Linear Approximations of Nonlinear Systems

(NASA-CR-170036) LINEAR APPROXIMATIONS OF NONLINEAR SYSTEMS (Texas Technological Univ.) 15 p HC A02/MF A01 CSCL 12A
N83-21845
Unclas
G3/64 03198

I.R. Hunt
Department of Mathematics
Texas Tech University
Lubbock, Texas 79409
(806) 742-1427

Renjeng Su
Department of Electrical Engineering
Texas Tech University
Lubbock, Texas 79409
(806) 742-3716

Regular Paper

22nd CDC

CDC presentation only

Linear Approximations of Nonlinear Systems

L.R. Hunt and Renjeng Su

Abstract

A method for designing an automatic flight controller for short and vertical take off aircraft is presently being developed at NASA Ames Research Center. This technique involves transformations of nonlinear systems to controllable linear systems and takes into account the nonlinearities of the aircraft. In general, the transformations cannot always be given in closed form. Using partial differential equations, an approximate linear system called the modified tangent model, was recently introduced. A linear transformation of this tangent model to Brunovsky canonical form can be constructed, and from this the linear part (about a state space point x_0) of an exact transformation for the nonlinear system can be found. Here we show that a canonical expansion in Lie brackets about the point x_0 yields the same modified tangent model.

ORIGINAL PAGE IS
OF POOR QUALITY

Linear Approximations of Nonlinear Systems

L.R. Hunt* and Renjeng Su**

I. Introduction

Suppose we have a nonlinear plant for which we are to design a control scheme. For example, George Meyer [1],[2],[3],[4],[5],[6] at NASA Ames Research Center is presently developing an automatic flight controller for the UH-1H helicopter which takes into account the nonlinearities involved. His design technique depends on a theory giving necessary and sufficient conditions for a nonlinear system to be transformed to a controllable linear system [7],[8],[9],[10],[11]. In other words state and control coordinate changes can be implemented to simplify the problem. Thus the method is to move

*Research supported by NASA Ames Research Center under grant NAG2-189 and the Joint Services Electronics Program under ONR Contract N0014-76-C1136.
**Research supported by NASA Ames Research Center under grant NAG2-203 and the Joint Services Electronics Program under ONR Contract N0014-76-C1136.

from the nonlinear model of the plant to the linear system in Brunovsky [12] canonical form. To have the helicopter fly a prescribed trajectory, no gain scheduling is needed in Meyer's technique, because through on-line computations of the transformation and its inverse, we always see the same trivial linear system.

Meyer's nonlinear system is in block triangular form, and it is not difficult to find a transformation and its inverse. However, the transformation theory in [8] applies to systems which are much more general than block triangular. Formally, the desired transformations can be constructed by considering a system of partial differential equations, which can be reduced to ordinary differential equations. It is not always possible to solve such equations in closed form, but cases where this can be accomplished are presented in the Ph.D. thesis of H. Ford [13]. Numerical techniques for constructing approximate transformations in certain situations (e.g. under the conditions due to Brockett [14]) are introduced in [13].

It is appropriate to develop a method to build an approximate transformation in all cases. In [15] we considered this problem in view of the partial differential equations from [8]. We found a related set of partial differential equations, the solution of which yielded a linear approximation to an actual transformation. In fact, given a point x_0 in state space, we are able to construct a mapping which is the linear part of an actual transformation about x_0, without knowing the transformation itself. In this process we introduce an approximating linear system called the modified tangent model. If one is working around an equilibruim point of the drift term in our nonlinear system, this modified tangent model is the linear system we obtain by truncating the Taylor series of our nonlinear system about the point. However, if we are away from such an equilibruim point, the modified tangent model can provide a different system than the usual linearization (an example is presented in [15]). If one is interested in tracking a certain trajectory, then modified tangent models are constructed at various points along the trajectory.

The purpose of this paper is to show how one obtains the modified tangent model by an expansion which parallels, but is unequal to in most cases, the Taylor approach. This technique is included in an article [16] by the authors giving canonical forms and canonical expansions for nonlinear systems.

We show that the partial differential equations and canonical expansion yield the same approximate controllable linear system. Moreover, the form of the approximate linear system and knowledge about nonlinear control theory convince us that it is the appropriate model for designing a controller for many nonlinear plants.

II. Expansions and the Tangent Model

In order that a nonlinear system be transformable to a linear system as in [8], it must be of the form

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t)), \tag{1}$$

where $f, g_1, \ldots, g_m$ are C^∞ vector fields on $\mathbb{R}^n$, and $g_1, g_2, \ldots, g_m$ are linearly independent (this is assumed for convenience). Now the results from [8] are local in nature (global theorems are presented in [9]), but for the sake of simplicity we assume that the system is transformable to a controllable linear system on all of $\mathbb{R}^n$. This linear system is in Brunovsky form,

$$\dot{y}(t) = A_0 y + B_0 v. \tag{2}$$

and with Kronecker indices $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_m$.

For vector fields f and g, $[f,g]$ denotes the well known Lie bracket, and

$$(ad^0 f,g) = g$$

$$(ad^1 f,g) = [f,g]$$

$$(ad^2 f,g) = [f,[f,g]]$$

$$\vdots$$

$$(ad^k f,g) = [f,(ad^{k-1} f,g)].$$

We define

$$C = \{g_1,[f,g_1], \dots , (ad^{\kappa_1 -1} f,g_1),g_2,[f,g_2], \dots ,(ad^{\kappa_2 -1} f,g_2), \dots ,g_m,[f,g_m], \dots ,(ad^{\kappa_m -1} f,g_m)\}$$

$$C_j = \{g_1,[f,g_1], \dots ,(ad^{\kappa_j -2} f,g_1),g_2,[f,g_2], \dots ,(ad^{\kappa_j -2} f,g_2), \dots ,g_m,[f,g_m], \dots ,(ad^{\kappa_j -2} f,g_m)\} \text{ for } j=1,2,\dots,m.$$

In [8] it is shown that system (1) is transformable to system (2) if and only if (with possible reordering of $g_1, g_2, \dots, g_m$)

1) the n vector fields in C are linearly independent,

2) the sets C_j are involutive for $j=1,2,\dots,m$, and

3) the span of C_j equals the span of $C_j \cap C$ for $j=1,2,\dots,m$.

We assume that our system (1) satisfies these three conditions. Using the partial differential equations approach from [8] we introduced the modified tangent model about a point x_0 in x-space

$$\dot{x}(t) = f(x_0) - Ax_0 + Ax + Bu. \tag{3}$$

It is shown in [15] how to construct an approximate transformation using this model. Here A is an nxn matrix and $B = (b_1, b_2, \ldots, b_m)$ is an m tuple of n vectors that satisfy the equations (take + for k even and - for k odd)

$$\begin{aligned} A^k b_1 &= \pm (ad^k f, g_1)(x_0),\ k = 0,1, \ldots, \kappa_1 \\ A^k b_2 &= \pm (ad^k f, g_2)(x_0),\ k = 0,1, \ldots, \kappa_2 \\ &\vdots \\ A^k b_m &= \pm (ad^k f, g_m)(x_0),\ k = 0,1, \ldots, \kappa_m. \end{aligned} \tag{4}$$

Equations (4) are nonlinear, but there is a simple method for computing A and B. Let D be the set of vector fields $\{(ad^{\kappa_1} f, g_1)(x_0), ad^{\kappa_1 - 1} f, g_1)(x_0), \ldots, (ad^{\kappa_2} f, g_1)(x_0), (ad^{\kappa_2} f, g_2)(x_0), (ad^{\kappa_2 - 1} f, g_1)(x_0), (ad^{\kappa_2 - 1} f, g_2)(x_0), \ldots, (ad^{\kappa_3} f, g_3)(x_0), \ldots, g_1(x_0), g_2(x_0), \ldots, g_m(x_0)\}$. Before forming this set checks such as $\kappa_1 = \kappa_2$ or $\kappa_1 > \kappa_2$, etc, should be made and no duplications should be included.

Now we introduce an interesting (n+m)x(n+m) matrix E. Let the first column be $(ad^{\kappa_1} f, g_1)(x_0)$ followed by m zeroes, the second column will be the second element of D followed by m zeroes, ... , the n^{th} column be the n^{th} element of D followed by m zeros, the $(n+1)^{th}$ column be $g_1(x_0)$ and m zeros, ... , the last column be $g_m(x_0)$ and m zeros.

Ignoring the last m components, the first column of E is $\pm A^{\kappa_1} b_1$, the second $\pm A^{\kappa_1 - 1} b_1$, ... , the $n^{\underline{th}}$ $\pm A b_m$, ... , and the last b_m. It is shown in [13] that there is an orthogonal coordinate change on $\mathbb{R}^n$ so that all entries above the first m superdiagonals are zero and the elements below the $n^{\underline{th}}$ row remain unchanged. For our purpose we can assume that E is initally in this "generalized lower Hessenberg" form, because knowing A and b in these coordinate, we can return to the original A and b through an orthogonal change of coordinates.

[More needs to be said here]

Hence we have by (4)

$$b_m = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ * \end{bmatrix}, \quad b_{m-1} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ * \\ * \end{bmatrix}, \quad \ldots, \quad b_1 = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ * \\ \cdot \\ \cdot \\ \cdot \\ * \end{bmatrix},$$

where * indicates a possible nonzero entry (recall that $g_m, g_{m-1}, \ldots, g_1$ were assumed to be linearly independent) and the first * in b_1 is in the $(n-m+1)^{\underline{th}}$ row.

We examine from (4)

$$Ab_m = -[f, g_m](x_0) = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ * \\ \cdot \\ \cdot \\ \cdot \\ * \end{bmatrix}$$

ORIGINAL PAGE IS
OF POOR QUALITY

with the first * in the column being at the $(n-m)^{\underline{th}}$ level. This is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ * \\ \cdot \\ \cdot \\ \cdot \\ * \end{bmatrix}$$

and we easily compute $a_{1n}, a_{2n}, \ldots, a_{nn}$. Similarly, $Ab_{m-1} = -[f,g_{m-1}](x_0)$ yields $a_{1(n-1)}, a_{2(n-1)}, \ldots, a_{n(n-1)}, \ldots,$ $Ab_1 = -[f,g_1](x_0)$ yields $a_{1(n-m+1)}, a_{2(n-m+1)}, \ldots, a_{n(n-m+1)}$.

Next we consider

$$A^2 b_m = (ad^2 f, g_m(x_0)$$

if the vector field on the right is in the set D. Writing this as

$$A^2 b_m = A(Ab_m) = (ad^2 f, g_m)(x_0),$$

and knowing Ab_m and $(ad^2 f, g_m)(x_0)$, we can compute $a_{1(n-m)}, a_{2(n-m)}, \ldots, a_{n(n-m)}$. Continuing in this way we can solve for every entry in A, and the method of solution is readily implemented on a computer (or by hand). We remark that the above equations can be solved because of our assumption that the set of vector fields in the set C are linearly independent. □

Now we show the canonical expansions like those in [16] give us the modified tangent model. First we rewrite the set C so that the vector fields appear in a different order.

$C = \{(ad^{\kappa_1 - 1} f, g_1), (ad^{\kappa_1 - 2} f, g_1), \ldots, (ad^{\kappa_2 - 1} f, g_1), (ad^{\kappa_2 - 1} f, g_2),$
$(ad^{\kappa_2 - 2} f, g_1), (ad^{\kappa_2 - 2} f, g_2), \ldots, (ad^{\kappa_3 - 1} f, g_1), (ad^{\kappa_3 - 1} f, g_2), (ad^{\kappa_3 - 1} f, g_3)$
$(ad^{\kappa_3 - 2} f, g_1), (ad^{\kappa_3 - 2} f, g_2), (ad^{\kappa_3 - 2} f, g_3), \ldots, g_1, g_2, \ldots, g_m\}.$

Note that $\kappa_1 > \kappa_2$ or $\kappa_1 = \kappa_2$, etc, should be checked before this set is formed. For example if $\kappa_1 = \kappa_2$, the first element is $(ad^{\kappa_1 - 1} f, g_1)$, the second $(ad^{\kappa_2 - 1} f, g_2)$, etc.

We introduce new independent variables $s_1, s_2, \ldots, s_n$ such that $s_0 = (s_{10}, s_{20}, \ldots, s_{n0})$ is the same point as $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})$ is our state space. The parameter s_1 is along the integral curves of the first vector field $(ad^{\kappa_1 - 1} f, g_1)$ in C, s_2 is along the integral curves of the second vector field in C, s_3 is along the curves of the third, ... , and s_n is along the integral curves of g_m. Notice that our system (1) is linear in the controls $u_1, u_2, \ldots, u_m$ and these control variables are treated at the same level at the $s_1, s_2, \ldots, s_n$. That is, in our linearization we do not want terms like $u_1 s_1, u_1 s_2$, etc, because these are not considered to be linear. In the following process all terms of degree greater than one in $u_1, u_2, \ldots, u_m, s_1, s_2, \ldots, s_n$ are included in the notation $+ \ldots$. In taking infinite expansions f and g are required to be real analytic, but we are only interested in finite truncations of such expansions.

First we rewrite our nonlinear system (1)

$$(5)\quad \dot{x} = g_1(s_1, s_2, \ldots, s_n) u_1 + g_2(s_1, s_2, \ldots, s_n) u_2 + \ldots + g_m(s_1, s_2, \ldots, s_n) u_m + f(s_1, s_2, \ldots, s_n).$$

Now we expand in the s_1 variable along the integral curves of $(ad^{\kappa_1 - 1} f, g_1)$

$$\dot{x} = g_1(s_{10}, s_2, \ldots, s_n)u_1 + g_2(s_{10}, s_2, \ldots, s_n)u_2 + \ldots +$$
$$g_m(s_{10}, s_{20}, \ldots, s_n)u_m + f(s_{10}, s_2, \ldots, s_n) +$$
$$[f, (ad^{\kappa_1-1}f, g_1)](s_{10}, s_2, \ldots, s_n)(s_1 - s_{10}) + \ldots$$

Expansion along the integral curves of the second vector field in C, which we assume is $(ad^{\kappa_1-2}f, g_1)$, gives

$$\dot{x} = g_1(s_{10}, s_{20}, \ldots, s_n)u_1 + g_2(s_{10}, s_{20}, \ldots, s_n)u_2 + \ldots +$$
$$g_m(s_{10}, s_{20}, \ldots, s_n)u_m + f(s_{10}, s_{20}, \ldots, s_n) +$$
$$[f, (ad^{\kappa_1-2}f, g_1)](s_{10}, s_{20}, \ldots, s_n)(s_2 - s_{20}) +$$
$$[f, (ad^{\kappa_1-1}f, g_1)](s_{10}, s_{20}, \ldots, s_n)(s_1 - s_{10}) + \ldots$$

Continuing in this way, we arrive at our last step which is an expansion in the s_n variable that provides (with $s_0 = (s_{10}, s_{20}, \ldots, s_{n0})$)

$$\dot{x} = g_1(s_0)u_1 + g_2(s_0)u_2 + \ldots + g_m(s_0)u_m + f(s_0) +$$
$$(6) \quad (ad^{\kappa_1}f, g_1)(s_0)(s_1 - s_{10}) + (ad^{\kappa_1-1}f, g_1)(s_0)(s_2 - s_{20}) + \ldots +$$
$$[f, g_m](s_0)(s_n - s_{n0}) + \ldots$$

Since s_0 corresponds to x_0, the important Lie brackets are $g_1(x_0), g_2(x_0), \ldots, g_m(x_0), [f, g_1](x_0), [f, g_2](x_0), \ldots, (ad^{\kappa_1}f, g_1)(x_0)$. Another way to find this set is to take the elements of C evaluated at x_0 plus $(ad^{\kappa_m}f, g_m)(x_0), (ad^{\kappa_{m-1}}f, g_{m-1})(x_0), \ldots, (ad^{\kappa_1}f, g_1)(x_0)$.

Thus if we wish to find a linear system

$$(7) \qquad \dot{x} = f(x_0) - Ax_0 + Ax + Bu$$

that emphasizes the linear part of the system in (6) we would need to solve

$$A^k b_1 = \pm (ad^k f, g_1)(x_0), \; k = 0,1, \dots, \kappa_1$$
$$A^k b_2 = \pm (ad^k f, g_2)(x_0), \; k = 0,1, \dots, \kappa_2$$
$$\vdots$$
$$A^k b_m = \pm (ad^k f, g_m)(x_0), \; k = 0,1, \dots, \kappa_m)$$

where $B = (b_1, b_2, \dots, b_m)$ and + is for k even and - is for k odd.

Thus (7) is exactly the modified tangent model that we defined in terms of the partial differential equations.

III. Conclusion

Given a nonlinear system (1) we can use expansions of the system about a point x_0 in terms of variables associated with the vector fields in the set C to produce an approximate linear system. This resulting linear system is the modified tangent model that is found by considering the partial differential equations that are solved in constructing an exact transformation of the nonlinear system to a controllable linear system.

ORIGINAL PAGE IS
OF POOR QUALITY

REFERENCES

[1] Meyer, G. and Cicolani, L., 1975, A formal structure for advanced automatic flight control systems, NASA TN D-7940.

[2] Meyer, G. and Cicolani, L., 1980, Application of nonlinear system inverses to automatic flight control design-system concepts and flight evaluations. AGARDograph 251 on Theory and Applications of Optimal Control in Aerospace Systems, P. Kant, ed., reprinted by NATO.

[3] Meyer, G., The design of exact nonlinear model followers, Proceedings of 1981 Joint Automatic Control Conference, FA3A.

[4] Meyer, G., Renjeng Su, and L.R. Hunt, Applications to aeronautics of the theory of transformations of nonlinear systems, 1982 CNRS Conference, pp. 153-162.

[5] Meyer, G., Renjeng Su, and L.R. Hunt, Application of nonlinear transformations to automatic flight control, submitted.

[6] Meyer, G., L.R. Hunt, and Renjeng Su, Design of helicopter autopilot by means of linearizing transformations, NASA Technical Memorandum 84295.

[7] Su, R., On the linear equivalents of nonlinear systems, Systems and Control Letters 2, No.1 (1982), pp. 48-52

[8] Hunt, L.R., Su, R., and Meyer, G., Design for multi-input systems, Differential Geometric Control Theory Conference, Birkhauser, Boston, R.W. Brockett, R.S. Millman, and H.J. Sussman, Eds., 27(1983), pp. 268-298.

[9] Hunt, L.R., Su, R. and Meyer, G., Global transformations of nonlinear systems, IEEE Trans. on Automatic Control, 28, No.1 (1983), pp. 24-31.

[10] Hunt, L.R. and Su, R., Control of nonlinear time-varying systems, 20th IEEE Conference on Decision and Control, San Diego, CA, 1981, pp. 558-563.

[11] Jakubczyk, B. and Respondek, W. On linearization of control systems., Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., 28 (1980), pp. 517-522.

[12] Brunovsky, P., A classification of linear controllable systems, Kibernetika (Praha) 6 (1970), pp. 173-188.

[13] H. Ford, Numerical and symbolic methods for transforming control systems to canonical form, Ph.D. thesis, Texas Tech University, in preparation.

[14] Brockett, R.W., Feedback invariants for nonlinear systems, IFAC Congress, Helsinki, 1978.

[15] H. Ford, L.R. Hunt, G. Meyer, and R. Su, The modified tangent model, submitted.

[16] R. Su and L.R. Hunt, A canonical form for nonlinear systems, submitted.

L.R. Hunt
Department of Mathematics
Texas Tech University
Lubbock, Texas 79409
(806) 742-1427

Renjeng Su
Department of Electrical Engineering
Texas Tech University
Lubbock, Texas 79409
(806) 742-3716