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March 22, 1983

Mr. J. F. Royall, Jr.
NASA-Langley Research Center
Mail Stop 126
Hampton, Virginia 23665

MAR 1983
RECEIVED
NASA STI FACILITY
ACCESS DEPT.

Re: NASA Grant NAG 1-158

Dear Mr. Royall:

Please find enclosed three copies of the End Of The Year Progress Report for the research grant, "The Effect Of Low Velocity Impact On The Strength Characteristics Of Composite Materials Laminates," NASA Grant NAG 1-158.

If additional information is needed, please do not hesitate to contact me.

Sincerely,

Harold Liebowitz
Dean

HL:jk

cc: Dr. J. R. Davidson, Fatigue and Fracture Branch, Head, NASA-LRC
Dr. Wolf Elber, Fatigue and Fracture Branch, NASA-LRC
Mr. R. H. Tolson, Chief Scientist, NASA-LRC
Mr. Edwin J. Prior, Office of the Director, NASA-LRC (Technical Officer)
Dr. J. L. Whitesides, GWU/JIAFS
NASA Scientific and Technical Information Facility
Post Office Box 8757
Baltimore and Washington International Airport, Maryland 21240 (2)

Enclosures

(NASA-CR-170059) THE EFFECT OF LOW VELOCITY IMPACT IN THE STRENGTH CHARACTERISTICS OF COMPOSITE MATERIALS LAMINATES Progress Report (George Washington Univ.) 29 p
HC A03/MF A01
N83-20282
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CSCL 20K G3/39 03112

THE EFFECT OF LOW VELOCITY IMPACT IN THE STRENGTH CHARACTERISTICS OF COMPOSITE MATERIALS LAMINATES

NASA Grant NAG 1-158

Principal Investigator: H. Liebowitz

End Of The Year Progress Report

School of Engineering and Applied Science
The George Washington University
Washington, D.C. 20052

The research work on NASA Grant NAG 1-158 has been directed toward a study of the effects of geometric nonlinearity on the vibration response of isotropic beams (Re: Our Interim Progress Report of January 12, 1983). Initial investigations have shown that a stable solution can only be resolved after an extremely fine meshed initial solution has been determined. This initial solution requires extremely small time steps with high computational resolution. An energetically stable solution for a beam with a sharp pulse initial velocity profile has been obtained. The results of this study are reported in the paper "On The Energetics Of Nonlinear Beam Vibrations" which has been submitted for publication to the Journal of Computers And Structures. A copy of this paper is enclosed. The example problem discussed produces very sharp deformation gradients even at moderately small deflections. This study has demonstrated many numerical difficulties often overlooked in vibration analyses. The insight provided by this study illustrates the numerical complexities which must be modeled accurately in future studies.

Currently the research involves studying the effects of different initial velocity profiles to determine an adequate model for impact loading. The effects of spatially distributed mass is being investigated to resolve the effect of an impact object in contact with a beam. Implicit time integration schemes are being employed to further solutions which are

started by the explicit methods summarized above allowing for larger time steps. To generate solutions of sufficient duration to practically investigate several fundamental periods is computationally prohibitive using an explicit technique. The implicit techniques are transiently unstable, therefore, a combination approach is indicated.

ON THE ENERGETICS OF NONLINEAR BEAM VIBRATIONS

By

E. Thomas Moyer, Jr.
Senior Research Engineer

March, 1983

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ABSTRACT

The nonlinear vibration response of a double cantilevered beam subjected to pulse loading over a central sector is studied. The initial response is generated in detail to ascertain the energetics of the response. The total energy is used as a gauge of the stability and accuracy of the solution. It is shown that to obtain accurate and stable initial solutions an extremely high spatial and time resolution is required. This requirement was only evident through an examination of the energy of the system. It is proposed, therefore, to use the total energy of the system as a necessary stability and accuracy criterion for the nonlinear response of conservative systems. The results also demonstrate that even for moderate nonlinearities, the effects of membrane forces have a significant influence on the system. It is also shown that while the fundamental response is contained in a first mode envelope, the fluctuations caused by the higher order modes must be resolved.

INTRODUCTION

Many problems in mechanics require an accurate modeling of nonlinear elastic vibrations. Many structural mechanics, machine design, and aerospace design problems involve the nonlinear vibration response of systems which can be decomposed into beams, plates or shells. Further, recent experimental work has shown that many problems dealing with the impact response of composite materials involve a nonlinear vibration response [1]. While many plate and shell solutions have been presented in the literature, accuracy and quantitative stability studies have been limited to modal response analysis [2]. Most applied analyses to the problems of nonlinear mechanics under pulse or impact loading have employed the techniques developed and tested under single or low order, multiple mode conditions. The purpose of this study is to analyze the energetics of the initial response of a nonlinear elastic beam to pulse loading (approximated by a sinusoidal velocity response). The total energy of the system has been chosen as the measure of the accuracy and stability of the solution. While the conservation of energy is only a necessary condition for accuracy, it is shown that the solution requirements to fulfill this criteria are extremely demanding. Further refinements of the time and spatial discretizations had no appreciable effect on the solution after energy convergence had been established.

Most problems of engineering interest (especially for composite application) involve the response of plates or combinations of plate elements. The plate equations, however, are significantly more complicated than the beam equations but still have the same characteristic properties. It was chosen, therefore, to analyze a nonlinear beam possessing the same bending stiffness and natural frequency as a typical composite plate.

There are various ways to formulate the governing equations for a nonlinear beam. For moderate deflections (i.e., less than 5 times the thickness of the beam), the assumptions of inextensibility and negligible shear appear reasonable. The resulting equations are mathematically equivalent to the standard nonlinear formulation for the moderately large deflection of plates [3]. This feature allows establishment of solution procedures applicable to plate problems without addressing the more complicated system directly. For simplicity, a square beam has been chosen with constant cross-section.

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FORMULATION OF THE GOVERNING EQUATIONS

Consider the free vibrations of a square beam of thickness h and length L. Let the origin, 0, be situated at the cross-sectional centroid at the left end of the beam as shown in Figure 1. Let x be the coordinate measure along the length of the beam in the undeformed state. The equilibrium conditions can then be written for a differential element as

$$\frac{\partial N}{\partial x} = \rho \frac{\partial^2 U}{\partial t^2} \tag{1a}$$

$$\frac{1}{h^2} \frac{\partial^2 M}{\partial x^2} + \frac{\partial}{\partial x}\left(N \frac{\partial W}{\partial x}\right) = \rho \frac{\partial^2 W}{\partial t^2} \tag{1b}$$

where N is the average normal stress across the cross-sectional face (i.e., the membrane stress) and M is the cross-sectional moment. It is explicitly assumed from the outset that shear terms can be neglected. U and W represent the longitudinal and transverse displacements, respectively. For an elastic, isotropic beam, assuming inextensibility, the stress-displacement relations can be written as [4]

$$M = - E I \frac{\partial^2 W}{\partial x^2} \tag{2a}$$

$$N = E \left[\frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right] \tag{2b}$$

where E is Young's modulus for the material and I is the

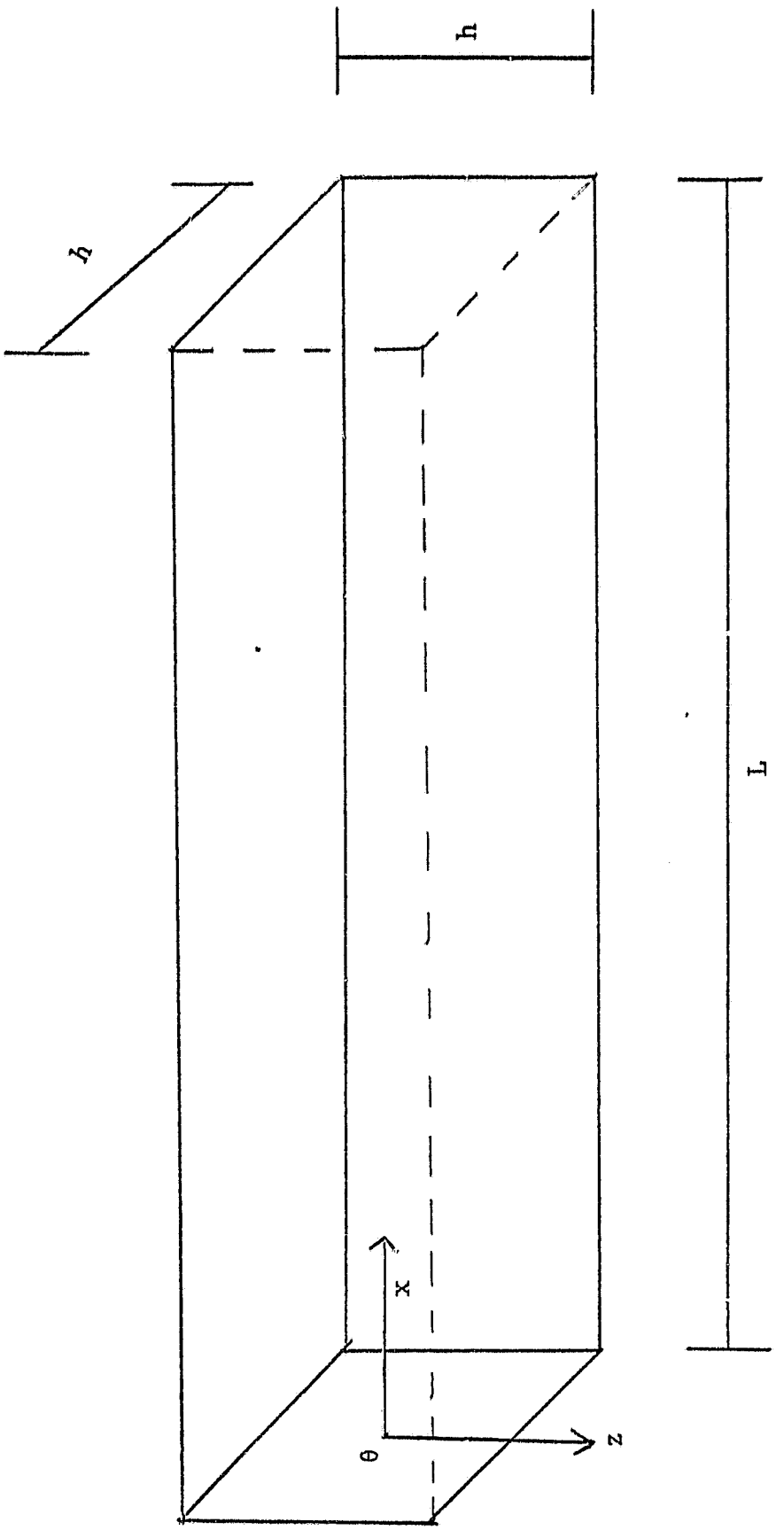

Figure 1: Geometry And Coordinates For Beam Vibration Studies.

cross-sectional moment of inertia. Utilizing the stress-displacement relations in the equilibrium conditions, the governing equations can be written as

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = -\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2} \tag{3a}$$

$$\frac{\partial^4 W}{\partial x^4} - \frac{1}{R^2}\left[\frac{3}{2}\left(\frac{\partial W}{\partial x}\right)^2 \frac{\partial^2 W}{\partial x^2} + \frac{\partial W}{\partial x}\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 W}{\partial x^2}\frac{\partial U}{\partial x}\right] = -\frac{1}{a^2}\frac{\partial^2 W}{\partial t^2} \tag{3b}$$

where the following definitions have been adopted

$$c^2 = E/\rho \tag{4a}$$

$$R^2 = I/h^2 \tag{4b}$$

$$a^2 = c^2 R^2 \tag{4c}$$

The quantity ρ is the mass density of the material and A is the cross-sectional area of the beam.

The total energy of the beam can be decomposed into bending energy, membrane energy and kinetic energy. By definition, these quantities can be written as

$$E_K = \left\{ \begin{matrix} \text{Kinetic} \\ \text{Energy} \end{matrix} \right\} = \frac{1}{2} \int_0^L \rho \left[\left(\frac{\partial U}{\partial t} \right)^2 + \left(\frac{\partial W}{\partial t} \right)^2 \right] A \, dx \qquad (5a)$$

$$E_B = \left\{ \begin{matrix} \text{Bending} \\ \text{Energy} \end{matrix} \right\} = \frac{1}{2} \int_0^L \frac{M^2}{E I} \, dx \qquad (5b)$$

$$E_M = \left\{ \begin{matrix} \text{Membrane} \\ \text{Energy} \end{matrix} \right\} = \frac{1}{2} \int_0^L \frac{N^2}{E} A \, dx \qquad (5c)$$

The total energy is given by the sum of these components as

$$E_T = \left\{ \begin{matrix} \text{Total} \\ \text{Energy} \end{matrix} \right\} = E_K + E_B + E_M \qquad (6)$$

Utilizing the stress-displacement relations, the bending and membrane components can be written as

$$E_B = \frac{E I}{2} \int_0^L \left(\frac{\partial^2 W}{\partial x^2} \right)^2 dx \qquad (7a)$$

$$E_M = \frac{E A}{2} \int_0^L \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial^2 W}{\partial x^2} \right)^2 \right]^2 dx \qquad (7b)$$

The displacement formulation will be utilized for computational convenience.

For the example problem in this study, the beam will be

assumed to be rigidly clamped at both ends. Symmetric loading will be applied to a central section of the beam to simulate central impact. The boundary conditions at the ends of the beam can be written as:

$$W(x = 0) = W(x = L) = 0 \tag{8a}$$

$$\frac{\partial W}{\partial x}(x = 0) = \frac{\partial W}{\partial x}(x = L) = 0 \tag{8b}$$

$$U(x = 0) = U(x = L) = 0 \tag{8c}$$

In general, four initial conditions must be specified in the form

$$U(x, t = 0) = f_1(x) \tag{9a}$$

$$\frac{\partial U}{\partial t}(x, t = 0) = g_1(x) \tag{9b}$$

$$W(x, t = 0) = f_2(x) \tag{9c}$$

$$\frac{\partial W}{\partial t}(x, t = 0) = g_2(x) \tag{9d}$$

RESPONSE TO AN INITIAL PULSE

The material properties used in this study were chosen to simulate the response of typical graphite-epoxy composite plates. Typical average quasi-isotropic properties are

$$E = 7.2135\ E + 10 \text{ Pascals}$$
$$\rho = 1.6000\ E + 03 \text{ Kg/m}^3 \qquad (10)$$
$$\nu = 0.33$$

For a circular plate with clamped edges the bending stiffness, K and natural frequency, ω_0 are given by [5]

$$K = \frac{4\pi\ E\ h^3}{3(1 - \nu^2)\ r^2} \qquad (11a)$$

$$\omega_0 = 3.125\ \frac{h}{r^2}\ \sqrt{\frac{E}{\rho}} \qquad (11b)$$

where r is the plate radius and h is the plate thickness. Assuming a plate with radius 2.54 cm. and thickness 1.03 mm., the bending stiffness and natural frequencies are given by

$$K = 6.0843\ E + 05 \text{ J/m}^3 \qquad (12a)$$
$$\omega_0 = 3.4150\ E + 04 \text{ /sec} \qquad (12b)$$

For a linear, double cantilevered beam of square crosssection, the bending stiffness and natural frequency are given by [6]

$$K = 192 \frac{E I}{L^3} \tag{13a}$$

$$\omega_0 = \frac{(4.730)^2}{L^2} \sqrt{\frac{E I}{\rho A}} \tag{13b}$$

Equating the beam and plate parameters, the equivalent beam length and thickness are

$$h = 3.5565E\text{-}03 \text{ meters}$$

$$L = 6.7203E\text{-}02 \text{ meters}$$

which are the dimensions employed in the present solution.

To simulate severe initial velocity response a sinusoidal variation over the central section of length L/10 was chosen. The initial conditions are then written as

$$U(x, t = 0) = \frac{\partial U}{\partial t}(x, t = 0) = 0 \tag{14a}$$

$$W(x, t = 0) = 0 \tag{14b}$$

$$\frac{\partial W}{\partial t}(x, t = 0) = \begin{cases} 0 & x < x_1 \\ V_0 \, \text{SIN}\left(\frac{10\pi(x - x_0)}{L}\right) & x_1 < x < x_2 \\ 0 & x > x_2 \end{cases} \tag{14c}$$

where

$$x_1 = 9L/20$$

$$x_2 = 11L/20 \tag{14d}$$

The center point velocity was chosen as 243 meters/second

which corresponds to an initial impact energy of 2.0077 Joules.

The governing equations were discretized by the central difference technique. Second order central differences were chosen for all spatial derivatives. Due to the steep gradients near the initial loading region, extremely fine meshes were required to obtain accuracy. Meshes consisting of 200, 500 and 1000 divisions along the length of the beam were employed. Several implicit and explicit time integration schemes were initially employed. To gauge the accuracy and stability of the solution, the energy components and total energy were calculated. Due to the conservative nature of the system, the total energy should remain constant and equal to the initial impact energy.

The second order central difference operators are summarized in the Appendix. Evaluation of all energy integrals in this study were performed using Simpson's 1/3 rule. Computations were performed on the CYBER 203 computer at NASA-Langley Research Center. The code was written in CYBER Vector FORTRAN. Some preliminary computations were also performed using scalar FORTRAN on a VAX 11/780 and an IBM 4341. For the solution range studied, neither of these machines had the resolution to produce accurate solutions (this is discussed in more detail later). The use of higher order difference operators is not indicated for multiple mode vibration problems due to the stability requirements necessary for convergence.

RESULTS AND DISCUSSION

Stability analysis for the linearized fourth order wave equation was performed for the explicit time integration formulation in order to establish an upper bound on the allowable time step for each order of spatial discretization. The results of this analysis are summarized in Table 1. There are no quantitatively accurate methods for predicting the numerical stability of nonlinear systems in general. For the nonlinear, fourth order wave equation, it can be shown that the nonlinearity causes a "Buoyancy" effect and the system is, in fact, more stable than the corresponding linear system [7]. It is therefore unnecessary to perform one of the stability approximation analyses discussed in the literature [8].

Initially, Newmark implicit time integration was chosen due to the unconditional stability for linear systems [9]. Computationally, however, the energy of the system oscillated by a minimum of 40% even for very small time steps and fine spatial discretizations. Similar results were obtained with a Crank-Nicholson scheme [10]. An explicit second order central difference scheme was chosen, therefore, to examine the energetics of the initiation of the solution.

Spatial discretizations of 200, 500 and 1000 divisions were chosen for comparison. The time steps initially chosen were 5.E-09 seconds, 1.E-09 second, and 1.E-10 seconds,

Table 1: Time Step Stability Limit (Δt) From Linear Analysis With N Spatial Divisions

N	Δx(meters)	Δt(seconds)
200	3.36015 E-04	8.1893 E-09
500	1.34406 E-04	1.3103 E-09
1000	6.72030 E-05	3.2757 E-10

Table 2: Error In Average Energy And Maximum Scatter From Constant Time Step Analysis With N Spatial Divisions

N	Δt(seconds)	% Error in $\bar{E}$	Maximum Deviation
200	5.E-09	92.13%	61.33%
500	1.E-09	46.02%	45.41%
1000	1.E-10	3.35%	8.12%

Table 3: Error In Average Energy And Maximum Scatter For N Spatial Divisions Using Time Marching Scheme

N	% Error in $\bar{E}$	Maximum Deviation
200	51.31%	26.41%
500	28.28%	9.37%
1000	0.11%	0.16%

respectively. The time steps were kept constant and the total energy was calculated as a function of time. The solution was generated for 50 microseconds which corresponds to approximately one-fourth of the natural period. The mean error and percentage of scatter in the total energy from these solutions is summarized in Table 2. While the results with 1000 spatial increments appear reasonable, a scatter of approximately 8% and a mean error of 3% could cause significant error later in the calculation depending on the propagation of the error caused by the integration algorithm employed to advance the solution.

To minimize the error incurred during the initiation of the solution, a time marching approach was adopted. An initial time step of 1.E-14 seconds was chosen for the first 100 time steps. The initial conditions and final step results were then employed to start the solution with a time step of 1.E-12 seconds. This solution was then advanced until the time reached 5.E-09 seconds (for a spatial discretization of 200 points), 1.E-09 seconds (for a spatial discretization of 500 points) or 1.E-10 seconds (for a spatial discretization of 100 points). These final time steps were then advanced until the elapsed time reached 50 microseconds. The total energy response is summarized in Table 3. For all spatial discretizations the predictions with the time marching approach are more accurate and more stable.

Due to the large gradients near the pulse load boundaries, 1000 spatial increments are required to obtain reasonable

accuracy during the initial response. To continue the solution with such a small time step, however, would be computationally prohibitive. Several approaches are currently being investigated. It is expected that the solution can be accurately advanced by one of the implicit techniques and a coarser spatial grid can be employed once the initial response is accurately determined.

It is of interest to investigate the component response of the energy during the initial response stage. At the start, all the energy is kinetic. As the pulse propagates, both longitudinal and transverse vibrations are established. While the first mode of vibration will dominate, many higher order modes will cause important energetic effects. Figure 2 is a plot of the Kinetic Energy as a function of time. While the envelope of this curve decreases as the first mode response would, several higher order vibrations are evident. These higher order effects can account for 15%-30% of the kinetic energy at any instant. Figure 3 is a plot of the bending energy as a function of time. While the modal response is less clear in this component, careful observation shows that the bending energy is approximately 180 degrees out of phase with the kinetic energy (as would be predicted by a modal analysis). These qualitative features of the kinetic and bending energy curves suggest the consistency of the present solution during the initial response.

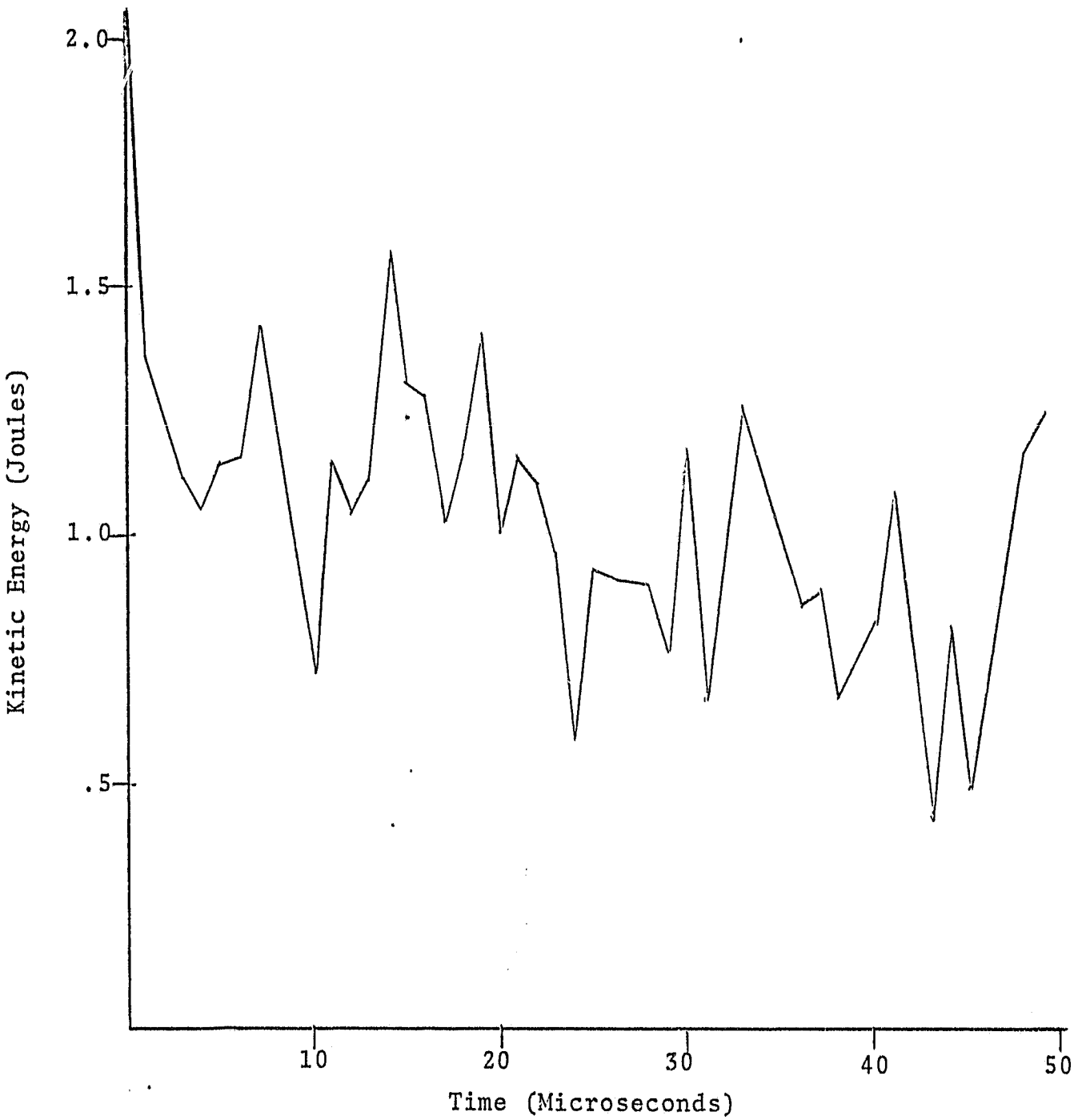

Figure 2: Kinetic Energy As A Function Of Time.

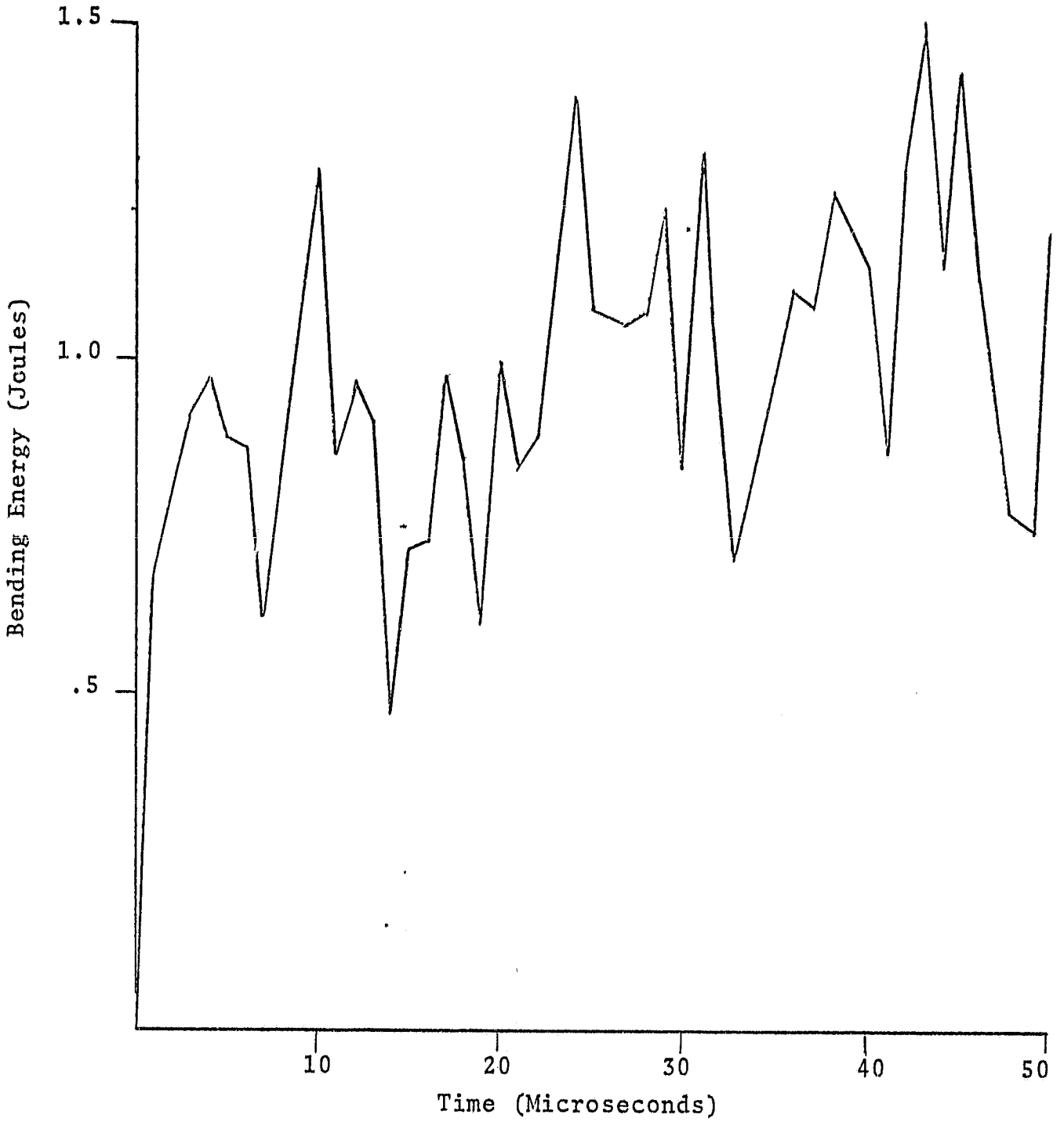

Figure 3: Bending Energy As A Function Of Time.

Figure 4 is a plot of the membrane energy as a function of time. During the first quarter period studied, the envelope response is monotonically increasing which is consistent with first mode predictions. The higher order effects, however, predict instantaneous fluctuations of up to 50%. These local fluctuations could have significant effect on the accuracy of failure prediction for nonlinear vibration problems.

Figure 5 is a plot of the center point displacement as a function of time. The envelope of this curve also follows the first mode predictions, however, higher order fluctuations cause significant instantaneous deviations (as high as 30%). The largest displacement in the time interval studied is approximately 37% of the thickness of the beam. This should be in the range of moderately small geometric linearity. This observation is consistent with the prediction that the maximum membrane energy is about 8% of the total energy.

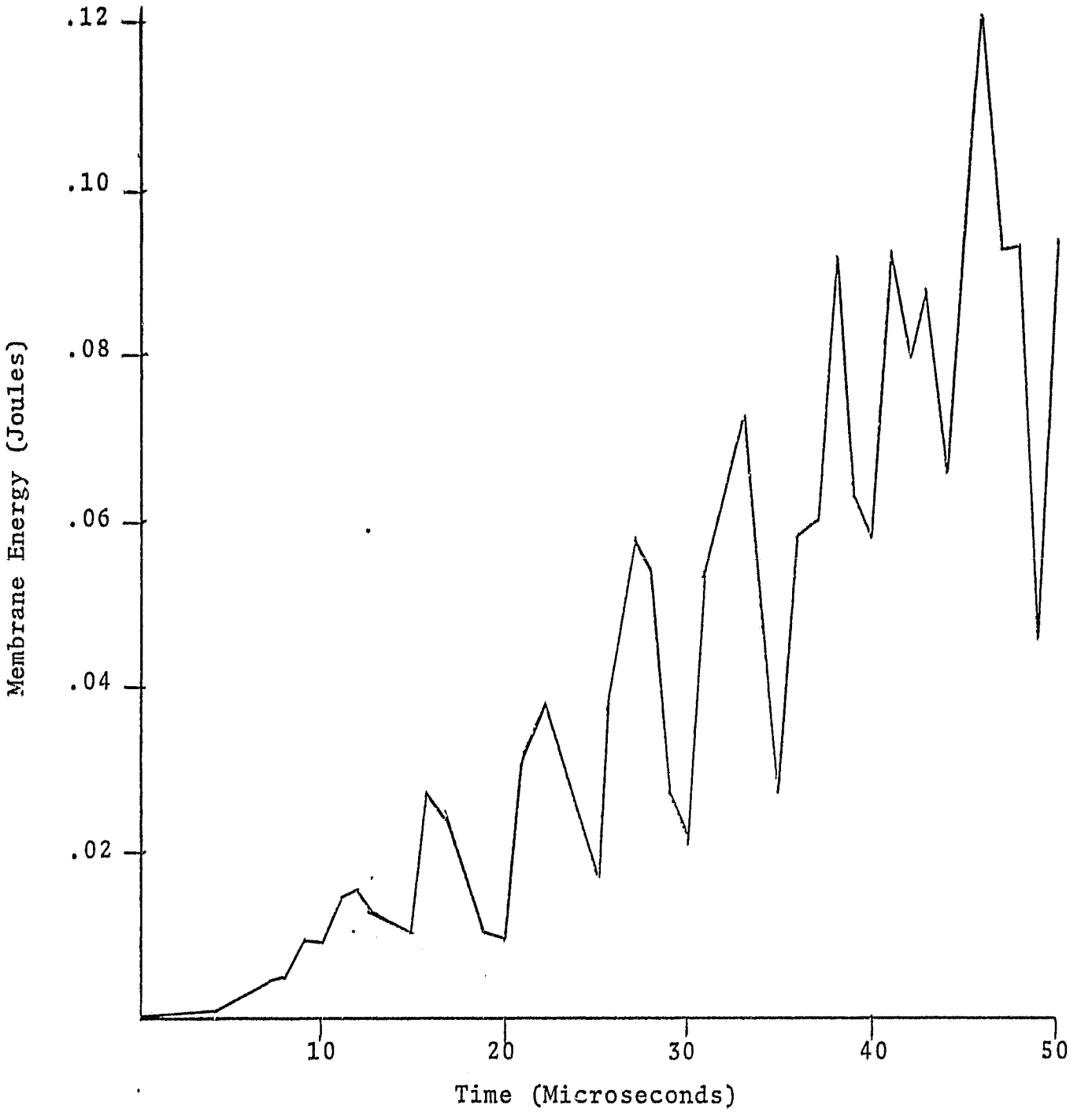

Figure 4: Membrane Energy As A Function Of Time.

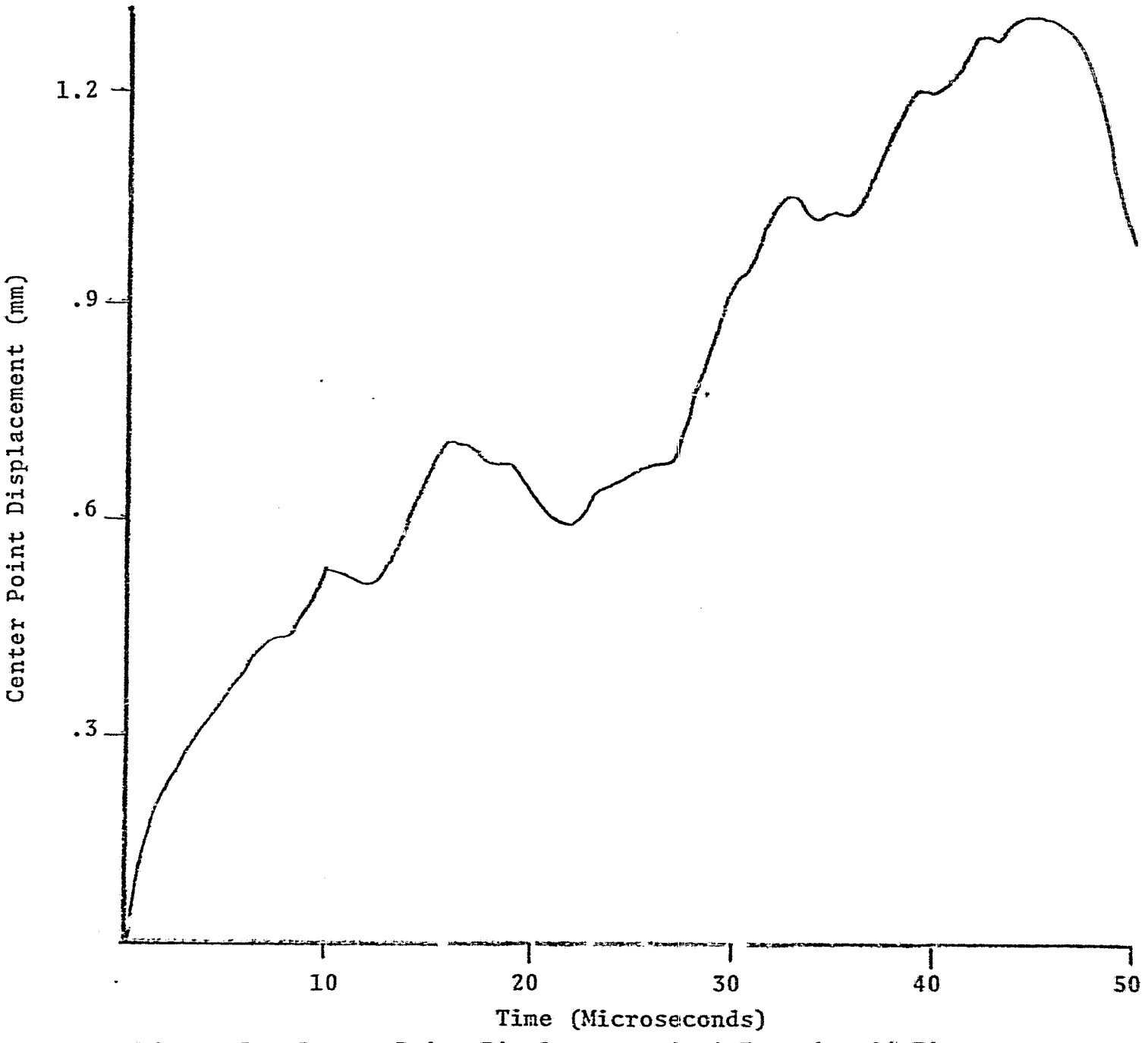

Figure 5: Center Point Displacement As A Function Of Time.

CONCLUSIONS

The results of this study demonstrate the importance of the initial response solution procedure to the solution of nonlinear vibration problems. The total energy of the system proved to be a good measure of the stability and accuracy of the solution. It was shown that the initial response is much more sensitive to time step size and spatial discretization than simple stability and accuracy analyses would predict.

A time marching explicit integration scheme was demonstrated to be an accurate and efficient way of initiating a stable solution. It is expected (and currently being investigated) that any of the implicit integration schemes can be used to accurately advance the solution after a good initial response is generated.

The results of this study also indicate that much caution should be exercised when solving vibration problems involving steep initial gradients. Not only are the solution procedure and discretization parameters important, but the precision of the calculation can play a significant role in the accuracy of the solution. All results presented in this study were generated on the CYBER 203 computer at NASA-Langley Research Center. The range of variables that can be resolved on that machine is from 9.54E + 8645 to 5.2E-8618 absolute. While this extreme range is unnecessary, calculations of a VAX 11/780 (absolute range of approximately 1.E + 32 to 1.E-37) and on

an IBM 4341 (absolute range of approximately 1.E + 76 to 1.E-79) produced erroneous results utilizing arithmetic with 14 digit precision. Extremely small displacements can cause significant gradients when entering into the nonlinear terms. It was discovered during this study that the solutions can be in error by as much as 100% due solely to the lack of resolution of the small initial displacements. Scaling of the data is not a total solution as the order of longitudinal and transverse displacements is significantly different. It is unknown at present the order of resolution necessary to resolve this problem. Once a stable initial solution is produced, however, the variation in the gradient parameters becomes resolvable on virtually any machine. The final calculation stage was checked on the VAX 11/780 using the initial conditions from the first two marching increments produced on the CYBER 203 and compared with the final results from the CYBER 203 calculation. Both solutions were identical proceeding from 1 to 50 microseconds. Alternative procedures for this problem are being investigated for use on machines without the precision of the CYBER machines.

The results of this study, while generated for a particular example problem, provide a consistent method for initiating solutions to nonlinear elasticity vibration problems in general. The constancy of the total energy is a necessary condition for establishing the accuracy of solution for any conservative system. More complicated systems and

spatial discretization schemes often obscure the fundamental problems which are present in nonlinear systems. The total energy is a much overlooked indicator of solution accuracy which is computationally simple and inexpensive. It is suggested that convergence of linear and nonlinear vibration solutions for conservative systems must satisfy conservation of energy in addition to local convergence criteria currently employed.

ACKNOWLEDGEMENTS

The author would like to thank Dr. Wolf Elber of NASA-Langley Research Center for his suggestions and encouragement during this study. This work was sponsored under NASA Grant # NAG-1-158.

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APPENDIX

The standard second order central difference operators are

$$\left.\frac{\partial Y}{\partial x}\right|_{x = x_i} = \frac{Y_{i+1} - Y_{i-1}}{2(\Delta x)}$$

$$\left.\frac{\partial^2 Y}{\partial x^2}\right|_{x = x_i} = \frac{Y_{i+1} - 2Y_i + Y_{i-1}}{(\Delta x)^2}$$

$$\left.\frac{\partial^4 Y}{\partial x^4}\right|_{x = x_i} = \frac{Y_{i+2} - 4Y_{i+1} + 6Y_i - 4Y_{i-1} + Y_{i-2}}{(\Delta x)^4}$$

for a function $Y(x,\ldots)$ where

$$Y(x = x_i) = Y_i$$

$$\Delta x = x_{i+1} - x_i$$

for all x_i in the domain.