NASA Contractor Report 172139

NASA-CR-172139
19830020607

ICASE

ENTROPY FUNCTIONS FOR SYMMETRIC SYSTEMS OF CONSERVATION LAWS

Eitan Tadmor

Contract No. NAS1-17070
June 1983

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

ENTROPY FUNCTIONS FOR SYMMETRIC SYSTEMS

OF CONSERVATION LAWS

Eitan Tadmor

Institute for Computer Applications in Science and Engineering

ABSTRACT

Using a simple symmetrizability criterion, we show that symmetric systems of conservation laws are equipped with a one-parameter family of entropy functions.

Received by the editors

1980 Mathematics Subject Classification. Primary 35L65; Secondary 35B35.

Research was supported by the National Aeronautics and Space Adminstration under NASA Contract No. NAS1-17070 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

N83-28878#

1. Introduction

An entropy function associated with a system of N conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(f(u)) = 0, \tag{1.1}$$

is a convex function, U, augmented by an entropy flux function, F, both taking values from $\mathbb{R}^N$ smoothly into $\mathbb{R}$, such that for any smooth $u \equiv u(x,t)$ satisfying (1.1) we have

$$\frac{\partial}{\partial t}(U(u)) + \frac{\partial}{\partial x}(F(u)) = 0. \tag{1.2}$$

Carrying out the differentation in (1.2) we find, on account of (1.1), that the above requirement amounts to the following <u>integrability condition</u>

$$U_u^T(u) f_u(u) = F_u^T(u). \tag{1.3}$$

Entropy functions play a significant role in the theory of systems of conservation laws. As observed by Friedrichs and Lax [1], if U is an entropy function for system (1.1), then its Hessian, U_{uu}, symmetrize that system, i.e., symmetrize f_u. It is fairly easy to see that the converse is also true; for future reference we can therefore state

Theorem 1. <u>A convex</u> U <u>serves as an entropy function for system</u> (1.1), <u>if and only if, its Hessian</u>, U_{uu}, <u>symmetrize</u> f_u,

$$U_{uu} f_u = \left(U_{uu} f_u\right)^T. \tag{1.4}$$

For the sake of completeness we include the proof. If U is an entropy in the sense that (1.3) holds, further differentation gives

$$U_{uu}f_u + U_u^T f_{uu} = F_{uu}.$$

The Hessian on the right is symmetric and so is the second matrix on the left, being the product of a vector and a 3-tensor; hence, their difference, $U_{uu}f_u$, is symmetric. Conversely, if $U_{uu}f_u$ is symmetric, so is $\left(U_u^T f_u\right)_u = U_{uu}f_u + U_u^T f_{uu}$. Hence $U_u f_u$ has a primitive

$$F(u) = \int^u U_w^T(w) f_w(w) \cdot dw \tag{1.5}$$

such that (1.3) holds; in other words, the symmetry of $U_{uu}f_u$ -- or what amounts to the same thing, of $\left(U_u^T f_u\right)_u$ -- is required as a compatibility conditon for $F(u)$ to the well-defined, i.e., for the integral on the RHS of (1.5) to be path independent.

We remark that the convexity of U did not enter into the proof, and was assumed just for the sake of complying with the definition of an entropy function being convex. Apart from it, the "if" part of the above theorem, (1.4), provides us -- unlike the integrability condition (1.3) -- with a <u>self-contained</u> criterion for U being an entropy function. The "only if" part of the theorem on the other hand, reveals the hyperbolic nature of systems equipped with entropy functions; indeed, multiplication of (1.1) by U_{uu} on the left, puts the system in symmetric hyperbolic form (in the sense of Friedrichs), for which the local well-posedness theory of smooth solutions, prevail, see [1].

It is well known that solutions for (1.1) may fail to be smooth at a finite time, after which one must admit these solutions in the weak sense. For the latter, the following entropy inequality is imposed as an admissibility criterion [3,4]

$$(1.6) \qquad \frac{\partial}{\partial t}(U(u)) + \frac{\partial}{\partial x}(F(u)) \leq 0 \qquad \text{(weakly)};$$

the inequality (1.6) follows from considerations of the regularized problem

$$(1.7_\varepsilon) \qquad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(f(u)) = \varepsilon\frac{\partial^2 u}{\partial x^2},$$

lettng ε goes to zero, $\varepsilon \downarrow 0$. Thus, the nonpositive LHS of (1.6) indicates the existence of vanishing viscosity in an admissible weak solution. In [4], Lax postulated a uniqueness criterion to single out the so called "physically relevant" solution of (1.1), requiring the entropy inequality (1.6) to hold for all entropy functions associated with (1.1). This brings us to the question of how rich is the family of such entropies.

In the scalar case, $N = 1$, this family consist of all smooth convex functions; in his penetrating paper [3], Kružkov has shown, that having the entropy inequality (1.6) for the one parameter family of convex functions $U(u;\lambda) = |u-\lambda|$, $\lambda \varepsilon \mathbf{R}$ -- which is in the convex hull of the former -- indeed single out the unique, physically relevant, stable solution in L^1. The situation with the general nonscalar case, is however less favorable: the integrability conditon is overdetermined unless $N = 2$, e.g., [4].

2. Symmetric System of Conservation Laws

In this section we restrict our attention to symmetric systems of conservation laws, i.e., systems of the form (1.1) with symmetric Jacobians, $f_u = f_u^T$. We will show that such systems are equipped with one-parameter family of entropy functions.

To this end we are making use of the symmetrizability criterion of Theorem 1, looking for Hessians which symmetrize f_u,

$$U_{uu} f_u = f_u U_{uu}. \tag{2.1}$$

An obvious first choice for such an Hessian will be the identity matrix, $U_{uu} = I_N$. This coinsides with Godunov's observation, [2], (see also [1]), that for symmetric systems, $U(u) = \frac{1}{2} u^T \cdot u$ serves as entropy function, augmented by an entropy flux, see (1.5), $F(u) = \int w^T f_w(w) \cdot dw = u^T f(u) - \int^u f(w) \cdot dw$. Our next choice for symmetrizing Hessian will be f_u: the assumed symmetry of f_u implies, as argued before, that it is indeed an Hessian, $f_u = U_{uu}$ with $U(u) = \int^u f(w) \cdot dw$, augmented by an entropy flux, see (1.5), $F(u) \equiv F(f(u)) = \frac{1}{2} f^T(u) \cdot f(u)$; furthermore, (2.1) is trivially satisfied with this choice (we note the identity $U_u^T(u) = F_f^T(f)$ in this case, from which (1.3) follows upon multiplication by f_u on the right). The function $U(u)$ so constructed is, however, not convex since its Hessian, f_u, is not necessarily positive definite. This can be easily overcome by considering a sufficiently small neighborhood of the first convex choice of an entropy function. thus we have shown

Theorem 2. <u>Any symmetric system of cnservatin laws</u>, (1.1), <u>is equipped with the following one-parameter family of entropy functions</u>

$$U(u;\lambda) = \tfrac{1}{2}u^T\cdot u - \lambda\int^u f(w)\cdot dw, \qquad \{\lambda\varepsilon\mathbf{R};\ \lambda f_u < I\} \tag{2.2a}$$

<u>with corresponding entropy fluxes</u>

$$F(u;\lambda) = u^T\cdot f(u) - \int^u f(w)\cdot dw - \frac{\lambda}{2}f^T(u)\cdot f(u). \tag{2.2b}$$

Let $u_\ell(u_r)$ denote the state on the left (respectively, right) of a discontinuity moving with speed s and governed by system (1.1). The entropy inequality (1.2) across such discontinuity amounts to

$$s\left[U(u_\ell;\lambda)-U(u_r;\lambda)\right] - \left[F(u_\ell;\lambda)-F(u_r;\lambda)\right] \leq 0. \tag{2.3}$$

Invoking the Rankine-Hugoniot (R-H) relation, $s(u_\ell-u_r) = f(u_\ell) - f(u_r)$, the inequality (2.3) reads, after little rearrangement,

$$(1-\lambda s)\left[\tfrac{1}{2}\left(f(u_\ell)+f(u_r)\right)\left(u_r-u_\ell\right)+\int_{u_r}^{u_\ell} f(w)\cdot dw\right] \leq 0.$$

Since λ was chosen so that $\lambda f_u < I$, the R-H relation implies that the first term on the left is positive; hence, the entropy inequality (2.3) for each of the λ-parameter members $U(u;\lambda)$ in (2.2), is consistent with that of $U(u;0)$. Thus, unfortunately (and, unsuprisingly), the one-parameter entropies' family provides us with no additional stability criteria than we have already gained from Godunov's original choice, $U(u) = \tfrac{1}{2}u^T\cdot u$.

3. A Note on the Regularized Problem

We have mentioned before the parabolic regularized problem (1.7_ε), in connection with the entropy inequality (1.6). The key of studying system (1.7_1) in the large, via standard energy methods, depends on obtaining a'priori information in the maximum norm $|u(\cdot,t)|_{L^\infty}$. Here we note that such information can be easily obtained when the symmetric system (1.1) is regularized via disspersive term

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(f(u)) = \frac{\partial^3 u}{\partial x^3}\,. \tag{3.1}$$

The proof is intimately related to the conserved entropies constructed in Section 2 above. Let $(\cdot,\cdot)_0$ denote the spatial L_2- inner product of compactly supported functions, $|\cdot|_0^2 = (\cdot,\cdot)_0$. Multiplying (3.1) by u^T on the left and integrating we find, as before, that $|u|_0^2$ is conserved in time. Next, differentiate (3.1), multiply by u_x^T and integrate to arrive at

$$\tfrac{1}{2}\frac{d}{dt}|u_x(\cdot,t)|_0^2 + (u_x,f_{xx})_0 = 0;$$

multiplying (3.1) by f^T on the left, integrating and adding to the above we find that $\tfrac{1}{2}\,|u_x(\cdot,t)|_0^2 + \int_x \int^u f(w)\cdot dw$ is also conserved. We remark that the last two conserved functionals are in exact agreement with the corresponding first two associated with KdV equation. Under appropriate growth assumption on the flux, f, they yield the required maximum norm bound.

REFERENCES

1. K. O. Friedrichs and P. D. Lax, Systems of Conservation equations with a convex extension, Proc. Nat. Acad. Sci. U.S.A. **68** (1971), 1636-1688.

2. S. K. Godunov, The problem of a generalized solution in the theory of quasi-linear equations and in gas dynamics, Russian Math. Surveys **17** (1962), 145-156.

3. S. N. Kružkov, First order quasi-linear equations in several independent variables, Math. U.S.S.R. Sbornik **10** (1970), 127-243.

4. P. D. Lax, Shock waves and entropy, "Contributions to Nonlinear Functional Analysis," ed. E. A. Zarantonello, pp. 603-634, New York, Academic Press, 1971.

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-172139		

4. Title and Subtitle	5. Report Date
Entropy Functions for Symmetric Systems of Conservation Laws	June 1983
	6. Performing Organization Code
7. Author(s)	8. Performing Organization Report No.
Eitan Tadmor	83-16
	10. Work Unit No.
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132-C, NASA Langley Research Center Hampton, VA 23665	11. Contract or Grant No. NAS1-17070
	13. Type of Report and Period Covered Contractor Report
12. Sponsoring Agency Name and Address National Aeronautics and Space Adminstration Washington, D.C. 20546	14. Sponsoring Agency Code

15. Supplementary Notes

Langely Technical Monitor: Robert H. Tolson
Final Report

16. Abstract

Using a simple symmetrizability criterion, we show that symmetric systems of conservation laws are equipped with a one-parameter family of entropy functions.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
conservation laws entropy functions symmetrizibility	Unclassified-Unlimited Subject Category 59

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	9	A02

End of Document