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OBSERVABILITY FOR  
TWO DIMENSIONAL SYSTEMS

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L.R. Hunt and Renjeng Su

ABSTRACT

Sufficient conditions that a 2 dimensional system with output is locally observable are presented. Known results depend on time derivatives of the output and the inverse function theorem. In some cases, no information is provided by these theories, and one must study observability by other methods. We dualize the observability problem to the controllability problem, and apply the deep results of Hermes on local controllability to prove a theorem concerning local observability.



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OBSERVABILITY FOR  
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L.R. Hunt\* and Renjeng Su\*\*

I. Introduction

Suppose we take a nonlinear system

$$\begin{aligned}\dot{x}(t) &= f(x(t)) \\ y &= h(x)\end{aligned}\tag{1}$$

where  $f$  is a real analytic vector field on  $\mathbb{R}^2$  (or on a 2 dimensional manifold in general) and  $h$  is a real analytic output function on  $\mathbb{R}^2$ . Given a point  $x_0 \in \mathbb{R}^2$ , under what conditions on  $f$  and  $h$  can we guarantee that there is an open neighborhood  $U$  of  $x_0$  so that knowledge of the observed output  $y$  of the trajectories of  $\dot{x} = f(x)$  starting at points in  $U$  allow us to distinguish between  $x_0$  and any other point in  $U$ ? We also want to distinguish between any two points  $x_1$  and  $x_2$  in  $U$  where  $h(x_1) = h(x_2) = h(x_0)$  when  $t = 0$ .

The known results in the literature (e.g. [1] and [2]) give sufficient conditions which involve the time derivatives of the output (or equivalently, the Lie derivatives of the output function  $h$  with respect to the vector field  $f$ ) and the inverse function theorem. The results of Kou, Elliott and Tarn [1] can be applied for  $n$  dimensional  $C^\infty$  systems with several outputs and those of Hermann and Krener [2] also involve a system with inputs, whereas in this paper,

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we consider the two dimensional system (1).

Easy examples like the following one are of interest for this problem. Take

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 \\ 0 \end{bmatrix} = f(x(t)) \quad (2)$$

$$y = h(x) = x_1$$

Computing the time derivatives of output we find

$$\begin{aligned} y &= x \\ \dot{y} &= x_2^3 \\ \ddot{y} &= 0 \\ \dddot{y} &= 0 \\ &\vdots \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

Thus the inverse function theory provides an answer only if  $x_2 \neq 0$ . No information is provided by this method at  $x_2 = 0$ , and if one is constructing a state estimator based on the above time derivatives, then one obtains bad results near  $x_2 = 0$ . However, if we draw a phase plane portrait of the trajectories of  $\dot{x} = f(x)$ , we realize that those trajectories with initial values in a particular level set of the output function (i.e.  $\{x: h(x) = \text{constant}\}$ ) and above the line  $x_2 = 0$  move to the right, and those below, to the left. Moreover, the trajectories starting at any two points in a level set move to different level sets in a given time  $t > 0$ . Hence, there should be some calculation (involving Lie derivatives at a point  $x_0$  where  $x_2 = 0$ ) that should let us know this is occurring, and also imply the ability to distinguish between  $x_0$  and the other points in some open

neighborhood of  $x_0$  in  $\mathbb{R}^2$ . In addition, we also want to differentiate any two points in the level of  $h$  through  $x_0$  by watching the output in time.

As emphasized in the paper of Hermann and Krener [2], the duality between controllability and observability is simply that between vector fields and differential forms. Since the gradient of the output  $y = h(x)$  in (2) is nonzero, we can find a nonvanishing vector field, say  $g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , so that the dual product of  $dy$  and  $g$  is zero. Then we consider the control problem

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 1 \end{bmatrix} = f + ug. \quad (3)$$

The local controllability along a reference trajectory results of Hermes [3] present a way to compute the precise Lie brackets involving  $f$  and  $g$  at a point where  $x_2 = 0$  that provide the needed information about the movement of the flow of  $\dot{x} = \begin{bmatrix} x_2^3 \\ 0 \end{bmatrix}$  on the level sets of  $y = x_1$  in (2). This is true because Hermes studies the attainable set from a point  $x_0$  at a time  $t$ .

If we compute Lie brackets for system (3) at a point  $x_0$  where  $x_2 = 0$  we find

$$g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [f, g] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, [g, [f, g]] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, [g, [g, [f, g]]] = \begin{bmatrix} -6 \\ 0 \end{bmatrix}.$$

The fact that  $[g, [g, [f, g]]]$  and  $g$  are linearly independent at  $x_0$ , and this is the first Lie bracket with this property, implies that the trajectories of  $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 \\ 0 \end{bmatrix}$  perform as previously described along an integral curve of  $g$  (i.e. a level set of the output  $y = h$ ). From this we can deduce the existence of an open neighborhood  $U$  of  $x_0$  in which

we can distinguish points in  $U - \{x_0\}$  from  $x_0$ , and any two points in the level set of  $h$  through  $x_0$  intersected with  $U$  can be distinguished.

The purpose of this paper is to present conditions under which system (1) has for a given point  $x_0$  such a neighborhood  $U$ . One assumption is that the gradient of  $h(x)$  at  $x_0$  is nonvanishing, implying the level set through  $x_0$  is a 1 dimensional manifold. In our example,  $f(x)$  vanishes at a point  $x_0$  where  $x_2 = 0$ . We will also provide a theory concerning observability in the case that  $f$  and  $g$  are linearly independent at  $x_0$ .

For other research into the problem of nonlinear observability we refer to [4], [5], [6], [7], [8], [9], [1], [10], [2], [11], [12], [13], and [14]. Open problems concerning observability are generated by this paper. Can the assumption that the gradient of  $y = h(x)$  at  $x_0$  is nonvanishing be reduced? What theory exists in  $n$  dimensions, and how does one handle several outputs and the introduction of inputs?

## II. Definitions and Results

First we motivate the notion of observability which is appropriate for our theory.

Suppose  $x_0 \in \mathbb{R}^2$  and the gradient of  $y = h(x)$  in (1) is nonzero at  $x_0$ . Then there is a neighborhood  $V$  of  $x_0$  in  $\mathbb{R}^2$  so that the level sets of  $y = h(x)$  form a 1 parameter family of 1 dimensional manifolds which foliate  $V$  as the parameter varies. Restricting to this set  $V$ , it is clear we can certainly distinguish between 2 points which are in distinct level sets. The problem is to differentiate between 2 points that start in the same level set by watching the movement under  $\dot{x} = f(x)$  as time advances. If for arbitrarily fixed small positive time, all points in the level set of  $h(x)$  through  $x_0$  are

carried to distinct level sets of  $h(x)$ , then we can distinguish between them.

Let  $C_{x_0}$  be the level set of  $h(x)$  in (1) through  $x_0$ .

Definition. The system (1) is locally level set observable at  $x_0$  if there is an open neighborhood  $U$  of  $x_0$  in  $\mathbb{R}^2$  and a one-to-one correspondence between the set  $U \cap C_{x_0}$  and the set of trajectories of the observed output  $y(t)$  for arbitrarily small time  $t > 0$ . Equivalently, for arbitrary small time  $t > 0$  the trajectories of  $\dot{x} = f(x(t))$  starting at any two distinct points in  $C_{x_0}$  lie in different level sets of  $h(x)$ .

Of course if (1) is locally level set observable at  $x_0$ , it is easy to distinguish  $x_0$  from all points in  $U - \{x_0\}$  for  $U$  sufficiently small.

Suppose we have  $C^\infty$  vector fields  $f$  and  $g$  on  $\mathbb{R}^2$ . The Lie bracket of  $f$  and  $g$  is

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g,$$

where  $\frac{\partial g}{\partial x}$  and  $\frac{\partial f}{\partial x}$  are Jacobian matrices. We can then define  $[f, [f, g]]$ ,  $[g, [f, g]]$ ,  $[f, f[[f, g]]]$ , etc. For notation we take

$$\begin{aligned} (\text{ad}^1 f, g) &= [f, g] \\ (\text{ad}^2 f, g) &= [f, [f, g]] \\ &\vdots \\ (\text{ad}^k f, g) &= [f, (\text{ad}^{k-1} f, g)] \end{aligned}$$

and similarly for  $(\text{ad}^k g, f)$ .

For  $h$  a  $C^\infty$  function on  $\mathbb{R}^2$  and  $f$  a  $C^\infty$  vector field we let

$$L_f(h) = \langle dh, f \rangle,$$

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with  $\langle \cdot, \cdot \rangle$  denoting the duality between one forms and vector fields, and we can inductively take  $L_f^2(h), L_f^3(h)$ , etc.

If  $w$  is a  $C^\infty$  one form on  $R^2$

$$L_f(w) = \left(\frac{\partial w^*}{\partial x} f\right)^* + w \frac{\partial f}{\partial x},$$

where  $*$  denotes transpose. Note that  $L_f(dh) = dL_f(h)$ . The three "Lie derivatives"  $[f, g], L_f(h)$  and  $L_f(w)$  are related by the rule

$$L_f \langle w, g \rangle = \langle L_f(w), g \rangle + \langle w, [f, g] \rangle \quad (4)$$

In system (1) we assume as before that the gradient of  $h$  is nonzero at  $x_0$ . Choose a real analytic vector field  $g$  such that  $\langle dh, g \rangle = 0$  and  $g$  is nonvanishing at  $x_0$ . Consider the 2 dimensional control system

$$\dot{x}(t) = f(x) + ug(x). \quad (5)$$

Using formula (4) we find

$$L_f \langle dh, g \rangle = \langle L_f(dh), g \rangle + \langle dh, [f, g] \rangle.$$

Since  $\langle dh, g \rangle = 0$ , we have  $\langle dh, [f, g] \rangle = -\langle L_f(dh), g \rangle = -\langle dL_f(h), g \rangle$ . Thus  $g$  and  $[f, g]$  are linearly independent at  $x_0$  if and only if  $dh$  and  $L_f(h)$  are linearly independent there. Similarly, if  $g$  and  $[f, g]$  are dependent at  $x_0$  (or equivalently,  $dh$  and  $dL_f(h)$  are), then by applying formula (4) again we find that  $g$  and  $(ad^2 f, g)$  are linearly independent at  $x_0$  if and only if  $dh$  and  $dL_f^2(h)$  are. This process can be continued indefinitely (and in some cases like systems (2) and (3) we obtain no linear independence). On the one hand if there is some  $L_f^k(h)$  satisfying the condition  $dh$  and  $dL_f^k(h)$  are independent at  $x_0$ , there is an open neighborhood  $U$  of  $x_0$  so system (1)



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is observable on  $U$  in the sense of [1] and [2]. Equivalently, if there is an  $(\text{ad}^k f, g)$  with  $g$  and  $(\text{ad}^k f, g)$  independent at  $x_0$ , then the results of Hermes [15] imply local controllability along a reference trajectory (starting at  $x_0$ ) at time  $t$  (to be defined momentarily). Thus the duality between observability and controllability is easily realized in this way. However, Hermes [3], [16], [17], [18], and [19] has results on controllability that are much more general than those that depend on the vector fields  $g, [f, g], (\text{ad}^2 f, g), \dots$  being linearly independent.

Let  $\varphi(t, x_0)$  be the solution of  $\dot{x} = f(x(t))$  in (1) or (5) at time  $t$  with  $\varphi(0, x_0) = x_0$ . We say that the system (5) is locally controllable along  $\varphi$  at time  $t > 0$  if all points in some 2-dimensional open neighborhood of  $\varphi(t, x_0)$  can be reached at time  $t$  by solutions of (5) initiating from  $x_0$ .

Now we define the following sets (see [16])

$$S^1 = \{g, [f, g], (\text{ad}^2 f, g), (\text{ad}^3 f, g), \dots\}$$

$$S^2 = \{g, (\text{ad}^2 g, f), [f, (\text{ad}^2 f, g)], (\text{ad}^2 f, (\text{ad}^2 f, g)), \dots\}$$

$$S^3 = \{g, (\text{ad}^3 g, f), [f, (\text{ad}^3 f, g)], (\text{ad}^2 f, (\text{ad}^3 f, g)), \dots\}$$

⋮  
⋮  
⋮

Let  $\dim \text{span } S_{x_0}^k$  = the dimension of the span of  $S^k$  at  $x_0$ .

The following result is of interest in our study of 2 dimensional observability. Assume the gradient of  $h$  in system (1) is nonvanishing at  $x_0 \in \mathbb{R}^2$  and let  $g$  be defined as in section (5).

Theorem. If either of the following conditions hold, then system (1) is locally level set observable at  $x_0$  in  $\mathbb{R}^2$ :

- 1)  $f$  and  $g$  are linearly independent at  $x_0$  and the smallest

integer  $m$  so that  $\dim \text{span } S_{x_0}^m = 2$  is odd.

- 2)  $f(x_0) = 0$  and the smallest integer  $m$  so that  $g$  and  $(\text{ad}^m g, f)$  are linearly independent at  $x_0$  is odd.

Remark. The corresponding results in [1] and [2], if dualized as we have done, consider only the set  $S^1$  in statement 1) and the Lie bracket  $[f, g]$ ,  $[g, L]$  in statement 2).

Proof. As stated before, since the gradient of  $h$  is nonvanishing at  $x_0$ , there is an open neighborhood  $V$  of  $x_0$  on which this gradient is nonzero, and  $V$  consists of a 1-parameter foliation of level sets of  $h$  (i.e. integral curves of  $g$ ).

For  $x \in V$ , we denote by  $(\exp t f)(x)$  (or  $\varphi(t, x)$ ) the integral curve (or solution curve) of  $f$  with initial value  $x$ . For fixed  $t$ ,  $(\exp t f)(x)$  also denotes the value of the solution at that time. For any  $t \geq 0$ , let  $L_t$  denote the integral curve of  $g$  through the point  $(\exp t f)(x_0)$ . Choose a point  $x \in L_0$  close to  $x_0$ , travel from  $x_0$  to  $x$  instantaneously along  $L_0$  (assuming unbounded controls) and then travel along  $(\exp t f)(x)$  for  $t$  units of time. If  $f$  and  $g$  are linearly independent at  $x_0$  and the integral curves of  $g$  and  $[f, g]$  through  $x_0$  cross at  $x_0$ , then Hermes shows in [19] that our final destination is a point in some  $L_\tau$  with  $\tau \neq t$ . If  $\tau = t$ , instantaneous movement along  $L_t$  to  $x_0$  contradicts the fact that  $\tau < t$  (or  $\tau > t$ ) as Hermes has indicated. In fact, we have  $\tau < t$  for those points  $x$  in  $L_0$  close to  $x_0$  and on one side of  $x_0$  in  $L_0$  and  $\tau > t$  for those  $x$  in  $L_0$  on the other side. By continuity arguments, for each  $x$  in  $L_0$  sufficiently close to  $x_0$ , we arrive in time  $t$  at a distinct  $L_{\tau_x}$ . Thus, by shrinking  $V$  to an open set  $U$ , if necessary, all points in

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$x_0 \cap U$  move in time  $t > 0$  (and  $t$  sufficiently small) to different level sets of  $h$ . Hence, system (1) is locally level set observable at  $x_0$ . Hermes proves in [16] that the integral curves of  $g$  and  $[f, g]$  cross at  $x_0$  if statement 1) holds. We remark that Hermes' labeling of the sets  $S^i$  is different from ours.

If condition 2) holds, then the work of Hermes in [3] applies. In this case  $x_0$  is an equilibrium point of  $f$ . Hermes shows that as we move along the integral curve of  $g$  at  $x_0$ , the vector field  $f$  "changes sides" as we pass through the point  $x_0$ , as in the example in the introduction. Thus if  $V$  and  $t > 0$  are sufficiently small, the trajectories of  $\dot{x} = f(x(t))$  starting at points in the integral curve of  $g$  through  $x_0$  (and contained in  $V$ ) move to different integral curves of  $g$  in the time  $t$ . Note that if we begin at  $x_0$  we stay there for all time  $t$ . We have the desired observability in an open neighborhood  $U$  of  $x_0$ .  $\square$

The example in the introduction has the property that  $f(x_0) = 0$  if  $x_0$  is a point where  $x_2 = 0$ . We now provide an example, similar to one in [16], where statement 1) of the Theorem applies.

$$\text{Let } f(x) = \begin{bmatrix} 4 + x_1 x_2^3 \\ 0 \end{bmatrix},$$

$$y = h(x) = x_1,$$

and  $x_0 =$  the origin. In this case  $g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . The smallest integer  $m$  so that  $\dim \text{span } S_{x_0}^m = 2$  is 3, and we have the local level set observability.

The problems of trying to extend the Theorem to  $n > 2$  dimensions are quite interesting. Let us consider the system

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$$\begin{aligned}\dot{x}(t) &= f(x(t)) \\ y &= h(x)\end{aligned}\tag{6}$$

with  $f$  and  $h$  being real analytic on  $\mathbb{R}^n$ . Here  $h$  is real-valued and has a nonvanishing gradient at  $x_0 \in \mathbb{R}^n$ . The level sets of  $h$  are real analytic  $(n-1)$  dimensional submanifolds of  $\mathbb{R}^n$  near  $x_0$ . Thus the dual system is

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{n-1} u_i(t) g_i(x(t)),\tag{7}$$

where  $g_1, g_2, \dots, g_{n-1}$  are real analytic vector fields forming an involutive set with integral manifolds being the level sets of  $h$ . The theory of Hermes in [17] can be applied to give conditions under which we can distinguish between certain points with initial values in the same level set of the output.

If  $h$  is a  $p$ -vector valued function in (6), then  $h = (h_1, h_2, \dots, h_p)$  and we assume their gradients are linearly independent at  $x_0$ . In this case the dual system becomes

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{n-p} u_i(t) g_i(x(t))\tag{8}$$

where the set  $\{g_1, g_2, \dots, g_{n-p}\}$  is involutive near  $x_0$ . If the results of Hermes [15] using linearization are not applicable, then the problems concerning observability of (6) appear to be very difficult.

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REFERENCES

- [1] S.R. Kou, D.L. Elliot, and T.J. Tarn, Observability of nonlinear systems, Infor. Control, Vol.22, pp.89-99, 1973.
- [2] R. Hermann and A.J. Krener, Nonlinear controllability and observability, IEEE Trans. Autom. Contr., Vol.22, pp.728-740, 1977.
- [3] H. Hermes, Controlled stability, Ann. Mat. Pur. Appl., Vol.114 pp.103-119, 1977.
- [4] Kostyukovskii, Yu. M.-L., Observability of nonlinear controlled systems, Automat. i Telemekh., Vol.9, pp.1384-1396, 1968.
- [5] Kostyukovskii, Yu. M.-L., Simple conditions of observability of nonlinear controlled systems, Automat. i Telemekh., Vol.10, pp.1575-1584, 1968.
- [6] J.M. Fitts, On the observability of nonlinear systems with applications to nonlinear regression analysis, Symposium on Non-linear Estimate Theory and its Applications, San Diego, 1970.
- [7] T. Fujisawa and E.S. Kuh, Some results on existence in uniqueness of solutions to nonlinear networks, IEEE Trans. Circuit Theory, Vol.18, pp.501-506, 1971.
- [8] E.S. Kuh and I. Hajj, Nonlinear circuit theory: resistive networks, Proc. IEEE, Vol.59, pp.340-355, 1971.
- [9] E.W. Griffith and K.S.P. Kumar, On the observability of nonlinear systems, I, J. Math. Anal. Appl., Vol.35, pp.135-147, 1971.
- [10] J.P. Gauthier and G. Bornard, Observability for any  $u(t)$  of a class of nonlinear systems, IEEE Trans. on Autom. Contr., Vol.26, pp.922-926.
- [11] M. Fliess, The unobservability ideal for nonlinear systems, IEEE Trans. Autom. Contr., Vol.26, pp.592-593, 1981.
- [12] A. Isidori, A.J. Krener, C. Gori-Giorgi and S. Nonaco, The observability of cascade connected nonlinear systems, IFAC Congress, Kyoto, 1981.
- [13] A. Isidori, Observabilité et observateurs des systèmes non-linéaires in: Outils et modèles Mathématique pour l'automatique-l'Analyse de systèmes et le traitement dus signal, I. D. Landau, Ed., C.N.R.S., Paris, pp. 295-305.
- [14] H. Nijmeijer, Observability of a class of nonlinear systems: A geometric approach, preprint.
- [15] H. Hermes, On local and global controllability, SIAM J. Contr., Vol.12, pp.252-261, 1974.

- [16] H. Hermes, On necessary and sufficient conditions for local controllability along a reference trajectory, *Geometric Methods in Systems Theory*, D.Q. Mayne and R.W. Brockett, Eds., Dordrecht, Holland: Reidel, 1973.
- [17] H. Hermes, Local controllability and sufficient conditions in singular problems, *J. Differential Equations*, Vol.20, pp.213-232, 1976.
- [18] H. Hermes, Local controllability and sufficient conditions in singular problems, II., *SIAM J. Control*, Vol.14, pp.1049-1062, 1976.
- [19] H. Hermes, On local controllability, 20th IEEE Conference on Decision and Control, San Diego, pp.543-550, 1981.