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A METHOD TO DETERMINE THE ACOUSTICAL PROPERTIES

OF LOCALLY AND NONLOCALLY REACTING DUCT LINERS

IN GRAZING FLOW

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1. INTRODUCTION

The description of the acoustical properties of duct liners depends on the measurement of properties in an impedance tube. However, in practical applications, these liners are usually mounted on a side wall of a flow duct. Thus, both the mean flow and acoustic wave graze over the surface of the liner. Lester and Parrott [1,2] find that these acoustical liners often act differently in grazing incidence than one would expect from the impedance tube measurements. The study of these differences is hindered by the lack of a simple technique to experimentally determine the properties of duct liners in grazing flow.

This paper describes a simple technique to measure the acoustical properties of duct liners in grazing flow. The technique is developed explicitly for rectangular ducts though it can be applied equally well to circular and annular ducts. The technique is to measure the axial wavenumber of the least attenuated mode. This wavenumber depends on the flow profile in the duct, on the boundary condition, and on the acoustical properties of the duct liner. Thus, the acoustical properties of the liner can be found by measuring the wavenumber and knowing the flow profile and boundary condition. The technique will be introduced for uniform flow ducts with point reacting liners. It is then applied to bulk reacting liners with both rigid and limp material structures. Finally the influence of shear flow is considered.

The acoustical properties of materials are usually measured in an impedance tube. An impedance tube is a hollow duct terminated by a duct filled with the unknown acoustical material. A sound wave impinges on the acoustical material and is reflected back. By measuring the interference pattern between the incident and reflected sound waves the acoustical properties of the material can be determined. If the material is locally reacting, a single measurement of the interference pattern is needed to

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find the admittance at each frequency. If the material is nonlocally reacting, two measurements must be made at each frequency. Two measurements are needed because bulk reacting materials are described by two parameters: the propagation constant and the plane wave impedance.

Three impedance tube techniques exist for measuring the acoustical properties of bulk materials. Scott [3] proposed a direct method for cases where the length of the bulk reacting material is effectively infinite. It is not precisely an impedance tube measurement because the propagation constant is determined directly by measuring the phase and amplitude of the pressure wave with a microphone traversing the length of the bulk material. The plane wave impedance is measured directly by simultaneously measuring the acoustical pressure and velocity at the surface of the bulk material. Dinardo [4] and Romero [5] propose an·impedance tube measurement called the two-cavity method. A finite length of bulk material is terminated first by a hard wall and then by a quarter wavelength resonator. The propagation constant and characteristic impedance of the material are deduced from a set of transcendental equations. Ferraro and Sacerdole [6] propose an impedance tube measurement known as the two-thickness method. Measurements are made using two lengths of bulk material terminated by a hard wall. Again, the acoustical properties of the bulk reacting material are deduced from a set of transcendental equations.

An experimental comparison of these three techniques is given by Smith [7]. There are two advantages to the two-thickness and two-cavity methods. First they avoid the need for measurements in the bulk material. Second, they give the acoustical properties by manipulation of simple transcendental equations. These advantages are retained in the method proposed herein.

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Consider a two-dimensional duct with a hard wall on one side and a soft wall treated with an acoustical material on the other side. The axial wavenumber for each mode propagating in this duct depends on the flow profile, the boundary conditions, and the acoustical properties of the liner material. Hence, by measuring the amplitude and phase of the least attenuated mode the liner properties can be deduced. The axial wavenumber can be determined from a pressure measurement made on the hard wall side of the duct. For point reacting materials a single measurement at each frequency is sufficient. For bulk reacting materials two measurements at each frequency are needed. These two measurements can be done with two thicknesses of the liner material, or by placing first a hard wall and then a quarter-wavelength resonator on the exterior, or with two duct widths. The important aspect is to create conditions so that two different values of the axial wavenumber are measured.

The method will be described in stages. First, it is described for point reacting liners used in uniform flow ducts. The influence of the boundary condition, the method for the phase and attenuation measurement, and special problems associated with the axial extent of the liner are discussed. Next, it is applied to bulk reacting liners; rigid liners are considered first, limp liners are considered next. Finally, the influence of shear in-flow is discussed.

Two descriptions of bulk reacting materials are possible. One approach, which we will not use, it to attempt to describe the material microscopically and determine sound propagation from first principles. Simplified models such as tubes [8], parallel fibers [9,10], and arrays of spheres [11] have been proposed. Ingard [12] points out that "If in such studies empirical para-

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meters are introduced, their value is questionable in comparison with what can be learned from dimensional analysis or from a macroscopic phenomenological approach."

   . This macroscopic approach is described by Ingard [12], Zwicher and Kosten [13], Beranek [14,15], and Tack and Lambert [16]. The approach is to write the mass, momentum, and compressibility equations for the air within the duct liner. These equations depend on macroscopic properties such as porosity, density, and specific flow-resistivity of the material. A wave equation is derived from these fundamental equations. The acoustical description of the waves depends on the propagation constant and characteristic impedance which, in turn, depend on the macroscopic properties of the material. The experimental technique described herein is one that relates the axial wavenumber in the flow duct to the propagation constant and characteristic impedance in the duct liner. The explicit relation is based on the macroscopic description of the liner. However, the method can be used equally well if other descriptions of the liner are used, provided that they admit wave-like solutions subject to a dispersion relation and a characteristic impedance.

2. THE UNIFORM FLOW DUCT WITH A POINT REACTING LINER

Consider a sound wave propagating in a two-dimensional duct. Let x be the axial direction and y be the transverse direction. At $y = 0$ let there be a hard wall, and at $y = 1$ let there be a point reacting liner characterized by an admittance a. Between the wall the motion of the air is described by the linearized mass and momentum conservation equations.

$$\rho_0 \frac{D}{Dt} \tilde{u} + \nabla p = 0 \tag{1}$$

$$\frac{D}{Dt} \rho + \rho_0 \tilde{\nabla} \cdot \tilde{u} = 0 \tag{2}$$

where

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \frac{Mc \partial}{\partial x} \right) \tag{3}$$

$$\frac{\partial p}{\partial \rho} = c^2 \tag{4}$$

Here, p ρ and $\tilde{u}$ are the first order perturberations in pressure density and velocity, Mc is the mean flow velocity in the axial direction, ρ_0 the mean air density, and c is the adiabatic sound speed. We will search for propagating waves in the axial direction. Hence, assuming all quantities vary as $e^{i(k_x x - \omega t)}$ gives

$$i k \rho_0 c (1 - K_x M) \tilde{u} = \tilde{\nabla} P \tag{5}$$

$$i k (1 - K_x M) p = \rho_0 c \tilde{\nabla} \cdot \tilde{u} \tag{6}$$

which, in turn, gives the wave equation

$$\frac{1}{k^2} \frac{\partial^2 p}{\partial y^2} + \left| (1 - K_x M)^2 - K_x^2 \right| p = 0 \tag{7}$$

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where $K_x = k_x/k$.

The sol   on can be constructed from elementary solutions proportional to $e^{i(k_x x + k_y y - \omega_0 t)}$. Thus, the dispersion relation is

$$(1-K_x M)^2 - (K_x^2 + K_y^2) = 0 \tag{8}$$

In particular, we must choose a solution that satisfies the zero normal velocity requirement at $y = 0$.

$$P = p_0 \cos k_y y \, e^{ik_x x} \tag{9a}$$

$$u_x = u_{ox} \cos k_y y \, e^{ik_x x} \tag{9b}$$

$$u_y = u_{oy} \sin k_y y \, e^{ik_x x} \tag{9c}$$

The relation between p_0, u_{ox} and u_{oy} is fixed by the momentum equation. Thus

$$\frac{p_0}{\rho c u_{ox}} = (1-K_x M) K_x \tag{10a}$$

$$\frac{p_0}{\rho c u_{oy}} = -i(1-K_x M) K_y \tag{10b}$$

The boundary condition at the soft wall is the particle displacement condition. This condition assumes an infinitesimally thin layer of still air just above the soft wall. The particle displacement (ξ) across the shear layer separating the still and moving air is the same. Here, the subscript f indicates a quantity in the moving fluid and the subscript w indicates a quantity in the still air just above the soft wall. Thus,

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$$\xi_f = \xi_w \cdot \qquad\qquad (11)$$

From the momentum equation it follows that in the moving fluid $u_{yf} = -ik(1-K_x M)\xi_f$ and in the still fluid just above the soft wall $u_{yw} = ik\xi_w$. Thus, at the wall, $(1-K_x M)u_{yw} = u_{yf}$. In turn, the momentum condition gives

$$ik\rho_0 c (1-K_x M)U_{yf} = -\frac{\partial p}{\partial y} \cdot$$

Hence, the admittance condition gives

$$a = \frac{\rho_0 cu}{p} \Big|\ y = 1$$

$$= \frac{1}{(1-K_x M)^2} \frac{\partial p/\partial y}{p} \Big|_{y\ =\ 1}$$

or

$$K_y \tan(K_y kL) = i(1-K_x M)^2 a \qquad\qquad (12)$$

Recall that the dispersion relation gives $K_y^2 = 1-K_x^2$. Thus, if we could measure K_x, and knowing k, L, and M, we could uniquely determine the admittance a at any frequency $\omega = ck$. For point reacting liners only one measurement needs to be done because there is only one unknown, the admittance a, in the characteristic equation.

Now consider the problem of measuring k_x. For the moment, assume the duct is infinitely long and only the least attenuated wave is propagating. Furthermore, assume that a pressure measurement can be made at any location x on the hard wall $y = 0$. This is an advantageous location because it is nonobstructive and also at a pressure antinode for all modes. Let the constant $P_0 = |p| e^{i\phi_0}$ in Eq. 9. There are two methods to determine the real, k_r, and imaginary, k_i, portions of the axial wavenumber k_x.

The first method is to define p_a and p_b where

$$p_a = \frac{2}{T} \int_0^T P(t) \cos \omega t \, dt = |p| \, e^{-k_i x} \cos(k_r x + \phi_0)$$

$$p_b = \frac{2}{T} \int_0^T P(t) \sin \omega t \, dt = |p| \, e^{-k_i x} \sin(k_r x + \phi_0)$$

(13)

It follows that the amplitude and phase are

$$\Lambda(x) = \ln \left(p_a^2 + p_b^2 \right)^{1/2} / P_{ref}$$

$$= \ln(|p|/P_{ref}) - k_i x$$

(14a)

$$\phi(x) = \tan^{-1}(p_b/p_a)$$

$$= \phi_0 + k_r x$$

(14b)

Thus the amplitude and phase vary linearly with distance. Hence, k_r and k_i may be found from the slopes of the experimentally determined amplitude and phase measurement.

The second method is to determine the temporal cross-correlation between two points separated axially. Let

$$R_{12}(\tau) = \frac{2}{T} \int_0^T p_1(t) p_2(t\tau) \, dt$$

(15)

$$= |p|^2 \, e^{-k_i(x_2 - x_1)} \cos(k_r(x_2 - x_1) - \omega\tau)$$

Now let R_{12} be normalized to the self correlation at point 1 to find

$$\overline{R}_{12}(\tau) = \frac{R_{12}(\tau)}{R_{11}(0)} = e^{-k_i(x_2 - x_1)} \cos(k_r(x_2 - x_1) - \omega\tau) \qquad (16)$$

Thus k_r is found by noting the phase shift of $\overline{R}_{12}$ and k_i by noting the amplitude of $\overline{R}_{12}$.

Finally, consider the practical problem that the length of lined duct is usually finite. There are two problems: the

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influence of high order modes and the influence of reflections. First consider the problem of high order modes. Let the duct be made of two parts. On one side is a hard wall duct in which a sound source is located. On the other side let one wall of the duct be lined. At the juncture there will be a reflected wave and a transmitted wave in the lined section (for the moment we neglect any reflected waves in the lined section). This transmitted wave will be composed of higher order modes as well as the least attenuated mode. These high order modes decay very rapidly. Thus, a plot of amplitude vs distance from the juncture will reveal a region of rapid decay, where the high order modes are influential, and a limit region controlled by the propagation of the least attenuated wave. The least attenuated wavenumber can be only be measured accurately in this limit region. However, the region is determined experimentally as indicated above. Furthermore, the region of influence of the high order modes can be diminished by making the duct width smaller. This observation follows by noticing in a hardwall duct the high order modes attenuate at a rate $k_x = i \left[(n\pi/L)^2 - k^2 \right]^{1/2}$, where L is the duct width. In general, only use the technique below the first cutoff frequency for a hard wall duct of identical width to the test section.

Now consider the problem of reflections in the lined section. Let the duct be made of three parts. On either end is a semi-infinite length of a hard wall duct. In the center is a limited region of duct having one wall lined. Near each juncture there will be a limited region where high order modes are influential. This region is determined as before. In addition there will be a reflected wave from the second junction. This reflected wave will be smaller than the incident wave in the lined section. In addition, it will be attenuated as it travels back to the inlet junction. If the liner length is sufficiently long, the reflected wave will have no influence on generation of sound

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in the lined section. In effect, the lined section is semi-
infinite. To determine experimentally if this condition is
reached, one must test (o liners of different axial lengths. If
the measured conditions at the inlet juncture are the same for
both lengths, then one can conclude that the reflected wave is
not influential. At some distance away from the juncture the two
measurements will begin to differ. Where they begin to differ is
the limit of the range in which k_x can be measured accurately.
This will be some short distance away from the exit juncture of
the shorter lined duct. This region of influence will be the
same for both the short and long lined duct. Hence the measure-
ments using the longer lined duct can be used to determine k_x.

Of course it is always possible to solve the equation of
motion exactly and fit the observations directly to some mathema-
tical model. However, that approach lacks the simplicity of a
direct measurement of the axial wavenumber in a lined duct con-
taining only the least attenuated wave. This least attenuated
wave depends directly on the flow profile in the duct, the boun-
dary condition at the wall, and the acoustical properties of the
lined material. Presumably, if a clean measurement is made, the
experimenter will be better able to determine which of these
three assumptions is incorrect, should discrepancies arise
between the predicted and observed values of the axial wave-
number.

3. EXTENSION OF THE METHOD TO RIGID BULK REACTING LINERS

Consider the liner by describing its macroscopic properties. Let the liner be isotropic and let the material structure of the liner be rigid. Furthermore, assume there is no mean flow in the liner. In the lined material, the linearized mass and momentum conservation equations are

$$\rho_o gh \frac{\partial \tilde{u}_s}{\partial t} + r \tilde{u}_s \tilde{\nabla} p_s = 0 \tag{17}$$

$$\frac{\partial \rho_s}{\partial t} + \rho_o \tilde{\nabla} \cdot \tilde{u}_s = 0 \tag{18}$$

where

$$\frac{\partial p_s}{\partial \rho_s} = c_s^2 \tag{19}$$

Here p_s and ρ_s and u_s are the first order perturbations of the pressure density and velocity in the lined material, ρ_o is the mean air density, g is the structure factor, h is the porosity, and c_s the complex phase velocity. Note that we use the average particle velocity $\tilde{u}_s$ in the porous space and not the volume velocity which is $h\tilde{u}_s$. The subscript s will be used to identify acoustic qualities in the lined material. The inertial mass density is larger than the average mass density $\rho_o h$ by the structure factor g which is normally between 1.2 and 2. The porosity h is the fraction of the liner not occupied by the material structure and is normally between 0.85 and 0.98 for practical acoustical materials. The flow resistance r is defined by the steady state relation $r\tilde{u}_s = -\tilde{\nabla} p_s$. The complex phase velocity c_s accounts for the nonadiabatic compression of air in the porous material due to heat tranfer from the entrapped air to the material structure.

We will search for propagating waves in the axial direction. Hence, assuming all quantities vary as $e^{i(k_{sx} x - \omega t)}$ gives

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$$-ik\rho_0 cgh \frac{K_s^2}{\mu^2} \tilde{u}_s = \nabla p_s \qquad (20)$$

$$\frac{ik}{\rho_0 c} \frac{gh}{\mu^2} p_s = \tilde{\nabla} \cdot \tilde{u}_s \qquad (21)$$

where the nondimensional quantities K_s and μ are given by

$$\mu^2 = gh \left(\frac{c_s}{c}\right)^2 \qquad (22)$$

$$K_s^2 = \mu^2 \left\{ 1 + \frac{ir}{k\rho_0 cgh} \right\} \qquad (23)$$

These equations give the wave equation

$$\frac{1}{k^2} \frac{\partial^2 p_s}{\partial y^2} + (K_s^2 - K_{sx}^2) p_s = 0 \qquad (24)$$

where $K_{sx} = k_{sx}/k$. The solution can be constructed from elementary solutions proportional to $e^{i(k_{sx} x + k_{sy} y - \omega_0 t)}$. Thus the dispersion relation is

$$K_s^2 - (K_{sx}^2 + K_{sy}^2) = 0 \qquad (25)$$

where $K_{sy} = k_{sy}/k$.

The possible values of K_{sy} are determined by the boundary conditions. We will show how to deduce the propagation constant k_s and the plane wave impedance $p_s/(h\rho_0 cu_s) = \mu^2/g$ from the axial wavenumber measured on the hard wall side of the duct.

There are two boundary conditions. One occurs at the juncture of the air within the duct and the liner material, the other depends on how the liner is terminated in the transverse direction. Three terminations are explored in detail.

The first termination is none at all. That 's to say the transverse extent of the liner is so great that the sound only propagates away from the juncture with the duct. Here

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$$P_s = P_{so} e^{i(k_{sx} x + k_s y)} \tag{26a}$$

$$u_{sx} = u_{sxo} e^{i(k_{sx}x + k_{sy}y)} \tag{26b}$$

$$u_{sy} = u_{syo} = e^{i(k_{sx} x + k_{sy}y)} \tag{26c}$$

The relation between the P_{so}, U_{sxo} and U_{syo} is given by the momentum equation. Thus

$$\frac{P_{so}}{\rho_o c u_{sxo}} = \frac{\mu^2}{gh} K_s K_{sx} \tag{27a}$$

$$\frac{P_{so}}{\rho_o c u_{syo}} = \frac{\mu^2}{gh} K_s K_{sy} \tag{27b}$$

Now consider a liner of depth Δ terminated by a hard wall. At the wall $y = L + \Delta$ the normal velocity u_{sy} must vanish. Thus

$$P_s = P_{so} \cos\left(k_{sy}(y-(L + \Delta))\right) e^{ik_{sx}x} \tag{28a}$$

$$u_{sx} = u_{sxo} \cos\left(k_{sy}(y-(L + \Delta))\right) e^{ik_{sx}x} \tag{28b}$$

$$u_{sy} = u_{syo} \sin\left(k_{sy}(y-(L + \Delta))\right) e^{ik_{sx}x} \tag{28c}$$

where $p_{so}/\rho c\, u_{sxo}$ is the same as before and

$$\frac{P_{so}}{\rho_o c u_{syo}} = -i \frac{\mu^2}{gh} K_s K_{sy} \tag{29}$$

Finally, consider a liner of depth Δ terminated by a quarter wavelength resonator. At the termination $y = L + \Delta$ the pressure must vanish. Thus

$$P_s = P_{so} \sin k_{sy} (y-(L + \Delta)) \; e^{ik_{sx}x} \tag{30a}$$

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$$u_{sx} = u_{sxo} \sin k_{sy}(y-(L + \Delta)) \; e^{ik_{sx}x} \tag{30b}$$

$$u_{sy} = u_{syo} \cos k_{sy}(y-(L + \Delta)) \; e^{ik_{sx}x} \tag{30c}$$

where $p_{so}/\rho c \; u_{sxo}$ is the same as before and

$$\frac{p_{so}}{\rho_o c u_{syo}} = i \frac{\mu^2}{gh} K_s K_{sy} \tag{31}$$

Again, the boundary condition at the soft wall is determined by the particle displacement condition. Again, we assume an infinitesimally thin layer of still air just above the soft wall. The particle displacement across this shear layer is $\xi_f = \xi_w$ as given in Eq. 11. Now the displacement in the linear equation is given by

$$\xi_w = h \; \xi_s \tag{32}$$

thus

$$u_y(x, \; L-\delta,t) = h(1-K_x M) \; u_{sy}(x,L-\delta,t) \tag{33}$$

To satisfy this requirement we must set $K_{sx} = K_x$. In addition, continuity of pressure requires that

$$p(x,L-\delta,t) = p(x,L+\delta,t) \tag{34}$$

Taking the ratio of Eqs. 33 and 34 gives

$$K_y \tan (K_y kL) = i(1-K_x M)^2 \; \beta \tag{35}$$

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and

value   termination

$$
\beta = \begin{cases}
\dfrac{\mu^2 \, K_{sy}}{q \, K_s^2} & \text{none} \\[3ex]
i \, \dfrac{\mu^2 \, K_{sy}}{qK_s^2} \tan(K_{sy} \, k\Lambda) & \text{hard wall} \qquad (36) \\[3ex]
-i \, \dfrac{\mu^2 \, K_{sy}}{q \, K_s^2} \cot(K_{sy} k\Lambda) & \text{quarter wave-} \\
 & \text{length resonator}
\end{cases}
$$

The first two forms of β in Eq. 36 are also given by Tack and Lambert [16]. Recall that the disperson relation gives $K_y^2 = (1-K_x^2)$ and that $K_{sy}^2 = (K_s^2 - K_{sx}^2) = (K_s^2 - K_x^2)$. Hence, just as before, a measurement of the axial wavenumber reveals the liner characteristics. Because we wish to extract two quantities, K_s and q/μ^2, in general two measurements must be made. These two measurements must yield two sufficiently different values of K_x, which we call K_{x_1} and K_{x_2}.

Given K_x the left hand side of equation 35 is determined. The right hand side contains the unknown β. By taking the ration of two successive measurements of β the value μ^2/q can be cancelled and a transcendental equation in this unknown value K_s results.

If two duct widths L_1 and L_2 are used, and the liner is effectively infinite in the transverse direction, then this equation is

$$
\frac{(1-K_{x_2}^2)^{1/2} \tan(1-K_{x_1}^2)^{1/2} \, kL_1}{(1-K_{x_2}^2)^{1/2} \tan(1-K_{x_2}^2)^{1/2} \, kL_2} = \left(\frac{K_s^2 - K_{x_1}^2}{K_s^2 - K_{x_2}^2} \right)^{1/2} \qquad (37)
$$

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If the duct width is constant and two liner depths Δ_1 and Δ_2 terminated by a hard wall are used, then this equation is

$$\frac{(1-K_{x1}^2)^{1/2} \tan\left((1-K_{x1}^2)kL\right)}{(1-K_{x2}^2)^{1/2} \tan\left((1-K_{x2}^2 \, kL)\right)}$$

$$= \frac{(K_s^2 - k_{x1}^2)^{1/2} \tan\left(k\Delta \, (K_s^2 - K_{x1}^2)\right)}{(K_s^2 - K_{x2}^2)^{1/2} \tan\left(k\Delta(K_s^2 - K_{x2}^2)\right)} \qquad (38)$$

If the duct width and liner depth are constant, and the liner is terminated first by a hard wall and then by a quarter wavelength resonator, then this equation is

$$\frac{(1 - K_{x1}^2)^{1/2} \tan\left((1 - K_{x1}^2)kL\right)}{(1 - K_{x2}^2)^{1/2} \tan\left((1 - K_{x2}^2)kL\right)}$$

$$= \frac{(K_s^2 - K_{x1}^2)^{1/2} \tan\left(k\Delta(K_s^2 - K_{x1}^2)\right)}{(K_s^2 - K_{x2}^2)^{1/2} \cot\left(k\Delta(K_s^2 - K_{x2}^2)\right)} \qquad (39)$$

In each case the left hand side is known and the right hand side depends only on the unknown K_s. Given K_s, one can back substitute into the original equation to find μ^2/g.

The most advantageous technique is to use two liners that are effectively infinite in the transverse direction. The reason is that it results in an equation for K_s that can be solved algebraically. To determine whether or not a liner is effectively infinite, simple increase the liner depth until K_x does not change anymore.

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4.   EXTENSION OF THE METHOD TO LIMP BULK-REACTING MATERIALS

Consider the possibility of motion in the solid material which forms the acoustic liner.   The bulk mass density of the liner ρ_1 is

$$\rho_1 = h\rho_0 + (1-h)\rho_m$$
$$\approx (1-h)\rho_m \tag{40}$$

where the subscript m refers to the material properties, and the subscript s to the properties of the entrapped air.   The motion of the air and material are coupled through the momentum equation

$$0 \quad = \rho_1 \frac{\partial \tilde{u}_m}{\partial t} + r(\tilde{u}_m - \tilde{u}_s) \tag{41}$$

$$-\nabla p_s = \rho_0 g h \frac{\partial \tilde{u}_s}{\partial t} + r(\tilde{u}_s - \tilde{u}_m) \tag{42}$$

In the momentum equation for the material structure, we have explicitly assumed that its motion is due only to resistive action of the entrapped air as it moves past the surrounding structure at a rate $\tilde{u}_s - \tilde{u}_m$.   This is not the only possibility. The structure also responds to the surrounding pressure gradients, and to its internal stiffness.   Zwicker and Kosten point out that even for zero resistance, coupling between the structure and entrapped air is possible by virture of the inertial coupling indicated by the g factor.   However, we wish only to illustrate how additional material effects alter the calculations described herein.   In keeping with this aim we limit our attention to simple resistive coupling.   From Eqs. 41 and 42 we find

$$\frac{u_m}{u_s} = \frac{r}{r - i\omega\rho_1}$$
$$= \frac{1}{1 - i(\omega/\omega_0)} \tag{43}$$

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where $\omega_0 = (r/\rho_1)$. At low frequencies $\omega/\omega_0 \sim 0$ and $u_m/u_s = 1$. That is to say, the material structure moves in phase with the acoustic wave. At high frequencies $u_m/u_s = r/-i\omega\rho_1$, i.e., u_m is 90° out of phase with u_s and its amplitude is in the ratio of ω_0/ω. This ratio can be reduced to zero as ω is increased without limit. Thus, if the material properties are known beforehand, a decision can be made as to the influence of the material motion.

Given the ratio u_m/u_s, we can derive the modified momentum relation for the air within the porous material

$$-\nabla p = \rho_0 gh \frac{\partial \tilde{u}_s}{\partial t} + \sigma \tilde{u}_s \qquad (44)$$

where $\sigma = -i\omega\rho_1 r/(r-i\omega\rho_1)$. Thus, wherever r appears in previous results, σ must now be used. The major change is in the expression for K_s, which is now

$$K_s^2 = \mu^2 (1 + \frac{i\sigma}{k\rho_0 cgh}) \qquad (45)$$

Ingard [12] points out that the effective resistance σ can be described as a flow resistance r in parallel with the structural mass density ρ_1.

There is an additional change that needs to be made. This change is to include the effects of the motion of the structural material on the boundary condition. Again, we assume an infinitesimally thin layer of still air at the soft wall. Mass continuity requires that

$$(\xi_w - \xi_m) = h(\xi_s - \xi_m) \qquad (46)$$

Noting that $\xi_m/\xi_s = u_m/u_s$ in the structural material gives

$$\xi_\omega = h(1+\epsilon)\xi_s$$

where

$$\epsilon = \frac{1-h}{h} \frac{1}{(1-i\omega/\omega_0)}$$

The change in the displacement condition alters the characteristic equation to

$$K_y \tan(K_y kL) = i(1-K_x M)^2(1+\epsilon)\beta \tag{47}$$

The expression for β remains the same as given in Eq. 36. Thus, even if the motion of the material becomes important, the same techniques may be used to discover the acoustic properties of the material.

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5. APPLICATION TO SHEAR FLOW PROFILES

Consider the effect of shear flow in the duct. Let the mean flow in the axial direction depend on the tansverse coordinate, $V(y)$. The linearized equations of mass and momentum continuity are

$$\frac{D}{Dt} \rho + \rho_0 \tilde{\nabla} \cdot \tilde{u} = 0 \qquad (48)$$

$$\rho_0 \frac{D}{Dt} u_x + u_y \frac{\partial}{\partial y} V + \frac{\partial p}{\partial y} = 0 \qquad (49)$$

$$\rho_0 \frac{D}{Dt} u_y + \frac{\partial p}{\partial y} = 0 \qquad (50)$$

where the adiabatic compressibility $\partial p / \partial \rho = c^2$ holds and the convective derivation $\frac{D}{Dt} = \frac{\partial}{\partial t} + V \frac{\partial}{\partial x}$ now depends on the transverse coordinate y. Assuming all quantities vary as $e^{i(k_x x - \omega t)}$ gives

$$i k (1-K_x M) p = \rho_0 c \nabla \cdot \tilde{u} \qquad (51)$$

$$i k \rho_0 c (1-K_x M) u_x + u_y \frac{\partial}{\partial y} V = i k_x P \qquad (52)$$

$$i k \rho_0 c (1-K_x M) u_y = \frac{\partial p}{\partial y} \qquad (53)$$

where M is a function of y. The wave equation can be derived using the commutation relation $(\nabla \frac{D}{Dt} - \frac{D}{Dt} \nabla) \tilde{u} = - \frac{\partial V}{\partial y} \frac{\partial}{\partial x} u_y$ and the momentum relation $u_y = \frac{\partial p}{\partial y} / (i k \rho_0 c (1-K_x M))$. Thus, we recover Hersch and Catton's [17] results

$$0 = \frac{\partial^2 p}{\partial y^2} + \frac{2 \, K_x}{(1-K_x M)} \frac{\partial M}{\partial y} \frac{\partial p}{\partial y} + k^2 \left[(1-K_x M)^2 - K_x^2 \right] p \qquad (54)$$

Because this equation is an ordinary differential equation we can use the Runge Kutta technique to develop a numerical eigenvalue equation. Mungar and Plumblee [18] use this approach to study annular ducts, here we apply their technique to two-dimensional rectangular ducts. At

$$a(y) = \frac{2 \, K_x}{1-K_x M(Y)} \frac{\partial M}{\partial y} \qquad (55a)$$

$$b(y) = (kL)^2 \left[(1-K_x M)^2 - K_x^2 \right] \qquad (55b)$$

and introduce the column vector $\tilde{f}$ with P and $\partial P/\partial Y$ as elements:

$$f_1 = P$$

$$f_2 = \frac{\partial P}{\partial Y} \qquad (56)$$

The second order wave equation can be rewritten as two coupled first order equations

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad (57)$$

or, equivalently

$$\frac{\partial}{\partial y} \tilde{f} = [F] \, \tilde{f} \qquad (58)$$

where [F] is the matrix operation defined by Eq. (57). The generalized Runga Kutta solution is to divide the duct width into

n subsections of length q = L/n. The solution at the j+1th point is related to that at the jth point by the rule

$$f_{j+1} = f_j + \frac{q}{6} \left[m_1 + 2m_2 + 2m_3 + m_4 \right] \tag{59}$$

where

$$m_1 = F(y_j, f_j)$$

$$m_2 = F(y_j + \frac{q}{2}, f_j + \frac{qm_1}{2})$$

$$m_3 = F(y_j + \frac{q}{2}, f_j + \frac{qm_2}{2})$$

$$m_4 = F(y_j + \frac{q}{2}, f_j + q m_3)$$

Each step is a linear transform of the form

$$\tilde{f}_{j+1} = [T_j] \, \tilde{f}_j \tag{60}$$

which can be combined to yield

$$\tilde{f}_n = \prod_{j=0}^{n-1} [T_j] \, \tilde{f}_0 \tag{61}$$

$$= [T] \, \tilde{f}_0$$

where $[T]$ is the transfer matrix. This matrix is uniquely defined by the function $M(y)$ and the constants K_x and kL. As an eigenvalue problem, K_x is the unknown parameter which is adjusted to satisfy the boundary conditions. In the application proposed herein, K_x is the measured axial wavenumber which is used to find the boundary conditions. This characteristic equation is found

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by noting $(\partial p/\partial y)/p = B_0$ at one wall and B_N at the other. Hence, eliminating $\partial p/\partial y$ in Eq. (61) gives

$$0 = [(T_{11} + B_0 T_{12}) - B_N (T_{21} + B_0 T_{22})] p_0 . \qquad (62)$$

For a hard wall at the point $y = 0$, we have $B_0 = 0$. For the soft wall at $y = L = nq$, we must again assume an infinitesimal layer of still air near the wall. Pressure and particle displacement are maintained across the layer. From the particle displacement condition, the acoustic velocity in the transverse direction is determined as before. For a point reacting liner $(\partial p/\partial y)/p = (1-K_x M)^2 \rho_0 c(u/p)$, where, at the wall, u_y/p is equal to the admittance α. For a bulk liner the velocity in the still air at the wall must again be related to the velocity in the bulk liner by the rule $u_{yw} = u_{ys} h(1+\varepsilon)$, where h is the porosity and ε the factor accounting for the motion of the material. Hence, the eigenvalue equation is

$$\frac{1}{k} \frac{T_{11}}{T_{12}} = i(1-K_x M(L))^2 \begin{cases} \alpha & \text{for point reacting liners} \\ (1+\varepsilon)\beta & \text{for bulk reacting liners} \end{cases}$$

$$(63)$$

The factor β is given in Eq. (36). Note that for uniform mean flow the factor T_{11}/kT_{12} is given explicitly as $K_y \tan (K_y kL)$. For shear flow (T_{11}/kT_{12}) must be determined numerically from the measured values kL, K_x, and $M(y)$.

If the flow is not two-dimensional, then the above mathematical technique cannot be used. In particular, if the axial velocity depends on both transverse coordinates $V(y,z)$, then the ordinary differential equation in y becomes a non-separable partial differential equation in y and z. Two additional correction terms appear in Eq. (54): $\partial^2 p/\partial z^2$ and

$[2K_x/(1-K_xM)]$ $[(\partial M/\partial z)(\partial p/\partial z)]$. An experimental test is to determine what extent p and M depend on the other transverse coordinate. For the pressure measurement, this can be done on the side of the duct with a probe flush-mounted on the hard wall.

An alternate experimental technique is to perform the measurement in an annular duct. The factors a(y) and b(y) are changed to functions of the radial coordinate as indicated by Mungar and Plumblee [18]. The hard wall at y = 0 is replaced by an interior circular cylinder at $r = r_0$. The soft wall becomes the exterior circular cylinder. For point reacting liners the admittance is defined as before. For bulk reacting liners the factor β may be redefined by replacing the exponential and trigonometric functions of the transverse coordinate with the appropriate Hankel and Bessel functions of the radial coordinate. In addition, the test must be done so that there is no azimuthal mean flow on pressure variation.

6. CONCLUSION

A method to determine the acoustical properties of both locally and nonlocally reacting acoustical materials in grazing flow is described. The method is to measure the axial wavenumber k/x in a flow duct. Details are presented for a two-dimensional duct lined on one side with a bulk reacting liner and on the other with a hard wall, though the technique also applies in cylindered and annular ducts. Because the axial wavenumber can be found from pressure measurements on the side of the duct, it should be easy to implement in the laboratory. Sample calculations have been illustrated for finite thickness liners backed by a hard wall or by a quarter wavelength resonator (a soft wall) and also for the special case when the liner may be regarded as infinite. The effects of mean flow and shear flow have also been considered. Furthermore, the influence of the motion of the material matrix forming the liner has been considered in the special case when liner stiffness can be neglected.

For point reacting liners the acoustical material is characterized by its admittance. Because one parameter describes the material, only one measurement must be done at each frequency. Bulk reacting liners are described by the propagation constant K_s, and plane wave impedance Z. Because two parameters must be determined, two measurements must be made at each frequency. These can be done with ducts of two widths, with two thicknesses of liner material backed by a hard wall, by a single thickness of liner material backed by a hard wall and quarter wavelength resonator, or to change condition by any permutation of the above.

The important point is to provide sufficiently different experimental conditions so that two significantly different values of the axial wavenumber are measured.

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