1(root element) through I_{10} (tip element)

W_1(weight on root element) through W_{10} (weight on tip element)

The problem is formulated as follows:

$$\text{Min } Z = 182.4\,\gamma \sum_{i=1}^{10} Q(I_i) + \sum_{i=1}^{10} W_i$$

$$\text{S.T.}\quad G_1 = f_{1L} - f_1 \leq 0$$

$$G_2 = f_1 - f_{1U} \leq 0$$

$$G_3 = f_{2L} - f_2 \leq 0$$

$$G_4 = f_2 - f_{2U} \leq 0$$

$$G_5 = f_{3L} - f_3 \leq 0$$

$$G_6 = f_3 - f_{3U} \leq 0$$

$$\text{and, } 0.83073 \leq I_i \leq 5.20833 \quad i=1,2,\ldots 10$$

$$0 \leq W_i \leq 100 \quad i=1,2,\ldots 10 \qquad (12)$$

The newly defined symbols in Eqn. 12 are as follows:

γ = element density, lbm/cu.in.

W_i = weight of lumped mass at i^{th} element center, lbm

Very minor modifications are necessary to include the analytical gradients of the objective function with respect to the lumped weights and of the constraints with respect to the lumped weights:

$$\frac{\partial Z}{\partial W_i} = 1 \qquad (i=1,2,\ldots 10)$$

$$\frac{\partial K}{\partial W_i} = [0]$$

$$\frac{\partial M}{\partial W_i} = \frac{1}{g}[N] \qquad (13)$$

where, Z is the objective function

K is the stiffness matrix

M is the mass matrix

I is the identity (unity) matrix

g is the acceleration of gravity, 386.4 inches/sec^2

The two matrix derivatives in Eqn. 13 are used to define the partial derivative of the eigenvalue with respect to the decision variable, $(\partial\lambda_i/\partial W_K)$. The remainder of the operations are outlined in Appendix I. Thus, the optimization can still be based on analytical gradients.

Three cases have been investigated. The results are shown in Tables 8A and 8B.

Case 10: A previous design with t = 0.10 (Case 7) is modified by reduction of the density from 0.10 lbf/cu.in. to 0.05 lbf/cu.in., and by replacement of lost weight by 1.512 lbf lumps at each element center. There is more freedom to choose the decision variables in Case 10 (i.e., the non-structural mass can be used efficiently to move frequencies). The final result is a constant-section beam of minimum thickness with lumped weights as shown in the Table. The optimal weight is only about two-thirds that of the optimal weight for Case 7.

Case 11: Two runs were made with only one difference in initial design. In 11A, 1 lbf lumped weights were used at each node, and in 11B lumped weights were not used. The final optimal designs were almost identical in all respects. Since all of the initial frequencies had to be raised, it was most efficient to remove all of the lumped weights of Case 11A.

Case 12: The initial design of case 11A was used with a compressed range of frequency constraints (from 1.8-35 to 2.6-32 Hz). The optimal design involved a combination of thicknesses greater than the minimum, and lumped weights. The tip 40% of the beam was made of minimum thickness elements without lumped weights.

Convergence for all the runs was excellent. Note that since the objective function is now weight (with numerical values of 20 to 30 lbs) previously used values of DABFUN are not appropriate. Herein, DABFUN was raised to 0.001. The other parameters were kept the same as before.

3.3 Addition of Auto-Rotational Constraint

This constraint is intended to be applied to rotating systems. However, the constraint is added here as the next step in developing the larger problem to be considered. Denoting the minimum mass moment of inertia about a vertical axis through the root of the beam as I_{min}, and the actual moment of inertia as I_m, it is required that $I_m \geq I_{min}$. Rearranging the information into a more usable form, the seventh constraint, to be added to Eqn. 12 becomes:

$$G_7 = 1 - (I_m/I_{min}) \leq 0 \qquad (14)$$

The gradients of G_7 with respect to the decision variables are analytically obtained, as shown in Appendix II.

Two optimization runs are made. In the first run, Case 10 is re-run with a I_{min} value of 500 lbf inches sec^2. This value is deliberately chosen to be low, such that the constraint remains inactive throughout the run. As expected, the results remained identical to that in Case 10. The optimal design has an I_m value of 807.7 lbf inches sec^2. The next case is reported in Table 9. A value of I_{min} = 1100 lbf inches sec^2 is demanded, such that the auto-rotational constraint is active during the optimization. For comparison, the new Case 13 and the previous Case 10 (without auto-rotational constraint) are included in Table 9. Although satisfaction of the formal convergence criteria is not met in 80 iterations, the optimization is close to being finished. (DELFUN and DABFUN $\leq$ 0.0004 and $\leq$ 0.005, respectively, for the last five iterations). As expected, the tip element is thickened and the lumped weight increased, since that is the most efficient way to satisfy the auto-rotational constraint. At optimum the auto-rotational constraint was, for all practical purposes, one of the active constraints.

4. OPTIMAL DESIGNS FOR A ROTATING CANTILEVER BEAM

If the cantilever beam is rotating in a horizontal plane, centrifugal forces are created which stiffen the system and increase the natural frequencies in vertical vibration. The first natural frequency has as a lower limit equal to the speed of rotation. Thus, for high speed of rotation, placement of frequencies must be done with due consideration of the centrifugal effects. These trends are shown in Table 10. For 300 RPM, one bound on the fundamental frequency is 5 Hz. Despite vast ranges in thickness and lumped weights, f_1 ranges only between 5.1 and 5.7 Hz for a uniform cantilever. Frequency placement is much less dependent on stiffness and mass distribution than for a non-rotating beam. The calculations in Table 10 are based on Ref. 8.

The optimization formulation remains the same as shown in Eqn. 12 and supplemented by Eqn. 14. The major modification required involves the contribution of element tension to the element stiffness matrices. The added contribution also causes a modification of the frequency constraint gradients. Some details of the derivations are shown in Appendix III. All eigenvalue calculations are based on double precison routines.

In Table 11A, results for three optimizations are shown. The single difference in input is the speed of rotation.

Case 14: This run is a repeat of the single precision run of Case 10. The non-rotating beam solutions are almost identical. The major difference is a slight re-arrangement of the lumped weights.

Case 15: The speed of rotation is the low value of 30 RPM. There is no difficulty in placing the frequencies in the same range as required for the non-rotating beam. The optimization results in a slight increase in weight over that for the previous case.

Case 16: With a speed of 100 RPM, the problem does not fully converge in 80 iterations. The objective function appears to be accurate to two (rather than the requested three) decimal places. The tip element and tip weight are now relatively large. The extra mass is needed to lower the frequency and counteract the effect of the high speed.

With speeds of 300 RPM, the requested frequency placement must be modified. Also, the convergence criteria is relaxed, as shown in Table 11B.

Case 17: This 300 RPM run converges in 39 iterations. The convergence criteria is satisfied by the DABFUN requirement (3 consecutive iterations with absolute change in objective function $\leq$ 0.01). Again, a large tip mass is needed, but that may be required to satisfy the more demanding auto-rotational constraint.

The use of lumped mass appears to be the more efficient way of controlling the fequency placement. In particular, masses are placed at the tip and near the zero points of the 2nd and 3rd mode shapes. In contrast, changes in thickness modify both stiffness and mass, and are less effective in perturbing the frequencies.

5. NUMERICAL ASPECTS OF THE PROBLEM

The results reported in Sections 2, 3 and 4 are encouraging, since they demonstrate that, given a mathematical representation of a cantilever beam under various conditions, the CONMIN program can produce an improved design. That is, at least for the problems considered thus far, numerical optimization is possible. Clearly, however, the design found through the use of CONMIN will be of no use if the mathematical representation of the structure is at fault. Thus, in the present section, several numerical aspects of the accuracy of the analysis model are considered.

5.1 Convergence with Increasing Number of Elements

One aspect of the mathematical representation of a structure with the use of finite elements is the question of how many elements are required to obtain an acceptable accuracy. For the present optimal design studies, this question takes two forms: 1) are enough elements used to predict the frequencies accurately, and 2) are enough elements used to describe the optimal design (that is, will essentially the same optimal design result if the mesh is refined)? The first question is addressed in Section 5.1a; the second in Section 5.1b.

5.1a Convergence of Frequency

To study convergence of frequency with increasing number of elements, a vibrating cantilever beam is considered. The beam is non-rotating. The analytical solution for the frequency in cps is known to be (Ref. 9)

$$f = \frac{N^2}{2\pi L^2} \sqrt{\frac{gEI}{W}} \qquad (15)$$

where $(NL)^2 = 3.515$ (first mode)

$= 22.4$ (second mode)

L = length of beam

g = acceleration of gravity

E = elastic modulus

I = moment of inertia of cross-sectional area

W = weight per length of beam

For the example under consideration,

$L = 20$ ft

$E = 10^7$ lbf/ft^2

$I = 1.3555$ inches^4

$W = 1.5120$ lbf/ft

The results of the finite element analysis for various numbers of elements are shown in Table 12. It can be seen that as few as six elements gives a good approximation (less than one percent error) for the first frequency. As would be expected, the approximations for the second frequency are not as accurate, but the error is only about two percent when ten elements are used. In general, these results indicate that the choice of ten elements in the optimization studies described in Sections 2, 3 and 4 is justified.

5.1b Convergence of Optimal Design

To study how the optimal design changes as the number of elements increases, a cantilever beam with "N" elements and with lumped weights added at the nodes but otherwise similar to the beam of Section 5.1a and Figure 1 is considered. The density, and the constraints on the natural frequencies, lumped weights, and moments of inertia are

$$\gamma = 0.05 \text{ lbf/in}^3$$
$$1.0 \leq f_1 \leq 1.3 \quad \text{(Hz)}$$
$$10.0 \leq f_2 \leq 11.2 \quad \text{(Hz)} \tag{16}$$
$$0.0 \leq W_i \leq 100.0 \quad \text{(lbf)}$$
$$0.83073 \leq I_i \leq 5.2083 \quad (\text{in}^4).$$

The initial design is

$$I_i = 1.3555$$
$$W_i = 15.120/N; \qquad I = 1,2,\ldots,N$$

which implies an initial value of OBJ (= the objective function = the total weight) of 30.2249 lbf.

Results of the study are shown in Figures 3-8. In all cases, the active frequency constraints were found to be

$$f_1 = 1.3 \text{ (Hz)}$$
$$f_2 = 11.2 \text{ (Hz)}$$

Figure 3 demonstrates, as one would expect, that the optimum weight does in fact decrease as more elements are added to the mesh. The change in optimum weight is quite small (note that the scale of the vertical axis begins at 20.0).

Figures 4 and 5 show the variation of the lumped weight and the moment of inertia (of the cross-sectional area) at the free end versus the total number of elements in the mesh. It appears that these quantities do not converge. This result can be explained, however, by referring to Figure 6, in which the upper curve represents the total weight at the free end.

(M(N) is the non-structural, or, lumped weight; $\overline{M}(N)$ is the structural weight associated with the mass distributed throughout element "N"). It can be seen from the figure that the total weight appears to converge smoothly as the mesh is refined. The explanation for the apparent non-convergence shown in figures 4 and 5 and the convergence shown in the top curve of Figure 6 is that the "structural weight" at the free end of the cantilever is not really structural, since there is no portion of the beam beyond the free end which needs to be supported. Thus, the optimization routine is indifferent to whether structural or non-structural weight is present at the free end - the only thing that counts is the total weight at that end.

Figure 6 also shows the variation of the lumped weight slightly beyond the middle of the beam. (All optimal designs had non-zero lumped weights there and at the free end of the beam.) The weight can be seen to decrease smoothly as the mesh is refined, although no asymptote appears present. A possible explanation for this behavior is that as the mesh is refined, the weight in the middle is being placed more efficiently - and thus less is needed.

The various sketches in Figure 7 show the distribution of mass and stiffness along the beam for increasing numbers of elements. It is interesting t observe that the optimization routine finds it most efficient to meet the contraints on frequency by varying the lumped weight rather than by varying the stiffness (moment of inertia), since this latter quantity is at its lower bound everywhere except near the end of the beam.

As was pointed out previously in reference to Figures 4 and 5, the optimization algorithm appears to treat the structural and non-structural mass at the end of the beam as interchangeable. To test this hypothesis further, the optimal design problem statement was altered slightly by decreasing the upper bound constraint on the moment of inertia from 5.2083 to 2.0. The resulting optimum design is shown in Figure 8, and should be compared with the design (for N = 10) shown in Figure 7. Note that the constraint on the moment of inertia for element 10 is not active in the optimal design of figure 8 (the constraint was active during the CONMIN iterations leading to this optimal design). Thus, the effect of the constraint is to lead the optimization algorithm along a different path than that followed when the constraint value was 5.2083. The design found, however, has about the same total weight at the free end (= 9.9705 lbf) as the previous ten-element optimum (= 9.9222 lbf). This result confirms the hypothesis that CONMIN increases the moment of inertia at the free end only as a means of increasing the mass there. Once that option is closed (that is, the upper bound constraint is reduced to a value of 2.0), CONMIN simply increases the lumped weight at the beam tip. This finding suggests that, in future optimization studies, a tight constraint be imposed on the moment of inertia at the free end, since little structural capability is needed there, and necessary end mass can be adequately represented by the lumped weight design variable.

5.2 Accuracy of Eigenvalue Calculations

Among finite element analysts, the problem of the static analysis of a cantilever beam subjected to an end load is notorious for being numerically ill-conditioned. This ill-conditioning also becomes apparent in the eigenvalue calculations associated with the present optimal design studies. Table 13 illustrates the magnitude of the errors arising in the eigenvalue calculations for a non-rotating cantilever like that of Figure 1, with thickness 0.1 inches, density 0.1 lbf/in^3, and no lumped mass. Column 1 in the table contains the first twenty frequencies of the beam, which were obtained by a double-precision version of a code based on the Sturm-sequence method with inverse iteration. The eigenvalue is defined as the square of the frequency ω (in rad/sec) in the equation

$$([K] - \omega^2[M])\{Y\} = 0 . \tag{17}$$

Here, K is the stiffness matrix, M the mass matrix, and Y the eigenvector. Columns 2 and 3 contain the eigenvalues for the same problem, but found by the IBM scientific subroutine program "NROOT" (based on the Jacobi method) in a single-precision version (column 2) and a double-precision version (column 3). The difference in the first entries in columns 1 and 2 is about three percent.

An alternative manner of formulating the eigenvalue problem is to write it as

$$([M] - \frac{1}{\omega^2}[K])\{Y\} = 0 . \tag{18}$$

The eigenvalue is now defined to be the reciprocal of the square of the circular frequency. For this formulation of the problem, column 4 gives the frequencies found by the Sturm-sequence method, and column 5 gives the frequencies found by the single-precision routine "NROOT". The two methods now give essentially the same frequencies. The improved performance of the single-precision "NROOT" routine is attributable to the fact that the accuracy of the Jacobi algorithm is dependent on the order (in terms of size) in which the eignevalues are found. By contrast, the Sturm-sequence method is independent of the "largeness" or "smallness" of the eigenvalues.

The significance of these errors in the eigenvalue calculations can be seen by inspection of Table 14, in which are given optimal designs found by CONMIN using both the inaccurate eigenvalue calculation and the accurate eigenvalue calculation. The example corresponds to Case 6 of Table 6. The designs are seen to differ appreciably.

5.3 Sensitivity of Frequency to Small Changes in Thickness

Because rotor blades can be manufactured only to within certain dimensional tolerances, the question naturally arises as to the sensitivity of the natural frequencies of the blade to small changes in blade dimensions. Clearly, if small changes in blade dimensions produce large changes in natural frequencies, then designing a theoretically optimum blade is futile: the small variations in blade dimensions introduced during manufacturing would destroy the optimally designed vibratory behavior. To study this question, the data of Table 15 were generated for the cantilever beam described in Section 4 (Case 14 of Table 11A). The first column in the table contains the derivative of the fundamental frequency with respect to the thickness of the first, second, third,..., and tenth (free end) element. If 0.01 inches is taken as a representative manufacturing tolerance, then the data in column 1 show that the maximum corresponding change in the first fundamental frequency is only about five percent (≈0.01 + 4.791). Another aspect of this sensitivity study is illustrated by the data in columns 2 and 3. Column 2 contains the derivative of the fundamental frequency with respect to the weight of the individual elements. Since

$$\frac{\partial f_1}{\partial \bar{m}_i} = \frac{\partial f_1}{\partial t_i} \frac{\partial t_i}{\partial \bar{m}_i} \tag{19}$$

and

$$\frac{\partial \bar{m}_i}{\partial t_i} = 9.12$$

(t_i is the thickness; see Fig. 1),

the entries in column 2 are derived by dividing the entries of column 1 by 9.12. Column 3 contains the derivative of the fundamental frequency with respect to the lumped weight of the individual elements. If 0.01 lbs is taken as a representative manufacturing tolerance, then the data in columns 2 and 3 show that the maximum corresponding change in the first fundamental frequency is about half a percent (=0.01 + 0.525).

It is also interesting to compare corresponding entries in columns 2 and 3 and to observe that they disagree near the blade root, with the discrepancy diminishing appreciably as the free end is approached. The explanation for this behavior lies in the fact that as the distributed weight of an element is increased, the cross-sectional area must of course also increase. Thus the second column in the table represents changes in stiffness as well as in mass. It follows that a unit increase in distributed weight (as represented by column 2) will produce a larger increase in frequency than is caused by a unit decrease of lumped weight (column 3), since, qualitatively speaking, stiffness appears in the numerator and mass in the denominator of the frequency expression, $\sqrt{K/M}$. Thus

$$\frac{\partial f_1}{\partial \tilde{m}_i} > \frac{\partial f_1}{\partial \bar{m}_i} \quad .$$

The inequality is large near the root, because increasing element stiffness has a large effect on structure stiffness. At the tip, the structure stiffness is barely changed (as witnessed by the last two entries in columns 2 and 3: - 0.227 and - 0.229).

6. CONCLUSIONS

For the types of beam vibration problems described in this report, the following conclusions may be drawn.

1. Either analytical or finite-difference gradients can be used. The finite-difference approach will of course require many more function evaluations, but it is easily implemented. Furthermore, the problems considered are not especially sensitive to numerical error in gradient calculations, therefore, we use analytic gradients; but we reserve the finite-difference gradients as a viable alternative should they be needed.

2. The frequency constraints should be formulated directly in terms of frequencies, rather than eigenvalues (frequency squared).

3. CONMIN can find an optimum design, after starting with an initially infeasible design, at least for the problems considered.

4. The default values for CONMIN appear adequate, with the exception of DABFUN ITMAX, for which a value of 40-80 works well.

5. Optimum designs can be found even for relatively tight constraints on frequencies.

6. The objective function is relatively flat near the optimum, and appreciably different distributions of stiffness may yield essentially the same value of the objective function.

7. Two and three frequency constraints can be handled.

8. Using both stiffness and lumped mass as decision variables presents no special difficulties. Indeed, the admittedly limited experience gained thus far with these types of problems indicates that the optimization seems to proceed more rapidly (fewer iterations to obtain convergenc) if CONMIN can add lumped mass rather than just add mass in the form of structural mass. Thus the greater complexity of the problem, i.e., the increased number of

decision variables, appears to be more than compensated by the greater freedom in choosing a design.

9. Optimal designs for rotating beams subject to frequency and auto-rotational constraints present no difficulties. Again, the use of lumped mass (rather than structural mass) appears to be a more efficient way of controlling frequencies.

7. NEAR FUTURE PLANS

At this writing, current research effort is concentrated on developing a generic cross-section sufficiently general that both the bending and torsional stiffnesses of currently existing blades can be matched. When a suitable generic section has been attained, it will be used in studies of the optimization of a non-rotating cantilever experiencing coupled flap, lag, and torsional vibrations. Some preliminary studies of torsional and in-plane vibration have already been done.

Preliminary work has also been done on the feasibility of using an objective function involving the sum of the squares of frequencies, rather than the weight. The results look promising, and this approach will be pursued, especially if difficulties arise with the weight-objective function approach, as more complex problems are analyzed.

Another topic of research in the immediate future will be the inclusion of a stress constraint in the problem of the optimization of a non-rotating cantilever. Preliminary examination of present optimum designs show no particular stress problems; but, in principle, the stress constraint should be necessary to prevent elimination of too much material from blade designs.

8. REFERENCES

1. Niebank, C. and Girvin, W., "Sikorsky S-76 Analysis, Design and Development for Successful Dynamic Characteristics." Proceedings of the 34th Annual Natural Forum of the American Helicopter Society, May 1978, pp 78-23-1 through 78-23-17.

2. Hirsh, Harold; Dutton, Robert E., and Rasumoff, Abner, "Effect of Spanwise and Chordwise Mass Distribution on Rotor Blade Cyclic Stresses," Journal of the American Helicopter Society, Vol. 1, No. 2, April 1956.

3. Ellis, C. W. et al., "Design, Development, and Testing of the Boeing Vertol/Army YUH-61A," Proceedings of the 32nd Annual National Forum of the American Helicopter Society, Preprint 1010, May 1976.

4. Fenanghty, Ronald R. and Noehren, William L., "Composite Bearingless Tail Rotor for UTTAS," Journal of the American Helicopter Society, Vol. 22, No. 3, July 1977, pp 19-26.

5. Hanson, H.W. and Calapodas, M.J., "Evaluation of the Practical Aspects of Vibration Reduction Using Structural Optimization Techniques," Proceedings of the 35th Annual National Forum of the American Helicopter Society, May 1979, pp 79-21-1 through 79-21-12.

6. Vanderplaats, G.N., CONMIN - A Fortran Program for Constrained Function Minimization. User's Manual, NASA TMX-62.282, August , 1973.

7. Zoutendijk, G., Method of Feasible Directions, Elsevier Publishing Co., Amsterdam, 1960.

8. Peters, D.A. An Approximate Solution for the Free Vibrations of Rotating Uniform Cantilever Beams, NASA TMX-62.299, August, 1978.

9. Thomson, W. T., Mechanical Vibrations, Prentice-Hall, Inc., New York, 1953, pp 36 and 217.

9. APPENDICES

APPENDIX I

GRADIENTS OF OBJECTIVE FUNCTION AND CONSTRAINTS

When analytical gradients are utilized in CONMIN, the following derivations are necessary.

Let Z = objective function

g_j = j^{th} frequency constraint function

t_i = i^{th} element thickness

I_i = i^{th} element moment of inertia

λ_i = i^{th} eigenvalue of the vibration problem

1) To find $\frac{\partial Z}{\partial I_i}$ we employ the chain rule

$$\frac{\partial Z}{\partial I_i} = \frac{\partial Z}{\partial t_i} \times \frac{\partial t_i}{\partial I_i} = \frac{\partial Z}{\partial t_i} / \frac{\partial I_i}{\partial t_i} \tag{1}$$

Since $$Z = \sum_{i=1}^{10} t_i \tag{2}$$

and, $$I_i = \frac{25}{96} + \frac{1}{24}\{285t_i - 228t_i^2 + 60.8t_i^3\} \tag{3}$$

Then, $$\frac{\partial Z}{\partial I_i} = 1\{\frac{1}{24}(285 - 456t_i + 182.4t_i^2\}^{-1} \tag{4}$$

Furthermore, $t_i = Q_{(I_i)}$

Finally, $$\frac{\partial Z}{\partial I_i} = [\frac{1}{24}\{285 - 456Q_{(I_i)} + 182.4Q_{(I_i)}^2\}]^{-1} \tag{5}$$

2) To find $\frac{\partial g_i}{\partial I_K}$ we also employ the chain rule

$$\frac{\partial g_i}{\partial I_K} = \frac{\partial g_i}{\partial f_i} \times \frac{\partial f_i}{\partial \lambda_i} \times \frac{\partial \lambda_i}{\partial I_K} \tag{6}$$

For constraints that are linear functions of frequency,

$$\frac{\partial g_i}{\partial f_i} = +1, -1, \text{ or } 0 \qquad (7)$$

Since $f_i = \frac{1}{2\pi} \lambda_i^{1/2}$

$$\frac{\partial f_i}{\partial \lambda_i} = \frac{1}{4\pi\sqrt{\lambda_i}} \qquad (8)$$

$\frac{\partial \lambda_i}{\partial I_K}$ is obtained as a function of the eigenvectors, mass matrix, and stiffness matrix of the problem [1]

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(1) Fox, R. L., Optimization Methods for Engineering Design, Addison-Wesley, Reading, PA, 1971.

APPENDIX II

GRADIENTS OF THE AUTO-ROTATATIONAL CONSTRAINT

For a system of uniform elements and lumped weights, the mass moment of inertia, I_m, can be written as:

$$I_m = \sum_{\text{all elements}} \{\int r^2 dm_i\} + \sum_{\text{all weights}} \{r_i^2 m_i\}$$

$$= \Sigma \{\int_{r_i}^{r_i} \rho A_i r^2 dr\} + \Sigma \{\frac{W_i}{2g} (r_U^2 + r_L^2)\} \qquad (1)$$

$$= \sum_{i=1}^{10} \{ \rho \frac{A_i}{3} (r_U^3 - r_L^3)\} + \sum_{i=1}^{10} \frac{W_i}{2g} (r_U^2 + r_L^2)$$

where, ρ = mass density

A_i = area of ith element = $0.5 + 7.6\ t_i$

W_i = weight of ith lump

g = gravitational constant

r_U, r_L = radius from beam root to the upper and lower end of the element respectively. (Note that half of the lumped weight has been placed at the lower end and upper end of the element).

1) To find $\frac{\partial g_7}{\partial I_i}$ we employ the chain rule

$$\frac{\partial g_7}{\partial I_i} = - \frac{1}{I_{min}} (\frac{\partial I_m}{\partial I_i}) \qquad (2)$$

but,

$$\frac{\partial I_m}{\partial I_i} = \frac{\partial I_m}{\partial A_i} \times \frac{\partial A_i}{\partial t_i} \times \frac{\partial t_i}{\partial I_i}$$

$$= \{ \frac{\rho}{3} (r_U^3 - r_L^3)\} \times (7.6) \times \{\frac{285-456t_i+182.4t_i^2}{24}\}^{-1} \qquad (3)$$

2) To find $\frac{\partial g_7}{\partial W_i}$, direct differentiation can be employed

$$\frac{\partial g_7}{\partial W_i} = \frac{r_U^2 + r_L^2}{2g} \qquad (4)$$

APPENDIX III

MODIFICATIONS TO ACCOMMODATE ELEMENTS IN TENSION

For an element under constant tension, T, the potential energy is given by

$$U = \frac{T}{2} \int_0^L \left(\frac{\partial W}{\partial X}\right)^2 dX \tag{1}$$

where W represents the vertical deflection coordinate in the Y direction, as shown in Fig. 2.

From Ref. 2, the displacement function for W is given by:

$$W(X) = \frac{V_1}{L^3} (2X^3 - 3LX^2 + L^3) + \frac{V_2}{L^2}(X^3 - 2LX^2 - L^2X)$$

$$+ \frac{V_3}{L^3} (3LX^2 - 2X^3) + \frac{V_4}{L^2}(X^3 - LX^2) \tag{2}$$

where, the nodal translations are V_1 and V_3, and the nodal rotations are V_2 and V_4.

Substituting the derivatives of (2) into (1), performing the integration over the length, and comparing the results to $U = \frac{1}{2}\{V\}^T [K_{(T)}] \{V\}$ leads to the identification of the <u>added</u> contribution to the element stiffness matrix, $[K_{(T)}]$

$$[K_{(T)}] = T \begin{bmatrix} 1.2/L & 0.1 & -1.2/L & 0.1 \\ & 2L/15 & -0.1 & -L/30 \\ \text{Symm.} & & 1.2/L & -0.1 \\ & & & 2L/15 \end{bmatrix} \tag{3}$$

The element tensions are easily defined by computing the centrigual force for each lumped mass, and accumulating the total tension from tip to root of

(2) Peters, D.A, Ko, T., Korn, A., and Rossow, M.P., First Semi-Annual Status Report on Design of Helicopter Rotor Blades for Optimum Dynamic Characteristics, NASA-Langley Grant No. NAG-1-250, Sept. 15, 1982.

the beam. To keep a constant tension in each element, the distributed and lumped weights were divided by two and placed at the element nodes.

To compute analytical gradients, the partial derivatives of $K_{(T)}$ with respect to the decision variables are needed.

1) To find $\frac{\partial K_{(T)}}{\partial I_K}$ for any element, the chain rule is used:

$$\frac{\partial K}{\partial I} = \frac{\partial K}{\partial T} \times \frac{\partial T}{\partial t} \times \frac{\partial t}{\partial I} \qquad (4)$$

The first term represents the matrix elements of Eqn. 3.

The middle term can be obtained by writing the element tension as the cumulated sum of $m_i \Omega^2 r_i$ terms, where, Ω = rotational speed in rad/sec. Since each mass is a function of thickness, the derivative of tension with respect to thickness can be obtained.

The final term has been defined in Eqn. 3 of Appendix II.

2) To find $\frac{\partial K_{(T)}}{\partial W_K}$ for any element, we use:

$$\frac{\partial K_{(T)}}{\partial W_K} = \frac{\partial K_{(T)}}{\partial T} \times \frac{\partial T}{\partial W_K} \qquad (5)$$

The second term can be obtained by writing the element tensions caused by the lumped masses, and then taking the appropriate partial derivatives.

10. Tables

Initial Values

Constant t inches: 0.05

Constant I $inches^4$: 0.831

(a) Objective Function, inches: 0.500

(b) Weight lbf: 21.12

f_1, Hz: 1.86

f_2, Hz: 11.6

Optimal Solution

(c) Analytical Gradients		(c) Finite Difference Gradients
15	No. of Iterations	21
0.540	Objective Function inches	0.539
21.85	Weight lbf	21.83
2.00	f_1 Hz	1.99
12.20	(d) f_2 Hz	12.20
1.250(o.0896)	I_1 $inches^4$ (t_1 inches)	1.253(0.0899)

(Elements 2 through 10 were essentially all at the lower bound value of side constraint on thickness: t = 0.05 inches, I = 0.831 $inches^4$)

Notes:

(a) Objective Function, inches $= \sum_{i=1}^{10} t_i$

(b) Weight lbf $= 12 + 18.24 \sum_{i=1}^{10} t_i$

(c) Convergence criteria were the same, except that ITRM was raised from 3 to 8 cycles when finite differences were used.

(d) Both solutions have an active frequency constraint: f_2 is at its lower bound value.

Table IA: Comparison of Analytical Gradient vs Finite Difference Solution

Initial Values

Constant t inches	:	0.10
Constant I $inches^4$	:	1.355
(a) Objective Function inches	:	1.000
(b) Weight lbf	:	30.24
f_1 Hz	:	1.98
f_2 Hz	:	12.4

Optimal Solution

(c) Analytical Gradients		(c) Finite Difference Gradients
15	No. of Iterations	18
0.541	Objective Function inches	0.543
21.87	Weight, lbf	21.90
1.94	f_1, Hz	1.94
12.20	(d) f_2, Hz	12.20
1.262(0.0908)	I_1 $inches^4$ (t_1 inches)	1.214(0.0861)
0.831(0.0500)	I_6 $inches^4$ (t_6 inches)	0.852(0.0519)
0.831(0.0500)	I_7 $inches^4$ (t_7 inches)	0.880(0.0545)

(Elements 2-5 and 8-10 were all at the lower bound value of side constraint (thickness): t = 0.05 inches, I = 0.831 $inches^4$)

Notes:

(a) Objective function, inches = $\sum_{i=1}^{10} t_i$

(b) Weight, lbf = $12 + 18.24 \sum_{i=1}^{10} t_i$

(c) Convergence criteria for the two runs were identical.

(d) Both Solutions have an active frequency constraint: f_2 is at it lower bound value.

Table 1B: Comparison of Analytical Gradient vs Finite Difference Solution

Initial Values

Constant t inches	:	0.05
Constant I $inches^4$	:	0.831
(a) Objective function inches	:	0.500
(b) Weight lbf	:	21.12
f_1 Hz	:	1.86
f_2 Hz	:	11.6

Optimal Solution

(c) Eigenvalue Constraints		(c) Frequency Constraints
12	No. of Iterations	21
0.551	Objective function inches	0.539
22.05	Weight lbf	21.83
1.94	f_1, Hz	1.99
12.20	(d) f_2, Hz	12.20
1.150(0.0799)	I_1, $inches^4$ (t_1 inches)	1.253(0.0899)
0.846(0.0514)	I_5, $inches^4$ (t_5 inches)	0.831(0.0500)
0.910(0.0573)	I_6, $inches^4$ (t_6 inches)	0.831(0.0500)
0.922(0.0584)	I_7, $inches^4$ (t_7 inches)	0.831(0.0500)
0.875(0.0541)	I_8, $inches^4$ (t_8 inches)	0.831(0.0500)

(Elements 2-4, 9, 10 were all at the lower bound value of side constraint on thickness: t = 0.05 inches, I = 0.831 $inches^4$)

Notes:

(a) Objective function, inches $= \sum_{i=1}^{10} t_i$

(b) Weight, lbf $= 12 + 18.24 \sum_{i=1}^{10} t_i$

(c) Convergence criteria for the two runs were identical. Gradients were computed by analytical techniques for the eigenvalue constrained run, whereas finite difference techniques were used for the frequency constrained run.

(d) Both solutions have an active frequency constraint: f_2 is at its lower bound value.

Table 2: Comparison of Eigenvalue and Frequency Constraints

Initial Values	Case 1	Case 2	Case 3 (f)
Constant t, inches	0.05	0.10	1.101
Constant I, inches^4	0.831	1.355	5.200
(a) Objective function inches	0.500	1.000	11.013
(b) Weight, lbf	21.12	30.24	212.87
f_1, Hz	1.86	1.98	1.46 (e)
f_2, Hz	11.6 (e)	12.4	9.17 (e)
Optimal Solution (c)			
No. of Iterations	15	15	18
Objective function inches	0.540	0.541	0.547
Weight lbf	21.85	21.87	21.98
f_1, Hz	2.00	1.94	1.92
(d) f_2, Hz	12.20	12.20	12.26
I_1 inches^4(t_1 inches)	1.250(0.0896)	1.262(0.0908)	1.327(0.0972)

(Elements 2 through 10 were essentially all at the lower bound value of the side constraint on thickness: t = 0.05 inches, I = 0.831 inches^4)

Notes:

(a) Objective function, inches $= \sum_{i=1}^{10} t_i$

(b) Weight, lbf $= 12 + 18.24 \sum_{i=1}^{10} t_i$

(c) All gradients were computed by analytical techniques.

(d) All solutions have active frequency contraint: f_2 is at its lower bound value.

(e) Initial frequency violates the constraint.

(f) In this case, a high value of the parameter DABFUN led to premature designation of convergence. The solution is nearly optimal, but the objective function was still changing in the 3rd significant figure.

Table 3: Comparison of Solutions Obtained for Various Initial Designs

Initial Values

Constant t, inches	:	0.10
Constant I, inches^4	:	1.355
Objective function, inches	:	1.000
Weight, lbf	:	30.24
f_1, Hz	:	1.98
f_2, Hz	:	12.4

Objective Function

(a) Using Default Parameters	(c)	(b) Using Tighter Convergence Criteria
0.54279	Iteration 10	0.54279
0.54212	Iteration 11	0.54279 (No change)
0.54076	Iteration 12	0.54083
0.54076 (No change)	Iteration 13	0.54083 (No change)
0.54076 (No change)	Iteration 14	0.54083 (No change)
0.54076 (No change)	Iteration 15	0.54072
	Iteration 16	0.54072 (No change)
	Iteration 17	0.54072 (No change)

Optimal Solution

15	No. of iterations	17
0.54076	Objective function inches	0.54072
21.87	Weight, lbf	31.87
1.94	f_1, Hz	1.96
12.20	f_2, Hz	12.20
1.2619(0.09076)	I_1 inches^4 (t_1 inches)	1.2615(0.09073)

Notes:

(a) Default values: ITRM = 3
DELFUN = 0.0001
DABFUN = 0.001

(b) Tighter Criteria: ITRM = 5
DELFUN = 0.00005
DABFUN = 0.0005

(c) The first ten iterations were identical.

Table 4: Convergence History for Two Different Convergence Criteria

θ [(a)]	No. of Iterations	Final Objective Function	I_1 inches^4 (t_1 inches)	f_1, Hz [(b)]
1	15	0.540	1.262(0.0908)	1.94
10	19	0.546	1.315(0.0960)	2.01
100	13	0.552	1.308(0.0953)	2.03
700	26	0.549	1.190(0.0838)	2.06

Notes:

(a) For all runs, Default Parameters were used:

PHI = 5.0

ITRM = 3

DELFUN = 0.0001

DABFUN = 0.001

(b) The second frequency, f_2, was at the lower bound constraint (f_2 = 12.2 Hz).

(c) The initial design was based on constant thickness, constant moment of inertia of 0.10 inches and 1.355 inches^4 respectively.

Table 5: The Effect of Parameter THETA on the Optimal Solution

Initial Values

Constant t, inches	:	0.25
Constant I, inches^4	:	2.675
(a) Objective function, inches	:	2.50
(b) Weight, lbf	:	57.60
f_1, Hz	:	2.02
f_2, Hz	:	12.6

Optimal Solutions (c)

	Case 4	Case 5	Case 6
	f_{1L} = 1.9 Hz	f_{1L} = 1.5 Hz	f_{1L} = 2.1 Hz
	f_{1U} = 2.1	f_{1U} = 1.9	f_{1U} = 2.5
	f_{2L} = 12.3	f_{2L} = 12.8	f_{2L} = 11.6
	f_{2U} = 12.5	f_{2U} = 13.2	f_{2U} = 12.0
No. of iterations	32	29	28
Objective function, inches	0.552	0.662	0.593
Weight, lbf	22.07	24.08	22.81
f_1, Hz	2.09 (e)	1.87	2.11 (d)
f_2, Hz	12.32 (d)	12.80 (d)	11.96
t_1, inches	0.1022	0.1122	0.0753
t_2, inches	0.0500 (f)	0.0500 (f)	0.0703
t_3, inches	↑	↑	0.0886
t_4, inches		↓	0.0586
t_5, inches		0.0500 (f)	0.0500 (f)
t_6, inches		0.0623	↑
t_7, inches		0.0791	
t_8, inches		0.1059	
t_9, inches	↓	0.0515	↓
t_{10}, inches	0.0500 (f)	0.0514	0.0500 (f)

Notes:

(a) Objective function, inches = $\sum_{i=1}^{10} t_i$ (b) Weight, lbf = $12 + 18.24 \sum_{i=1}^{10} t_i$

(c) ITMAX = 40, ITRM = 5, DELFUN = 0.0001, DABFUN = 0.0025 for case 5 and 6. For Case 4, ITRM = 3.

(d) Active lower bound contraint. (e) Active upper bound constraint.

(f) Active side constraint.

Table 6: Placement of Frequencies

Initial Values	Case 7	Case 8A	Case 8B
Constant t, inches	0.10	0.10	0.05
Constant I, inches^4	1.355	1.355	0.831
(a) Objective function, inches	1.000	1.000	0.500
(b) Weight, lbf	30.24	30.24	21.12
f_1, Hz	1.98	1.98	1.86
f_2, Hz	12.4	12.4	11.6
f_3, Hz	34.8	34.8	32.7
ITMAX	40	80.	
ITRM	3	5.	
DELFUN	0.0001	0.0001	
DABFUN	.001	0.0005	
f_{1L}, f_{1U}, Hz	1.8,2.2	2.1,2.5	
f_{2L}, f_{2U}, Hz	12.2,12.6	11.6,12.0	
f_{3L}, f_{3U}, Hz	34.6,35.0	33.6,34.0	
(c) Optimal Solution			
No. of iterations	20	57	44
Objective function inches	0.579	0.623	0.611
Weight lbf	22.56	23.37	23.13
f_1, Hz	2.02	2.11	2.10 (d)
f_2, Hz	12.51	12.00 (e)	12.00 (e)
f_3, Hz	(d) 34.60	33.62	33.60 (d)
t_1 inches	0.1178	0.0906	0.0884
t_2 inches	(f) 0.0500	0.0560	0.0526
t_3 inches	↑	0.0771	0.0744
t_4 inches	↕	0.0995	0.0951
t_5 inches	↓	0.0500 (f)	0.0500 (f)
t_6 inches	(f) 0.0500	↑	↑
t_7 inches	0.0501	↕	↕
t_8 inches	0.0608	↕	↕
t_9 inches	(f) 0.0500	↓	↓
t_{10} inches	(f) 0.0500	0.0500 (f)	0.0500 (f)

(continued)

Notes:

(a) Objective function, inches = $\sum_{i=1}^{10} t_i$

(b) Weight, lbf = $12 + 18.24 \sum_{i=1}^{10} t_i$

(c) All gradients were computed by analytical techniques

(d) Active lower bound constraint

(e) Active upper bound constraint

(f) Active side constraint

Table 7A: Optimization with Three Frequency Constraints

Initial Values	Case 9A	Case 9B	Case 9C
Constant t, inches	0.10	0.10	0.25
Constant I, inches^4	1.355	1.355	2.675
(a) Objective function, inches	1.000	1.000	2.500
(b) Weight, lbf	30.24	30.24	57.60
f_1, Hz	1.98	1.98	2.02
f_2, Hz	12.4	12.4	12.6
f_3, Hz	34.8	34.8	35.5
ITMAX	40	40	80
ITRM	3	5	5
DELFUN	0.0001	0.0001	0.0001
DABFUN	0.001	0.0005	0.0005
f_{1L}, f_{1U}, Hz	←	1.5, 1.9	→
f_{2L}, f_{2U}, Hz	←	12.8, 13.2	→
f_{3L}, f_{3U}, Hz	←	35.6, 36.0	→
(c) Optimal Solution			
No. of iterations	15	25	47
Objective function inches	0.697	0.684	0.684
Weight, lbf	24.72	24.47	24.47
f_1, Hz	1.87 (e)	1.86	1.85
f_2, Hz	12.80 (d)	12.83	12.82
f_3, Hz	35.70	35.79	35.73
t_1, inches	0.0927	0.1025	0.1074
t_2, inches	0.0500 (f)	0.0500 (f)	0.0500 (f)
t_3, inches	↕	↕	↕
t_4, inches	0.0500 (f)	↕	↕
t_5, inches	0.0637	0.0500 (f)	0.0500 (f)
t_6, inches	0.0827	0.0705	0.0621
t_7, inches	0.0917	0.0943	0.0808
t_8, inches	0.1064	0.1166	0.1335
t_9, inches	0.0602	0.0500 (f)	0.0500 (f)
t_{10}, inches	0.0500 (f)	0.0500 (f)	0.0500 (f)

(continued)

Notes:

(a) Objective function, inches $= \sum_{i=1}^{10} t_i$

(b) Weight, lbf $= 12 + 18.24 \sum_{i=1}^{10} t_i$

(c) All gradients were computed by analtyical techniques

(d) Active lower bound constraint

(e) Active upper bound constraint

(f) Active side constraint

Table 7B: Optimization with Three Frequency Constraints

Initial Values	Case 10	Case 11A	Case 11B
Constant t, inches	←	0.10	→
Constant I, inches[4]	←	1.355	→
Density, γ, lbm/inches[3]	0.05	0.10	0.10
Constant W_1, lbf	1.512	1.000	0.000
(a)Objective function, lbf	24.24	28.24	18.24
(b)Weight, lbf	30.24	40.24	30.24
f_1, Hz	1.98	1.66	1.98
f_2, Hz	12.4	10.7	12.4
f_3, Hz	34.8	29.9	34.8
f_{1L}, f_{1U}, Hz	←	1.8,2.2	→
f_{2L}, f_{2U}, Hz	←	12.2,12.6	→
f_{3L}, f_{3U}, Hz	←	34.6,35.0	→
(c)Optimal Solution			
No. of iterations	53	38	19
Objective function lbf	9.252	10.557	10.557
Weight lbf	15.25	22.56	22.56
f_1, Hz	2.12	1.96	2.03
f_2, Hz	12.60(e)	12.51	12.51
f_3, Hz	35.00(e)	34.60(d)	34.60(d)
t_1 inches and W_1 lbf	0.0500(f) & 0(f)	0.1150 & 0(f)	0.1147 & 0(f)
t_2 inches and W_2 lbf	↑ 0(f)	0.0500(f) ↑	0.0500(f) ↑
t_3 inches and W_3 lbf	0.056	↑	↑
t_4 inches and W_4 lbf	2.214		
t_5 inches and W_5 lbf	0.001	↓	↓
t_6 inches and W_6 lbf	0(f)	0.0500(f)	0.0500(f)
t_7 inches and W_7 lbf	1.194	0.0535	0.0525
t_8 inches and W_8 lbf	0(f)	0.0618	0.0616
t_9 inches and W_9 lbf	↓ 0(f)	0.0500(f) ↓	0.0500(f) ↓
t_{10} inches and W_{10} lbf	0.0500(f) 1.227	0.0500(f) & 0(f)	0.0500(f) & 0(f)

(continued)

Notes:

(a) Objective function, lbf = $182.4 \gamma \sum_{i=1}^{10} t_i + \sum_{i=1}^{10} W_i$

(b) Weight, lbf = 120γ + objective function

(c) All gradients were computed by analytical techniques

For all cases, ITMAX = 80, ITRM = 5, DELFUN = 0.0001, DABFUN = 0.001

(d) Active lower bound constraint

(e) Active upper bound constraint

(f) Active side constraint

Table 8A: Optimization Including Lumped Weights

Initial Values	Case 12		
Constant t, inches	0.10		
Constant I, inches^4	1.355		
Density, γ, lbm/inches^3	0.10		
Constant W_i, lbf	1.000		
(a) Objective function, lbf	28.24		
(b) Weight, lbf	40.24		
f_1, Hz	1.66		
f_2, Hz	10.7		
f_3, Hz	29.9		
f_{1L}, f_{1U}, Hz	2.6, 3.0		
f_{2L}, f_{2U}, Hz	10.5, 11.0		
f_{3L}, f_{3U}, Hz	31.6, 32.0		
(c) Optimal Solution			
No. of iterations	46		
Objective function, lbf	23.697		
Weight, lbf	35.70		
f_1, Hz	2.60 (d)		
f_2, Hz	10.99		
f_3, Hz	31.99		
t_1 inches and W_1, lbf	0.1360	&	0.527
t_2 inches and W_2, lbf	0.1725		0.435
t_3 inches and W_3, lbf	0.1976		0.426
t_4 inches and W_4, lbf	0.2053		0.839
t_5 inches and W_5, lbf	0.0769		1.395
t_6 inches and W_6, lbf	0.0500 (f)		1.135
t_7 inches and W_7, lbf	↑		0 (f)
t_8 inches and W_8, lbf			↑
t_9 inches and W_9, lbf	↓		↓
t_{10} inches and W_{10}, lbf	0.0500 (f)		0 (f)

(continued)

Notes:

(a) Objective function, lbf = $182.4\gamma \sum_{i=1}^{10} t_i + \sum_{i=1}^{10} W_i$

(b) Weight, lbf = 120γ + Objective function

(c) All gradients were computed by analytical techniques

ITMAX = 80, ITRM = 5, DELFUN = 0.0001, DABFUN = 0.001

(d) Active lower bound constraint

(e) Active upper bound constraint

(f) Active side constraint

Table 8B: Optimization Including Lumped Weights

Initial Values	Case 10		Case 13	
Constant t inches, I inches^4, γ lbm/inches^3	0.10, 1.355, 0.05			
Constant W_1, lbf.	1.512			
(a) Objective function, lbf	24.24			
(b) Weight, lbf	30.24			
f_1, Hz	1.98			
f_2, Hz	12.4			
f_3, Hz	34.8			
f_{1L}, f_{1U}, Hz	1.8,2.2			
f_{2L}, f_{2U}, Hz	12.2,12.6			
f_{3L}, f_{3U}, Hz	34.6,35.0			
I_{min} lbf inches sec^2	0.		1100	
(c) Optimal Solution				
No. of iterations	53.		80	
Objective function lbf	9.252		11.596	
Weight lbf	15.25		17.60	
f_1, Hz	2.12		1.80 (d)	
f_2, Hz	12.60 (e)		12.59	
f_3, Hz	35.00 (e)		35.00 (e)	
I_m lbf inches sec^2	807.7		1144 (g)	
t_1 inches and W_1 lbf	0.0500 (f) &	0 (f)	0.0500 (f) &	0(f)
t_2 inches and W_2 lbf	↑	0 (f)	0.0869	0(f)
t_3 inches and W_3 lbf		0.056	0.0757	0.131
t_4 inches and W_4 lbf		2.214	0.0500(f)	0.516
t_5 inches and W_5 lbf		0.001		0.235
t_6 inches and W_6 lbf		0 (f)		0.142
t_7 inches and W_7 lbf		1.194		0.719
t_8 inches and W_8 lbf		0 (f)		1.175
t_9 inches and W_9 lbf	↓	0 (f)	0.0500(f)	1.174
t_{10} inches and W_{10} lbf	0.0500(f)	1.227	0.1094	1.833

(continued)

Notes:

(a) Objective function, lbf = $182.4\gamma \sum_{i=1}^{10} t_i + \sum_{i=1}^{10} W_i$

(b) Weight lbf = 120γ + objective function

(c) All gradients were computed by analytical techniques

For all cases, ITMAX = 80, ITRM = 5, DELFUN = 0.0001, DABFUN = 0.001

(d) Active lower bound constraint

(e) Active upper bound constraint

(f) Active side constraint

(g) Very close to being active auto-rotational constraint

Table 9: Optimization Including Auto-Rotational Constraint

γ = 0.05 lbm/inches^3, t = 0.10 inches	Ω = 0 RPM	Ω = 300 RPM
(a) W_1 = 1.5 lbf	f_1 = 2.0 Hz	f_1 = 5.7 Hz
(a) W_1 = 100 lbf	f_1 = 0.3 Hz	f_1 = 5.1 Hz

γ = 0.05 lbm/inches^3, t = 1.25 inches	Ω = 0 RPM	Ω = 300 RPM
(a) W_1 = 1.5 lbf	f_1 = 1.4 Hz	f_1 = 5.5 Hz
(a) W_1 = 100 lbf	f_1 = 0.7 Hz	f_1 = 5.2 Hz

Note: (a) Lumped weights were assumed to be uniformly distributed for purposes of computation.

Table 10: Influence of Rotational Speed, Mass, and Stiffness on the Fundamental Frequency of a Uniform Cantilever

Initial Values	Case 14		Case 15		Case 16	
Constant, t, inches, I inches^4, γ lbf/cu.in	← 0.10,1.355,0.05 →					
Constant W_1, lbf	← 1.512 →					
(a) Objective function, lbf.	← 24.24 →					
(b) Weight, lbf	← 30.24 →					
f_{1L}, f_{1U}, Hz	← 1.8,2.2 →					
f_{2L}, f_{2U}, Hz	← 12.2,12.6 →					
f_{3L}, f_{3U}, Hz	← 34.6,35.0 →					
I_{min}, lbf inches sec^2	← 500 →					
Ω, RPM	0		30		100	
(c) Optimal Solution						
No. of iterations	68		76		80	
Objective function, lbf	9.252		9.303		16.718	
Weight, lbf	15.25		15.30		22.72	
f_1, Hz	2.12		2.16		2.20(e)	
f_2, Hz	12.60(e)		12.60(e)		12.59	
f_3, Hz	35.00(e)		35.00(e)		34.97	
I_m, lbf inches sec^2	796.7		811.7		1957	
t_1, inches and W_1, lbf	0.0500(f) &	0 (f)	0.0500(f) &	0 (f)	0.0500(f) &	0(f)
t_2, inches and W_2, lbf	↕	0 (f)	↕	0 (f)	↕	↓
t_3, inches and W_3, lbf	↕	0.022	↕	0 (f)	↕	↓
t_4, inches and W_4, lbf	↕	2.377	↕	2.345	↕	0(f)
t_5, inches and W_5, lbf	↕	0 (f)	↕	0 (f)	↕	1.00[illegible]
t_6, inches and W_6, lbf	↕	0 (f)	↕	0 (f)	↕	0 (f)
t_7, inches and W_7, lbf	↕	1.145	↕	1.121	↕	0.40[illegible]
t_8, inches and W_8, lbf	↕	0 (f)	↕	0 (f)	↓	1.44[illegible]
t_9, inches and W_9, lbf	↓	0 (f)	↓	0 (f)	0.0500(f)	0(f)
t_{10}, inches and W_{10}, lbf	0.0500(f)	1.149	0.0500(f)	1.277	0.3685	6.39[illegible]

(continued)

Notes:

(a) Objective function lbf = $182.4\gamma \sum_{i=1}^{10} t_i + \sum_{i=1}^{10} W_i$

(b) Weight lbf = 120γ + objective function

(c) All gradients were computed by analytical techniques

Eigenvalues were computed with double precision routines

For all cases, ITMAX = 80, ITRM = 5, DELFUN = 0.0001, DABFUN = 0.001

(d) Active lower bound constraint

(e) Active upper bound constraint

(f) Active side constraint

(g) Active auto-rotational constraint

Table 11A: Optimization of Rotating Beam

Initial Values		Case 17		
Constant t, inches, I $inches^4$, γ lbm/cu.in.		0.10,1.355,0.05		
Constant W_1, lbf		1.512		
(a) Objective function, lbf.		24.24		
(b) Weight, lbf		30.24		
f_{1L}, f_{1U}, Hz		5.2,5.6		
f_{2L}, f_{2U}, Hz		18.0,18.6		
f_{3L}, f_{3U}, Hz		42.0,43.0		
I_{min} lbf inches sec^2		1100		
Ω RPM		300		
(c) Optimal Solution				
No. of iterations		39.		
Objective function lbf		10.539		
Weight lbf		16.54		
f_1, Hz		5.59 (e)		
f_2, Hz		18.59 (e)		
f_3, Hz		42.99 (e)		
I_m lbf inches sec^2		1113. (g)		
t_1, inches and W_1, lbf	0.0500 (f) ↑	&	0	(f)
t_2, inches and W_2, lbf	↕		0	(f)
t_3, inches and W_3, lbf	↕		0.002	
t_4, inches and W_4, lbf	↕		0.591	
t_5, inches and W_5, lbf	↕		0.712	
t_6, inches and W_6, lbf	↕		0	(f)
t_7, inches and W_7, lbf	↕		0.745	
t_8, inches and W_8, lbf	↕		0.438	
t_9, inches and W_9, lbf	↓		0	(f)
t_{10}, inches and W_{10}, lbf	0.0500 (f)		3.492	

(continued)

Notes:

(a) Objective function, lbf = $182.4\gamma \sum_{i=1}^{10} t_i + \sum_{i=1}^{10} W_i$

(b) Weight, lbf = 120γ + objective function

(c) All gradients were computed by analytical techniques

Eigenvalues were computed with double precision routines

For all cases, ITMAX = 80, ITRM = 3, DELFUN = 0.0001, DABFUN = 0.01

(d) Active lower bound constraint

(e) Active upper bound constraint

(f) Active side constraint

(g) Active auto-rotational constraint

Table 11B: Optimization of Rotating Beam

No. of Elements	Frequency: First Mode (Hz)	Error (%)	Second Mode (Hz)	Error (%)
6	1.9678	0.6	12.142	3.6
8	1.9733	0.3	12.258	2.7
10	1.9761	0.2	12.312	2.3
12	1.9780	0.1	12.342	2.0
14	1.9789	0.06	12.360	1.9
20	1.9805	0.03	12.385	1.7
Analytical	1.980		12.6	

Table 12: Frequency vs. Number of Elements

Frequency

(cps)

Mode No.	(1)	(2)	(3)	(4)	(5)
1	2.01622	1.95462	2.01614	2.01614	2.01511
2	12.6354	12.6354	12.6355		
3	35.3872	35.3916		35.3875	
4	69.3934			69.3937	
5	114.892			114.892	
6	172.121	Rest are essentially		172.122	rest are
7	241.495	identical to (1)		241.495	essentially
8	323.537			323.536	identical
9	418.329			418.329	to (1)
10	520.042			520.043	
11	692.152			692.155	
12	836.210			836.210	
13	1013.68			1013.68	
14	1223.45			1223.44	
15	1470.15			1470.15	
16	1758.30			1758.31	
17	2086.84			2086.85	
18	2436.29			2436.30	
19	2743.53			2743.53	
20	3433.47			3433.43	

Table 13: Frequencies Calculated by Various Methods

Element No.	Element Thicknesses (inches) Single-Precision Calculation	Double-Precision Calculation
1	0.0753	0.0724
2	0.0703	0.0881
3	0.0886	0.0821
4	0.0856	0.0564
5	0.0500	0.0500
6	0.0500	0.0500
7	0.0500	0.0500
8	0.0500	0.0500
9	0.0500	0.0500
10	0.0500	0.0500
Objective Function (LBF)	0.593	0.599

Table 14: Difference in Optimal Designs Caused by Errors in Eigenvalue Calculation

Element No.	(1) $\partial f_1/\partial t_i$ (cps/in.)	(2) $\partial f_1/\partial \bar{m}_i$ (cps/lbf)	(3) $\partial f_1/\partial m_i$ (cps/lbf)
1	4.791	0.525	0.000
2	3.492	0.383	-0.001
3	2.388	0.262	-0.003
4	1.472	0.161	-0.009
5	0.729	0.080	-0.022
6	0.105	0.012	-0.043
7	-0.443	-0.049	-0.073
8	-0.957	-0.105	-0.114
9	-1.489	-0.163	-0.166
10	-2.074	-0.227	-0.229

Table 15: Sensitivity of Frequency to Changes in Thickness and Weight

11. FIGURES

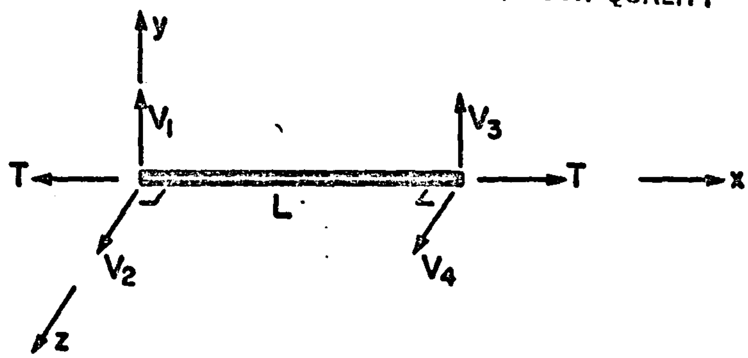

Figure 2: Element under Constant Tension.

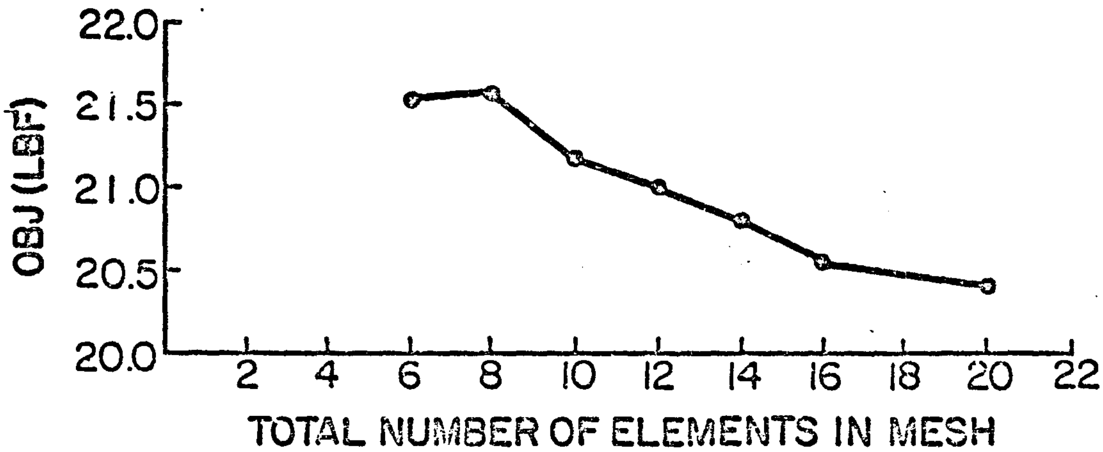

Figure 3: Obj vs. No. of Elements.

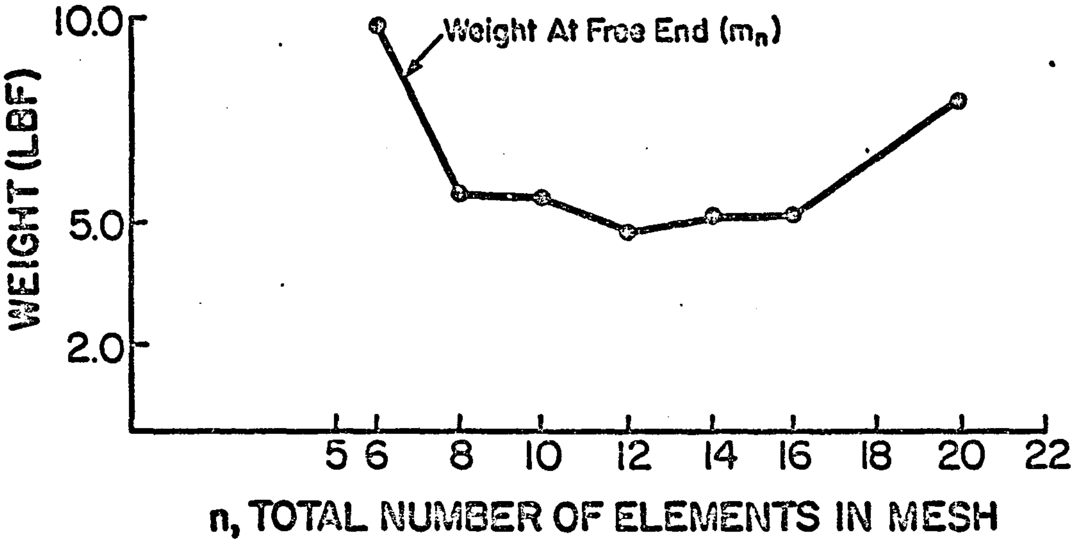

Figure 4: Lumped Weight Versus No. of Elements.

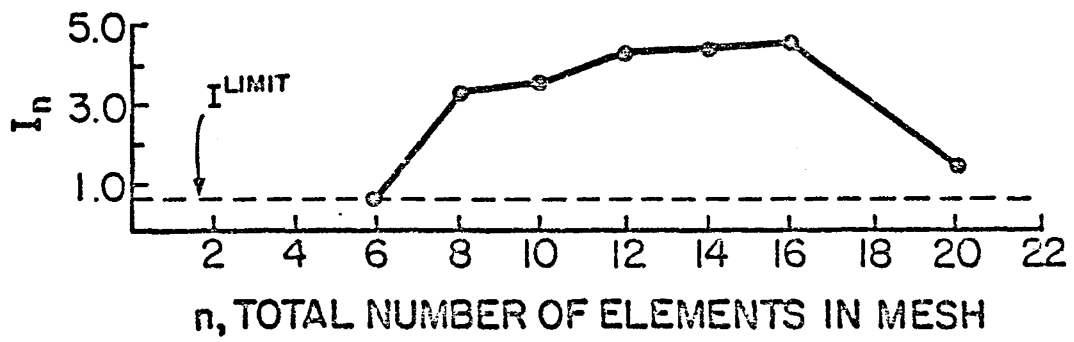

Figure 5: Moment of Inertia at Free End Versus No. of Elements in Mesh.

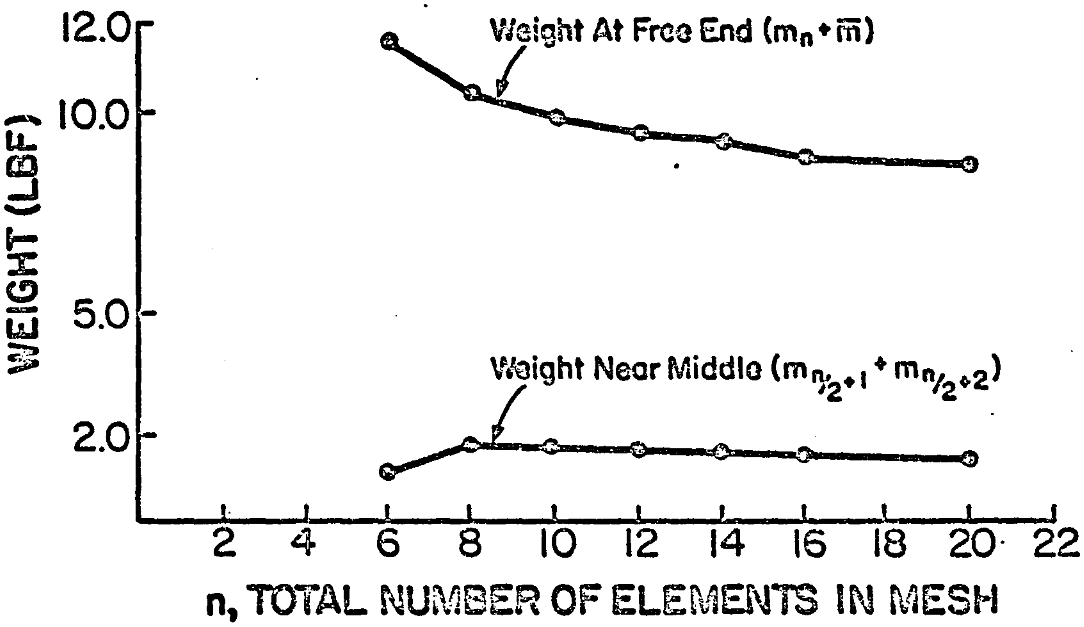

Figure 6: Combined Weight vs. No. of Elements.

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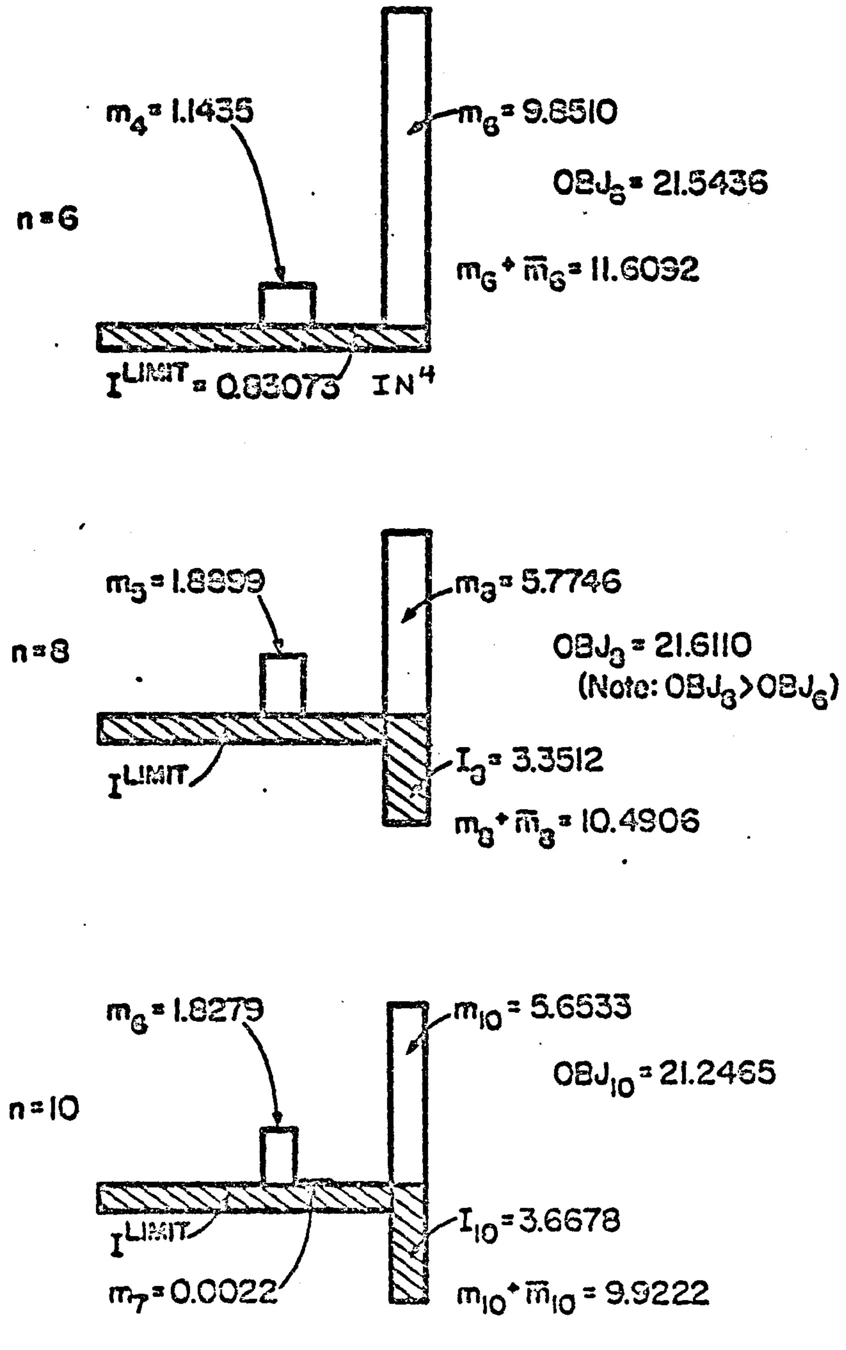
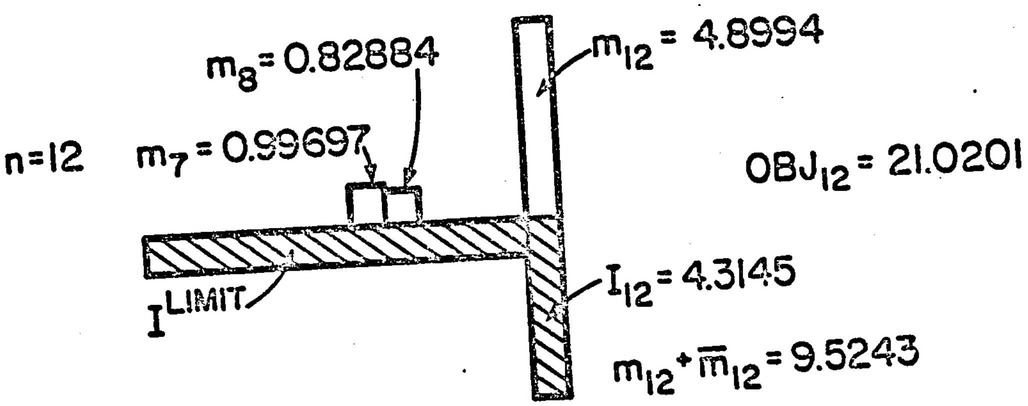


Figure 7: Optimal Designs for Various Values of n.



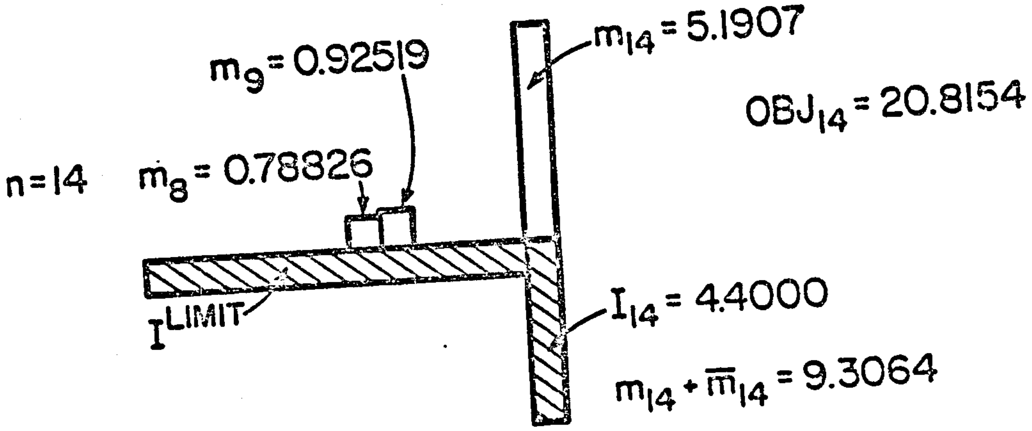
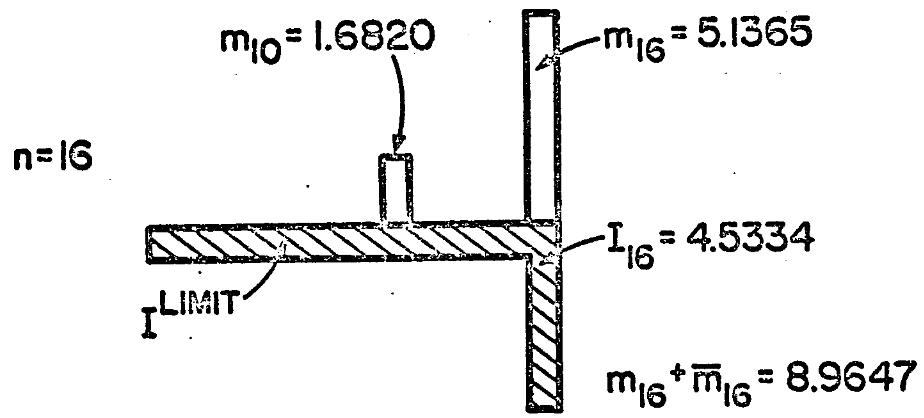


Figure 7: (Continued)



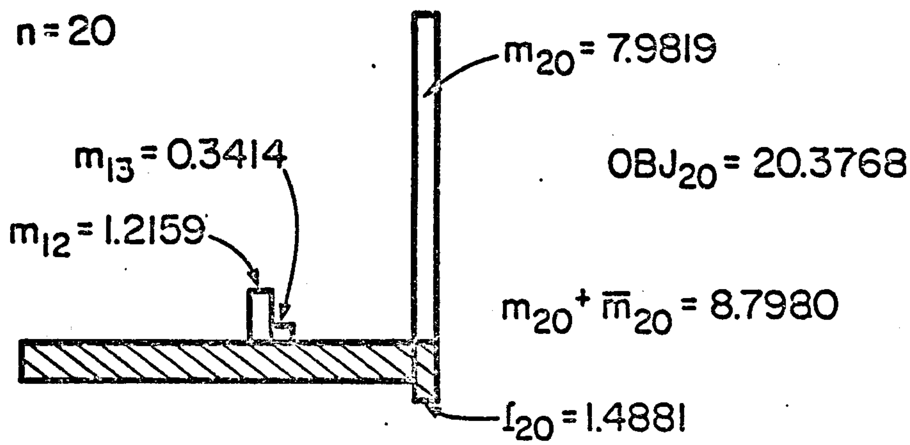


Figure 7: (Continued)

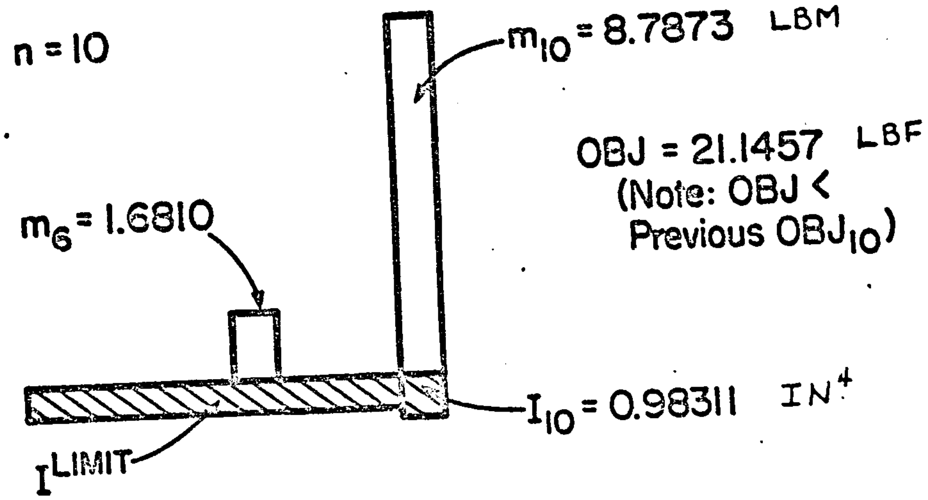


Figure 8: Alternative Optimum - Found by Imposing Constraint $0.83073 \leq I_{10} \leq 2.0$.

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