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NUMERICAL COMPUTATIONS ON ONE-DIMENSIONAL INVERSE SCATTERING PROBLEMS

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and

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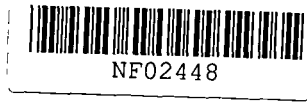

NUMERICAL COMPUTATIONS ON ONE-DIMENSIONAL INVERSE SCATTERING PROBLEMS

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ABSTRACT

In this note we present an approximate method to determine the index of refraction of a dielectric obstacle. For simplicity we treat one-dimensional models of electromagnetic scattering. The governing equations yield a second order boundary value problem, in which the index of refraction appears as a functional parameter. The availability of reflection coefficients yield two additional boundary conditons. We approximate the index of refraction by a k-th order spline which can be written as a linear combination of B-splines. For N distinct reflection coefficients, the resulting N boundary value problems yield a system of N non-linear equations in N unknowns which are the coefficients of the B-splines.

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Introduction

In this note we treat a class of inverse scattering problems in one dimension. Similar problems hae been studied and motivated by Hagin [1] and Gray and Hagin [2]. Our goal, like theirs is to reconstruct an unknown function, which we refer to as the index of refraction, to moderate accuracy and by simple means.

The problem we solve is as follows:

We seek a function $n(x)$ such that the solution u of the boundary value problem

$$u''(x) + n^2(x)\omega^2 u(x) = 0 \qquad x\in(0,1)$$

$$\left(u'+in_0\omega u\right)(0) = 2in_0\omega \qquad (P)$$

$$\left(u'-in_0\omega u\right)(1) = 0$$

satisfies $u(0) = f(\omega)$ and $u'(0) = g(\omega)$, where f and g are known functions of the real parameter ω.

This class of problems arises in several situations. In order to motivate the above problem we shall describe one such situation arising from electromagnetic wave propagation.

For a general graded index medium, whose refractive index is $n(x)$, the time harmonic Maxwell's equation, with single angular frequency k, can be written as [3]

$$\text{curl } \underline{H} = ikn^2(\underline{x})\varepsilon_0\underline{E}$$

$$\text{curl } \underline{E} = -ik\mu_0\underline{H} \qquad (M)$$

$$\text{div } \left(n^2(\underline{x})\varepsilon_0\underline{E}\right) = 0$$

$$\text{div } \underline{H} = 0$$

where $\underline{H}$ and $\underline{E}$ are the electric and magnetic fields and ε_0 and μ_0 are the permittivity and magnetic permeability. Consider a dielectric slab of thickness L (Figure 1) parallel to the y-axis

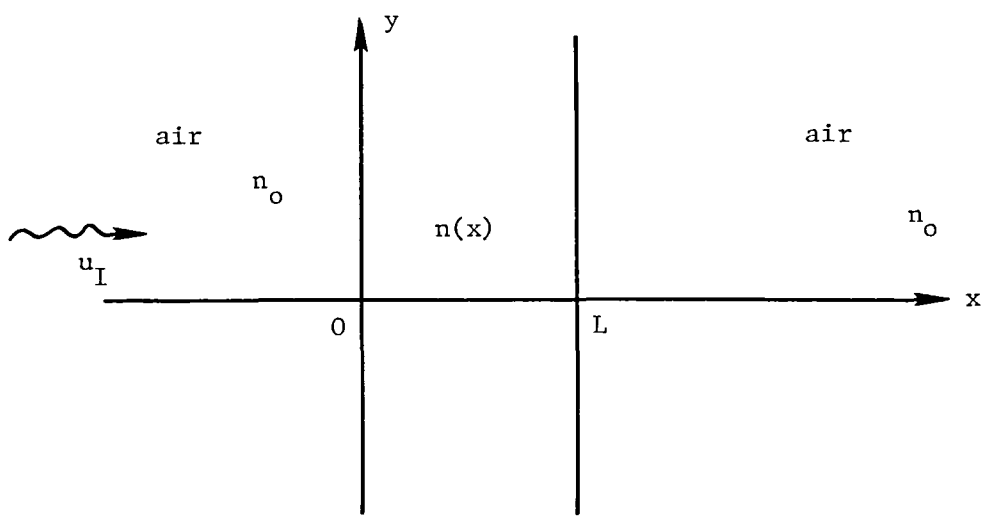

Figure 1

We assume the magnetic permeability of the slab is the same as for the ambient medium (air). Suppose there is a transverse electromagnetic wave u_I present. That is to say the electric field $\underline{E}$ and the magnetic field $\underline{H}$ are at right angles to each other and to the direction of propagation which is

taken as the x-axis. $\underline{E} = (0,u(x),0)^T$ satisfies $(M)_3$. $\underline{H} = \left(0,0,\frac{iu'(x)}{k\mu_0}\right)^T$ satisfies $(M)_2$ and $(M)_4$. Demanding $\underline{H}$ and $\underline{E}$ to satisfy $(M)_1$ gives $u''(x) + k^2\mu_0\varepsilon_0 n^2(x)u(x) = 0$. Upon nondimensionalization, in particular replacing x with x/L, the above becomes

$$u''(x) + \omega^2 n^2(x)u(x) = 0 \tag{1.1}$$

where $\omega^2 = \mu_0\varepsilon_0 k^2 L^2$. The bondary conditions are that the tangential components of $\underline{E}$ and $\underline{H}$ are continuous across the interfaces of the slab. This implies that u and u' are continuous at $x = 0$ and $x = 1$. The problem that we consider is that the index of refraction of the ambient medium is a constant n_0. The process of wave propagation is initiated by a wave coming from $x = -\infty$ of the form $e^{in_0\omega x}$. Then the solution of (1.1) for $x < 0$ can be written as

$$u(x) = e^{in_0\omega x} + R(\omega)e^{-in_0\omega x}, \tag{1.2}$$

where $R(\omega)$ is called the reflection coefficient. For $x > L$ part of this wave is transmitted and the solution has the form

$$u(x) = T(\omega)e^{in_0\omega x}, \tag{1.3}$$

where $T(\omega)$ is called the transmission coefficient. Since u and u' are continuous at $x = 0$ and $x = 1$, then (1.2) and (1.3) imply that

$$\left(u'+in_0\omega u\right)(0) = 2in_0\omega \tag{1.4}$$

and

$$(u'-in_0\omega u)(1) = 0 \tag{1.5}$$

In practice the reflection coefficient $R(\omega)$ can be measured for any desired value of ω. The continuity of u and u' at $x = 0$ together with (1.2) imply that

$$u(0) = 1 + R(\omega)$$

and (1.6)

$$u'(0) = in_0\omega(1-R(\omega))$$

Thus $u(0)$ and $u'(0)$ are known functions of ω.

2. Solution Procedure

Our solution procedure consists of approximating $n(x)$ with a k-th order spline $\bar{n}(x)$. This choice of approximation is motivated by the need to solve the problem (P) quickly and accurately. We will expound on this statement after the description of the scheme.

We begin with a brief description of the spline spaces $S_k(\underline{x})$. Let $\underline{x} = \{x_i\}_{i=1}^{N+1}$ be a partition of $[0,1]$, and let $S_k(\underline{x})$ denote the space of k-th order splines with knots at each x_i. $s\in S_k(\underline{x})$ implies that in each interval $[x_i,x_{i+1}]$ s is a polynomial of degree at most $k-1$ and $s\in C^{(k-2)}[0,1]$. Let $\{B_{i,k}(x)\}_{i=1}^{N+k-1}$ denote the B-spline basis for $S_k(\underline{x})$. The general properties of B-splines are well known and can be found in [4] for example. A property for which we will have occassion to utilize is that $B_{i,k}(x) = 0$ if $x\notin[x_{i-k+1},x_{i+1}]$. As a consequence, if $f(x) = \sum_{i=1}^{N+K-1} \alpha_i B_{i,k}(x)$ and if $x\in[x_m,x_{m+1}]$ then $f(x)$ is simply given by

$$f(x) = \sum_{i=m}^{m+k-1} \alpha_i B_{i,k}(x). \tag{2.1}$$

The approximation of n proceeds as follows:

For given N and k we seek an approximation

$$\bar{n}_{(\bar{N})}(x) = \sum_{i=1}^{\bar{N}} \lambda_i^{(\bar{N})} B_{i,k}(x), \tag{2.2}$$

where $\bar{N} = N + k - 1$, such that, given the distinct pairs $(\omega_j, R(\omega_j))$ $j = 1,\cdots,\ell$, where $\ell = \lceil \bar{N}/2 \rceil$, the $\lambda_i^{(\bar{N})}$'s are chosen such that the solutions $\bar{u}_j$ of the initial value problems

$$\bar{u}_j'' + \omega_j^2 \bar{n}_{(\bar{N})}^2(x)\bar{u}_j = 0$$

$$\bar{u}_j(0) = 1 + R(\omega_j) \qquad \text{(I)}$$

$$\bar{u}_j'(0) = i\omega_j(1-R(\omega_j))$$

satisfy $(\bar{u}_j' - i\omega_j \bar{u}_j)(1) = 0$ for $j = 1,2,\cdots,\ell$. This is equivalent to solving

$$\underline{F}(\underline{\lambda}) = \underline{0} \tag{2.3}$$

where $\underline{\lambda} = (\lambda_1^{(\bar{N})},\cdots,\lambda_{\bar{N}}^{(\bar{N})})^T$ and $\underline{F}(\underline{\lambda}) = (f_1(\underline{\lambda}),\cdots,f_{\bar{N}}(\underline{\lambda}))^T$ and where $f_{2j}(\underline{\lambda}) = \text{Real}((\bar{u}_j' - i\omega_j\bar{u}_j)(1))$ and $f_{2j-1}(\underline{\lambda}) = \text{Imag}((\bar{u}_j' - i\omega_j\bar{u}_j)(1))$ $j = 1,2,\cdots,\ell$. Before proceeding to the motivation we note that the system (2.3) is highly nonlinear.

In order to solve (2.3) one must solve (I) many times. In general this requires the numerical solution of these problems which leads to a large cost

in computer time. Our choice of approximation reduces this time to a great extent. For example, when $k = 1$ one can obtain the solution of (I) in closed form, and when $k \geqslant 2$ one can easily obtain truncated Taylor expansions, whose accuracy is comparable to the numerical integrations of (I), but the time taken to compute the expansions is, in general, much less than the time taken to integrate (I) numerically. The Taylor expansions are computed as follows: For $x\in[x_m,x_{m+1}]$, $\bar{n}_{(\bar{N})}(x)$ is a polynomial of degree at most $k - 1$ and is given by

$$\bar{n}_{(\bar{N})}(x) = \sum_{i=m}^{m+k-1} \lambda_i^{(\bar{N})} B_{i,k}(x).$$

This follows from (2.1). Let $S_m(x)$ and $H_m(x)$ denote two linearly independent Taylor series solutions, expanded about x_m, of $(I)_1$. Then for $x\in[x_m,x_{m+1}]$ the solution of (I) is given by $\bar{u}_j(x) = a_m S_m(x) + b_m H_m(x)$, where the a_m's and b_m's are chosen so that $\bar{u}_j$ and $\bar{u}_j'$ are continuous at each x_m and also to satisfy the initial conditions.

It is known [4], that if $f\in C^{(j)}[0,1]$ for $j = 0,1,\cdots,k-1$, then

$$\inf_{s\in S_k(\underline{x})} \|f-s\|_\infty \leqslant C_{k,j} h^j w(f^{(j)},h)$$

where $h = \max_{1\leqslant i\leqslant N} (x_{i+1}-x_i)$, $C_{k,j}$ is a constant which depends only on k and j, and

$$w(f^{(j)};h) := \sup\{|f^{(j)}(x)-f^{(j)}(y)|: \ x,y\in[0,1], \ |x-y|\leqslant h\}$$

is the modulus of continuity of $f^{(j)}$ at h. It is also known that the above estimate cannot be improved. Thus, if $n\in C^{(k)}[0,1]$, then

$w(n^{(k-1)};h) \leq h\|n^{(k)}\|_\infty$ and the best that we can hope for is $\|n-\bar{n}_{(\bar{N})}\|_\infty = O(h^k)$. As will be seen, in several numerical examples tested the optimal convergence rate was attained.

3. Numerical Results

In this section we present some numerical examples for $k = 1,2$ and 3 with knot sequence $\{\frac{i-1}{N}\}_{i=1}^{N+1}$. In all examples the Levenberg-Marquardt algorithm [5] was used to solve (2.3). The $R(\omega_j)$'s were computed by inputting the exact solution $n(x)$ into (P).

To solve a large non-linear system of equations it is usually necessary to have a good initial approximation to the solution. For fixed $\bar{N}$ and k our solution was built up according to the following algorithm:

(i) Select a sequence of integers $\underline{m} = \{m_i\}_{i=1}^{M}$ with $m_1 = k$, $m_M = \bar{N}$ and $m_i < m_{i+1}$.

(ii) Compute $\bar{n}_{(m_1)}$ by solving (2.3) with an initial guess

$$\lambda_1^{(m_1)} = \cdots = \lambda_{m_1}^{(m_1)} = 1.$$

(iii) For $j = 1,\cdots,M-1$ compute $\bar{n}_{(m_{j+1})}$ by solving (2.3) with the initial guess chosen so that $\bar{n}_{(m_{j+1})}$ interpolates $\bar{n}_{(m_j)}$ at the m_{j+1} distinct points $\{y_\ell\}_{\ell=1}^{m_{j+1}}$. This gives a linear system for the initial $\lambda_i^{(m_j+1)}$'s that is invertible if and only if

$x_\ell < y_\ell < x_{\ell+k}$ $\ell = 1,\cdots,m_{j+1}$ [4].

We found that the best results were obtained by setting $\underline{m} = \{2^i\}_{i=1}^{M}$ for $k = 1$ and $\underline{m} = \{2i\}_{i=1}^{M}$ for $k = 2$ and 3.

The maximum attainable $\overline{N}$ is limited by considerations of computer time and accuracy. In the examples tried we found that satisfactory results were obtained if $\overline{N} = 32$ for $k = 1$ and $\overline{N}$ varied between 10 and 14 for $k = 2$ and 3.

Example 1. (Figures 2, 6, 10)

$n(x) = 1 + x^2$

Table I

k	1	2	3
$\overline{N}$	32	12	10
$\|n-\overline{n}_{(\overline{N})}\|_\infty$	.036	.0064	2.4×10^{-6}
computational time (cpu sec.)	123	31.2	8.30
estimated convergence rate	1.0	2.1	____

Example 2. (Figures 3, 7, 11)

$$n(x) = \begin{cases} 1 & x \in [0, \frac{1}{4}) \cup (\frac{3}{4}, 1] \\ 1 + \sin^2 2\pi(x - \frac{1}{4}) & x \in [\frac{1}{4}, \frac{3}{4}] \end{cases}$$

Table II

k	1	2	3[a]
$\overline{N}$	32	12	10
$\|n-\overline{n}_{(\overline{N})}\|_\infty$	.16	.058	.021
computational time (cpu sec.)	195	101	35.7
estimated convergence rate	1.1	.96	____

Example 3. (Figures 4, 8, 12)

$$n(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ 2 & x \in [\frac{1}{2}, 1] \end{cases}$$

Table III

k	1	2	3
$\overline{N}$	16	8	10
$\|n-\overline{n}_{(\overline{N})}\|_\infty$	.50	.46	.47
computational time (cpu sec.)	25.6	20.2	104
estimated convergence rate	____	____	____

(a)These results were obtained by demanding the initial approximation to interpolate the final approximation for k = 2.

Example 4. (Figures 5, 9, 13)

$$n(x) = \begin{cases} 1 & x \in [0, \frac{1}{4}) \\ \frac{1}{2} + 2x & x \in [\frac{1}{4}, \frac{3}{4}] \\ 2 & x \in (\frac{3}{4}, 1] \end{cases}$$

Table IV

k	1	2	3
$\overline{N}$	32	14	10
$\|n-\overline{n}_{(\overline{N})}\|_\infty$	.050	.028	.025
computational time (cpu sec.)	09.2	117	51.1
estimated convergence rate	1.1	1.1	____

4. Discussion

In all examples with $k = 1$ and 2, except for example 3, the optimal convergence rate was attained.

It is obvious from the results listed in Tables 1 - 4 that if a desired accuracy is specified, then the minimum computational time is achieved by increasing the order of the splines rather than increasing the number of unknowns for a fixed order.

In practice the measurement of the reflection coefficients is subject to experimental error. To simulate this situation we introduced Gaussian type noise in the values of $R(\omega)$. Stable results were obtained if the amplitude of the noise was less than 10^{-6}.

Our results are comparable in accuracy and computational time to those obtained in [1] and [2]. However, our method is applicable to any boundary value problem in which the unknown appears as a parameter and sufficient additional conditions are known.

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K-1

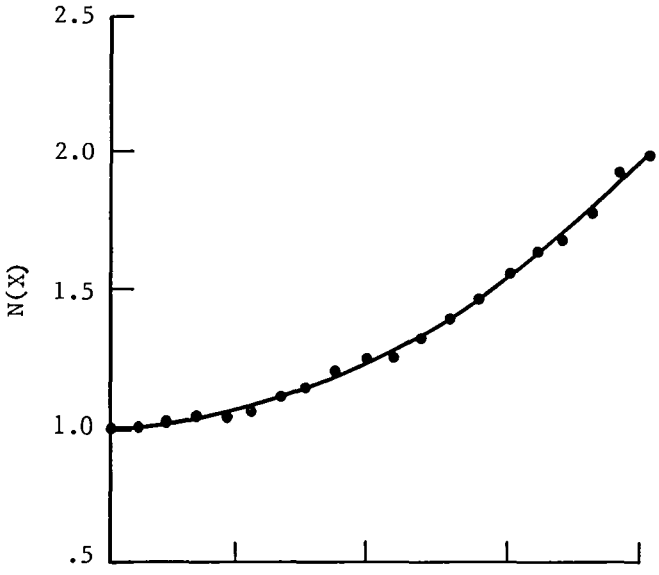
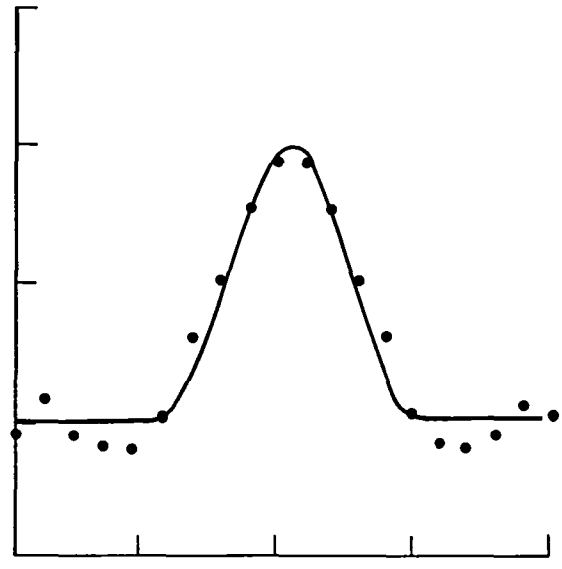


Figure 2



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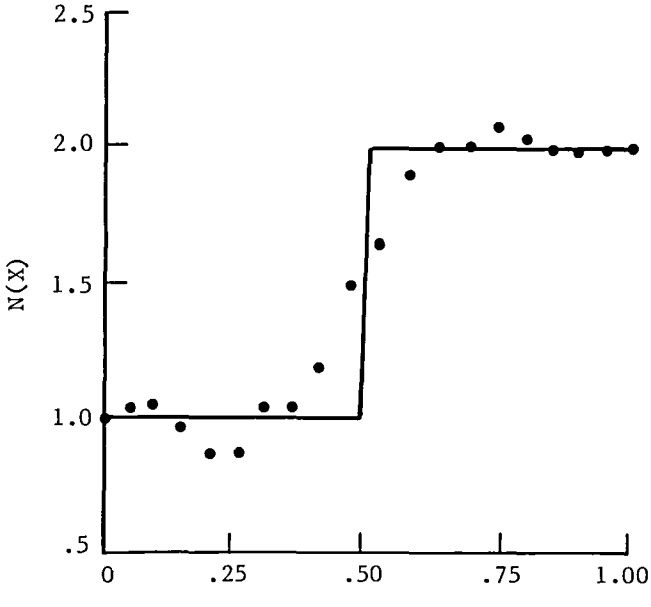

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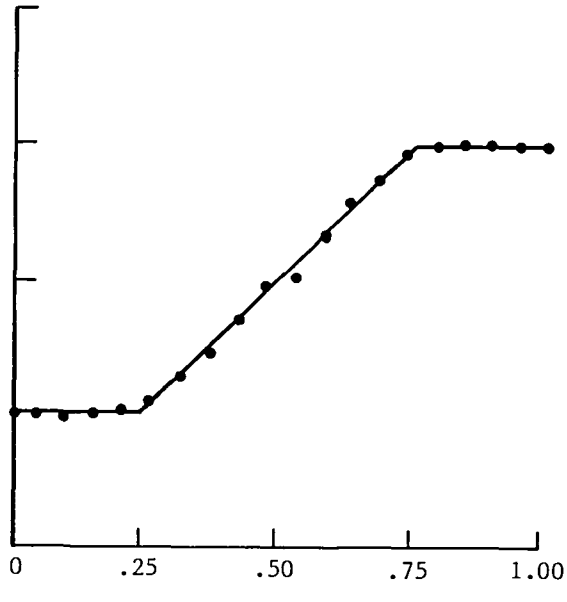


Figure 5

____ exact solution

• computed solution

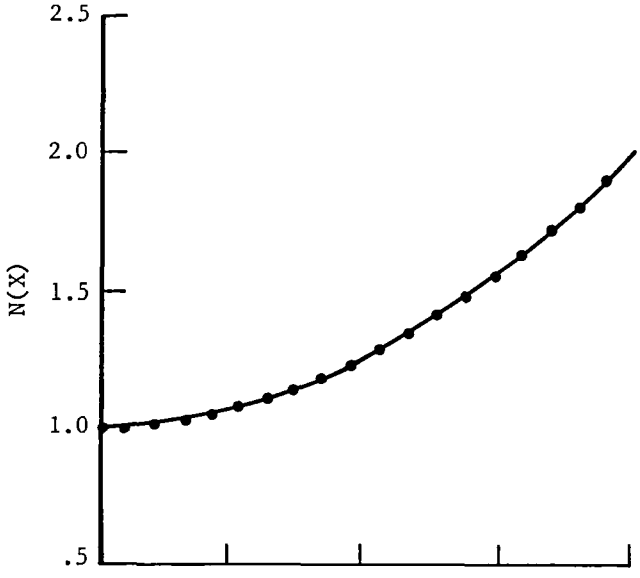
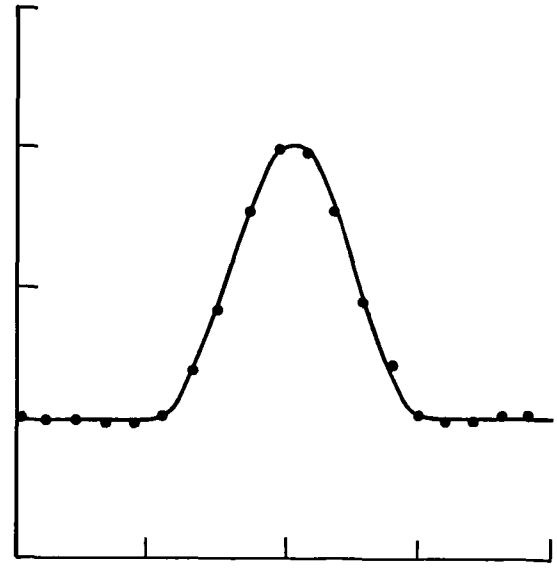


Figure 6



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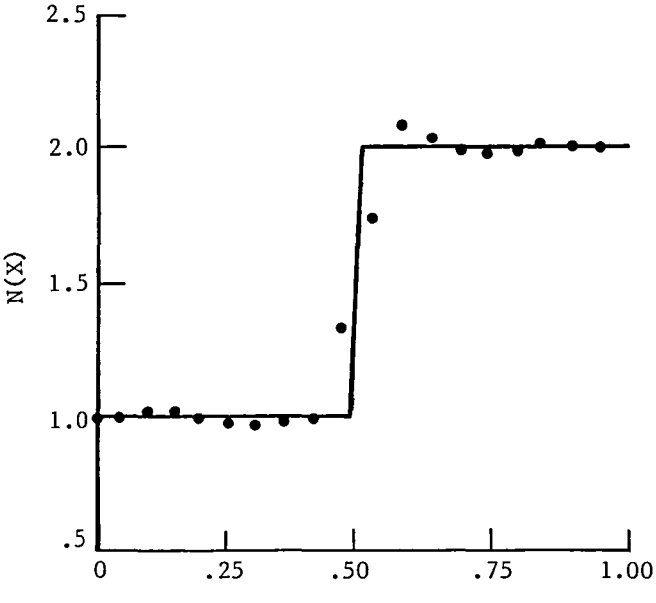

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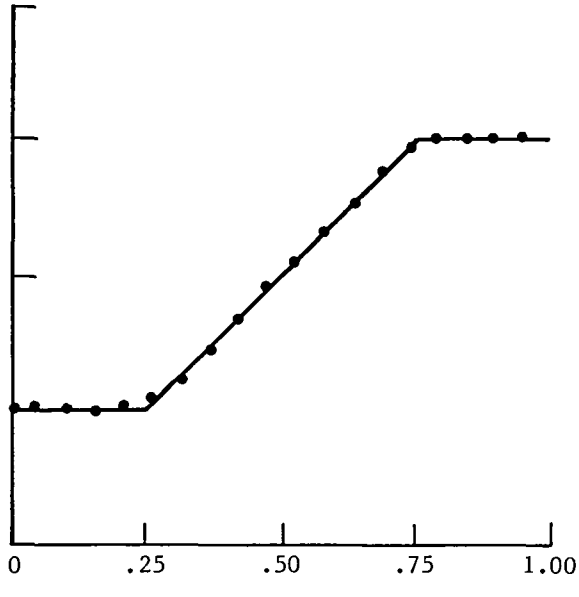
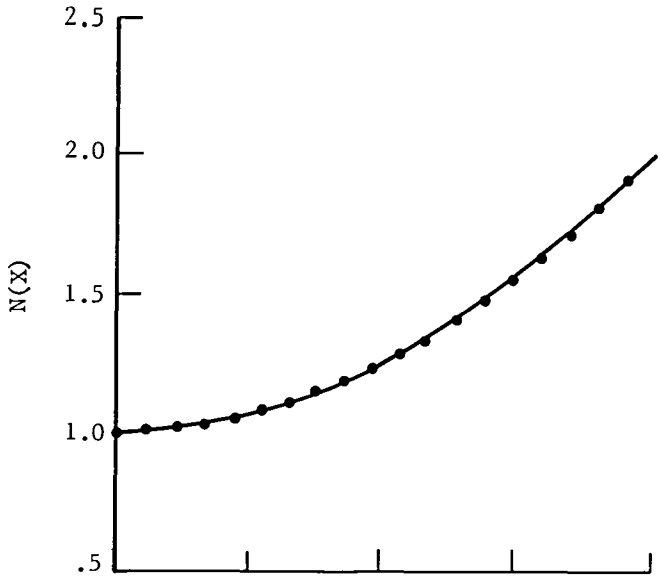


Figure 9

____ exact solution

• computed solution



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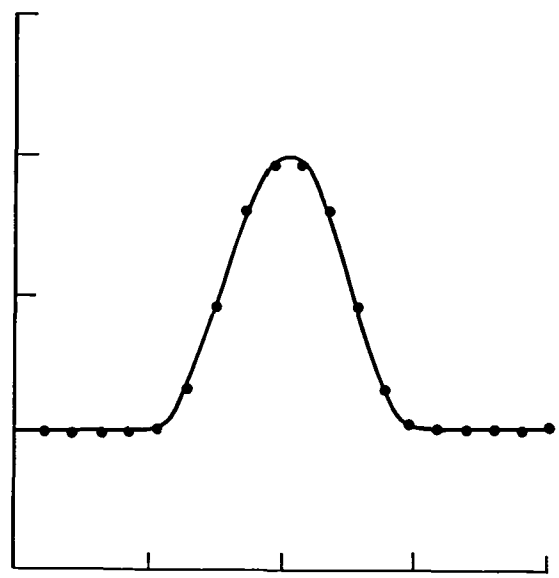

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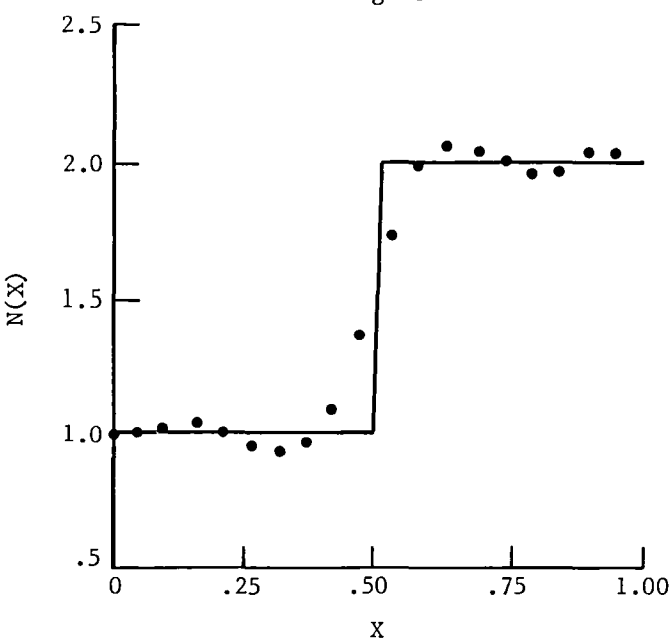

Figure 12

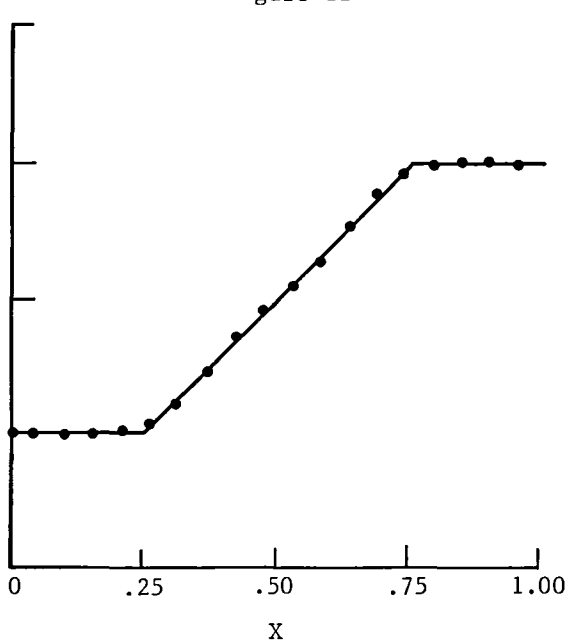

Figure 13

____ exact solution

• computed solution

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15 Supplementary Notes

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16 Abstract

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