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EMISSION, ABSORPTION AND POLARIZATION OF
GYROSYNCHROTRON RADIATION OF MILDLY
RELATIVISTIC PARTICLES

BY

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EMISSION, ABSORPTION AND POLARIZATION OF GYROSYNCHROTRON
RADIATION OF MILDLY RELATIVISTIC PARTICLES

by

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## ABSTRACT

This paper presents approximate analytic expressions for the emissivity and absorption coefficient of synchrotron radiation of mildly relativistic particles with an arbitrary energy spectrum and pitch angle distribution. From these, an expression for the degree of polarization is derived. The analytic results are compared with numerical results for both thermal and non-thermal (power law) distributions of particles.

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## I. INTRODUCTION

The formulas for evaluation of emissivity and absorption coefficient of synchrotron radiation in the ultra-relativistic (synchrotron) and non-relativistic (cyclotron) limits have been known for decades.[1,2]. In the intermediate energy range, however, no simple formula exists for an arbitrary distribution of particles. Earlier Trubnikov[2] derived formulas for emissivity and absorption limited to a thermal gas and convenient only for propagation perpendicular to the magnetic field. As a result, the usual practice has been to use lengthy numerical calculations.[3,4].

In a recent paper[5] (referred to as Paper I, hereafter) we presented simple approximate methods for evaluation of the frequency spectrum and angular variation of the synchrotron radiation at high harmonics from an (essentially) arbitrary distribution of particles in a given magnetic field. In this paper, we shall use the same methods to calculate the emissivity and absorption coefficient of the extraordinary and ordinary modes of synchrotron radiation separately. Also, we will derive the degree of circular polarization from these expressions. The general expressions presented in Sec. II were derived for high harmonics of the cyclotron frequency which are the most useful since the low harmonics are usually self-absorbed, absorbed by the surrounding plasma, or suppressed by the Razin-Tsytovich effect. In Sec. III we use the above results to find emissivity, absorption and polarization of the radiation from particles with Maxwellian and power law energy distributions. Here, through comparison with numerical results, we also show that our formulas provide a good approximation even at low harmonics. In Sec. IV we present a final summary.

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## II. GENERAL RESULTS

Consider particles with charge e, mass $m_e$, and distribution $f(\mu,\gamma)$ where $f d\mu d\gamma$ is the number density of particles in the energy interval (in units of $m_e c^2$) from $\gamma$ to $\gamma + d\gamma$ and with pitch angle cosine between $\mu$ and $\mu + d\mu$. The emissivity and absorption coefficients for the ordinary (+) and extraordinary (−) modes at frequency $\nu$ and at angle $\theta$ with respect to the magnetic field B are[1,4],

$$
\left\{ \begin{array}{c} j_{\pm}(\nu,\theta) \\ \\ \nu^2 \kappa_{\pm}(\nu,\theta) \end{array} \right\} = \frac{2\pi e^2 \nu_b}{c} \frac{\nu}{\nu_b \sin^2\theta} \int_1^{\infty} d\gamma \int_{-1}^{1} d\mu f(\mu,\gamma)\, \eta_{\pm}(\theta,\gamma,\mu,\nu) \left\{ \begin{array}{c} 1 \\ \\ \omega \end{array} \right\}, \quad (1)
$$

where

$$
\eta_{\pm} = \frac{1}{T_{\pm}^2 + 1} \sum_{m=1}^{\infty} \left\{ T_{\pm}(\cos\theta - \beta\mu) J_m(x) - (1-\beta\mu\cos\theta) \frac{x J'_m(x)}{m} \right\}^2 \delta(y), \quad (2)
$$

$$
\omega = -\partial \ln[f(\mu,\gamma)/\beta\gamma^2]/\partial\gamma + \frac{\beta\cos\theta - \mu}{\beta\gamma^2(1-\mu^2)^{1/2}} \, \partial\ln f(\mu,\gamma)/\partial\mu , \quad (3)
$$

$$
y = \frac{m\nu_b}{\gamma\nu} - (1-\beta\mu\cos\theta), \quad x = (\nu\gamma/\nu_b)\beta\sin\theta(1-\mu^2)^{1/2}, \quad \nu_b = eB/2m_e c . \quad (4)
$$

The quantity T represents the ratio of the major to the minor axis of the polarization ellipse and, in general, is a complicated function of angle $\theta$, $\nu$, $\nu_b$, and the plasma frequency $\nu_p$. The ordinary (+) and extraordinary (−) modes are distinguished by their respective values of T, which in vacuum or for $\nu_p/\nu \ll 1$ are

$$
T_{+} = -T_{-}^{-1} = T_0 = \xi + (1+\xi^2)^{1/2}, \quad \xi = \nu_b \sin^2\theta/(2\nu\cos\theta) . \quad (5)
$$

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Note that for $\nu/\nu_b \gg 1$, the quantity $\xi \ll 1$ and $T_\pm \to \pm 1$ except for a small range of angles $(\pi/2 - \theta) < \nu_b/2$ where $\xi > 1$. And in the limit $\theta \to \pi/2$, $\xi \to \infty$ so that $T_+ \to \infty$ and $T_- \to 0$.

Since we are only dealing with the higher harmonics, m is large and we can replace the sum in equation (2) with an integral. This integral can then be done using the $\delta$- function. Also, since m is large, we can use the Carlini approximation for the Bessel's function as[6].

$$J_m(mz) = \frac{(1-z^2)^{-1/4}z^m}{(2\pi m)^{1/2}} \quad , \quad z^2 = \frac{[1-(1-z^2)^{1/2}]}{[1+(1-z^2)^{1/2}]} e^{2(1-z^2)^{\frac{1}{2}}} \tag{6}$$

which is valid for $(1-z^2)^{-3/2}/m \ll 1$.

We evaluate the integrals to first order in this quantity.

**Integration over Pitch Angle**

As can be seen from Eq. (6), the Bessel function and its derivative have sharp maxima at $\mu = \beta\cos\theta$ and tend to zero rapidly at $\mu = \pm 1$. The rest of the integrand is positive and varies slowly with $\mu$ if the pitch angle distribution of the particles is not extremely anisotropic. We use the method of steepest descent to evaluate the integrals over $\mu$. The only exception to this is the coefficient of $J_m(x)$ in Eq. (3) which becomes zero at $\mu = \cos\theta/\beta$. For a general $\cos\theta$ this will give rise to a secondary maximum whose contribution is of second order in our expansion parameter. In particular, for perpendicular propagation $(\cos\theta = 0)$ this term becomes zero at the maximum of the function $Z^m$. This gives rise to two maxima, each with a second order contribution to the integral.

The use of the method of the steepest descent amounts to setting $\mu = \beta\cos\theta$ in the integrand and multiplying it by $(\pi\nu_b/\nu)^{1/2}(1 - \beta^2\cos^2\theta)^{3/4}$.

This means that most of the contribution to the radiation at an angle $\theta$ comes from particles with pitch angle cosines in a narrow range of the order of $(\pi \nu_b/\nu)^{1/2}$ centered at $\mu = \beta\cos\theta$.

This approximation is valid only for pitch angle distributions which are not extremely anisotropic, i.e., distributions with $\partial\ln f/\partial\mu \ll \nu/\nu_b$ [cf. Paper I, Eqs. (7) and (19)]. Note that this enables us to drop the second term in the expression for $\omega$ in Eq. (3) so that the pitch angle dependence becomes the same for $j$ and $\kappa$.

### Integration over Energy

The resultant integrand for integration over energy also has a sharp maximum for particle distributions which fall rapidly with increasing energy [e.g., a power law $f \propto (\gamma - 1)^{-\delta}$ or a thermal distribution $f \propto \exp(-\gamma/kT)$]. Consequently, we can use the method of steepest descent to carry out this integration also. This gives

$$\left.\begin{matrix} j_\pm \\ \\ \\ \nu^2\kappa_\pm \end{matrix}\right\} = \frac{\pi e^2 \nu_b}{c}\left(\frac{\nu}{\nu_b}\right)^{\frac{1}{2}}\left(\frac{u_o}{\gamma_o}\right)^{\frac{1}{2}} Y(\theta,\gamma_o)\left[Z_{mx}(t_o)\right]^{2\nu u_o^2/\nu_b\gamma_o}\left\{\begin{matrix} X_j(\gamma_o)f_j(\beta_o\cos\theta,\gamma_o) \\ \\ \\ X_\kappa(\gamma_o)f_\kappa(\beta_o\cos\theta,\gamma_o) \end{matrix}\right. \tag{7}$$

where

$$X_i^{-2} = -\gamma^2 d^2\ln f_i/d\gamma^2 - W\gamma d\ln f_i/d\gamma \tag{8}$$

$$Z_{mx}^2 = \left(\frac{u-1}{u+1}\right)e^{2/u}, \quad u^2 = 1 + \beta^2\gamma^2\sin^2\theta \tag{9}$$

$$Y(\theta,\gamma) = \left(\frac{u - T\cos\theta}{\gamma\sin\theta}\right)^2 \bigg/ (1 + T^2) . \tag{10}$$

Note that all these expressions are evaluated at critical energy $\gamma_0$ (and the corresponding $\beta_0$ and $u_0^2 = 1 + \beta_0^2 \gamma_0^2 \sin^2\theta$), where most of the contribution to the integral comes from. There are two such critical energies; one for emission and one for absorption. These are obtained from the transcendental equation;

$$u^{-1}(u^2 - 1)^{-1} + (1 - \cot^2\theta/\gamma^2)\ln Z_{mx} = -(\omega_b/2\nu\sin^2\theta)d\ln f_{j,\kappa}/d\gamma \tag{11}$$

$$\equiv \epsilon/\sin^2\theta .$$

Here and in equation (7) the function $f_j$ and $f_\kappa$ are related to the particle distribution $f$ as

$$f_j = f(\beta\cos\theta, \gamma)/\gamma , \qquad f_\kappa = -\beta\gamma d[f(\beta\cos\theta,\gamma)/\beta\gamma^2]/d\gamma . \tag{12}$$

In Eq. (8) the function $W$ is a complicated function of $u_0$ and $\theta$. However, as shown in paper I, $W$ can be approximated by the following simple expression,

$$W = 3/2 + 1/(\gamma_0^2 - 1), \tag{13}$$

which is an excellent approximation in the entire region where equations (6) to (11) are valid.

Ordinary and Extraordinary Modes

The emission and absorption coefficient of each mode is obtained by the substitution of $T_\pm$ in Eq. (7). Thus, $j_\pm$ and $\kappa_\pm$ become proportional to $Y_\pm$ evaluated at the respective values of the critical energy $\gamma_0$. Since $T_+T_- = -1$, it can be shown that

$$Y_+ = (u^2 - 2T_0 u\cos\theta + T_0^2\cos^2\theta)/[\gamma^2\sin^2\theta(1 + T_0^2)],$$

$$Y_- = (T_0^2 u^2 + 2T_0 u\cos\theta + \cos^2\theta)/[\gamma^2\sin^2\theta(1 + T_0^2)]. \tag{14}$$

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The total emissivity then becomes

$$j_{tot} = j_+ + j_- \propto (Y_+ + Y_-) = (u_o^2 + \cos^2\theta)/\gamma^2\sin\theta = 1 + 2\cot^2\theta/\gamma^2 \qquad (15)$$

in agreement with Paper I. Similarly, with the help of Eq. (5) the difference of emissivities is given by

$$j_- - j_+ \propto Y_- - Y_+ = (2u\cos\theta/\gamma^2\sin^2\theta + \xi)/(1 + \xi^2)^{1/2}. \qquad (16)$$

For $\nu/\nu_b \gg 1$ and away from the direction perpendicular to the magnetic field, $\xi \ll 1$ and $T_o = 1 + \xi$ so that the ratio of $j$ and $\kappa$ of the ordinary to that of the extraordinary mode becomes

$$\kappa_+/\kappa_- = j_+/j_- = \left(\frac{u_o - \cos\theta}{u_o + \cos\theta}\right)^2 \left[1 - 2\xi\left(\frac{u_o^2 + \cos^2\theta}{u_o^2 - \cos^2\theta}\right)\right] \qquad . \qquad (17)$$

This approximation breaks down for $(\theta - \pi/2) < \gamma_o/2\nu$. As $\theta \to \pi/2$, $u \to \gamma$, $T_o \simeq 2\xi \to \infty$ and

$$Y_+ \to (u - \nu_b/\nu)^2 \nu^2\cos^2\theta/\nu_b^2\gamma^2 \qquad ,$$

$$\qquad (18)$$

$$Y_- \to 1 + 2\nu\cot^2\theta/\nu_b\gamma \qquad .$$

In the limit $\theta = \pi/2$, $Y_+ = 0$ and $Y_- = 1$. This, of course, means that for evaluating the $j_+$ and $\kappa_+$ at $\theta = \pi/2$ one must consider higher order terms which we have neglected. At $\theta = \pi/2$ the ratio $Y_+/Y_-$ is of higher order of our expansion parameter $m^{-1}(1-z^2)^{-3/2} = u_o\gamma_o\nu_b/\nu$.

## 1. Polarization

The polarization of the radiation can be obtained from the Stokes parameters which are (cf. references 1 and 4)

$$j_Q = [(1-T_o^2)(j_+-j_-) + 4\cos\delta T_o(j_+j_-)^{1/2}](1 + T_o^2),$$

$$j_u = 2\sin\delta(j_+j_-)^{1/2},$$

(19)

$$j_v = [2T_o(j_+-j_-) - 2\cos\delta(1-T_o^2)(j_+j_-)^{1/2}]/(1+T_o^2).$$

With the help of Eq. (5) and using Eqs. (15) and (16) and if we define

$$P_o = \frac{Y_+-Y_-}{Y_++Y_-} = \frac{(2u\cos\theta/\gamma^2\sin^2\theta+\xi)}{(1+\xi)^{1/2}(1+2\cot^2\theta/\gamma^2)},$$

(20)

these become

$$j_Q/j_{tot} = [\xi P_o + \cos\delta(1 - P_o^2)^{1/2}]/(1 +\xi^2)^{1/2},$$

$$j_U/j_{tot} = \sin\delta(1 - P_o^2)^{1/2},$$

(21)

$$j_V/j_{tot} = [P_o - 2\cos\delta(1 - P_o^2)^{1/2}\xi]/(1 + \xi^2)^{1/2}.$$

Here $\delta$ is the phase difference between the ordinary and the extraordinary modes. For a definite value of $\delta$ the degree of Polarization $P = (j_Q^2 + j_u^2 + j_v^2)^{1/2}/j_{tot} = 1$. However, for radiation from many particles in a large source with large Faraday rotation, the phase relations are randomized so that the average values $\langle\sin\delta\rangle = \langle\cos\delta\rangle = 0$. In this case $j_U = 0$ and the degrees of linear, circular and total (elliptical) polarization are simply

$$P_{lin} = \xi P_o/(1 + \xi^2)^{1/2}, \quad P_{circ} = P_o/(1 + \xi^2)^{1/2}, \quad P_{tot} = P_o.$$

(22)

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When $\nu/\nu_b \gg 1$ and $\xi \ll 1$, $P_{lin} \ll P_{circ}$ and the radiation is circularly polarized except near $\theta = \pi/2$ where $\xi \to \infty$ and radiation becomes linearly polarized. As evident from Eqs. (20) and (22) the degree of polarization decreases with increasing frequency (or the energy of the particles).

## 2. Summary

Eqs. (7) to (13) along with (21) and (22) give our results in their general forms and are valid for all particle distributions which are not extremely anisotropic. The results for the extremely anisotropic situation are more complicated and were described in Paper I. The modification for obtaining the emission and absorption coefficients of the two modes separately and the polarization is similar to the modification described above. We will not present these results here because of their complexity and their limited usefulness.

Given a distribution function subject to this limitation, the first step is evaluation of the critical energies $\gamma_0$ from Eq. (11). Then Eqs. (6) to (10) and (12) and (13) evaluated at the appropriate $\gamma_0$'s give the desired results. The most complicated part of this procedure is the solution of Eq. (11) for $\gamma_0$. It turns out that for most practical cases it is not necessary to solve this equation.

In the next section we shall show how this step of the calculation is simplified considerably for the two most commonly used particle distributions. Before doing so we consider the asymptotic limits of these equations.

3. Asymptotic limits. Let us consider first the case when angle $\theta$ is not too small (i.e., radiation away from the direction of the field). Then in the two extreme limiting cases, Eq. (11) simplifies

i) $\epsilon \ll 1$, $\gamma_0 \gg 1$, $\beta_0^3 \gamma_0^3 = 2/3 \epsilon \sin\theta$, $W = 3/2$;

$$(23)$$

ii) $\epsilon \gg 1$, $\beta_0 \ll 1$, $\gamma_0 \simeq 1$, $\beta_0^2 \gamma_0^2 = 4/\epsilon$, $W = \epsilon/4$.

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The first case is realized at high frequencies and particle distributions which are not extremely non-relativistic which is the case of interest here. The second case is valid for non-relativistic particles and at low frequencies and has limited usefulness except for a low temperature thermal gas.

### III. EMISSIVITY AND ABSORPTION COEFFICIENT OF TWO COMMONLY USED PARTICLE DISTRIBUTIONS

The two particle distributions we use as examples are i) the distribution from a thermal gas, i.e. a Maxwellian distribution in energy and isotropic pitch angle distribution; and ii) the distribution with a power law spectrum at high energies and with a slowly varying pitch angle distribution.

### A. Thermal Spectrum

In this case, the distribution $f$ of particles at temperature $kT$ (in units of $m_e c^2$) is

$$f(\mu,\gamma) = Ce^{-(\gamma-1)/kT\gamma}(\gamma^2 - 1)^{1/2} , \qquad (24)$$

where for $kT \lesssim 1$

$$C \approx (n/2) [2\pi(kT)^3]^{-1/2} (1 - 15 kT/8 + ...), \qquad (25)$$

and $n$ is the number density of particles. From these and Eq. (12), it is clear that $kT\kappa_k = f_j = f/\gamma$ so that [as is evident from Eq. (11)] the critical $\gamma_0$ is the same for both $j$ and $\kappa$ and, in fact, according to Eq. (7), $j_\pm = kT\kappa_\pm$. We also find that

$$\frac{d\ln(f/\gamma)}{d\gamma} = \frac{-1}{kT}\left(1 - \frac{kT\gamma}{\gamma^2-1}\right) , \quad \frac{d^2\ln(f/\gamma)}{d\gamma^2} = \frac{-(\gamma^2 + 1)}{(\gamma^2 - 1)^2} . \qquad (26)$$

Using these equations we can calculate $\gamma_0$ and $X$ from Eqs. (11) and (8) respectively. As shown in Paper I, these expressions can be considerably simplified. We find that the following expressions

$$(\gamma_0^2 - 1) = \begin{cases} (2\nu kT/\nu_b)(1 + 4.5 \ \nu kT \sin^2\theta/\nu_b)^{-1/3} & kT < 1 \\ \\ (4\nu kT/3\nu_b \sin\theta)^{2/3} & kT \simeq 1 \end{cases}$$

$$(27)$$

$$X^2 = (2kT/\gamma_0)(\gamma_0^2 - 1)/(3\gamma_0^2 - 1), \qquad kT \lesssim 1$$

have the correct asymptotic limits in agreement with Eq. (23) and agree with the exact results from Eqs. (8) and (11) to within 30% for most relevant ranges of angles, frequencies and temperatures and better than 10% in the majority of the interesting cases.

The above equations and Eqs. (7), (8), and (10) give a complete description of the emissivity and absorption coefficient from a Maxwellian gas at all temperatures and frequencies. They are valid for $kT \lesssim 1$ because at temperatures $kT > 1$ the use of the method of steepest descent for integration over the energy becomes less accurate. However, the existence of the extremely relativistic thermal gas is in doubt[8]. On Fig. 1 we compare the total absorption coefficient $\kappa = \kappa_+ + \kappa_-$ obtained from these relationships with numerical results from Lamb and Masters[3]. As evident, our analytic results give excellent agreement to the detailed numerical results even at low harmonics. A more detailed comparison with similar results was presented by Marsh and Dulk[8].

In the two limiting cases described in the previous section, these equations are considerably simpler. The interesting case, $\epsilon \ll 1$ corresponds to $\nu kT/\nu_b \gg 1$, $\gamma_0^2 = 4\nu kT/3\nu_b \sin\theta$, so that

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$$j_{\pm} = \nu^2 kT\kappa_{\pm} = (2^{3/2}\pi e^2 \nu_b/3c)C(\nu kT/\nu_b) \exp\left\{\frac{-\nu}{\nu_b}\left[\frac{4.5}{\sin\theta}\left(\frac{\nu_b}{\nu kT}\right)^2\right]^{1/3}\right\}Y_{\pm}$$

$$\tag{28}$$

$$Y_{\pm} = 1 + 2(3\nu_b\sin^4\theta/4\nu kT)^{1/3}$$

Similarly, the degree of polarization becomes $P = (48\nu_b\sin^4\theta/\nu kT)^{1/3}$ .

### B. Power Law Energy Spectrum

Power law spectra are commonly used spectra in astrophysical problems and in other problems when the tail of the Maxwellian distribution begins to deviate from the exponential form. Usually power law spectra are defined with a low energy cut-off. To avoid such discontinuities and the divergence of the number of particles, we assume a spectrum of the form

$$f(\mu\gamma) = + \frac{n(\delta - 1)}{\varepsilon_c}\,[1 + (\gamma - 1)/\varepsilon_c]^{-\delta}\,g(\mu), \qquad \int_{-1}^{+1} g(\mu)d\mu = 1 \ . \tag{29}$$

Here $\varepsilon_c$ plays the role of the low energy cutoff (in units of $m_e c^2$). For energies much greater than $\varepsilon_c$ the spectrum is a power law with index $-\delta$ but it tends to a constant value at lower energies. The particles can be classified as ultra-relativistic or non-relativistic if $\varepsilon_c \gg 1$ or $\varepsilon_c \ll 1$. We are interested primarily in cases with $\varepsilon_c \simeq 1$.

For distributions which are not highly anisotropic (i.e. $d\ln g(\mu)/d\mu \ll \nu/\nu_b$), we can carry out a calculation similar to that for a thermal gas. From Eqs. (12) and (29) we find that

$$f_{\kappa} = \left[\frac{\delta}{\gamma - 1 + \varepsilon_c} + \frac{2\gamma^2 - 1}{\gamma(\gamma^2 - 1)}\right] f_j \tag{30}$$

so that the quantities $\gamma^2$ and $X^{-2}$ in Eqs. (8) and (11) can all be calculated for exact evaluation of the emissivity and the absorption coefficient in Eq. (7).

As shown in Paper I for semi-relativistic particle energies, $\varepsilon_c \gtrsim 1$, and for $\nu/\nu_b \gg 1$ all these complicated expressions can be simplified

considerably. This is because it turns out that the high energy asymptotic limit of Eq. (23) provides a good approximation throughout most of the relevant ranges of frequency and angle $\theta$. In this limit Eqs. (29) and (30) give

$$\beta_o^2 \gamma_o^2 = X^2 4\nu/(3\nu_b \sin\theta), \quad X^{-2} = \begin{cases} 1 + \delta & \text{for } j \\ 2 + \delta & \text{for } \kappa \end{cases} . \tag{31}$$

Values of $\gamma_o$ obtained from this simple expression agree to within 30% of the exact values derived from Eq. (11) for $1 \gtrsim \epsilon_c \gtrsim 0.5$ and $\sin\theta > \epsilon^{\frac{1}{2}}$. Note also that in this limit the exact form of the energy distribution is not important as long as it tends to a power law $f \propto \gamma^{-\delta}$ at high $\gamma$. In the extreme-relativistic limit $\nu \gg \nu_b$ and $\gamma_o \gg 1$, substitution of Eq. (31) into Eq. (7) gives

$$\left.\begin{matrix} j_\pm(\nu,\theta) \\ \\ \nu^2 \kappa_\pm(\nu,\theta) \end{matrix}\right\} = \frac{\sqrt{3}\pi e^2 \nu_b \sin\theta}{4c} C\epsilon_c g(\beta_o \cos\theta) Y_\pm(\theta) \begin{cases} (3\epsilon_c^2 \nu_b(\delta+1)\sin\theta/4\nu)^{(\delta-1)/2} e^{-(\delta+1)/2} \\ \\ (3\epsilon_c^2 \nu_b(\delta+2)\sin\theta/4\nu)^{\delta/2} e^{-(\delta+2)/2} \left(\frac{\delta+2}{\epsilon_c}\right) \end{cases} . \tag{32}$$

Note that in this limit $Y_\pm = 1 \mp 2 \cot\theta/\gamma_o$ so that the ratios of

$$\kappa_-/\kappa_+ = j_-/j_+ = (1 + 4 \cot\theta/\gamma_o) \tag{33}$$

tend to unity with increasing frequency. This, and the dependence on $\nu/\nu_b$, and $\delta$ of Eq. (32) is identical to the results for emissivity obtained from ultra-relativistic expressions.

It is interesting that with a quite different method and approximation we have obtained the same expression. The reason for this agreement can be seen by examination of the expansion parameter $(1 - z^2)^{-3/2} /m = u_o \gamma_o \nu_b/\nu$. which in this limit is equal to $4/[3(1 + \delta)]$ or $4/[3(2 + \delta)]$. For $\delta > 3$ the

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expansion parameter is less than 0.3 which, although not extremely small, nevertheless is less than unity. Thus, for an ultra-relativistic expression more accurate than 30% to 50%, one must include higher order terms in our analysis.

For extremely non-relativistic particles, that is, for $\epsilon_c \ll 1$, the above expressions are valid as long as $\nu_b \delta / \nu \ll 1$. For the unlikely case of $\nu_b \delta / \nu \gg 1$, we find that $\gamma_0 \simeq 1$ and $\beta_0^2 \simeq \nu \epsilon_c / \nu_b \delta$. Substitution of this in (7) gives the emissivity identical to that expected from a thermal gas if one identifies $\epsilon_c / \delta$ with the temperature $kT$ (cf. Paper I).

In Figs. 2, 3 and 4 we compare results obtained from substitution of Eqs. (30) and (32) in Eqs. (7) and (20) for $j_{\pm}$, $\nu^2 \kappa_{\pm}$, and $P$, at $\theta \simeq 60^0$ and $\delta = 4$ with the results of numerical integration kindly provided by Marsh and Dulk[8], who used a power law spectrum with a sharp cutoff at $\epsilon_c = 1.02$.

The values for $j_{\pm}$ and $\kappa_{\pm}$ are within 60% at high frequencies and the approximation is better at higher $\nu / \nu_b$. The polarization looks better at lower frequencies (although it still is better than 50% even when $P$ is small and errors can be magnified). Also our results are (nearly) systematically higher than the numerical results at high harmonics, which could be the effect of higher order terms. However, the percentage error never exceeds 30% for $\nu / \nu_b \gtrsim 6$.

## IV. SUMMARY

Using a simple method of integration developed previously, we have derived expressions for the emissivity, absorption coefficient and polarization of synchrotron radiation for an arbitrary distribution of particles.

Equations (7) to (13) and (21) give our results in their most general form. And we find that Eqs. (8) and (11) can be simplified considerably, as

in Eqs. (27) for a Maxwellian distribution and in (31) for a power law distribution.

Our results do agree with previous analytic results, and they give good approximations to detailed numerical results. Although our results were derived for high harmonics, they give good agreement down to lower harmonics: to $\nu \simeq 6\nu_b$ for $j_\pm$, to $\nu \simeq 10\nu_b$ for $\kappa_+$, and to even lower harmonics, $\nu = 2\nu_b$, for the total $j$ and $\kappa$. These results are limited to pitch angle distributions which are not extremely anisotropic and energy spectra that decrease rapidly with increasing energy. They also are only applicable for emissivities and absorption coefficients away from the direction of the magnetic field lines.

Our equations are intended for semi-relativistic particles, but they also give excellent approximations for the extreme non-relativistic and ultra-relativistic particle distributions.

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# FIGURE CAPTIONS

Figure 1. The total synchrotron absorption for a thermal source at $\theta = \pi/2$ for kT = 0.04 (20 keV electrons). Points are from analytic expressions; the solid lines are numerical results of Lamb and Masters[3].

Figure 2. Synchrotron emissivity of each mode divided by magnetic field B and total particle number N. Log $(j_{\pm}/BN)$ vs Log $(\nu/\nu_b)$, at $\theta = 60^{\circ}$, $\delta = 4$. [$j_{\pm}$ in units erg(cm$^3$ sec sterad H )$^{-1}$.] The ordinary mode has been shifted down by a factor of 10 for clarity. The solid lines are numerical results of Dulk and Marsh (private communication). The O's are our analytic results.

Figure 3. Same as Fig. 2 except for the absorption coefficient $\nu^2\kappa_{\pm}$ in units of erg sec$^?$ (cm$^3$ sterad)$^{-1}$.

Figure 4. Degree of Circular Polarization vs Log $(\nu/\nu_b)$ in the limit $\xi \to 0$. Circles from Eq. (22). Solid line from numerical results.
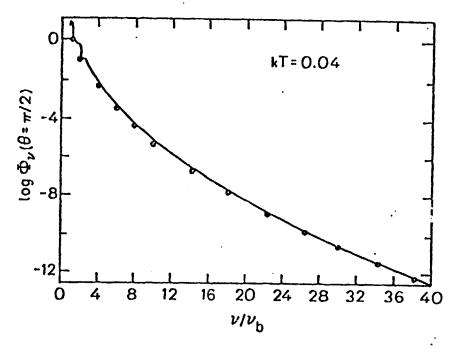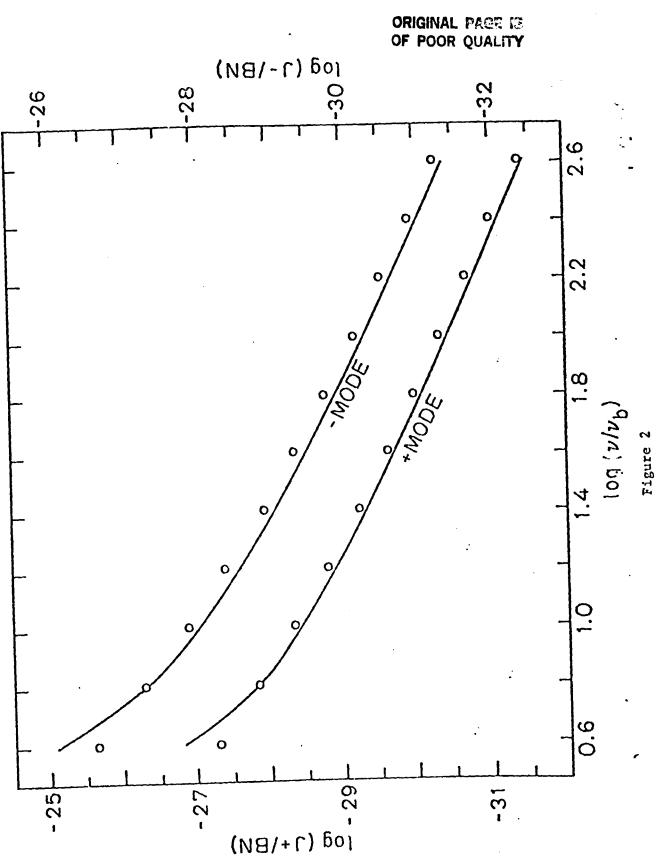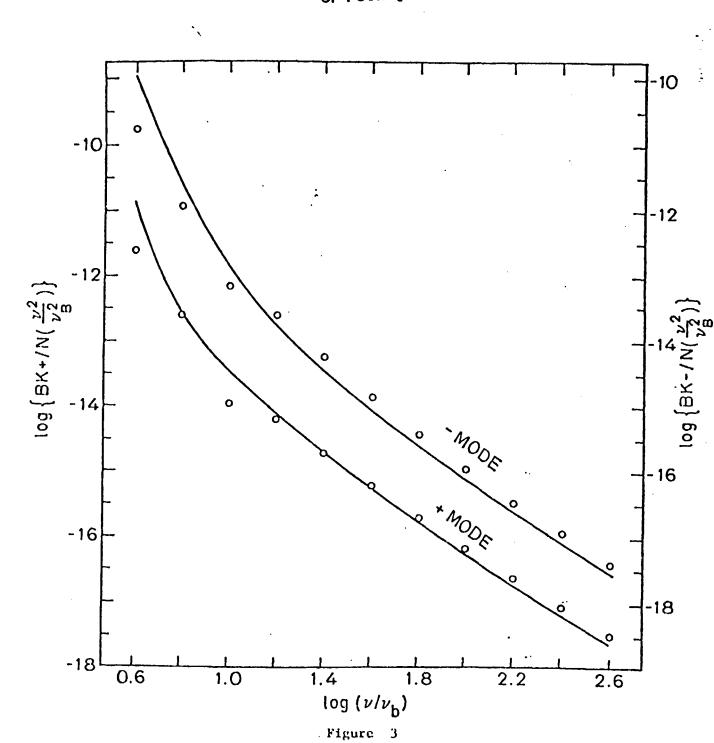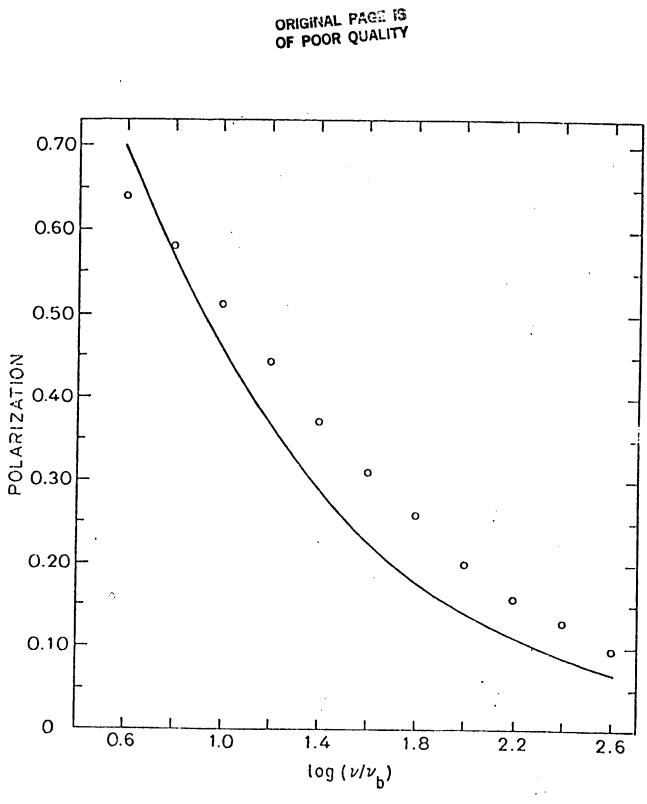
Figure 1

Figure 2

Figure  3

Figure 4