

FINITE ELEMENT MODELING AND ANALYSIS OF TIRES

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Abstract

Although the problem of tire modeling and analysis has been a subject of continuing concern for the tire industry, to date no simple and general tire model exists for predicting the response of the tire under various loading conditions. Much of the recent progress in finite element technology has not been exploited for tire modeling and analysis. The present paper focuses on this issue. Specifically, the paper reviews some of the recent advances in finite element technology which have high potential for application to tire modeling problems. It also identifies the analysis and modeling needs for tires.

The topics covered include: 1) reduction methods for large-scale nonlinear analysis, with particular emphasis on treatment of combined loads, displacement-dependent and nonconservative loadings; 2) development of simple and efficient mixed finite element models for shell analysis, identification of equivalent mixed and purely displacement models, and determination of the advantages of using mixed models; and 3) effective computational models for large-rotation nonlinear problems, based on a total Lagrangian description of the deformation.

INTRODUCTION

The problem of tire modeling and analysis has long been an area of major concern to the tire and aircraft industries. A hierarchy of models varying in the degree of sophistication has been proposed. Some of these models are listed in Fig. 1 and are sketched in Fig. 2. For a detailed description of the models see Ref. 1. The models are grouped into six groups as follows:

The first group consists of the early tire models which are characterized by their simplicity. Among these models are the *string, beam, and ring on elastic (or viscoelastic) foundations*. These models were used by Clark and co-workers (Ref. 2). Their major drawbacks are: 1) they require extensive experiments to evaluate the equivalent properties, and 2) their accuracy and range of validity are not known in advance.

The second group consists of the *cord-network models*, which are sometimes referred to as *netting analysis*, wherein the inflation pressure is assumed to be carried exclusively by the cords (see Ref. 3). These models have the drawback of neglecting both the bending in the tire and the stiffening effect of the rubber.

The third group of models are the *membrane models*, which are based on the use of a linear or nonlinear momentless theory of shells (Refs. 4, 5 and 6). Their major drawback is that they cannot handle discontinuities in loading, geometry or material properties.

The fourth group is the *two-dimensional axisymmetric models* (Ref. 7), which are limited to axisymmetric loadings.

The fifth group is the *three-dimensional continuum models*. Two approaches have been proposed for the analysis of these models. The first approach is based on using semi-analytic techniques to reduce the dimensionality of the problem (e.g., Fourier expansions in the circumferential direction). The second approach is based on using three-dimensional isoparametric solid elements.

The sixth group of models includes a variety of *two-dimensional thin and thick shell models* (see, for example, Refs. 8 and 9). Thin shell models neglect transverse shear deformation, and their use for modeling tires is therefore questionable. Anisotropy results in increasing the size of the analysis model, and consequently many investigators neglect its effects by using an orthotropic model.

The present paper focuses on the use of two-dimensional thick shell models.

DIFFERENT TIRE MODELS

MODEL	DRAWBACKS
<ul style="list-style-type: none"> • SIMPLE MODELS <li style="margin-left: 20px;">STRING } ON { ELASTIC } FOUNDATION <li style="margin-left: 20px;">BEAM } OR <li style="margin-left: 20px;">RING } VISCO-ELASTIC 	<ul style="list-style-type: none"> • REQUIRE EXTENSIVE EXPERIMENTS TO EVALUATE EQUIVALENT PROPERTIES • ACCURACY AND RANGE OF VALIDITY NOT KNOWN IN ADVANCE
<ul style="list-style-type: none"> • CORD-NETWORK MODELS 	<ul style="list-style-type: none"> • NEGLECT BENDING IN TIRE AND STIFFENING EFFECTS OF RUBBER
<ul style="list-style-type: none"> • MEMBRANE MODELS (BASED ON MOMENTLESS THEORY OF SHELLS) 	<ul style="list-style-type: none"> • CANNOT HANDLE DISCONTINUITIES (OR SHARP CHANGES) IN LOADING, GEOMETRY (CURVATURE) OR MATERIAL PROPERTIES
<ul style="list-style-type: none"> • TWO-DIMENSIONAL AXISYMMETRIC MODELS 	<ul style="list-style-type: none"> • LIMITED TO AXISYMMETRIC LOADING
<ul style="list-style-type: none"> • THREE-DIMENSIONAL MODELS <li style="margin-left: 20px;">• SEMI-ANALYTIC SOLUTIONS <li style="margin-left: 20px;">• THREE-DIMENSIONAL SOLID ELEMENTS 	<ul style="list-style-type: none"> • COMPUTATIONALLY EXPENSIVE
<ul style="list-style-type: none"> • LAMINATED ANISOTROPIC SHELL MODELS (THIN AND THICK) 	<ul style="list-style-type: none"> • THIN SHELL MODELS NOT ADEQUATE • EFFECT OF ANISOTROPY CAN BE SIGNIFICANT

Figure 1

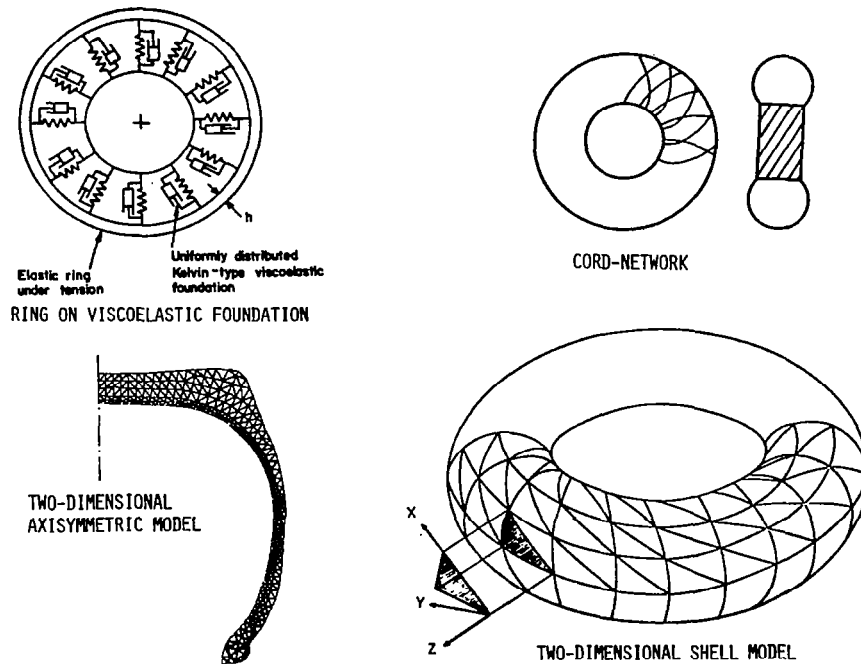


Figure 2

TIRE CONFIGURATION AND COMPONENTS

Typical configurations and components of modern tires are shown in Fig. 3 (see Ref. 10). Commercially successful tires are now built as a series of layers of flexible high-modulus cords encased in a low-modulus rubber or rubber-like material. Hence, a laminated (or layered) model is needed.

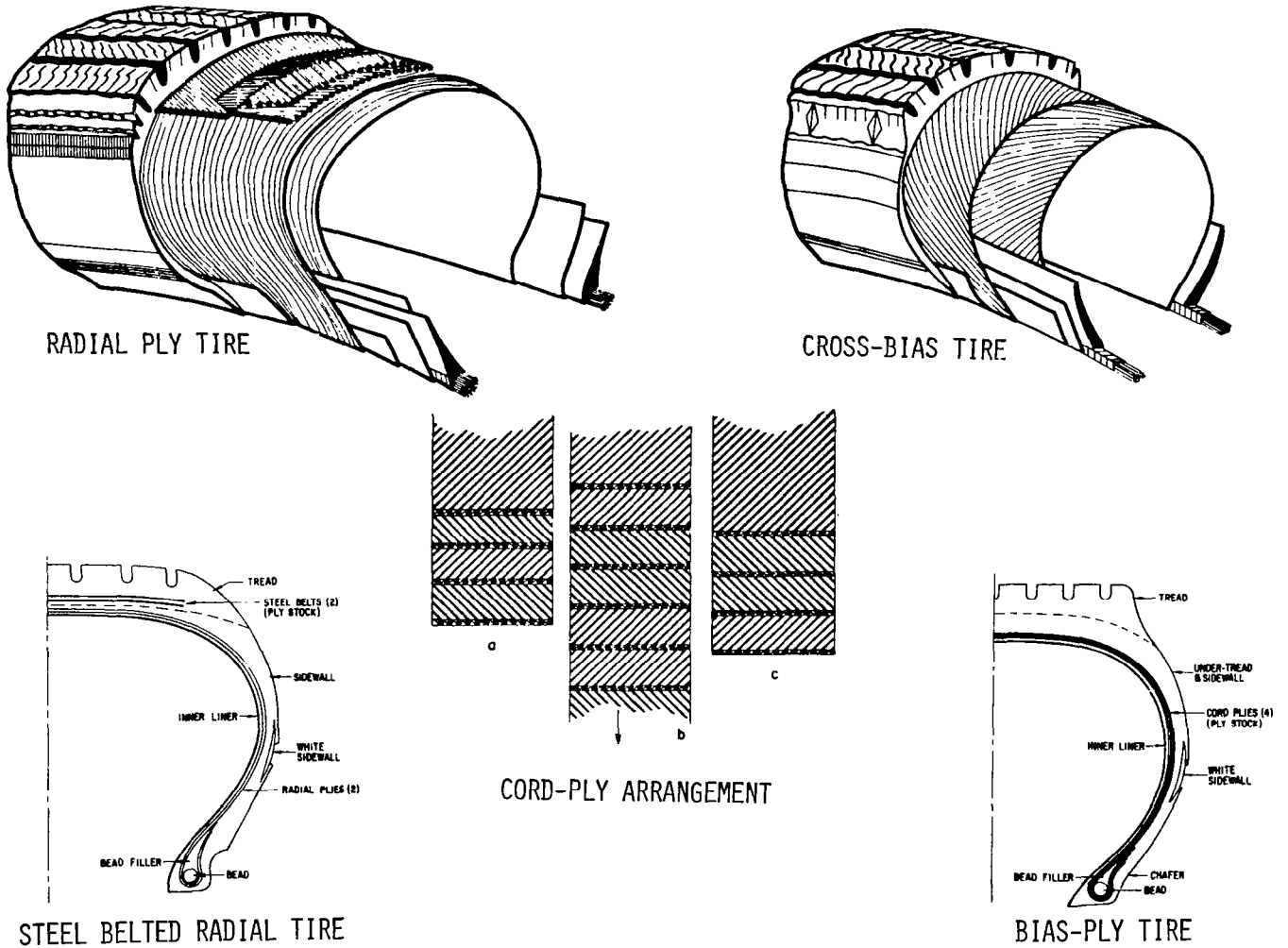


Figure 3

TIRE LOADS

The three types of loads applied to the tire and their major characteristics are listed in Fig. 4. The three load types are:

- 1) *Inflation pressure*, which is axisymmetric but is displacement dependent
- 2) *Mechanical loads* which include centrifugal force, impact loading, contact forces, and frictional forces; except for the centrifugal force, which is axisymmetric (and displacement dependent), all the other loads are symmetric
- 3) *Thermal loads*, which arise due to various manufacturing and operating conditions, such as unequal expansion and contraction of rubber and cord, hysteretic heating, sliding of the tread on a rough surface, and cord shrinking after molding

LOADS	CHARACTERISTICS
<ul style="list-style-type: none"> • INFLATION PRESSURE 	<ul style="list-style-type: none"> • AXISYMMETRIC BUT DISPLACEMENT-DEPENDENT
<ul style="list-style-type: none"> • MECHANICAL LOADS <ul style="list-style-type: none"> • CENTRIFUGAL FORCE • IMPACT LOADING • CONTACT FORCES • FRICTIONAL FORCES 	<ul style="list-style-type: none"> • AXISYMMETRIC BUT DISPLACEMENT-DEPENDENT • ASYMMETRIC • • •
<ul style="list-style-type: none"> • THERMAL LOADS <ul style="list-style-type: none"> • UNEQUAL EXPANSION AND CONTRACTION OF RUBBER AND CHORD • HYSTERETIC HEATING • CHORD SHRINKING AFTER MOLDING 	<ul style="list-style-type: none"> • ASYMMETRIC • • • • • •

Figure 4

CHARACTERISTICS OF EFFECTIVE SHELL ELEMENTS
FOR ANALYZING TIRES

The characteristics of an effective shell finite element model for analyzing tires are listed in Fig. 5. The shell element is developed using either a consistent two-dimensional shell theory or a three-dimensional continuum theory with proper interpolation functions in the thickness direction. The elements obtained by using the latter approach are referred to as *degenerate shell elements*. If a two-dimensional shell theory is used, the elements need to be deep and curved and must account for each of the following effects:

- 1) Laminated construction and anisotropic material behavior
- 2) Variation in geometry (e.g., curvature and thickness) as well as of other lamination parameters
- 3) Transverse shear deformation
- 4) Large rotations
- 5) Pressure stiffness (for displacement-dependent loadings such as inflation pressure)
- 6) Thermoviscoelastic material response

- BASED ON EITHER
 - CONSISTENT TWO-DIMENSIONAL SHELL THEORY, OR
 - THREE-DIMENSIONAL CONTINUUM THEORY WITH PROPER INTERPOLATION FUNCTIONS IN THE THICKNESS DIRECTION (DEGENERATE SHELL ELEMENTS)
- DEEP, CURVED ELEMENTS
- INCLUDE EFFECTS OF:
 - LAMINATED CONSTRUCTION AND ANISOTROPIC MATERIAL BEHAVIOR
 - VARIATION IN GEOMETRY (E.G., CURVATURE AND THICKNESS), LAMINATION PARAMETERS
 - TRANSVERSE SHEAR DEFORMATION
 - LARGE ROTATIONS
 - PRESSURE STIFFNESS (FOR DISPLACEMENT-DEPENDENT LOADING; E.G., INFLATION PRESSURE)
 - THERMOVISCOELASTIC MATERIAL RESPONSE

Figure 5

OBJECTIVES AND SCOPE

The objectives of this paper are listed in Fig. 6. They are:

- 1) To review some recent developments in finite element technology which are applicable to the analysis and modeling of tires
- 2) To identify some of the analysis and modeling needs for tires

The paper is divided into four parts. The first part deals with new developments in reduction methods for nonlinear problems. These include computational procedures for handling combined, displacement-dependent, and nonconservative loads. The second part of the paper deals with mixed finite element models for tires in which the fundamental unknowns consist of both force and displacement parameters. The equivalence of some of these models with some of the purely displacement models is discussed.

The third part of the paper deals with large-rotation nonlinear problems. Two formulations are presented; namely, a mixed formulation and a penalty formulation. Both formulations are based on the total Lagrangian description of the deformation. The fourth and last part of the paper deals with analysis and modeling needs for tires.

OBJECTIVES

- REVIEW SOME RECENT DEVELOPMENTS IN FINITE ELEMENT TECHNOLOGY WHICH ARE APPLICABLE TO ANALYSIS AND MODELING OF TIRES
- IDENTIFY ANALYSIS AND MODELING NEEDS FOR TIRES

SCOPE

- REDUCTION METHODS FOR NONLINEAR PROBLEMS
 - COMBINED LOADING PROBLEMS
 - DISPLACEMENT-DEPENDENT AND NONCONSERVATIVE LOADING PROBLEMS
 - DYNAMIC PROBLEMS
- MIXED FINITE ELEMENT MODELS
 - EFFICIENT AND ACCURATE MIXED MODELS
 - EQUIVALENT CLASSES OF MIXED MODELS AND REDUCED/SELECTIVE INTEGRATION DISPLACEMENT MODELS
 - MERITS OF MIXED MODELS OVER EQUIVALENT DISPLACEMENT MODELS
- LARGE-ROTATION NONLINEAR PROBLEMS
 - MIXED FORMULATION
 - PENALTY FORMULATION

Figure 6

REDUCTION METHODS FOR NONLINEAR TIRE PROBLEMS

The first topic considered in this paper is reduction methods for nonlinear analysis. The basic features of reduction methods are outlined in Fig. 7. They are techniques for reducing the number of degrees of freedom through the transformation shown in the figure. The vector $\{X\}$ represents the original displacement degrees of freedom. The vector $\{\psi\}$ refers to amplitudes of displacement modes and $[\Gamma]$ is a transformation matrix whose columns represent a priori chosen global displacement modes.

As is to be expected, the effectiveness of reduction methods depends to a great extent on the proper selection of the displacement modes. In a number of studies it was shown that an effective choice of the displacement modes includes the various-order derivatives of the displacement vector with respect to the load parameter (see Refs. 11 and 12). These vectors are generated by using the finite element model of the tire. The recursion formulas for evaluating the derivatives $\left\{\frac{\partial X}{\partial p}\right\}$, $\left\{\frac{\partial^2 X}{\partial p^2}\right\}$, ... are obtained by successive differentiation of the original finite element equations. The left-hand sides of the recursion formulas are the same (see Ref. 12). Therefore, only one matrix factorization is required for the generation of all the global approximation vectors. Several numerical experiments have demonstrated the effectiveness of this choice (see Refs. 12 and 13).

DEFINITION: ARE TECHNIQUES FOR REDUCING THE NUMBER OF D.O.F. THROUGH THE TRANSFORMATION

$$\{X\}_{n,1} = [\Gamma]\{\psi\}_{r,1} \quad r \ll n$$

$\{X\}$ = ORIGINAL D.O.F. IN THE FINITE ELEMENT MODEL

$[\Gamma]$ = MATRIX OF GLOBAL DISPLACEMENT MODES

$\{\psi\}$ = REDUCED D.O.F. - AMPLITUDES OF DISPLACEMENT MODES

JUSTIFICATION: FOR MANY TIRE PROBLEMS THE LARGE NUMBER OF D.O.F. $\{X\}$ IS DICTATED BY THE COMPLEX TOPOLOGY OF THE TIRE (DISCONTINUITIES IN GEOMETRY, LAMINATION, ETC.) RATHER THAN BY EXPECTED COMPLEXITY OF BEHAVIOR

SELECTION OF GLOBAL DISPLACEMENT MODES :

$$[\Gamma] = \left[\begin{array}{c|c|c} \left\{\frac{\partial X}{\partial p}\right\} & \left\{\frac{\partial^2 X}{\partial p^2}\right\} & \left\{\frac{\partial^3 X}{\partial p^3}\right\} \dots \end{array} \right]$$

p = LOAD PARAMETER

- COLUMNS OF $[\Gamma]$ GENERATED BY USING THE ORIGINAL FINITE ELEMENT MODEL OF THE TIRE
- THEIR GENERATION REQUIRES ONLY ONE 'LARGE MATRIX' FACTORIZATION
- NUMERICAL EXPERIMENTS HAVE DEMONSTRATED THEIR EFFECTIVENESS

Figure 7

BASIC EQUATIONS USED IN REDUCTION METHODS
FOR NONLINEAR TIRE PROBLEMS

The basic equations used in the reduction methods for geometrically nonlinear tire problems are given in Fig. 8. It is worth noting that the original displacement unknowns $\{X\}$ can be on the order of thousands whereas the reduced unknowns $\{\psi\}$ are typically 20 or less. This is true regardless of the complexity of the structure and/or loading. The details of the computational procedure for tracing the load-deflection paths in geometrically nonlinear static analysis are given in Refs. 11 and 12.

	ACTUAL (LARGE) PROBLEM	REDUCED (SMALL) PROBLEM
FUNDAMENTAL UNKNOWNNS	$\{X\} = \text{INDIVIDUAL DISPLACEMENTS}$ <ul style="list-style-type: none"> • THOUSANDS OF UNKNOWNNS 	$\{X\} = [\Gamma] \{\psi\}$ → AMPLITUDES OF DISPLACEMENT MODES <ul style="list-style-type: none"> • TWENTY OR LESS
GOVERNING EQUATIONS	$[K]\{X\} + \{G(X)\} - p\{P\} = 0$ ~ 1000 EQUATIONS	$[\tilde{K}]\{\psi\} + \{\tilde{G}(\psi)\} - p\{\tilde{P}\} = 0$ ~ 20 EQUATIONS
HOW TO TRACE LOAD-DEFLECTION PATH	<ul style="list-style-type: none"> • REPEATED SOLUTION OF LARGE SYSTEMS OF SIMULTANEOUS NONLINEAR ALGEBRAIC EQUATIONS 	<ul style="list-style-type: none"> • GENERATION OF $[\Gamma]$ • MARCHING WITH SMALL SYSTEM OF EQUATIONS • ERROR SENSING AND CONTROL (UPDATING $[\Gamma]$ WHENEVER NEEDED)

$$[\tilde{K}] = [\Gamma]^T [K] [\Gamma] \quad , \quad \{\tilde{G}(\psi)\} = [\Gamma]^T \{G(X)\} \quad , \quad \{\tilde{P}\} = [\Gamma]^T \{P\}$$

Figure 8

APPLICATION OF REDUCTION METHODS TO GEOMETRICALLY NONLINEAR
ANALYSIS OF A TIRE SUBJECTED TO UNIFORM INTERNAL PRESSURE

As a simple application of reduction methods to the geometrically nonlinear analysis of tires, consider the laminated anisotropic elliptic toroidal shell shown in Fig. 9. Due to axial symmetry, only one meridian was modeled using four-noded elements with cubic Lagrangian interpolation functions for all the displacement and rotation degrees of freedom. The high accuracy of the total strain energy obtained by using six basis vectors is demonstrated in Fig. 9.

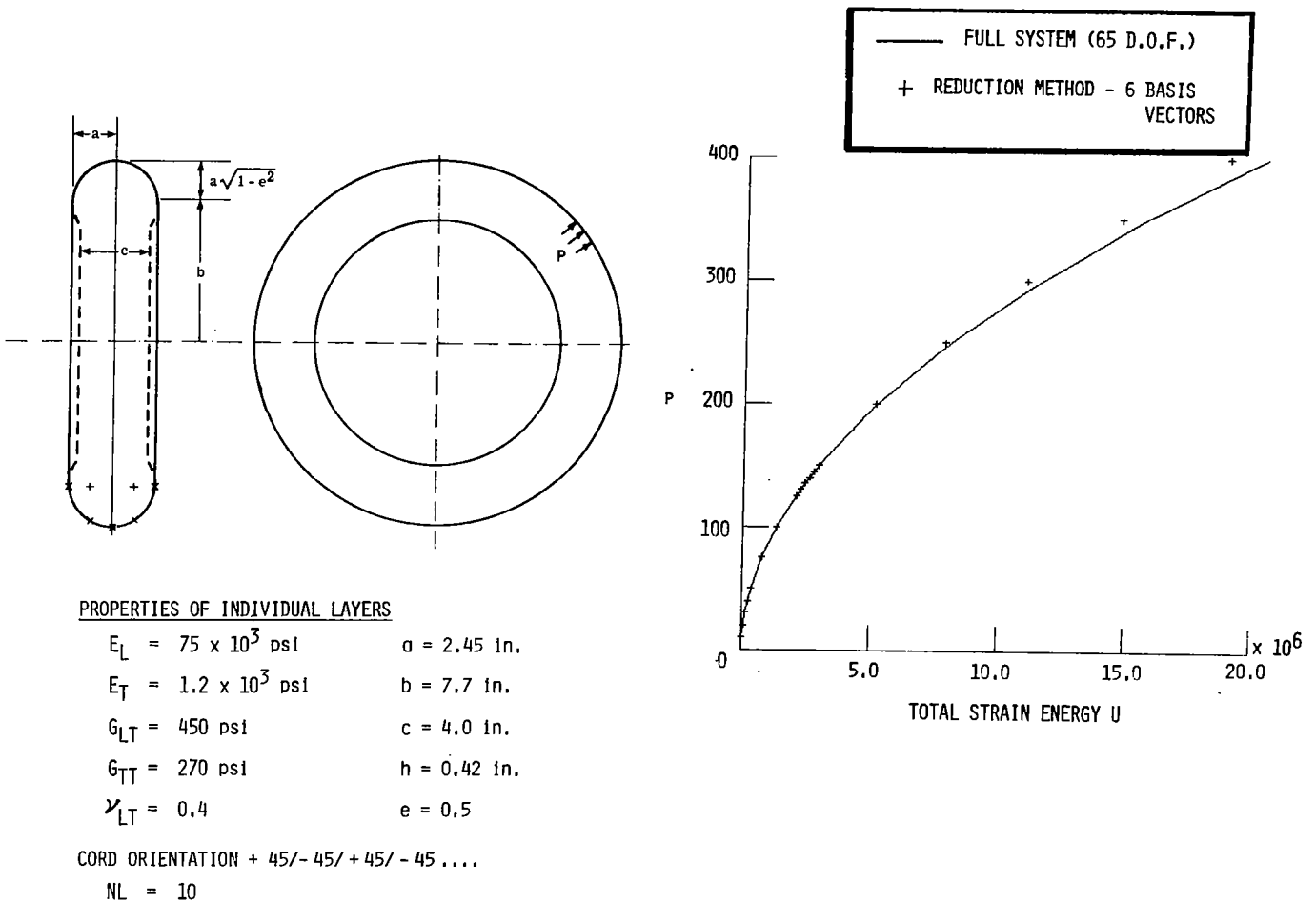


Figure 9

TREATMENT OF COMBINED LOADS

The basic equations and the computational procedure used in applying reduction methods to the analysis of tires subjected to combined loads are highlighted in Figs. 10 and 11. For simplicity, only two independent loads are considered.

First, the original finite element equations are given. The external loading is normalized with respect to two independent load parameters p_1 and p_2 . The basis reduction is done as before, via the transformation shown in Fig. 10. Then the Rayleigh-Ritz technique is used to approximate the original set of finite element equations by a reduced system of equations in the new unknown parameters $\{\psi\}$. The number of these equations is considerably less than that of the original equations.

As previously noted, the crux of reduction methods is the proper selection of the transformation matrix $[\Gamma]$. In the case of combined loading, the columns of the matrix $[\Gamma]$ are selected to be the various-order derivatives of the displacement vector $\{X\}$ with respect to the two independent parameters p_1 and p_2 .

To trace the different nonlinear paths, corresponding to different combinations of the independent load parameters, the basis vectors are evaluated for the unloaded structure ($p_1 = p_2 = 0$), and the corresponding reduced equations are generated. The different nonlinear paths of the tire are obtained by fixing one of the load parameters, varying the other, and repeating the process with different values of the first load parameter. This is all done using the *same set of reduced equations*. The total cost of the analysis, to a first approximation, is little more than the cost of one linear solution of the original, full system of finite element equations. The procedure is described in detail in Ref. 14.

As a by-product of this technique, a considerable reduction can be made in the size of the analysis model used in studying the nonlinear response of tires subjected to asymmetric loading. This can be accomplished by decomposing the loading into symmetric and antisymmetric components and treating each as an independent loading.

REDUCTION METHODS FOR NONLINEAR PROBLEMS TREATMENT OF COMBINED LOADS

GOVERNING FINITE ELEMENT EQUATIONS

$$[K]\{x\} + \{G(x)\} - p_1\{P^{(1)}\} - p_2\{P^{(2)}\} = 0$$

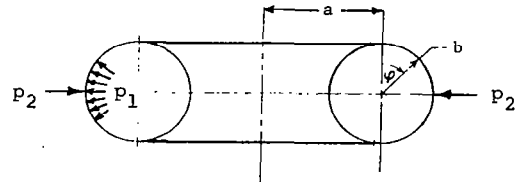
$[K]$ = LINEAR GLOBAL STIFFNESS MATRIX

$\{x\}$ = VECTOR OF NODAL DISPLACEMENTS

$\{G(x)\}$ = VECTOR OF NONLINEAR TERMS

$\{P^{(1)}\}, \{P^{(2)}\}$ = NORMALIZED LOAD VECTORS

p_1, p_2 = INDEPENDENT LOAD PARAMETERS



BASIS REDUCTION

$$\{x\}_{n,1} = [\Gamma]_{n,r} \{\psi\}_{r,1} \quad , \quad r \ll n$$

REDUCED SYSTEM OF EQUATIONS

$$[\tilde{K}]\{\psi\} + \{\tilde{G}(\psi)\} - p_1\{\tilde{P}^{(1)}\} - p_2\{\tilde{P}^{(2)}\} = 0$$

Figure 10

TREATMENT OF COMBINED LOADS

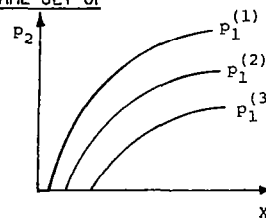
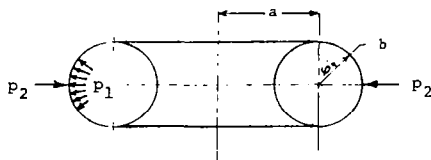
SELECTION OF BASIS VECTORS

$$\{x\} = [\Gamma]\{\psi\}$$

$$[\Gamma] = \left[\left\{ \frac{\partial x}{\partial p_1} \right\} \left\{ \frac{\partial x}{\partial p_2} \right\} \left\{ \frac{\partial^2 x}{\partial p_1^2} \right\} \left\{ \frac{\partial^2 x}{\partial p_1 \partial p_2} \right\} \left\{ \frac{\partial^2 x}{\partial p_2^2} \right\} \dots \right]$$

COMPUTATIONAL PROCEDURE

- EVALUATE BASIS VECTORS AT $p_1 = p_2 = 0$ (UNLOADED TIRE) AND GENERATE REDUCED EQUATIONS
- TRACE DIFFERENT EQUILIBRIUM PATH BY FIXING ONE OF THE LOAD PARAMETERS AND VARYING THE OTHER (USING THE SAME SET OF REDUCED EQUATIONS)



NOTE: THIS APPROACH CAN BE USED TO REDUCE THE SIZE OF ANALYSIS MODELS FOR THE CASE OF UNSYMMETRIC LOADINGS.

Figure 11

TREATMENT OF DISPLACEMENT-DEPENDENT
AND NONCONSERVATIVE LOADING

The basic equations used in applying reduction methods to the nonlinear analysis of tires subjected to displacement-dependent loading are given in Fig. 12.

First, the governing finite element equations for the total Lagrangian formulation are shown. The only new term in these equations is the pressure stiffness matrix which represents the follower-load effect and is unsymmetric for nonconservative loadings. The basis reduction is done and the reduced equations are obtained in the manner outlined previously. The following two important facts are to be noted:

1. The basis vectors are evaluated for the unloaded structure. Hence, the pressure stiffness matrix does not enter into the left-hand side and *only the linear symmetric global stiffness matrix needs to be decomposed.*

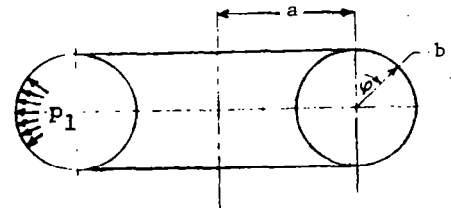
2. Since the reduced equations are small in number (on the order of ten or less) no symmetrization is needed in the case of nonconservative loadings.

GOVERNING FINITE ELEMENT EQUATIONS

FOR A TOTAL LAGRANGIAN FORMULATION

$$[K] - p[K^{(p)}] \{x\} + \{G(x)\} - p\{P\} = 0$$

$[K^{(p)}]$ = PRESSURE STIFFNESS MATRIX (UNSYMMETRIC FOR
NONCONSERVATIVE LOADING)



BASIS REDUCTION

$$\{x\} = [\Gamma]\{\psi\}$$

REDUCED SYSTEM OF EQUATIONS

$$[\tilde{K}] - p[\tilde{K}^{(p)}]\{\psi\} + \{\tilde{G}(\psi)\} - p\{\tilde{P}\} = 0$$

NOTES:

- BASIS VECTORS ARE EVALUATED AT $p = 0$. THEREFORE, ONLY THE SYMMETRIC $[K]$ MATRIX NEEDS TO BE DECOMPOSED.
- REDUCED EQUATIONS ARE SMALL IN NUMBER (~ 10). THEREFORE, NO SYMMETRIZATION IS NEEDED.

Figure 12

APPLICATION OF REDUCTION METHODS TO THE BIFURCATION BUCKLING
ANALYSIS OF A RING SUBJECTED TO HYDROSTATIC PRESSURE

As a simple application of reduction methods to structures subjected to displacement-dependent loadings, consider the circular ring subjected to hydrostatic pressure shown in Fig. 13.

Doubly-symmetric buckling modes are considered; hence, only one quadrant of the ring was analyzed using higher-order shear-flexible elements with a total of 59 non-zero degrees of freedom. The lowest three buckling loads obtained using three, four and five vectors are listed in Fig. 13. The lowest buckling load obtained by using four vectors agrees, to five significant digits, with that obtained using the full system of equations. With five vectors, the error in the third buckling load is less than 3%.

NUMBER OF BASIS VECTORS	EIGENVALUES $\hat{p} = \frac{p_0 R^3}{EI}$		
	\hat{p}_1	\hat{p}_2	\hat{p}_3
3	2.9998	15.1888	
4	2.9997	14.9964	36.5126
5	2.9997	14.9957	35.9668
FULL SYSTEM (59 D.O.F.)	2.9997	14.9947	34.9751

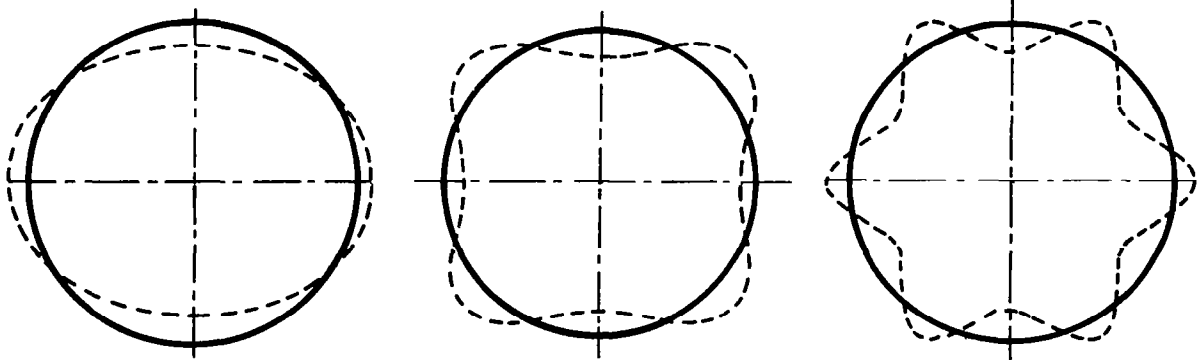
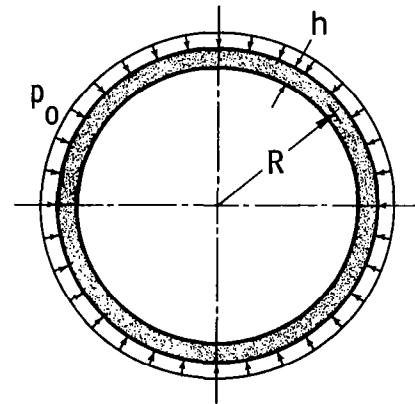


Figure 13

REDUCTION METHODS FOR NONLINEAR DYNAMIC PROBLEMS

The application of reduction methods to transient tire problem is highlighted in Fig. 14. First, the governing semi-discrete finite element equations are given for the case of no damping. Then the key elements for an effective reduction method are listed (see Ref. 13). They include:

- 1) The proper selection of basis vectors (the columns of the matrix $[\Gamma]$)
- 2) Characterization of nonlinear dynamic response by means of one or few scalars
- 3) Sensing and controlling the error in the reduced system of equations

GOVERNING SEMI-DISCRETE FINITE ELEMENT EQUATIONS

$$[M] \{\ddot{X}\} = \{P\}_t - [K] \{X\} - \{G(X)\}$$

KEY ELEMENTS FOR EFFECTIVE REDUCTION METHOD

● PROPER SELECTION OF BASIS VECTORS

$$\{X\}_{n,1} = [\Gamma]_{n,r} \{\psi\}_{r,1} \cdot r \ll n$$

● CHARACTERIZATION OF NONLINEAR DYNAMIC RESPONSE BY MEANS OF ONE OR FEW SCALARS

● SENSING AND CONTROLLING THE ERROR IN THE REDUCED SYSTEM OF EQUATIONS

Figure 14

SELECTION OF BASIS VECTORS FOR THE CASE OF STEP LOADING

A particular choice of basis vectors which was found to work well for the case of step loading is shown in Fig. 15. The vectors consist of a few eigenvectors of the linear problem and a few eigenvectors of the steady-state (static) nonlinear problem. The matrix $\left[\frac{\partial G_i}{\partial X_j} \right]$ is obtained by using the steady-state (static) nonlinear solution. Reduction methods can be used to reduce the computational effort required for generating the steady-state nonlinear solution.

BASIS VECTORS CONSIST OF:

- FEW EIGENVECTORS OF LINEAR PROBLEM

$$[K] \{X\} = \lambda [M] \{X\}$$

- FEW EIGENVECTORS OF STEADY-STATE (STATIC) NONLINEAR PROBLEM

$$\left[[K] + \left[\frac{\partial G_i}{\partial X_j} \right] \right] \{X\} = \lambda [M] \{X\}$$

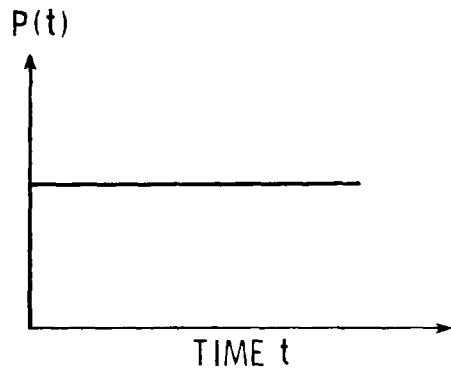


Figure 15

APPLICATION TO CLAMPED SHALLOW SPHERICAL CAP SUBJECTED
TO A CONCENTRATED LOAD AT THE APEX

As a simple application of reduction methods to nonlinear dynamic problems, consider the clamped spherical cap subjected to a concentrated load which has a step variation in time (see Fig. 16). The displacement time history obtained using the full system of finite element equations, the reduced system with ten linear vibration modes, and the reduced system with the proposed set of modes are shown in Fig. 16. The basis vectors (eigenmodes) were not updated throughout the analysis. As can be seen from Fig. 16, the proposed set of basis vectors predicts qualitatively the correct response. The phase shift was almost eliminated by increasing the number of basis vectors to 14.

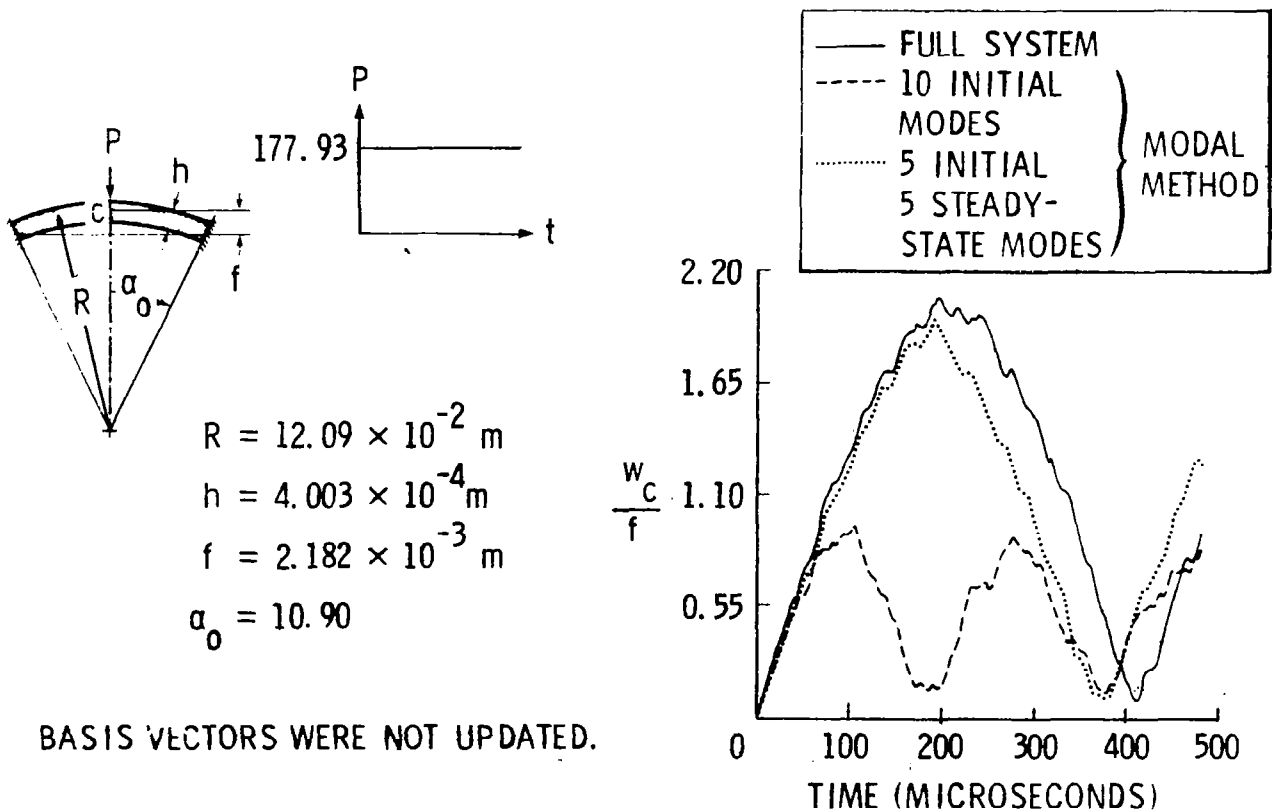


Figure 16

MIXED FINITE ELEMENT MODELS FOR TIRES

The second topic considered in this paper is mixed finite element models for tires. The basic features of the mixed models are outlined in Fig. 17. The finite element models include the effects of both laminated anisotropic construction and transverse shear deformation, and allow the geometric and material properties to vary within individual elements. The fundamental unknowns consist of the eight stress resultants and the five generalized displacements. The stress resultants are discontinuous at element interfaces, and therefore can be eliminated on the element level.

- TIRE MODELED USING LAMINATED ANISOTROPIC, SHEAR-FLEXIBLE, DEEP SHELL ELEMENTS WITH VARIABLE GEOMETRIC AND MATERIAL PROPERTIES

- FUNDAMENTAL UNKNOWNNS ARE:

STRESS RESULTANTS

$N_{\alpha\beta}$, $M_{\alpha\beta}$, Q_{α}

GENERALIZED DISPLACEMENTS

u_{α} , w , ϕ_{α}

- STRESS RESULTANTS ARE DISCONTINUOUS AT ELEMENT INTERFACES - ELIMINATED ON ELEMENT LEVEL

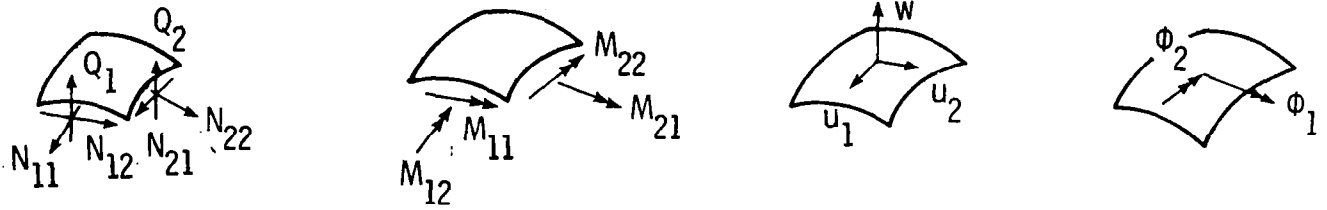


Figure 17

MATHEMATICAL FORMULATION FOR THE MIXED MODEL

The mathematical formulation for the two-field mixed model is based on the use of a moderate-rotation nonlinear shell theory in conjunction with the Hellinger-Reissner mixed variational principle. The basic features of this formulation are outlined in Fig. 18. Different approximation functions are used for each of the stress-resultant fields and the generalized displacement field. The governing finite element equations for individual elements can be partitioned as shown in Fig. 18. The vector $\{M\}$ is quadratic in $\{X\}$ and the vector $\{G\}$ is bilinear in $\{H\}$ and $\{X\}$.

For mixed models with discontinuous stress resultants at element interfaces, the stress resultants can be eliminated on the element level and the governing finite element equations reduce to cubic equations in $\{X\}$ (see Ref. 15).

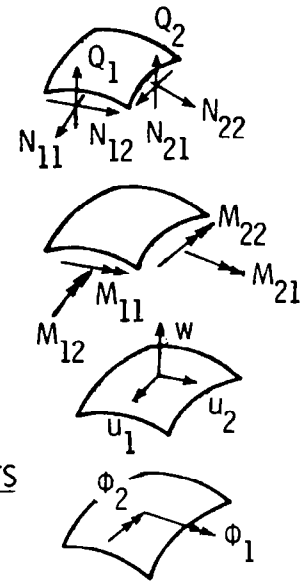
APPROXIMATION FUNCTIONS

STRESS RESULTANTS

$$\begin{Bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ Q_{\alpha} \end{Bmatrix} = [\bar{N}]\{H\} \quad , \quad \{H\} = \text{VECTOR OF STRESS RESULTANT PARAMETERS}$$

DISPLACEMENTS

$$\begin{Bmatrix} u_{\alpha} \\ w \\ \phi_{\alpha} \end{Bmatrix} = [N]\{X\} \quad , \quad \{X\} = \text{VECTOR OF NODAL DISPLACEMENTS}$$



GOVERNING FINITE ELEMENT EQUATIONS FOR INDIVIDUAL ELEMENTS

$$\begin{bmatrix} -[F] & [S] \\ [S]^T & [0] \end{bmatrix} \begin{Bmatrix} \{H\} \\ \{X\} \end{Bmatrix} + \begin{Bmatrix} \{M(X)\} \\ \{G(H, X)\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{P\} \end{Bmatrix}$$

WHERE $\{M(X)\}$ AND $\{G(H, X)\}$ ARE VECTORS OF NONLINEAR TERMS (QUADRATIC AND BILINEAR IN $\{H\}$ AND $\{X\}$).

DISCONTINUOUS STRESS RESULTANTS AT ELEMENT INTERFACES

$$\{H\} = [F]^{-1} [S]\{X\} + [F]^{-1} \{M(X)\}$$

AND GOVERNING FINITE ELEMENT EQUATIONS REDUCE TO

$$[S]^T [F]^{-1} [S]\{X\} + \{\bar{G}(X)\} = \{P\}$$

WHERE $\{\bar{G}(X)\} =$ VECTOR OF NONLINEAR TERMS (CUBIC IN $\{X\}$).

Figure 18

REDUCED/SELECTIVE INTEGRATION DISPLACEMENT MODELS

In recent years a class of displacement models with a performance comparable to that of the mixed model has been developed. These are referred to as reduced/selective integration displacement models (see, for example, Refs. 15 and 16). The major features of these models are outlined in Fig. 19. The governing finite element equations for the individual elements are cubic in $\{X\}$. The definitions of full, reduced, and selective integration are given in Fig. 19. If an $n \times n$ Gauss-Legendre formula is used to integrate the linear stiffness matrix $[K]$ exactly for parallelogram elements, then in full integration $n \times n$ quadrature points are used. In reduced integration $(n-1) \times (n-1)$ quadrature points are used, and in selective integration $n \times n$ quadrature points are used for some terms of $[K]$ and $(n-1) \times (n-1)$ points for other terms.

GOVERNING FINITE ELEMENT EQUATIONS FOR INDIVIDUAL ELEMENTS

$$[K]\{X\} + \{G(X)\} = \{P\}$$

$$\{X\} = \text{VECTOR OF NODAL DISPLACEMENTS}$$

$$\{G(X)\} = \text{VECTOR OF NONLINEAR TERMS (CUBIC IN } \{X\})$$

FULL (NORMAL), REDUCED AND SELECTIVE INTEGRATION

- IF $n \times n$ GAUSS-LEGENDRE FORMULA IS USED TO INTEGRATE $[K]$ EXACTLY FOR RECTANGULAR (OR PARALLELOGRAM) ELEMENTS
 - **FULL (NORMAL) INTEGRATION** USES $n \times n$ QUADRATURE POINTS
 - **REDUCED INTEGRATION** USES $(n-1) \times (n-1)$ QUADRATURE POINTS
 - **SELECTIVE INTEGRATION** USES $n \times n$ POINTS FOR SOME TERMS AND $(n-1) \times (n-1)$ POINTS FOR OTHER TERMS OF $[K]$

Figure 19

EQUIVALENT FINITE ELEMENT MODELS

The equivalence between finite element models is defined in Fig. 20. Finite elements are equivalent if their individual governing equations, when expressed in terms of a common set of nodal variables and/or parameters, are identical (see Ref. 16). It is important to note that the other parameters not contained in the common set are local to the individual elements. For nearly equivalent models, the finite element equations are almost identical.

DEFINITION OF EQUIVALENCE

- TWO FINITE ELEMENT MODELS ARE EQUIVALENT IF THEIR GOVERNING FINITE ELEMENT EQUATIONS, WHEN EXPRESSED IN TERMS OF A COMMON SET OF NODAL VARIABLES AND/OR PARAMETERS, ARE IDENTICAL.

THE OTHER NODAL VARIABLES AND/OR PARAMETERS NOT CONTAINED IN THE COMMON SET MUST BE LOCAL TO THE INDIVIDUAL ELEMENTS (I.E., DO NOT AFFECT THE ASSEMBLY PROCESS).

- NEARLY EQUIVALENT MODELS ARE ONES FOR WHICH THE GOVERNING FINITE ELEMENT EQUATIONS ARE ALMOST IDENTICAL.

Figure 20

EQUIVALENT MIXED AND DISPLACEMENT MODELS

The governing finite element equations of both mixed and displacement models are shown in Fig. 21 and the mathematical requirements for the equivalence of the two models are listed. The table given in Fig. 21 lists examples of equivalent quadrilateral (in planform) mixed and displacement models. The following can be noted:

1) Equivalent mixed and displacement models have the same number of displacement nodes, use the same approximation functions for the generalized displacements, and use the same number of numerical quadrature points.

2) The symbol (F) refers to full integration and (R) refers to reduced integration.

3) If the geometric and material characteristics within the individual elements are constants, the number of quadrature points listed in the table generates exact integrals for the mixed models and only approximate integrals for the displacement models.

EQUIVALENCE

MIXED MODEL WITH DISCONTINUOUS STRESS RESULTANTS	DISPLACEMENT MODEL
GOVERNING FINITE ELEMENT EQUATIONS	
$[S]^T [F]^{-1} [S] \{X\} + \{\bar{\sigma}(X)\} = \{P\}$	$[K] \{X\} + \{G(X)\} = \{P\}$
FROM WHICH	
$[S]^T [F]^{-1} [S] = [K]$	$\{\bar{\sigma}(X)\} = \{G(X)\}$

NEAR EQUIVALENCE = IS REPLACED BY \approx

EXAMPLES OF EQUIVALENT QUADRILATERAL ELEMENTS




NUMBER OF DISPLACEMENT NODES	QUADRATURE POINTS	NUMBER OF STRESS RESULTANT PARAMETERS	
4	2 × 2 (F) 1 (R)	4 1	
9	3 × 3 (F) 2 × 2 (R)	9 4	
16	4 × 4 (F) 3 × 3 (R)	16 9	

Figure 21

NONLINEAR RESPONSE OF CIRCULAR TOROIDAL SHELL
SUBJECTED TO UNIFORM EXTERNAL PRESSURE

To assess the accuracy of the mixed models with discontinuous stress resultants at interelement boundaries, the nonlinear response of the circular toroidal shell shown in Fig. 22 is analyzed using these models. The solutions obtained using six and eight finite elements with nine displacement nodes and four stress nodes are compared with the converged solution in Fig. 22.

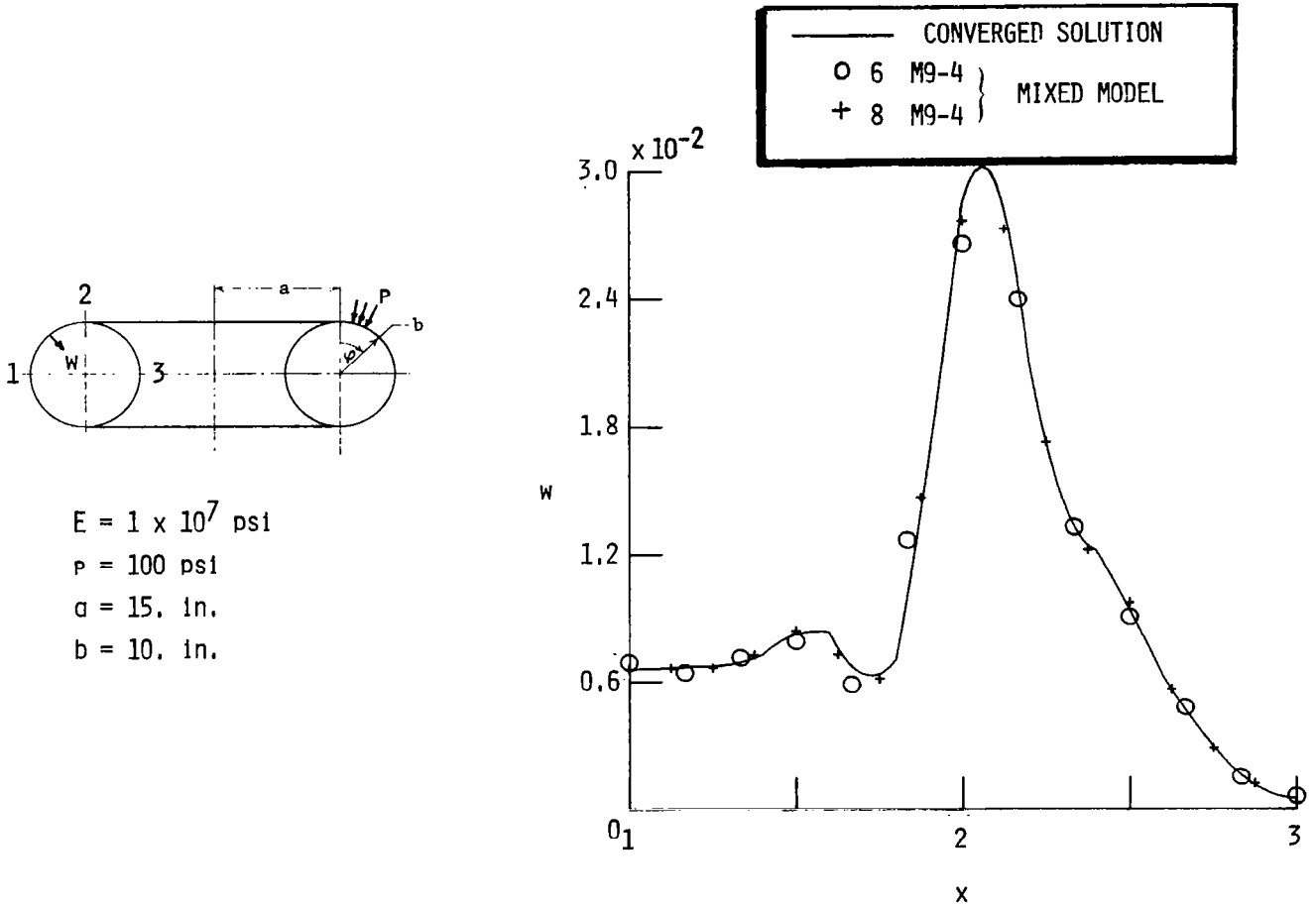


Figure 22

CLAMPED CYLINDRICAL SHELLS SUBJECTED
TO UNIFORM PRESSURE LOADING

To assess the accuracy of the different displacement and mixed models, the large-deflection nonlinear response of the clamped cylindrical panel shown in Fig. 23 is analyzed using these models. The solutions obtained using a 4 x 4 grid of four-noded quadrilateral elements are shown in Fig. 23. The solutions obtained using a 2 x 2 grid of nine-noded quadrilateral elements are shown in Fig. 24.

As is to be expected, the full-integration four-noded displacement model is too stiff. The full-integration nine-noded displacement model (with the same total number of degrees of freedom), though less stiff than the four-noded model, overestimates the stiffness, particularly at higher loads. The mixed model with discontinuous stress resultants is more accurate than the mixed models with continuous stress resultants developed in Refs. 17 and 18.

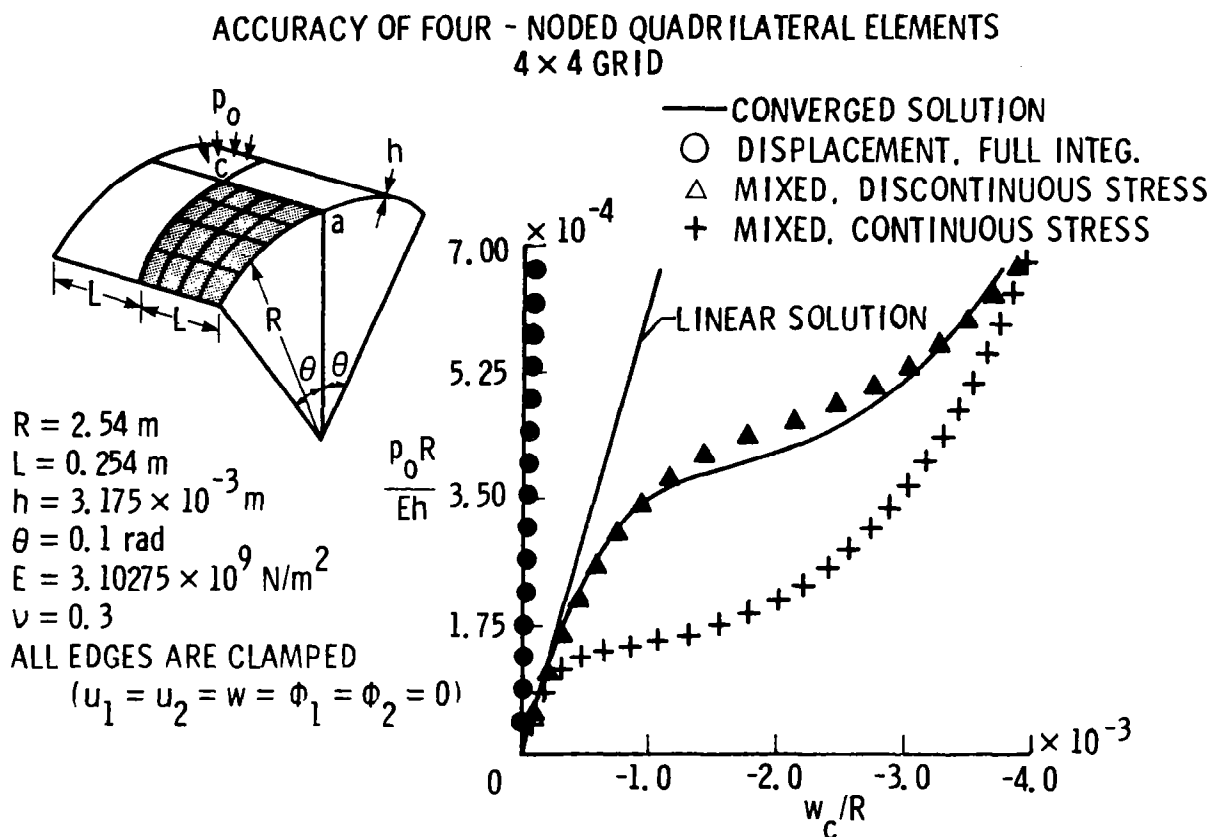


Figure 23

CLAMPED CYLINDRICAL PANEL

ACCURACY OF NINE - NODED QUADRILATERAL ELEMENTS

2 × 2 GRID

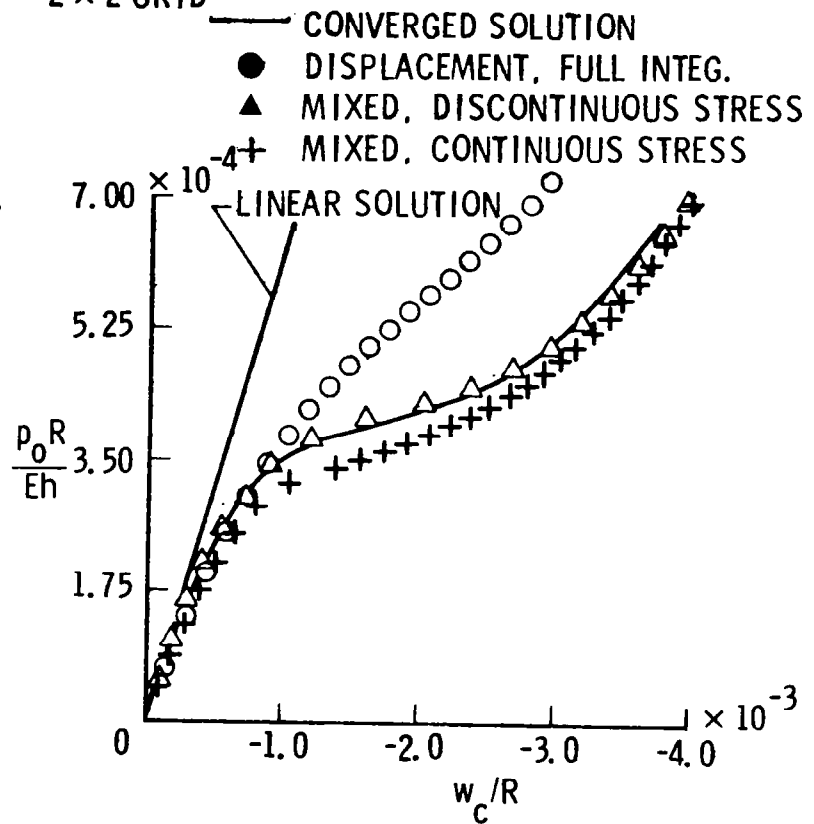
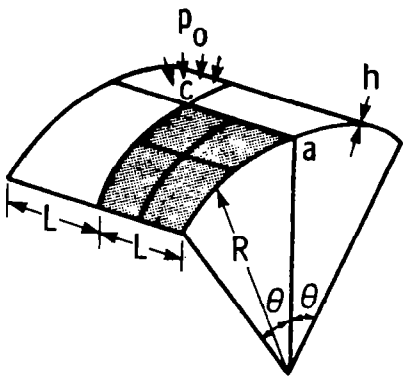


Figure 24

USE OF REDUCTION METHODS IN CONJUNCTION
WITH MIXED MODELS

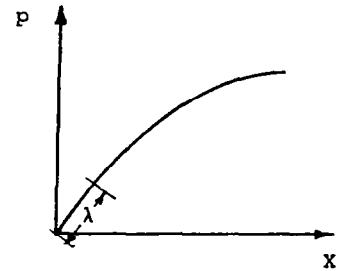
The use of reduction methods in conjunction with mixed models is outlined in Fig. 25. First, the governing finite element equations for the individual elements are given. Then, the vectors of fundamental unknowns (stress resultants and displacements) are expressed as linear combinations of a small number of vectors. The basis reduction and reduced system of equations are obtained in the manner outlined previously. It is important to note that *the reduced equations are quadratic in the reduced unknowns* $\{\phi\}$.

GOVERNING FINITE ELEMENT EQUATIONS FOR INDIVIDUAL ELEMENTS

$$\begin{bmatrix} -[F] & [S] \\ [S]^T & [0] \end{bmatrix} \begin{Bmatrix} H \\ X \end{Bmatrix} + \begin{Bmatrix} M(X) \\ G(H, X) \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

BASIS REDUCTION

$$\begin{Bmatrix} H \\ X \end{Bmatrix}_{n,1} = \begin{bmatrix} \Gamma_H \\ \Gamma_X \end{bmatrix}_{n,r} \{\phi\}_{r,1}, \quad r \ll n$$



WHERE

$$\begin{bmatrix} \Gamma_H \\ \Gamma_X \end{bmatrix} = \begin{bmatrix} \begin{Bmatrix} H \\ X \end{Bmatrix} \\ \frac{\partial}{\partial P} \begin{Bmatrix} H \\ X \end{Bmatrix} \\ \dots \\ \frac{\partial^{r-1}}{\partial P^{r-1}} \begin{Bmatrix} H \\ X \end{Bmatrix} \end{bmatrix}$$

$\{\phi\}$ = VECTOR OF UNDETERMINED COEFFICIENTS

P = LOAD PARAMETER

REDUCED SYSTEM OF EQUATIONS

$$\tilde{[F]}\{\phi\} + \tilde{[M]}\{\phi\} = P\tilde{[P]}$$

WHERE

$$\tilde{[F]} = \sum_{\text{elements}} -[\Gamma_H]^T [F] [\Gamma_H] + [\Gamma_H]^T [S] [\Gamma_X] + [\Gamma_X]^T [S]^T [\Gamma_H]$$

$$\tilde{[M]} = \sum_{\text{elements}} [\Gamma_H]^T \{M(X)\} + [\Gamma_X]^T \{G(H, X)\}$$

$\{\phi\}$ = VECTOR OF UNDETERMINED COEFFICIENTS OR PARAMETERS

Figure 25

USE OF REDUCTION METHODS IN CONJUNCTION
WITH DISPLACEMENT MODELS

The use of reduction methods in conjunction with displacement models, which are equivalent to the proposed mixed models, is outlined in Fig. 26. Note that the resulting reduced system of equations is cubic in the reduced unknowns $\{\phi\}$. The implication of this is that even if the mixed model and the displacement model are equivalent their reduced systems are not equivalent.

GOVERNING FINITE ELEMENT EQUATIONS FOR INDIVIDUAL ELEMENTS

$$[K] \{X\} + \{G(X)\} = p \{P\}$$

BASIS REDUCTION

$$\{X\}_{n,1} = [\Gamma_X]_{n,r} \{\phi\}_{r,1} \quad r \ll n$$

$$[\Gamma_X] = \left[\begin{array}{c} \{X\} \\ \frac{\partial}{\partial p} \{X\} \dots \dots \frac{\partial^{r-1}}{\partial p^{r-1}} \{X\} \end{array} \right]$$

$\{\phi\}$ = VECTOR OF UNDETERMINED COEFFICIENTS
OR PARAMETERS

p = LOAD PARAMETER

REDUCED SYSTEM OF EQUATIONS

$$[\tilde{K}]\{\phi\} + \{\tilde{G}(\phi)\} = p \{\tilde{P}\}$$

WHERE $[\tilde{K}] = [\Gamma_X]^T [K] [\Gamma_X]$

$$\{\tilde{G}(\phi)\} = [\Gamma_X]^T \{G(\phi)\}$$

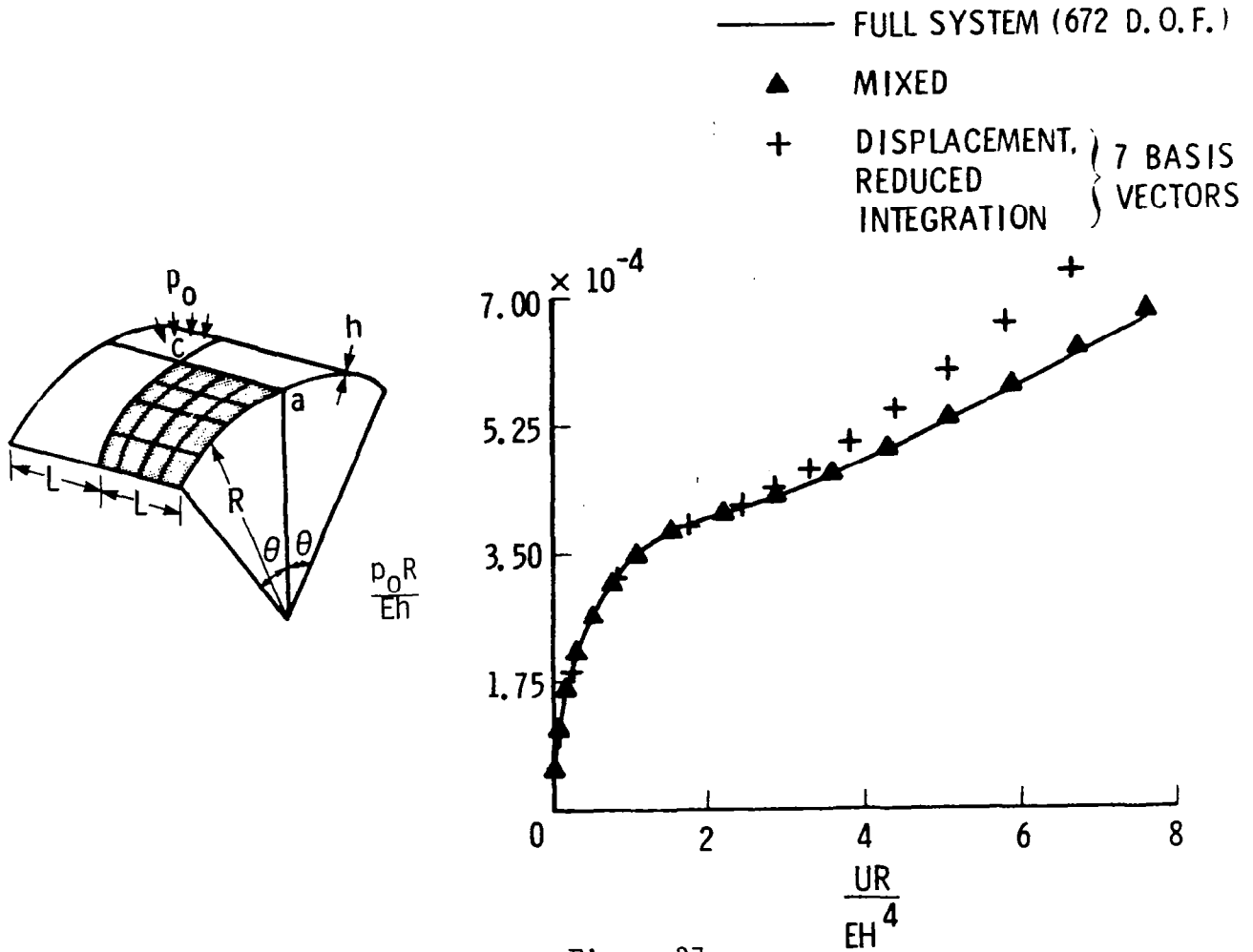
= VECTOR OF NONLINEAR TERMS (CUBIC IN (ϕ))

**NOTE: EVEN IF MIXED MODEL AND DISPLACEMENT MODEL ARE EQUIVALENT,
THEIR REDUCED SYSTEMS ARE NOT EQUIVALENT.**

Figure 26

ACCURACY OF REDUCTION METHOD - MIXED AND DISPLACEMENT MODELS

The nonlinear solutions obtained using the reduction method in conjunction with the equivalent mixed and reduced-integration displacement models are compared in Fig. 27 for the case of a clamped cylindrical shell subjected to uniform pressure loading. Seven basis vectors were generated for the unloaded shell. The variations of the strain energy with the loading, as predicted by the reduction method mixed and displacement models, are shown in Fig. 27. The high accuracy of the predictions of the mixed model is clearly seen in this figure.



ADVANTAGES OF MIXED MODELS OVER EQUIVALENT DISPLACEMENT MODELS

The advantages of mixed models over the equivalent displacement models are listed in Fig. 28. These include:

- 1) *Simplicity of formulation.* Only quadratic and bilinear terms appear in the governing finite element equations. By contrast, the governing equations of the displacement model include cubic terms.
- 2) If the geometric and material characteristics are constants within each element, then *most of the integrals can be evaluated exactly* for the mixed elements (even when the element has curved faces and edges).
- 3) The mixed models are *better suited for use with reduction methods* in non-linear problems, in the sense that:
 - a) The basis vectors are simpler to generate.
 - b) The mixed models lead to higher accuracy of the solutions obtained by the reduced system. This is especially true for stress resultants.

- SIMPLICITY OF FORMULATION (ONLY QUADRATIC AND BILINEAR TERMS APPEAR IN GOVERNING FINITE ELEMENT EQUATIONS)
- MOST OF THE INTEGRALS CAN BE EVALUATED EXACTLY (EVEN FOR ELEMENTS WITH CURVED FACES AND EDGES)
- BETTER SUITED FOR USE WITH REDUCTION METHODS
 - BASIS VECTORS ARE SIMPLER TO GENERATE
 - BETTER APPROXIMATION PROPERTIES (HIGHER ACCURACY OF REDUCED SYSTEM, ESPECIALLY FOR STRESS RESULTANTS)

Figure 28

LARGE ROTATION NONLINEAR PROBLEMS

The third topic considered in this paper is the large rotation nonlinear problems. The basic features of two effective computational models are outlined in Fig. 29. In both models a *total Lagrangian* description of the deformation is used. Consequently, the strain-displacement relations contain *trigonometric functions of the rotation components*.

The first computational model is a two-field mixed model with discontinuous stress-resultant fields at interelement boundaries. The second model is based on the use of the penalty method for handling the trigonometric functions, thereby simplifying the analysis.

FORMULATION

- TOTAL LAGRANGIAN DESCRIPTION OF DEFORMATION
- STRAIN - DISPLACEMENT RELATIONS CONTAIN TRIGONOMETRIC FUNCTIONS OF ROTATION COMPONENTS

FINITE ELEMENT MODELING

- MIXED MODELS WITH DISCONTINUOUS STRESS RESULTANTS
- PENALTY METHOD FOR HANDLING TRIGONOMETRIC FUNCTIONS

Figure 29

ELASTICA PROBLEM - FORMULATION

As an application of the proposed computational models, consider the elastica problem shown in Fig. 30. In the mixed formulation, the Hellinger-Reissner two-field mixed variational principle is used. Transverse shear deformation, though small, is included to simplify the formulation. The extensional strain ϵ and the transverse shear strain γ are trigonometric functions of the rotation ϕ .

The penalty formulation, on the other hand, is based on the Euler-Bernoulli type beam theory with both the extensional and transverse shearing strains neglected. The axial and transverse displacements u and w are incorporated into the functional through the use of constraints and penalty numbers.

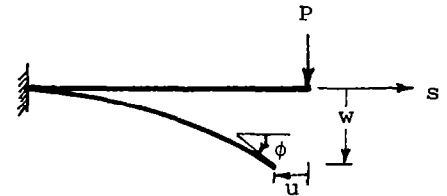
MIXED FORMULATION

$$\pi = \int [(N \epsilon + M \kappa + Q \gamma) - \frac{1}{2} (\frac{N^2}{EA} + \frac{M^2}{EI} + \frac{Q^2}{GA})] ds$$

ϵ, γ INCLUDE TRIGONOMETRIC FUNCTIONS OF ϕ

$$\kappa = \frac{d\phi}{ds}$$

N, M, Q ARE DISCONTINUOUS AT INTERELEMENT BOUNDARIES



PENALTY FORMULATION

$$\pi = \int [\frac{1}{2} EI (\frac{d\phi}{ds})^2 + \lambda_1 (\frac{dw}{ds} - \sin \phi)^2 + \lambda_2 (\frac{du}{ds} - 1 + \cos \phi)^2] ds$$

WHERE λ_1, λ_2 ARE PENALTY NUMBERS

THE RESULTING STIFFNESS MATRIX IS POSITIVE DEFINITE.

Figure 30

ELASTICA PROBLEM - NUMERICAL RESULTS

The displacements, rotations and total strain energy obtained by using *four two-noded elements and two three-noded elements* are depicted in Figs. 31 and 32. Also, the deformed configurations of the beam for various values of the transverse load P are shown in Fig. 31. Both exact-integration displacement models (DE models) and mixed models with discontinuous forces (MD models) are used. As to be expected, the displacement models are too stiff. This is particularly true for the two-noded elements. By contrast, the predictions of the mixed models are highly accurate.

LARGE-ROTATION ELASTICA PROBLEM

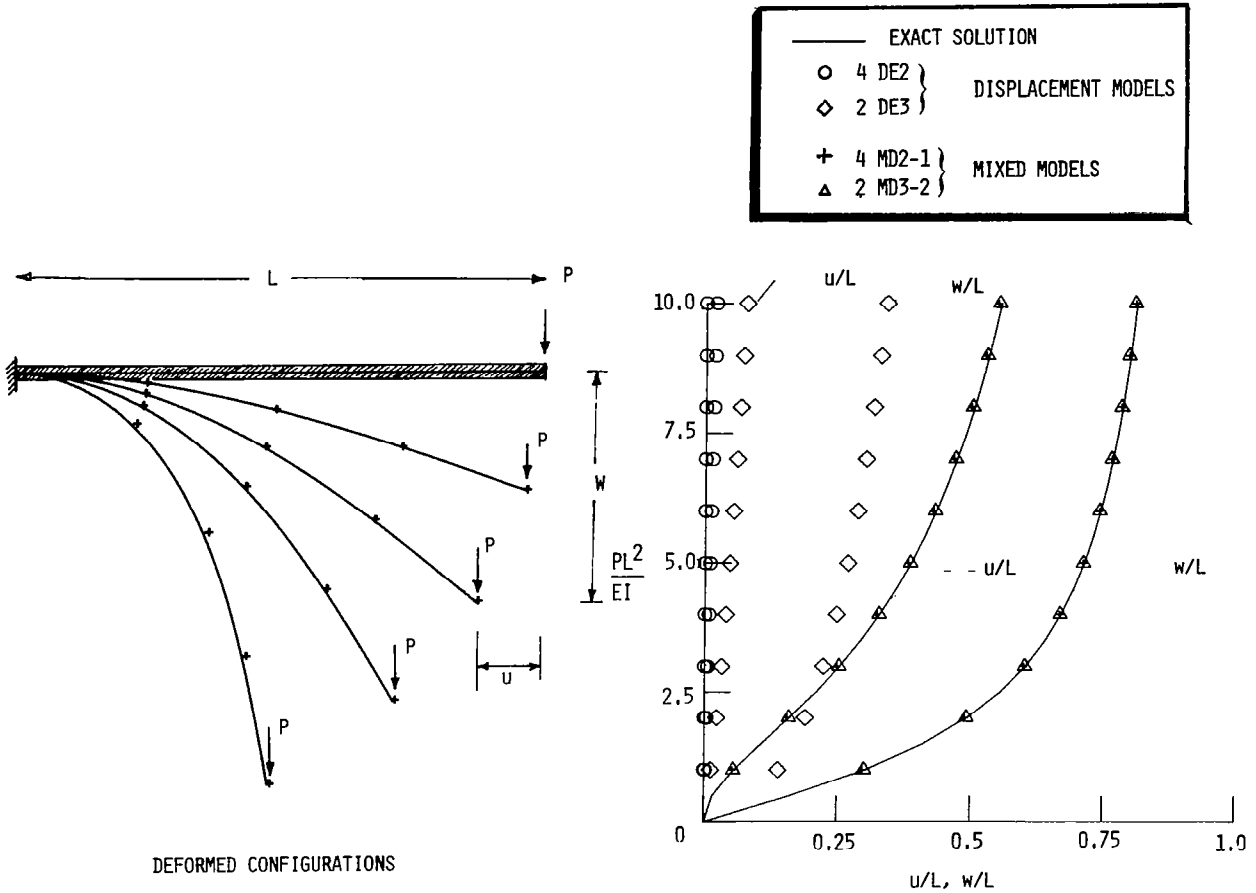


Figure 31

LARGE-ROTATION ELASTICA PROBLEM (Cont'd.)

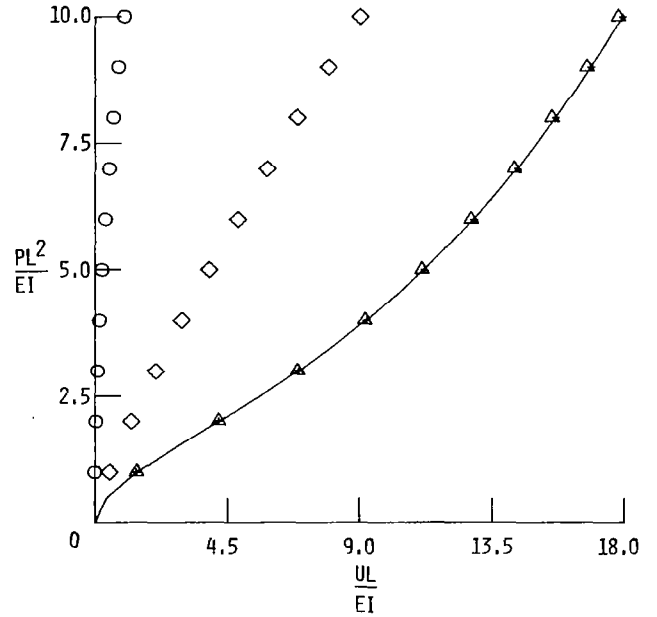
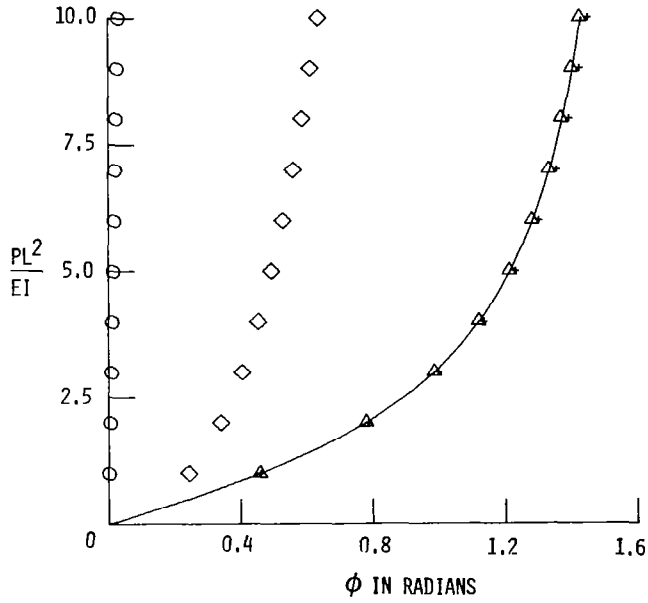
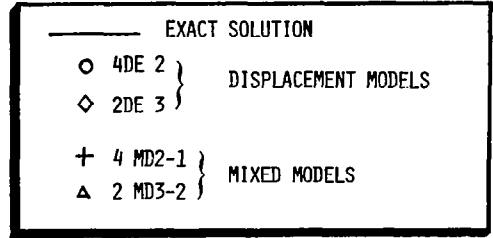
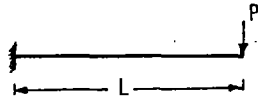


Figure 32

ANALYSIS AND MODELING NEEDS FOR TIRES

The fourth part of the paper deals with future analysis and modeling needs for tires. The overall goal is to develop a general tire analysis capability which includes (see Fig. 33):

- 1) Accurate representation of the tire configuration and construction
- 2) Reliable material characterization including thermo-viscoelastic response
- 3) Capability for predicting the stresses and deformations due to footprint loading; this also includes the prediction of the contact area

Since there is a certain degree of uncertainty in the accuracy of the various elements of the tire model, considerable work should be directed towards assessing the sensitivity of the tire response to various modeling details such as material characteristics, surface inaccuracies, and variations in the tire design variables. The result of such sensitivity study would allow the identification of the minimum degree of sophistication of the model required to achieve a prescribed level of accuracy.

There is also a need for identifying failure mechanisms and developing a verifiable failure analysis capability for tires. Use can be made of the considerable experience gained in damage tolerance design concepts for fibrous composite structures.

- GENERAL TIRE ANALYSIS CAPABILITY
 - ACCURATE REPRESENTATION OF TIRE CONFIGURATION AND CONSTRUCTION
 - RELIABLE MATERIAL CHARACTERIZATION INCLUDING THERMO-VISCOELASTIC RESPONSE
 - PREDICTION OF STRESSES AND DEFORMATIONS DUE TO FOOTPRINT LOADING (CONTACT AREA, STRESSES AND SLIP)
- SENSITIVITY ANALYSIS - SENSITIVITY OF RESPONSE TO:
 - MATERIAL CHARACTERISTICS
 - SURFACE INACCURACIES
 - VARIATIONS IN DESIGN VARIABLES (REQUIRED FOR EVALUATION OF STRUCTURAL CONCEPTS AND FOR OPTIMIZATION)
 - MODELING DETAILS (IN ORDER TO DEVELOP SIMPLE TIRE MODELS)
- FAILURE MECHANISMS AND FAILURE ANALYSIS OF TIRES
 - DAMAGE TOLERANCE

Figure 33

SUMMARY

In summary, four topics are covered in this paper; namely, recent advances in reduction methods for nonlinear problems, mixed models for tires, computational models for large-rotation nonlinear problems, and analysis and modeling needs for tires. (See Fig. 34.)

Reduction methods have proven to be very effective for the nonlinear static analysis of structures subjected to either combined loads or displacement-dependent loads. However, more work is needed to realize their full potential for nonlinear dynamic and time-dependent problems.

Mixed shell models with discontinuous stress resultants at element interfaces have high potential for nonlinear analysis of tires. These models can be easily incorporated into existing general-purpose finite element programs based on the displacement formulation.

Two computational models are presented for the large-rotation nonlinear problems. Both models use a total Lagrangian description of the deformation. The first model uses a mixed formulation, and the second model uses a penalty formulation. Both models appear to have high potential.

As far as analysis and modeling needs are concerned, three areas have been identified. As is to be expected, the modeling and analysis of tires will be strongly impacted by new advances in materials technology, computer hardware, software, integrated analysis, and CAD/CAM systems.

- REDUCTION METHODS
 - VERIFIED FOR STATIC NONLINEAR PROBLEMS INCLUDING CASES OF COMBINED LOADS AND DISPLACEMENT-DEPENDENT LOADS
 - FURTHER DEVELOPMENT NEEDED FOR NONLINEAR DYNAMIC PROBLEMS
- MIXED MODELS WITH DISCONTINUOUS STRESS RESULTANTS
 - HAVE HIGH POTENTIAL FOR ANALYZING TIRES
 - CAN BE EASILY INCORPORATED INTO EXISTING GENERAL-PURPOSE FINITE ELEMENT PROGRAMS
- LARGE ROTATION NONLINEAR PROBLEMS
 - BOTH MIXED AND PENALTY FORMULATIONS PROVIDE EFFECTIVE ANALYSIS TECHNIQUES
- ANALYSIS AND MODELING NEEDS
 - GENERAL ANALYSIS CAPABILITY FOR TIRES
 - SENSITIVITY ANALYSIS
 - FAILURE MECHANISMS AND FAILURE ANALYSIS

Figure 34

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