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APPLICATIONS OF THE CONTRAVARIANT FORM OF THE NAVIER-STOKES EQUATIONS

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SUMMARY

The contravariant Navier-Stokes equations in weak conservation form are well suited to certain fluid flow analysis problems. Three-dimensional contravariant momentum equations may be used to obtain Navier-Stokes equations in weak conservation form on a nonplanar two-dimensional surface with varying streamsheet thickness. Thus a three-dimensional flow can be simulated with two-dimensional equations to obtain a quasi-three-dimensional solution for viscous flow. When the Navier-Stokes equations on the two-dimensional nonplanar surface are transformed to a generalized body-fitted mesh coordinate system, the resulting equations are similar to the equations for a body-fitted mesh coordinate system on the Euclidean plane.

Contravariant momentum components are also useful for analyzing compressible, three-dimensional viscous flow through an internal duct by parabolic marching. This type of flow can be efficiently analyzed by parabolic marching methods, where the streamwise momentum equation is uncoupled from the two crossflow momentum equations. This can be done, even for ducts with a large amount of turning, if the Navier-Stokes equations are written with contravariant components.

INTRODUCTION

The Navier-Stokes equations are often transformed to special coordinate systems for computational purposes. However, if it is desired to have the equations in conservation form, it is generally necessary to use Cartesian momentum components. On the other hand, there are cases where it is useful to use the contravariant momentum components in weak conservation form. Two examples are considered here: one example is a nonplanar flow surface embedded in a three-dimensional flow field; the other example is three-dimensional flow in a turning duct. Both examples have applications in turbomachinery.

There are many instances of two-dimensional flow where the flow is not planar; i.e., either the two-dimensional surface is curved or the streamsheet thickness is not constant (e.g., axisymmetric flow). In such cases the three-dimensional Navier-Stokes equations may be reduced to two dimensions. A general method for doing this is to transform the three-dimensional equation to a new coordinate system such that two coordinate lines lie on the surface of interest, with the third coordinate normal to the surface. The momentum equations are then combined so as to obtain contravariant momentum components. Thus two components of momentum lie on the surface and the third component is omitted; the equation will involve only the two contravariant components. As an illustration, equations are presented here for viscous flow on a rotating surface of revolution. A flow solution on such a surface would be useful for analyzing flow through a turbomachine.

Ducts of complex geometry may be analyzed efficiently if the flow is of a "boundary layer" type; i.e., flow with a predominant direction (no reverse flow), and negligible streamwise diffusion. This kind of flow can be analyzed by parabolic marching methods. Patankar and Spalding (ref. 4), Briley (ref. 8), Roberts and Forester (ref. 7), Moore and Moore (ref. 6), and others have calculated three-dimensional viscous flow in ducts by means of parabolic space marching. For this purpose the momentum equations are uncoupled. The streamwise momentum equation is used to advance streamwise velocity components, and cross-stream momentum equations are used to advance the cross-stream velocity components. If a body-fitted coordinate system is used, the contravariant components of the momentum equation (with contravariant velocity components as dependent variables) provide a natural method of uncoupling the streamwise momentum equation from the cross-stream momentum equation. As an example, equations are presented for flow through a rotating passage, e.g., a blade passage through a turbomachine blade row.

NAVIER-STOKES EQUATIONS

The two-dimensional, planar, nonsteady Navier-Stokes equations in conservation form are

$$\partial_t q + \partial_x E + \partial_y F = \partial_x R + \partial_y S \tag{1}$$

where

$$q = \begin{bmatrix} \rho \\ \rho V_x \\ \rho V_y \\ e \end{bmatrix} \qquad E = \begin{bmatrix} \rho V_x \\ \rho V_x^2 + p \\ \rho V_x V_y \\ V_x(e + p) \end{bmatrix} \qquad F = \begin{bmatrix} \rho V_y \\ \rho V_x V_y \\ \rho V_y^2 + p \\ V_y(e + p) \end{bmatrix}$$

$$R = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ V_x \tau_{xx} + V_y \tau_{xy} + k\, \partial_x T \end{bmatrix} \qquad S = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ V_x \tau_{xy} + V_y \tau_{yy} + k\, \partial_y T \end{bmatrix}$$

Here ρ is density, V is velocity, p is pressure, e is total internal energy per unit volume, T is temperature, and τ is shear stress tensor. The x and y subscripts refer to the x and y coordinates. For numerical solutions, the equation may be transformed to an arbitrary $\xi - \eta$ coordinate system (ref. 1). These are usually body-fitted coordinates, such that constant values of ξ or η correspond to the body surface. The transformed equations in conservation form are

$$\partial_t \hat{q} + \partial_\xi \hat{E} + \partial_\eta \hat{F} = \partial_\eta \hat{S} \tag{2}$$

where

$$\hat{q} = \frac{q}{J} \qquad \hat{E} = \frac{1}{J}\begin{bmatrix} \rho V^{\xi} \\ \rho V_x V^{\xi} + \xi_x p \\ \rho V_y V^{\xi} + \xi_y p \\ (e + p)V^{\xi} \end{bmatrix} \qquad \hat{F} = \frac{1}{J}\begin{bmatrix} \rho V^{\eta} \\ \rho V_x V^{\eta} + \eta_x p \\ \rho V_y V^{\eta} + \eta_y p \\ (e + p)V^{\eta} \end{bmatrix}$$

$$\hat{S} = \frac{1}{J}\begin{bmatrix} 0 \\ \eta_x \tau_{xx} + \eta_y \tau_{xy} \\ \eta_x \tau_{xy} + \eta_y \tau_{yy} \\ \eta_x \beta_x - \eta_y \beta_y \end{bmatrix}$$

$$\beta_x = V_x \tau_{xx} + V_y \tau_{xy} + k\, \partial_x T$$

$$\beta_y = V_x \tau_{xy} + V_y \tau_{yy} + k\, \partial_y T$$

The superscripts ξ and η refer to contravariant velocity components, and J is the Jacobian of the transformation. The thin-layer approximation is used here (viscous derivatives in the streamwise direction are neglected). Note that the momentum components (the second and third equations) are still x and y components. If ξ and η (either contravariant or covariant) momentum components are used the equation cannot be written in conservation form if there is any curvature of the ξ or η coordinate lines.

It is desired to obtain two-dimensional Navier-Stokes equations for nonplanar surfaces (i.e., quasi-three-dimensional). The desired equations can be obtained from the three-dimensional version of Eq. (2), written with contravariant momentum components in a ξ, η, ζ coordinate system. The ξ and η coordinates lie in the desired nonplanar surface, and ζ is chosen in a special way to be orthogonal to the surface. It is assumed that the chosen nonplanar surface is a stream surface, i.e., the contravariant velocity component $V^{\zeta} = 0$. When this is done the third momentum equation can be eliminated and the two-dimensional Navier-Stokes equation for flow on a two-dimensional surface is obtained. Rather than give the general equation, an example will be given which is of interest for turbomachinery flow analysis.

NAVIER-STOKES EQUATIONS ON A ROTATING SURFACE OF REVOLUTION

For analysis of flow through turbomachinery blade rows it is useful to consider flow on a blade-to-blade surface as shown in figure 1. The coordinates are m (physical distance) and θ (radians). In addition the stream-sheet thickness h must be specified, as shown in figure 2. The thickness h and the radius r are both functions of m only. The three-dimensional contravariant momentum equations are obtained in m, θ, n coordinates, where n is normal to the blade-to-blade surface. The n coordinate is defined such that the metric coefficient is $1/h$, so that not only is the normal velocity zero, but also the derivative in the n direction is zero. The surface of revolution is rotating at a constant rotational velocity ω. The relative velocity is W; m and θ subscripts will denote physical (not covariant) velocities.

The Navier-Stokes equation obtained is

$$\partial_t q' + \partial_m E' + \partial_r F' = \partial_m R' + \partial_r S' + K' \tag{3}$$

where

$$q' = \frac{1}{J_1}\begin{bmatrix} \rho \\ \rho W_m \\ \rho W_\theta \\ e \end{bmatrix} \qquad E' = \frac{1}{J_1}\begin{bmatrix} \rho W_m \\ \rho W_m^2 + p \\ \rho W_m W_\theta \\ W_m (e + p) \end{bmatrix} \qquad F' = \frac{1}{rJ_1}\begin{bmatrix} \rho W_\theta \\ \rho W_m W_\theta \\ \rho W_\theta^2 + p \\ W_\theta (e + p) + \omega r p \end{bmatrix}$$

$$R' = \frac{1}{J_1}\begin{bmatrix} 0 \\ \tau_{mm} \\ \tau_{m\theta} \\ W_m \tau_{mm} + W_\theta \tau_{m\theta} + k\, \partial_m T \end{bmatrix} \qquad S' = \frac{1}{rJ_1}\begin{bmatrix} 0 \\ \tau_{m\theta} \\ \tau_{\theta\theta} \\ W_m \tau_{m\theta} + W_\theta \tau_{\theta\theta} + k/r\, \partial_\theta T \end{bmatrix}$$

$$K' = \frac{1}{rJ_1}\begin{bmatrix} 0 \\ \sin\alpha \left(\rho W_\theta^2 + p - \tau_{\theta\theta}\right) - r\, \partial_m hp \\ \sin\alpha (\tau_{m\theta} - \rho W_\theta W_m) \\ 0 \end{bmatrix}$$

$J_1 = 1/rh$ is the Jacobian and m and θ momentum components are used. Except for the source term, K', Eq. (3) is similar to Eq. (1). Eq. (3) can now be transformed to body-fitted coordinates, similar to Eq. (2). The transformed equation is

$$\partial_t \hat{q}' + \partial_u \hat{E}' + \partial_n \hat{F}' = \partial_n \hat{S}' + \hat{K}' \tag{4}$$

where

$$\hat{q}' = \frac{q'}{J_2} \qquad \hat{E}' = \frac{1}{J}\begin{bmatrix} \rho W^{\xi} \\ \rho W_m W^{\xi} + \xi_m p \\ \rho W_\theta W^{\xi} + \xi_\theta p/r \\ (e + p)W^{\xi} \end{bmatrix} \qquad \hat{F}' = \frac{1}{J}\begin{bmatrix} \rho W^{\eta} \\ \rho W_m W^{\eta} + \eta_m p \\ \rho W_\theta W^{\eta} + \eta_\theta p/r \\ (e + p)W^{\eta} \end{bmatrix}$$

$$\hat{S}' = \frac{1}{J}\begin{bmatrix} 0 \\ \eta_m \tau_{mm} + \eta_\theta \tau_{m\theta}/r \\ \eta_m \tau_{m\theta} + \eta_\theta \tau_{\theta\theta}/r \\ \eta_m \beta_m + \eta_\theta \beta_\theta /r \end{bmatrix} \qquad \hat{K}' = \frac{K'}{J_2}$$

$$\beta_m = W_m \tau_{mm} + W_\theta \tau_{m\theta} + k\, \partial_m T$$

$$\beta_\theta = W_m \tau_{m\theta} + W_\theta \tau_{\theta\theta} + k/r\, \partial_\theta T$$

$$\left.\begin{aligned} W^{\xi} &= \xi_m W_m + \xi_\theta W_\theta /r \\ W^{\xi} &= \eta_m W_m + \eta_\theta W_\theta /r \end{aligned}\right\} \text{Contravariant relative velocity components}$$

$J = J_1 J_2$ is the Jacobian $(x, y, z) \rightarrow (\xi, \eta, n)$, J_2 is the Jacobian $(m, \theta) \rightarrow (\xi, \eta)$, and m and θ momentum components are used. Equation (4) uses m and θ momentum components, and is similar to Eq. (2), except for the source term K' which is required because of the curvature of the m and θ coordinates. The techniques and computer codes for solving Eq. (2) (ref. 1) should be able to be extended to Eq. (4).

THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS FOR A ROTATING COORDINATE SYSTEM

An example of a viscous, three-dimensional internal flow that can be analyzed by parabolic space marching is flow through a turbomachinery blade row. The desired equations are obtained by transforming the Navier-Stokes equations in cylindrical (r, θ, z) coordinates to a body-fitted (ξ, η, ζ) coordinate system. Then the contravariant components of the momentum equations are obtained. These equations involve only the contravariant velocity components. (The covariant components of momentum have a simpler form, but involve both covariant and contravariant velocity components (ref. 5).) This permits the streamwise momentum calculation to be uncoupled from the cross-stream momentum calculation. Continuity is satisfied indirectly by modifying an inviscid pressure field to correct for viscous effects. Modifying the streamwise pressure gradient will affect streamwise velocity components so that global continuity is satisfied. Modifying the pressure over a cross section will affect the cross-stream velocities so that local continuity can be satisfied. The steady-state three-dimensional Navier-Stokes equations for a rotating coordinate system (transformed to a body-fitted (ξ, η, ζ) coordinate system, and with contravariant momentum components) are:

$$\partial_\xi \hat{\hat{E}} + \partial_\eta \hat{\hat{F}} + \partial_\zeta \hat{\hat{G}} = \partial_\eta \hat{\hat{S}} + \partial_\zeta \hat{\hat{T}} + \hat{\hat{K}} \tag{5}$$

where

$$\hat{\hat{E}} = \frac{1}{J}\begin{bmatrix} \rho W^\xi \\ \rho W^\xi W^\xi \\ \rho W^\xi W^\eta \\ \rho W^\xi W^\zeta \\ \rho W^\xi I \end{bmatrix} \qquad \hat{\hat{F}} = \frac{1}{J}\begin{bmatrix} \rho W^\eta \\ \rho W^\eta W^\xi \\ \rho W^\eta W^\eta \\ \rho W^\eta W^\zeta \\ \rho W^\eta I \end{bmatrix} \qquad \hat{\hat{G}} = \frac{1}{J}\begin{bmatrix} \rho W^\zeta \\ \rho W^\zeta W^\xi \\ \rho W^\zeta W^\eta \\ \rho W^\zeta W^\zeta \\ \rho W^\zeta I \end{bmatrix}$$

$$\hat{\hat{S}} = \frac{1}{J}\begin{bmatrix} 0 \\ \xi_r \tau_r^\eta + \xi_\theta \tau_\theta^\eta / r + \xi_z \tau_z^\eta \\ \eta_r \tau_r^\eta + \eta_\theta \tau_\theta^\eta / r + \eta_z \tau_z^\eta \\ \zeta_r \tau_r^\eta + \zeta_\theta \tau_\theta^\eta / r + \zeta_z \tau_z^\eta \\ \eta_r \beta_r + \eta_\theta \beta_\theta / r + \eta_z \beta_z \end{bmatrix} \qquad \hat{\hat{T}} = \frac{1}{J}\begin{bmatrix} 0 \\ \xi_r \tau_r^\zeta + \xi_\theta \tau_\theta^\zeta / r + \xi_z \tau_z^\zeta \\ \eta_r \tau_r^\zeta + \eta_\theta \tau_\theta^\zeta / r + \eta_z \tau_z^\zeta \\ \zeta_r \tau_r^\zeta + \zeta_\theta \tau_\theta^\zeta / r + \zeta_z \tau_z^\zeta \\ \zeta_r \beta_r + \zeta_\theta \beta_\theta / r + \zeta_z \beta_z \end{bmatrix}$$

$$\hat{\hat{K}} = \frac{1}{J}\begin{bmatrix} 0 \\ \rho G_{\xi\rho} - g^{\xi\xi} p_\xi - g^{\xi\eta} p_\eta - g^{\xi\zeta} p_\zeta - \xi_r \tau_{\theta\theta}/r + \xi_\theta \tau_{r\theta}/r^2 \\ \rho G_{\eta\rho} - g^{\eta\xi} p_\xi - g^{\eta\eta} p_\eta - g^{\eta\zeta} p_\zeta - \eta_r \tau_{\theta\theta}/r + \eta_\theta \tau_{r\theta}/r^2 \\ \rho G_{\zeta\rho} - g^{\zeta\xi} p_\xi - g^{\zeta\eta} p_\eta - g^{\zeta\zeta} p_\zeta - \zeta_r \tau_{\theta\theta}/r + \zeta_\theta \tau_{r\theta}/r^2 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} W^\xi &= \xi_r W_r + \xi_\theta W_\theta / r + \xi_z W_z \\ W^\eta & \\ W^\zeta & \end{aligned}\right\} \text{Contravariant relative velocity componets}$$

$$\tau_r^\eta = \eta_r \tau_{rr} + \eta_\theta \tau_{r\theta} + \eta_z \tau_{rz}, \text{ etc.}$$

$$\beta_r = W_r \tau_{rr} + W_\theta \tau_{r\theta} + W_z \tau_{rz} + k \, \partial_r T, \text{ etc.}$$

$$I = \frac{e + p}{\rho} - \omega r V_\theta = c_p T' - \omega r V_\theta, \text{ Rothalpy}$$

$G_{\xi\rho}$, $G_{\eta\rho}$, $G_{\zeta\rho}$ are functions of velocity and geometry

The flow is in the ξ direction, and η, ζ are the cross-stream directions. The thin layer approximation is used.

CONCLUDING REMARKS

Two examples of applications of the contravariant Navier-Stokes equations have been presented.

The first example shows that the contravariant Navier-Stokes equations can be used to obtain two-dimensional equations for a rotating surface of revolution with varying streamsheet thickness. The equations are similar in form to the conservation-law form of the Cartesian two-dimensional Navier-Stokes equations. The methods already developed for solving the Cartesian two-dimensional equations can thus be used for analyzing flow on a blade-to-blade surface of a turbomachine. Other applications for nonplanar surfaces seem possible.

The second example is for three-dimensional viscous flow through a rotating duct. The contravariant form of the Navier-Stokes equations provides a method for uncoupling the streamwise momentum from the cross-stream momentum. This makes it easier to use parabolic marching methods for turbomachinery blade passages with a large amount of turning. A similar technique could be used with other types of internal passages.

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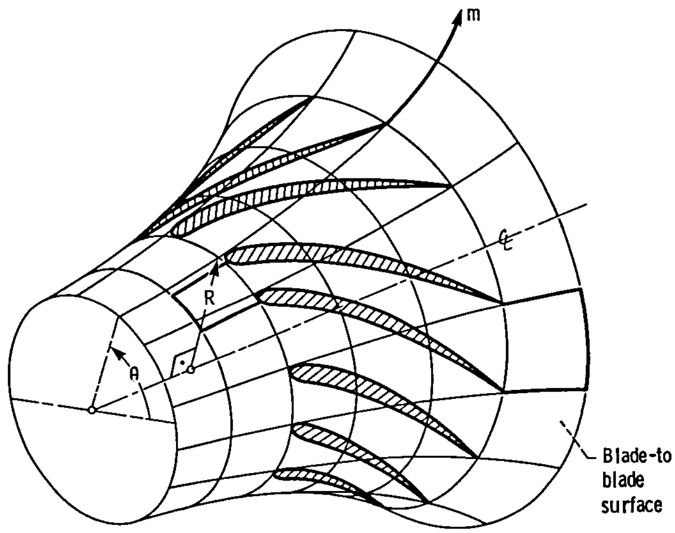

Figure 1. - Blade-to-blade surface of revolution showing m - θ coordinates.

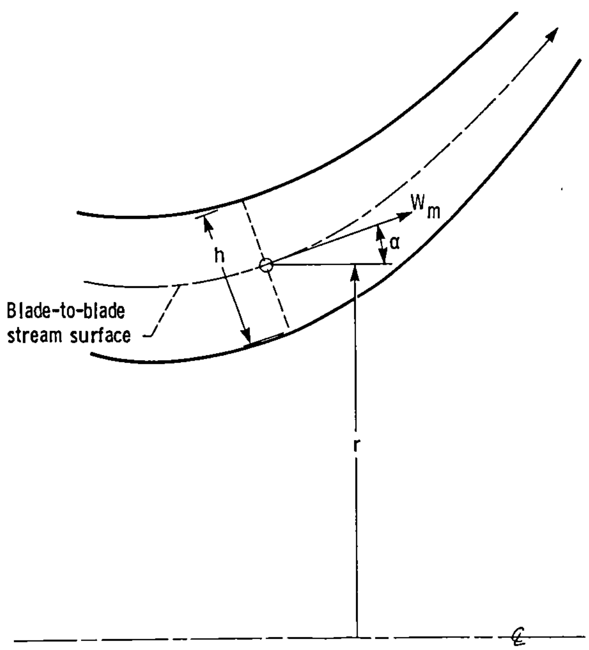

Figure 2. - Flow in a mixed-flow stream channel.

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16. Abstract

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