NASA Contractor Report 172165

ICASE

NASA-CR-172165
19830021841

A COMPACT FINITE DIFFERENCE SCHEME FOR $\mathrm{div}(\rho \ \mathrm{grad}\ u) - q^2 u = 0$

T. N. Phillips

and

M. E. Rose

Contract No. NAS1-17070
June 1983

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

A COMPACT FINITE DIFFERENCE SCHEME FOR

$$\mathrm{div}(\rho \ \mathrm{grad}\ u) - q^2 u = 0$$

T. N. Phillips

and

M. E. Rose

Institute for Computer Applications in Science and Engineering

ABSTRACT

A representative class of elliptic equations is treated by a dissipative compact finite difference scheme and a general solution technique by relaxation methods is discussed in detail for the Laplace equation.

Research was supported by the National Aeronautics and Space Administration under NASA Contract NAS1-17070 while the authors were in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

Introduction

This paper studies a compact finite difference scheme for the solution of the elliptic system

$$\text{div } \underline{v} - q^2 u = 0$$

(1.1)

$$\underline{v} - \rho \text{ grad } u = 0$$

when u or its normal derivative are prescribed on the boundary of a domain. While important in its own right, this system also provides a prototype for the equilibrium equations for elastic materials. We shall show elsewhere that the methods described in this paper can be applied without essential modification to such equilibria problems.

The compact finite difference scheme to be described represents a finite-volume method which expresses algebraic relationships between average values of u on the sides of a computational cell and the average values of the flux normal to the sides. The term "compact" refers to the fact that these relationships hold without reference to neighboring cells. An advantage of such schemes is that any prescribed boundary data may be incorporated with the same accuracy as the scheme itself if irregularly shaped cells are employed.

Unfortunately, the algebraic problem presented by such schemes is difficult to treat, especially if fast iterative methods are sought. By a process analogous to eliminating the flux $\underline{v}$ in (1.1) it is possible to obtain algebraic relationships solely between the solution variables u in neighboring cells. This provides a two-stage iterative process, the first concerning cell neighbors in one direction, the second the neighbors in the other direction. These equations may be conveniently solved by a Gauss-Seidel

type of iterative process or, as will be shown elsewhere, by a multigrid method. Both stages may also be combined into one stage.

We illustrate these features by treating first Laplace's equation $\nabla^2 u = 0$. The combined one-stage method is seen to yield a second-order accurate nine-point finite difference scheme for the Laplacian.

We conclude our discussion by describing the scheme for the more general problem (1.1) and indicating the sense in which the scheme is dissipative by means of an energy estimate.

The method described in this paper has its origin in an earlier approach (Rose [1]) to which the reader is referred.

2. A Compact Scheme for $\nabla^2 u = 0$

Let $(x_i,y_j) = (i\Delta x, j\Delta y)$, $h_x = \Delta x/2$, $h_y = \Delta y/2$ and let $\pi_{i,j}$ denote the rectangular cell $\{|x-x_i| \leq h_x, |y-y_j| \leq h_y\}$. We describe variables associated with the sides of $\pi_{i,j}$ by referencing the center point of the side; thus $u_{i\pm 1/2,j}$, $u_{i,j\pm 1/2}$ indicate the average values of u associated with the sides of $\pi_{i,j}$.

The translation operators s and t are defined by

$$su_{i,j} \equiv u_{i+1/2,j}, \qquad tu_{i,j} \equiv u_{i,j+1/2}$$

and we define central average and difference operators by

$$\mu_x = (s + s^{-1})/2 \quad , \qquad \mu_y = (t + t^{-1})/2$$

$$\delta_x = (s - s^{-1})/2h_x, \qquad \delta_y = (t - t^{-1})/2h_y.$$

Consider the Dirichlet problem for $\nabla^2 u = 0$ in a square domain. In system form: if $\underline{v} = (v,w)'$, then

$$\text{div } \underline{v} = 0$$

(2.1)

$$\underline{v} - \text{grad } u = 0.$$

Corresponding to this system we consider the following compact finite difference scheme: in any cell π, u, v, w are related by the algebraic equations

$$\text{a)} \quad \delta_x v + \delta_y w = 0$$

(2.2) $$\text{b)} \quad \mu_x v - \delta_x u = 0, \qquad \mu_y w - \delta_y u = 0$$

$$\text{c)} \quad h_y^2 \delta_y w - (\mu_y - \mu_x)u = 0.$$

Equations (2.2a,b) are clearly consistent with (2.1). Equation (2.2c) expresses an $0(h^2)$ approximation to the value of u at the center of the cell; motivation for this particular approximation will be given in Section 6 where energy estimates are discussed. We may expect this scheme to yield u to second-order and, noting (2.2b), (v,w) to first-order accuracy.

Write

$$U^T \equiv (\mu_x u, \delta_x u, \mu_y u, \delta_y u),$$

(2.3)

$$V^T \equiv (\mu_x v, \delta_x v, \mu_y w, \delta_y w).$$

Then (2.2) may be written in matrix form as

$$P\,V = Q\,U \tag{2.4}$$

where

$$P = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & h_x^2/2 & 0 & -h_y^2/2 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

If

$$R = P^{-1}Q \tag{2.5}$$

the fluxes V may be expressed in terms of U by

$$V = R\,U, \tag{2.6}$$

where,

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ h^{-2} & 0 & -h^{-2} & 0 \\ 0 & 0 & 0 & 1 \\ -h^{-2} & 0 & h^{-2} & 0 \end{bmatrix} \equiv \begin{bmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \underline{r}_3 \\ \underline{r}_4 \end{bmatrix}, \tag{2.7}$$

and

$$h^2 = (h_x^2 + h_y^2)/2.$$

Now consider the contiguous cells $\pi_{i-1/2,j}$ and $\pi_{i+1/2,j}$. The value $v_{i,j}$ which is associated with the side common to these cells is given by

$$\begin{aligned} v_{i,j} &= (\mu_x + h_x \delta_x) v_{i-1/2,j} \\ &= (\mu_x - h_x \delta_x) v_{i+1/2,j}, \end{aligned} \tag{2.8}$$

i.e.,

$$\left((\mu_x - h_x \delta_x) s - (\mu_x + h_x \delta_x) s^{-1} \right) v_{ij} = 0. \tag{2.9}$$

Recalling the definitions of μ_x, δ_x in terms of s as well as the definition of V in (2.3) we may write (2.9) as

$$2h_x (\delta_x, -\mu_x, 0, 0) V = 0$$

so that, employing (2.6) and dropping the factor $2h_x$, we obtain, in terms of the rows $\underline{r}_1$, $\underline{r}_2$ of R,

$$(\delta_x \underline{r}_1 - \mu_x \underline{r}_2)U = 0, \tag{2.10}$$

i.e.,

$$(h^2 \delta_x^2 - \mu_x(\mu_x - \mu_y))u = 0. \tag{2.10}'$$

Similarly, by eliminating the value $w_{i,j}$ common to the cells $\pi_{i,j-1/2}$ and $\pi_{i,j+1/2}$ we obtain

$$(h^2 \delta_y^2 - \mu_y(\mu_y - \mu_x))u = 0. \tag{2.11}$$

For simplicity let us now assume $h = h_x = h_y$. Since

$$h^2 \delta_x^2 = \mu_x^2 - 1, \qquad h^2 \delta_y^2 = \mu_y^2 - 1$$

(2.10) and (2.11) assume the simpler form

$$u_{i,j} = \mu_x \mu_y u_{i,j} \tag{2.12}$$

where $u_{i,j}$ is a value on the side common to either $\pi_{i-1/2,j}$, $\pi_{i+1/2,j}$ or $\pi_{i,j-1/2}$, $\pi_{i,j+1/2}$. Referring to the stencil indicated in Figure 1

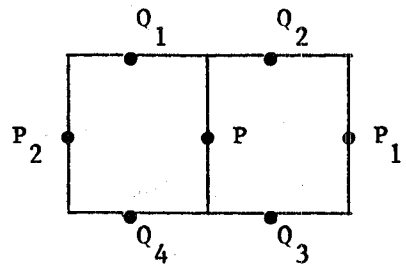

Figure 1

(2.12) indicates that

$$4u(P) = u(Q_1) + u(Q_2) + u(Q_3) + u(Q_4). \tag{2.13}$$

Utilizing (2.12) to express $u(Q)$ in terms of its neighbors as well we obtain

$$u_{i,j} = \mu_x^2 \mu_y^2 u_{i,j}, \tag{2.14}$$

which is a familiar second-order accurate nine-point expression for $\nabla^2 u = 0$.

In the more general case to be described in a later section (2.13) appears in the more complex form

$$u(P) = \sum_{i=1}^{4} \alpha_i u(Q_i) + \beta_1 u(P_1) + \beta_2 u(P_2). \tag{2.15}$$

In this case the simplified discussion of iterative solutions of (2.12) to be given in the next section will, of course, not apply. Nevertheless the analysis is indicative of the more general situation.

3. Iterative Methods

Here the structure of the difference equations obtained in the previous section is investigated in order to devise appropriate iterative methods for their numerical solution.

We first rewrite (2.12) in the form

$$\text{a)} \quad (1 - \mu_x \mu_y) u_{i',j} = 0,$$

(3.1)

$$\text{b)} \quad (1 - \mu_x \mu_y) u_{i,j'} = 0.$$

Equations (3.1a) and (3.1b) represent difference equations for values of u defined respectively at the vertical and horizontal mid-points of the sides of the computational cells $\pi_{i,j}$. The problem is certainly fully determined since at each grid point we have either an equation or a boundary value.

We define a partition of the system (3.1) as follows:

(i) Let p denote the vector of those unknowns defined at mid-points of vertical mesh lines. Equations (3.1a) represent finite difference equations defined at these points.

(ii) Let q denote the vector of these unknowns defined at mid-points of horizontal mesh lines. Equations (3.1b) represent finite difference equations defined at these points.

With this partition (3.1) may be written in the following matrix form

(3.2)
$$\left[\begin{array}{c|c} 4I & B \\ \hline B^T & 4I \end{array}\right] \left[\begin{array}{c} P \\ \hline q \end{array}\right] = \left[\begin{array}{c} b \\ \hline c \end{array}\right],$$

where B is an $N(N-1) \times N(N-1)$ matrix and I is the $N(N-1) \times N(N-1)$ identity matrix. The elements of B are negative and they contain at most four non-zero entries per row, these all being -1. The vectors b and c contain boundary values.

The coefficient matrix of the system (3.2) is symmetric, positive definite and consistently ordered. The first two properties are important since then the convergence of the S.O.R. method is obtained for any value of the relaxation factor ω in the range $0 < \omega < 2$ (see Young [3]). The property of consistent ordering is important in the theory of the S.O.R. method because at present the calculation of the optimum relaxation factor is possible only for consistently ordered matrices. Results related to the determination of the optimum relaxation factor for the S.O.R. method are proved by Young [3].

Equation (3.2) illustrates the fact that each equation defined at a horizontal mid-point is not coupled to any other unknowns at horizontal mid-points. The same is true for equations defined at vertical mid-points.

We note that this system is not equivalent to the one obtained by discretizing Laplace's equation using the rotated five-point formula on a uniform mesh. One reason for this is that the coefficient matrix of this system is reducible whereas that of (3.2) is not.

Equation (3.1) may be written in component form as follows

$$4p + Bq = b,$$

(3.3)

$$B^T p + 4q = c.$$

These equations have thus been cast in a form which is amenable to many of the standard iterative methods of solution. Equation (3.2) may be solved

using the following two-stage iterative process:

$$4p^{(n+1)} + Bq^{(n)} = b \qquad \left(u^{(n+1)}_{i',j} = \mu_x \mu_y u^{(n)}_{i',j}\right),$$

(3.4)

$$B^T p^{(n+1)} + 4q^{(n+1)} = c \qquad \left(u^{(n+1)}_{i,j'} = \mu_x \mu_y u^{(n+1)}_{i,j'}\right).$$

This is the Gauss-Seidel method for solving the system (3.2).

Each of the equations in the system (3.2) defines a rotated Laplacian operator acting on the value of u at the mid-points of the cells. The authors have had some experience with this operator while working with the Cauchy-Riemann equations. In that application serious problems with convergence were encountered. These difficulties appear to be due to treating only the first stage of a two-stage process. A similar effect is seen in A.D.I. methods for parabolic equations where the use of only one of the two steps leads to instabilities.

As an alternative to the scheme (3.4) we may eliminate p from (3.3) to obtain the following nine-point scheme for q

$$(3.5) \qquad (16I - B^T B)q = 4c - B^T b \qquad \left(u = \mu_x^2 \mu_y^2 u\right)$$

$$\text{i.e.,} \quad Aq = d$$

At interior points we have the following finite difference equation

$$(3.6) \quad 2\left(q_{i+1,j+\frac{1}{2}} + q_{i-1,j+\frac{1}{2}} + q_{i,j+3/2} + q_{i,j-\frac{1}{2}}\right)$$

$$+ \left(q_{i+1,j+3/2} + q_{i+1,j-\frac{1}{2}} + q_{i-1,j+3/2} + q_{i-1,j-\frac{1}{2}}\right) - 12q_{i,j+\frac{1}{2}} = 0,$$

at the point $(x_i, y_{j+1/2})$. We note that this is not the higher order nine-point approximation to Laplace's equation; rather (3.6) is second-order accurate. Boundary conditions are to be incorporated into interior equations in the usual fashion. The matrix A appearing in (3.5) is irreducibly diagonally dominant and so both the point Jacobi and point Gauss-Seidel iteration matrices are convergent and the associated iterative methods are convergent for any initial approximation $q^{(0)}$ (see Varga [2]). This is evident, also, from the fact that (3.6) expresses $q_{i,j+1/2}$ as a weighted average of its eight neighbors.

It is interesting to note that although the coefficient matrix of the two-stage scheme (3.4) is consistently ordered, the coefficient matrix for the nine-point scheme (3.6) is not. An estimate of the optimum relaxation factor for the higher-order nine-point formula has been obtained by van de Vooren and Vliegenthart [4] using separation of variables.

The rate of convergence of the two-stage scheme (3.4) is $0(h^2)$. Convergence may be accelerated by employing the S.O.R. method with optimum relaxation factor, in which case the rate of convergence increases to be $0(h)$.

When $h_x \neq h_y$ or in the more general treatment of (1.1) to be described below the observation that the equations defined at the two sets of points are decoupled will no longer be true. In this case the equations corresponding to (2.8a) and (2.8b) will assume the more general form (2.15). This structure suggests the use of line relaxation as a method of solution; this would involve relaxing (2.8a) along horizontal lines and (2.8b) along vertical lines.

4. Numerical Results

Here we demonstrate the performance of the two-stage method on a simple model problem. We consider Laplace's equation defined in the unit square with Dirichlet boundary conditions such that the true solution is given by

$$u(x,y) = e^{-\pi y} \sin(\pi x).$$

The S.O.R. method was applied to this problem with the optimum choice of relaxation factor. This choice of relaxation factor is given by

$$\omega = \frac{2}{1 + \sqrt{[1 - \sigma^2]}},$$

where

$$\sigma = \cos\left(\frac{\pi h}{\sqrt{2}}\right).$$

The reader is referred to Young [3] for the derivation of this formula.

The algorithm was terminated when the magnitude of the maximum difference between successive iterates was less than 10^{-4}. Table I shows the dependence of the relaxation factor, ω, and the number of iterations on the mesh length.

The behavior observed is typical of that of S.O.R. methods. We now have an effective relaxation method for solving the compact scheme in a form which should permit the development of a multigrid algorithm and thereby obtain an efficient means for solving the class of problems in this paper. We will report on this development separately.

Table I

h	ω	No. of Iterations
1/8	1.57	19
1/16	1.76	34
1/32	1.87	67
1/64	1.93	129

5. The General Problem

The discussion of $\nabla^2 u = 0$ given in Section 2 can be easily adapted to the more general system (1.1).

Assume $h_x = h_y = h$ and introduce the abbreviations

$$\sigma_x = \mu_x \rho, \qquad \sigma_y = \mu_y \rho. \tag{5.1}$$

Consider the compact scheme

$$\begin{aligned}
&\text{a)} && \delta_x v + \delta_y w - \tfrac{1}{2} q^2(\mu_x + \mu_y)u = 0 \\
&\text{b)} && \mu_x v - \sigma_x \delta_x u = 0 \\
&\text{c)} && \mu_y w - \sigma_y \delta_y u = 0 \\
&\text{d)} && h^2(\delta_y w - \tfrac{1}{2} q^2 \mu_y u) - (\sigma_y \mu_y - \sigma_x \mu_x)u = 0.
\end{aligned} \tag{5.2}$$

As before, (a), (b), and (c) are clearly consistent with (1.1) and (d) is a consistent center-point approximation of u to terms $0(h^2)$.

Using the definitions of U, V given by (2.3) we may again express (5.2) in matrix form and, as a result, express V in terms of U by $V = R\,U$

where R is the matrix described below. The result of eliminating $v_{i,j}$ and $w_{i,j}$ at points common to neighboring cells is again of the form

$$\text{a)} \quad (\delta_x \underline{r}_1 - \mu_x \underline{r}_2)U = 0$$

(5.3)

$$\text{b)} \quad (\delta_y \underline{r}_3 - \mu_y \underline{r}_4)U = 0,$$

(c.f. (2.20)) in which $\underline{r}_i$ is a row of R.

From (5.2) we find

$$R = \begin{bmatrix} 0 & \sigma_x & 0 & 0 \\ (\frac{1}{2} q^2 + h^{-2}\sigma_x) & 0 & -h^{-2} \sigma_y & 0 \\ 0 & 0 & 0 & \sigma_y \\ -h^{-2}\sigma_x & 0 & (\frac{1}{2} q^2 + h^{-2} \sigma_y) & 0 \end{bmatrix}. \tag{5.4}$$

Thus (5.3) leads explicitly to

$$(h^2 \delta_x \sigma_x \delta_x - \mu_x \sigma_x \mu_x - \tfrac{1}{2} h^2 q^2 \mu_x^2 + \mu_x \sigma_y \mu_y)u = 0$$

$$(h^2 \delta_y \sigma_y \delta_y - \mu_y \sigma_y \mu_y - \tfrac{1}{2} h^2 q^2 \mu_y^2 + \mu_y \sigma_x \mu_x)u = 0,$$

or, using the identity

$$h^2(\delta \sigma \delta)u = (\mu \sigma \mu)u - (\mu \sigma)u,$$

and the definitions of σ_x, σ_y, to

(5.5)

$$\text{a)} \quad (\mu_x^2 \rho)u + \frac{1}{2} q^2 h^2 \mu_x^2 u = \mu_x((\mu_y \rho)\mu_y)u$$

$$\text{b)} \quad (\mu_y^2 \rho)u + \frac{1}{2} q^2 h^2 \mu_y^2 u = \mu_y((\mu_x \rho)\mu_x)u.$$

Reference to Figure 1 shows that eq. (5.5) involves u at all of the center points of the sides of $\pi_{i\pm 1/2,j}$ and $\pi_{i,j\pm 1/2}$. Clearly, (5.5) reduces to (2.10) and (2.11) when $\rho = 1$ and $q = 0$ and retains the essential properties required to adapt to these equations the same iterative methods described earlier.

6. An Energy Estimate

Let

$$Lu \equiv - \operatorname{div} \rho \operatorname{grad} u + q^2 u$$

and

$$\|u\|^2 \equiv \int_D (\rho \operatorname{grad}^2 u + q^2 u)dxdy.$$

Green's theorem applied to (1.1) yields, after multiplication by u, the familiar estimate for the energy norm $\|u\|$ given by

(6.1)
$$\|u\|^2 = \int_D u\, Lu\, dxdy + \int_\Gamma u u_n\, ds.$$

In addition to providing a uniqueness argument for the solution of $Lu = 0$, (6.1) forms the basis for many important properties of the solution of this elliptic problem.

In this section we shall show that the compact difference scheme (5.2) (or (2.2)) yields an identity closely related to (6.1).

With respect to the operators μ, δ in a cell we shall write, if $\underline{v} = (v,w)$,

$$\text{(6.2)} \qquad \operatorname{div}_h \underline{v} = \delta_x v + \delta_y w,$$

and also write the discrete form of Gauss' theorem as

$$\text{(6.3)} \qquad \sum_D \operatorname{div}_h v \, \Delta\pi = \sum_\Gamma \underline{v}\cdot\underline{n} \, \Delta s.$$

Summation-by-parts is accomplished by the use of the identity

$$\text{(6.4)} \qquad \delta(\phi\psi) = (\mu\phi)\delta\psi + (\mu\psi)\delta\phi.$$

Finally, recall the definitions

$$\sigma_x \equiv \mu_x \rho, \qquad \sigma_y \equiv \mu_y \rho$$

from (5.1) and set

$$\text{(6.5)} \qquad L_h u \equiv - \operatorname{div}_h v + \frac{1}{2} q^2(\mu_x + \mu_y)u,$$

$$\text{(6.6)} \qquad \|u\|_h^2 \equiv \sum_D \left[\sigma_x(\delta_x u)^2 + \sigma_y(\delta_y u)^2 + \frac{1}{2} q^2\left((\mu_x u)^2 + (\mu_y u)^2\right)\right]\Delta\pi.$$

We proceed as follows: first, multiply $L_h u$ by $\mu_x u$ and employ (5.2d) to obtain

$$\mu_x u\, L_h u = \frac{1}{2} q^2\left((\mu_x u)^2 + (\mu_y u)^2\right) + h^2\left(\delta_y w - \frac{1}{2} q^2 \mu_y u\right)^2$$

$$- \left(\mu_x u\, \delta_x v + \mu_y u\, \delta_y w\right).$$

Next, using (5.2b,c) (6.3), and (6.7), we may sum over cells in D to obtain

$$\|u\|_h^2 + h^2 \sum_D (\sigma_x/\sigma_y)\left(\delta_y w - \frac{1}{2} q^2 \mu_y u\right)^2 \Delta\pi \tag{6.7}$$

$$= \sum_\Gamma u u_n \Delta s + \sum_D \mu_x u\, L_j u \Delta\pi,$$

so that

$$\|u\|_h^2 \leq \sum_\Gamma u u_n \Delta s + \sum_D \mu_x u\, L_h u \Delta\pi, \tag{6.8}$$

where the inequality is strict unless $u = 0$.

This inequality implies the uniqueness and existence of the solution to (5.2). We leave it to the reader to adopt standard arguments to (6.8) to verify that the solution of the compact scheme provides a second-order approximation to (1.1)

In order to help understand the effect of the dissipative term

$$h^2 \sum_D (\sigma_x/\sigma_y)\left(\delta_y w - \frac{1}{2} q^2 \mu_y u\right)^2 \Delta\pi \tag{6.10}$$

in (6.7) again consider (2.2) where $\rho = 1$, $q = 0$. In place of (2.2c) consider

$$\frac{1}{2} \alpha h^2 \delta_y w - (\mu_y - \mu_x)u = 0 \tag{2.2c)'}$$

together with (2.2a,b). Let $\varepsilon_{h,\alpha}$ denote the error between the solution of the modified compact scheme and the solution of the example discussed in Section 4.

Table II compares $\|\varepsilon_{h,\alpha}\|$ as a function of h and also of α. For this example the dissipative term (6.10) has the form

$$\alpha\, h^2 \sum_{D} (\delta_y\, w)^2\, \Delta\pi,$$

i.e., the dissipation is directly proportional to α.

Table II

Values of $\|\varepsilon_{h,\alpha}\|$ for different values of h and α

α	$h = 1/8$	$h = 1/16$	$h = 1/32$
8	0.0581	0.0169	0.0044
1	0.0123	0.0030	0.0009
1/8	0.0048	0.0012	0.0003
1/16	0.0042	0.0012	0.0009
1/32	0.0031	0.0024	0.0025
0	0.2798	0.2858	0.2871

The effect of the dissipative term is clearly evident in the results given in Table II. For values of α of $0(1)$ the convergence of the scheme is $0(h^2)$. However, for small values of α for which the dissipative term becomes less than $0(h^2)$ the convergence deteriorates. A particularly interesting feature occurs when $\alpha = 0$, in which case the scheme fails to

converge. These results indicate that the dissipative term is required for convergence of the scheme and should be $0(h^2)$ in magnitude in order to obtain $0(h^2)$ convergence.

A closer examination of the transmission matrix for this example shows that R becomes singular for $\alpha \to 0$. As a result the compact scheme and the flux-elimination scheme are no longer equivalent in this limit.

Conclusions

We have described a compact system of finite difference equations for treating (1.1) and have shown how a related noncompact finite difference system provides an equivalent formulation. An energy estimate explains that the compact scheme is dissipative and also can serve to show that the solution approximates the solution of (1.1) to second-order accuracy.

For $\nabla^2 u = 0$ standard relaxation methods can be adopted to solve the noncompact scheme either as a two-stage method or, more directly, as a one-stage method. Both appear to be adaptable to multigrid solution methods.

The methods described in this paper also apply, with little modification, to the equilibrium equations of elastic materials and appear to offer an interesting approach to convective-diffusion equations as well. We plan to report on these applications elsewhere.

Acknowledgment:

To R. Nicolaides for helpful comments and criticism.

References

[1] M. E. ROSE, "Weak-element approximations to elliptic differential equations," Numer. Math., 24 (1975), pp. 185-204.

[2] R. S. VARGA, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

[3] D. M. YOUNG, Iterative Solution of Large Linear Systems, Academic Press, London and New York, 1971.

[4] A. I. VAN DE VOOREN and A. C. VLIEGENTHART, "On the nine-point difference formula for Laplace's equation," J. Engrg. Math., 1 (1967), pp. 187-202.

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-172165		

4. Title and Subtitle	5. Report Date
A Compact Finite Difference Scheme for $\mathrm{div}(\rho \ \mathrm{grad}\ u) - q^2 u = 0$	June 1983
	6. Performing Organization Code
7. Author(s)	8. Performing Organization Report No.
T. N. Phillips and M. E. Rose	83-26
	10. Work Unit No.
9. Performing Organization Name and Address	
Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665	11. Contract or Grant No. NAS1-17070
	13. Type of Report and Period Covered Contractor Report
12. Sponsoring Agency Name and Address	
National Aeronautics and Space Administration Washington, DC 20546	14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor: Robert H. Tolson
Final Report

16. Abstract

A representative class of elliptic equations is treated by a dissipative compact finite difference scheme and a general solution technique by relaxation methods is discussed in detail for the Laplace equation.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
finite difference elliptic iterative methods	64 Numerical Analysis Unclassified-Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	21	A02

End of Document