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DEPARTMENT OF MECHANICAL ENGINEERING AND MECHANICS
SCHOOL OF ENGINEERING
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA

LINEARIZED POTENTIAL SOLUTION FOR AN
AIRFOIL IN NONUNIFORM PARALLEL STREAMS

By

R. K. Prabhu

and

S. N. Tiwari, Principal Investigator

Progress Report
For the period January 1 to May 31, 1983

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PREFACE

This report covers the work completed on the research project "Wing-Propeller Interference Studies" during the period January 1 through May 31, 1983. The work was supported by the NASA/Langley Research Center through the Cooperative Agreement NCC1-65. The cooperative agreement was monitored by Dr. Chen-Huei Liu of the Low Speed Aerodynamics Division (Mail Stop ~~286~~ 152).

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LINEARIZED POTENTIAL SOLUTION FOR AN
AIRFOIL IN NONUNIFORM PARALLEL STREAMS

By

R. K. Prabhu[1] and S. N. Tiwari[2]

SUMMARY

A small perturbation potential flow theory is applied to the problem of determining the chordwise pressure distribution, lift and pitching moment of a thin airfoil in the middle of five parallel streams. This theory is then extended to the case of an undisturbed stream having a given smooth velocity profile. Two typical examples are considered and the results obtained are compared with available numerical solutions of Euler's equations. The agreement between these two results is not quite satisfactory. Possible reasons for the differences are indicated.

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LIST OF SYMBOLS

A_m	unknown constants
a, b	distance of the airfoil above or below a surface of velocity discontinuity
c	airfoil chord
c_1	airfoil lift coefficient
c_m	airfoil pitching moment coefficient
C_J	jet momentum coefficient
d	distance representing the width of velocity nonuniformity
d_m	unknown constants
i, j, k, n, s	running indices
I, J, K, L	circulation of the image vortices
h	width of a stream in which the velocity is uniform
$m(\theta)$	slope of the mean camber line
u, v	perturbation velocity components
U	undisturbed free stream velocity
x, x'	distance from the airfoil leading edge along the chord
y	height of the image vortex above the airfoil chord
y(x)	shape of the deflected jet sheet
α	angle of attack; factor defined by $(U_0^2 - U_1^2)/(U_0^2 + U_1^2)$
β	factor defined by $(U_0^2 - U_{-1}^2)/(U_0^2 + U_{-1}^2)$
$\gamma(x)$	vortex distribution representing the airfoil
Γ	circulation around the airfoil
θ	defined by $x = (1-\cos\theta) \, c/2$
ϕ	defined by $x' = (1-\cos\phi) \, c/2$

1. INTRODUCTION

The study of aerodynamic characteristics of lifting surfaces in non-uniform flow is of considerable practical interest. Wing sections behind a propeller experiencing a jet-like velocity profile, and tailplane sections of a conventional airplane experiencing a wake-like velocity profile are two examples of such problems. Solutions of these problems are complex and require simplifying assumptions. Even if the viscous and compressibility effects are neglected, the presence of vorticity in the approaching stream necessitates the solution of Euler's equations. Being nonlinear, Euler's equations require numerical treatment. This has been done by several workers, see for example Chow et al. (ref. 1) and Whitfield (ref. 2).

This problem can be simplified considerably by replacing the approaching nonuniform stream by an equivalent stream having a stepped velocity profile with a finite number of discontinuities. This problem can be solved by neglecting viscosity and compressibility and making a potential flow analysis. Karman (ref. 3) gave the basis for a linearized potential flow analysis for such problems. Glauert (ref. 4) employed this method to solve the problem of an airfoil in the presence of a jet. He replaced the airfoil by a single vortex and, by computing the increment in axial velocity and streamline curvature at the airfoil, determined the increase in lift of the airfoil. Ting and Liu (ref. 5) extended Karman's method further and determined the chordwise pressure distribution on a thin airfoil in a nonuniform stream.

In this note, the basis for linearized potential flow analysis for the problem of an airfoil in a nonuniform stream is reviewed. The method of solution of the integral equation for the unknown vortex distribution of ref. 5 is simplified. The analysis is then extended to cover the case of five streams with four surfaces of discontinuity. Finally, the problem of an airfoil in a stream of smooth velocity profile is also treated by the linearized potential flow analysis.

2. FUNDAMENTAL BASIS FOR THE ANALYSIS

The present analysis is an extension of Karman's method of representing the flow past an airfoil in the proximity of a surface of velocity discontinuity. This method is based on a linearized potential flow analysis. Consider two parallel streams with velocities U_0 and U_1 with the x-axis being the undisturbed streamline separating the two streams. Let an airfoil be located in the lower stream and let (u_0, v_0) and (u_1, v_1) be the perturbation velocity components in the lower and upper streams respectively. If the distorted streamline that separates the two streams makes an angle β with the undisturbed streamline, then we have (see figure 1)

$$\tan \beta = \frac{v_0}{U_0 + u_0} = \frac{v_1}{U_1 + u_1}$$

Retaining only first order terms, we obtain

$$\frac{v_0}{U_0} = \frac{v_1}{U_1} \tag{1}$$

We also require the static pressure to vary continuously across the surface, i.e.,

$$P_0 + 1/2 \, \rho \left[U_0^2 - (U_0 + u_0)^2 - v_0^2 \right] = P_0 + 1/2 \, \rho \left[U_1^2 - (U_1 + u_1)^2 - u_1^2 \right]$$

Again retaining only first order terms, we obtain

$$u_0 \, U_0 = u_1 \, U_1 \tag{2}$$

These are the two basic conditions that must be satisfied across the undisturbed surface of discontinuity, and form the basis for the entire analysis that follows.

2.1 Airfoil Near a Surface of Discontinuity

Consider two parallel streams with velocities U_0 and U_1 with the x-axis as the undisturbed streamline separating the two streams. Let an airfoil be located at the point P (with OP = a) in the lower stream (see figure 1). Glauert (ref. 4) replaced the airfoil by a vortex of unknown circulation Γ. He then showed that equations (1) and (2) are satisfied if the flow in the

upper stream is that due to a point vortex of strength $(\Gamma + K) U_1/U_0$ at P, and the flow in the lower stream is that due to the vortex Γ at P together with its defracted image of strength K at the point P' (see figure 2) where

$$K = \Gamma (U_0^2 - U_1^2)/(U_0^2 + U_1^2) = \Gamma\alpha.$$

He then computed the increase in the axial velocity and the streamline curvature induced by the image vortex and determined the change in lift of the airfoil.

A logical extension of this approach is to replace the airfoil by a vortex distribution $\gamma(x)$, $0 \leqslant x \leqslant c$, instead of a single vortex Γ. Each of the vortex elements $\gamma(x)dx$ forms images as described above. The downwash at the mean camber line of the airfoil can then be determined in terms of $\gamma(x)$ and its image $\alpha\gamma(x)$ and, by satisfying the flow tangency boundary condition on the airfoil, the unknown $\gamma(x)$ can be determined. This problem is treated as a particular case of a more general problem of an airfoil in a jet of finite width which is considered in the following section.

2.2   Airfoil in a Jet of Finite Width

A more interesting problem is the flow past an airfoil placed in a jet of finite width. Let us consider the general case of three parallel streams of velocities U_{-1}, U_0, and U_1 with two undisturbed surfaces of discontinuity (AA & BB) as shown in figure 3. Let an airfoil of chord c be placed in the middle stream at a distance a below the undisturbed surface AA. In this case the conditions (1) and (2) have to be satisfied at both the surfaces AA & BB. Ting and Liu (ref. 5) represented the airfoil by a vortex distribution of unknown strength $\gamma(x)$, $0 < x < c$, and by satisfying equations (1) and (2) repeatedly across the surfaces AA & BB, obtained an infinite set of image vortex distributions (see figure 3). The downwash at the airfoil chord is then given by

$$v(x) = \frac{1}{2\pi} \int_0^c \left\{ \frac{1}{x-x'} + \sum_{j=0}^{\infty} (\alpha\beta)^j \left[\frac{\alpha(x-x')}{(x-x')^2+4(jh+a)^2} + \frac{\beta(x-x')}{(x-x')^2+4(jh+b)^2} \right. \right.$$

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$$+ \frac{2\alpha\ \beta(x-x')}{(x-x')^2 +4(j+1)^2\ h^2}]\}\ \gamma(x')dx' \tag{3}$$

The flow tangency condition at the airfoil requires

$$v(x) = U_0\ \left(\alpha - m(x)\right) \tag{4}$$

where α is the angle of attack and $m(x)$ is the slope of the mean camber line of the airfoil. Equations (3) and (4) form an integral equation for the unknown $\gamma(x)$ subject to the boundary condition (Kutta condition) $\gamma(c) = 0$. Note in equation (3) the first term in the integrand is the familiar singular term that appears in the classical airfoil theory; the other terms are not singular.

As the first step in the solution of (3), x and x' are replaced by θ and ϕ using the following transformation:

$$x = (1-\cos\theta)c/2; \quad x' = (1-\cos\phi)c/2$$

Equations (3) and (4) together then result in

$$\alpha - m(\theta) = \frac{U_0}{2\pi} \int_0^\pi \{\frac{1}{\cos\phi-\cos\theta} + \sum_{j=0}^\infty (\alpha\beta)^j [\frac{\alpha(\cos\phi-\cos\theta)}{(\cos\phi-\cos\theta)^2+16(jh+a)^2/c^2}$$

$$+ \frac{\beta(\cos\phi-\cos\theta)}{(\cos\phi-\cos\theta)^2+16(jh+b)^2/c^2} + \frac{2\alpha\beta(\cos\phi-\cos\theta)}{(\cos\phi-\cos\theta)^2+16(j+1)h^2/c^2}]\}$$

$$\gamma(\phi)\ \sin\phi\ d\phi \tag{5}$$

Ting and Liu (ref. 3) then expressed $\gamma(\phi)$ as the following series:

$$\gamma(\phi) = 2\ U_0\ (A_0\ \cot\ \phi/2 + \sum_{m=1}^\infty A_m\ \sin\ m\phi) \tag{6}$$

where A_m, $m = 0,1,2,\ldots,M$ are the unknown constants. Similarly they expanded the expression $\alpha - m(\theta)$ into a Fourier series as follows:

$$\alpha - m(\theta) = d_0 + \sum_{m=1}^{\infty} d_m \cos m\theta \qquad (7)$$

where d_m, $m = 0,1,2,\ldots,M$, are the known constants which can be determined for given $\alpha - m(\theta)$. These expansions were then substituted in equation (5). The right side of the resulting equation was resolved into a cosine Fourier series and, by equating the coefficients of like Fourier components, a set of algebraic equations for the unknowns A_m, $m = 0,1,2,\ldots,M$, was obtained. The solution of these equations gave the values of A_m and hence the distribution $\gamma(\phi)$.

This method of solution is rather lengthy. Since the integrand in equation (5) is no more singular than the integrand in the classical airfoil theory, all the methods of solving the classical airfoil integral would be applicable in the present case also. In particular discretization of $\gamma(\phi)$ would be possible.

We therefore use Lan's method (ref. 6) and discretize $\gamma(\phi)$. In this method vortex points ϕ_k and the control points θ_i are chosen as follows:

$$\phi_k = (2k-1)\pi/2N, \quad k = 1,2\ldots,N \qquad (8)$$

$$\theta_i = i\pi/N, \quad i=1,2\ldots,N \qquad (9)$$

With this, equation (5) reduces to

$$\alpha - m(\theta_i) = \frac{1}{2N} \sum_{k=1}^{N} \left\{ \frac{1}{\cos\phi_k-\cos\theta_i} + \sum_{j=0}^{\infty} (\alpha\beta)^j \left[\frac{\alpha(\cos\phi_k-\cos\theta_i)}{(\cos\phi_k-\cos\theta_i)^2+16(j+1)h^2/c^2} \right. \right.$$

$$\left. \left. + \frac{\beta(\cos\phi_k-\cos\theta_i)}{(\cos\phi_k-\cos\theta_i)^2+16(jh+b)^2/c^2} + \frac{2\alpha\beta(\cos\phi_k-\cos\theta_i)}{(\cos\phi_k-\cos\theta_i)^2+16(j+1)h^2/c^2} \right] \right\}$$

$$\gamma(\phi_k) \sin\phi_k, \quad i=1, 2,\ldots, M \qquad (10)$$

This is a set of linear simultaneous equations for the unknowns $\gamma(\phi_k)$, $k = 1,2,\ldots,N$ which can be solved easily. The lift and pitching moment of the airfoil are then given by

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$$c_1 = \frac{\pi}{N} \sum_{k=1}^{N} \gamma(\phi_k) \sin \phi_k \tag{11}$$

$$c_m = \frac{-\pi}{2N} \sum_{k=1}^{N} \gamma(\phi_k) \sin \phi_k (1-\cos \phi_k) \tag{12}$$

2.3 Airfoil in the Middle of Five Streams

Following Glauert (ref. 4) let us consider the problem of an airfoil in the presence of an infinite series of jets of the same width b. Let the velocity in the nth jet be denoted by U_n, $-\infty < n < \infty$. Let an airfoil represented by a vortex of circulation Γ be located on the axis of the principal jet in which the velocity is U_0. Then the flow in any jet can be represented by an infinite column of equispaced point vortices at the centers of the jets. Flow in the nth jet is due to the vortex system ${}^n K_s$ as shown in figure 4. In general

$$ {}^n K_n = 0, \quad n \neq 0 \tag{13} $$

$$ {}^0 K_0 = \Gamma. \tag{14} $$

Now by applying conditions (1) and (2) at the surface of discontinuity between the nth and the (n+1)th jets, the following fundamental relation for the strength of vortices can be obtained.

$$ \sqrt{(1-\alpha_{n+1}^2)} \; {}^{n+1} K_{n+s+1} = {}^n K_{n+s+1} - \alpha_{n+1} \; {}^n K_{n-s}, \quad \begin{array}{l} -\infty < n < \infty \\ -\infty < s < \infty \end{array} \tag{15} $$

This equation can be solved; but as pointed out by Glauert, the complete solution is very complex. If, however, we consider only 5 streams as shown in figure 5, the problem gets simplified to some extent. When the conditions (1) and (2) are applied to the four surfaces of discontinuity, we obtain a relation similar to relation (15), except that in this case n assumes the values -2, -1, 0 and 1 only. Denoting ${}^2 K_s$, ${}^1 K_s$, ${}^0 K_s$, ${}^{-1} K_s$ and

$^{-2}K_s$ by I_s, J_s, Γ_s, K_s and L_s respectively, equation (15) for the four different values of n can be written as follows.

$$\beta_2 \; I_{s+2} = J_{s+2} - \alpha_2 \; J_{1-s}$$

$$\beta_1 \; J_{s+1} = \Gamma_{s+1} - \alpha_1 \; \Gamma_{-s}$$

$$\beta_0 \; \Gamma_s = K_s - \alpha_0 \; K_{-s-1}$$

$$\beta_{-1} \; K_{s-1} = L_{s-1} - \alpha_{-1} \; L_{-s-2}$$

where $\alpha_n = (U_n^2 - U_{n+1}^2)/(U_n^2 + U_{n+1}^2)$; $\beta_n = \sqrt{(1-\alpha_n^2)}$, n=2, 1,0,-1.　　　　(17)

$$I_s = 0, \; s > 2.$$

$$J_1 = 0, \; \Gamma_0 = \Gamma, \; K_{-1} = 0,$$

and $L_s = 0$, $s < -2$.

The solution of these equations can be obtained by substituting successively, positive and negative values of s. However, with some algebraic manipulations, it is possible to obtain recurrence relations for Γ_s; see Appendix A for details. These relations are:

$$\Gamma_s = \alpha_1 \; \Gamma_{1-s} + \alpha_2 \; \Gamma_{3-s} - \alpha_1\alpha_2\Gamma_{s-2}, \; s > 2,$$

and $\Gamma_s = - \alpha_0 \; \Gamma_{-1-s} - \alpha_{-1} \; \Gamma_{-3-s} - \alpha_0 \; \alpha_{-1} \; \Gamma_{s+2}$, $s < -2$,　　　　(18)

with $\Gamma_0 = \Gamma$, $\Gamma_1 = \alpha_1\Gamma$, and $\Gamma_{-1} = \alpha_0 \; \Gamma$.

Some of the values of Γ_s computed using these relations are given below:

$$\Gamma_2 = - (\alpha_0\alpha_{-1})\Gamma = \Gamma_{-2}$$

$$\Gamma_3 = (\alpha_2\beta_1^2 - \alpha_0\alpha_1^2)\Gamma,$$

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$$\Gamma_{-3} = (\alpha_0^2 \alpha_1 - \alpha_{-1}^2 \beta_0)\Gamma,$$

$$\Gamma_4 = (\alpha_0^2 \alpha_1^2 - \alpha_0 \alpha_2^2 \beta_1 - \alpha_1 \alpha_{-1}^2 \beta_0)\Gamma = \Gamma_{-4}, \text{ etc.}$$

In general $\Gamma_n = a_n \Gamma$ where a_n's are constants. Now, following the procedure adopted in the previous section we replace the airfoil by a vortex distribution $\gamma(x)$, $0 < x < c$ rather than by a single vortex. Then the defracted images will also be vortex distributions $a_n \gamma(x)$. We note that it is not necessary to place the airfoil on the axis of the central stream. The images formed when the airfoil is offset from the centerline will be shown in figure 5.

The downwash at the airfoil is then given by

$$v(x) = \frac{1}{2\pi} \int_0^c \left\{ \frac{1}{x-x'} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{n=\infty} \frac{a_n(x-x')}{(x-x')^2 + 4n^2h^2} \right\} \gamma(x') \, dx' \tag{19}$$

Transforming x and x' into θ and ϕ respectively as before, we obtain

$$v(\theta) = \frac{1}{2\pi} \int_0^\pi \left\{ \frac{1}{\cos\phi-\cos\theta} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{n=\infty} \frac{a_n(\cos\phi-\cos\theta)}{(\cos\phi-\cos\theta)^2 + 16n^2h^2/c^2} \right\} \gamma(\phi)\sin\phi \, d\phi \tag{20}$$

On discretizing $\gamma(\phi)$, and choosing ϕ_k as the vortex points and θ_i as the control points, as in equations (8) and (9) we obtain:

$$v(\theta_i) = \frac{1}{2N} \sum_{k=1}^N \left\{ \frac{1}{\cos\phi_k-\cos\theta_i} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{n=\infty} \frac{a_n(\cos\phi_k-\cos\theta_i)}{(\cos\phi_k-\cos\theta_i)^2 + 16n^2h^2/c^2} \right\} \gamma(\phi_k)\sin\phi_k$$

$$i = 1,2,\ldots,N \tag{21}$$

For the linearized boundary condition on the airfoil at the control points, we have

$$v(\theta_i) = U_0 \left(\alpha - m(\theta_i) \right), \quad i = 1,2,\ldots,N \tag{22}$$

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Now this set of linear simultaneous equations can be solved for the unknowns $\gamma(\phi_k)$, $k = 1,2,\ldots,N$. The lift and pitching moment coefficients can then be computed using the relations (11) and (12) respectively.

2.4 Airfoil in a Stream of Smooth Velocity Profile

So far we have considered an airfoil in a stream with a stepped velocity profile. We can extend our analysis to the case of a stream with a smooth velocity profile. Let us consider, as in the previous section, an infinite series of jets of the same width h, the velocity in the nth jet being denoted by U_n. In the limit as h tends to zero, the velocity profile tends to the given smooth velocity profile. Let the airfoil be placed on the axis of the primary jet in which the velocity is U_0 (see Figure 6). The strength of image vortices is then given by the relation (15). Glauert attempted to extend this to the case of a stream with continuous variation of velocity; unfortunately, since he had only a single vortex representing the airfoil, he ended up with a simple logarithmic singularity in his expression for the increase in axial velocity as well as for the streamline curvature at the airfoil. However, if we represent the airfoil by a vortex distribution as was done in the previous sections, and satisfy the flow tangency boundary condition at the airfoil, we do not encounter any difficulty. This has been done in the following.

As we noted earlier, the solution of equation (15) for the case of an infinite set of jets of the same width h is very complex. However, if we assume small variation of velocity from jet to jet, we can write α_n as

$$\alpha_n = \frac{U_n^2 - U_{n+1}^2}{U_n^2 + U_{n+1}^2} = -\frac{2U_n u_n + u_n^2}{2U_n^2 + 2U_n u_n + u_n^2} \approx -\frac{u_n}{U_n} , \qquad (23)$$

where $u_n = (U_{n+1} - U_n) \ll U_n$.

With $\alpha_n \ll 1$, we can obtain a first order solution (consistent with the linearization done elsewhere in the analysis) for equation (15). The resulting image system for the primary stream is then found to be

$$\Gamma_0 = \Gamma, \text{ at } y = 0,$$

$$\Gamma_{2n} = 0,$$

$$\Gamma_{2n-1} = \alpha_n \Gamma, \text{ at } y_n = (2n-1)h, \ n > 1$$

$$= \alpha_n \Gamma, \text{ at } y_n = (2n-1)h, \ n < 0.$$

Replacing the airfoil by a vortex distribution $\gamma(x)$ instead of a single vortex Γ, we end up with a similar image system. The downwash at the airfoil with this image vortex system is then given by

$$v(x) = \frac{1}{2\pi} \int_0^c \left\{ \frac{1}{x-x'} + \sum_{n=1,3,\ldots} \frac{a_n(x-x')}{(x-x')^2+y_n^2} \right.$$

$$\left. - \sum_{n=-1,-3,\ldots} \frac{a_n(x-x')}{(x-x')^2+y_n^2} \right\} \gamma(x')dx' \tag{24}$$

This is the governing equation for the downwash velocity $v(x)$ at a point x on the airfoil chord in terms of the unknown vortex distribution $\gamma(x)$. For a given airfoil at a given (small) angle of attack, the slope of the mean camber line is known. Then, by satisfying the flows tangency condition on the camber line, the unknown $\gamma(x)$, $0 < x < c$ can be determined.

The stepped velocity profile is only an approximation to a smooth velocity profile. We can formally proceed to the limit of a smooth velocity profile by increasing the number of steps (N) and correspondingly decreasing the width of the step (h). For small h (=dy), $u_n = (dU/dy) \, dy$, and the expression (23) for α_n becomes

$$\alpha_n = -\frac{1}{U} \frac{dU}{dy} \cdot dy \tag{25}$$

where U and dU/dy are measured at $(2n-1) h/2 = ndy - dy/2$, and the corresponding image is located at $(2n-1) h = 2ndy - dy$. In the limit as N tends to infinity, the summations in the integrand in equation (24) are replaced by the corresponding integrals. With this, the downwash equation (24) can be rewritten as

$$v(x) = \frac{1}{2\pi} \int_0^c \left\{ \frac{1}{x-x'} - \int_0^\infty \frac{1}{U} \frac{dU}{dy} \frac{(x-x')dy}{(x-x')^2+4y^2} \right.$$

$$\left. + \int_{-\infty}^0 \frac{1}{U} \frac{dU}{dy} \frac{(x-x')dy}{(x-x')^2+4y^2} \right\} \gamma(x') \, dx' \qquad (26)$$

It can be shown that for U and U' of the order of unity, the linearized boundary condition is

$$v(x)/U(0) = \alpha - m(x) \qquad (26a)$$

For any given smooth velocity profile $U(y)$ with $U(y) \neq 0$, $-\infty < y < \infty$, the integrals within the brackets can be evaluated using any standard technique and the vortex distribution $\gamma(x)$ can be determined by the usual procedure. Relations (11) and (12) can then be used to determine the lift and pitching moment coefficients.

If the given velocity profile $U(y)$ is symmetric and the airfoil is placed on the line of symmetry, equation (26) reduces to

$$v(x) = \frac{1}{2\pi} \int_0^c \left\{ \frac{1}{x-x'} - 2(x-x') \int_0^\infty \frac{1}{U} \frac{dU}{dy} \frac{dy}{(x-x')^2+4y^2} \right\} \gamma(x')dx' \qquad (27)$$

The unknown vortex distribution can now be determined by the usual procedure after which relations (11) and (12) yield the lift and pitching moment co-efficients.

If the given velocity profile $U(y)$ is symmetric and the airfoil is placed on the line of symmetry, equation (26) reduces to

$$v(x) = \frac{1}{2\pi} \int_0^c \left\{ \frac{1}{x-x'} - 2(x-x') \int_0^\infty \frac{1}{U} \frac{dU}{dy} \frac{dy}{(x-x')^2+4y^2} \right\} \gamma(x')dx' \qquad (27)$$

The unknown vortex distribution can now be determined by the usual procedure after which relations (11) and (12) yield the lift and pitching moment co-efficients.

2.5 Jet-Flapped Airfoil in a Nonuniform Stream

Let us consider a thin jet flapped airfoil in the middle of five

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parallel streams. Following the method of Spence (ref. 7) the airfoil is represented by a distribution of vortex with density $\gamma(x)$, $0 < x < c$, and the jet emerging from the trailing edge of the airfoil is represented by a distribution of vortex with density $\gamma_j(x)$, $c < x < \infty$. As in the previous examples, this vortex distribution also forms a system of images. Then the expression for the downwash on the x-axis can be written as follows:

$$v(x) = \frac{1}{2\pi} \int_0^c \left\{ \frac{1}{x-x'} + \sum_{\substack{n \neq 0 \\ -\infty}}^{\infty} \frac{a_n(x-x')}{(x-x')^2 + 4n^2h^2} \right\} \gamma(x')dx'$$

$$+ \frac{1}{2\pi} \int_c^\infty \left\{ \frac{1}{x-x'} + \sum_{\substack{n \neq 0 \\ -\infty}}^{\infty} \frac{a_n(x-x')}{(x-x')^2 + 4n^2h^2} \right\} \gamma_j(x')dx' \qquad (28)$$

The coefficients a_n's are those described in section 2.3. The boundary conditions are:

on the airfoil $\quad v(x) = U_0 \left(\alpha - m(x)\right)$, $0 < x < c$, $\qquad (29)$

and on the jet $\quad v(x) = -U_0\, y'(x)$, $c < x < \infty$. $\qquad (30)$

Now γ_j is related to the curvature of the jet by the relation

$$\gamma_j(x) = -1/2\ U_0 C_J y'' \qquad (31)$$

where C_J is the jet momentum coefficient and $y(x)$ is the shape of the jet. Combining these, we obtain the following pair of integro-differential equations:

$$\frac{1}{2\pi U_0} \int_0^c \{F\}\gamma(x')dx' - \frac{C_J}{2\pi} \int_c^\infty \{F\}\, y''\, dx' = \alpha - m(x),\ 0<x<c$$
$$= -y'(x),\ c<x<\infty. \qquad (32)$$

where $\quad F = \frac{1}{x-x'} + \sum_{\substack{n \neq 0 \\ -\infty}}^{\infty} \frac{a_n(x-x')}{(x-x')^2 + 4n^2h^2}$

It should be noted that $\gamma(x)$ for $0 < x < c$ and $y(x)$ for $c < x < \infty$ are the unknowns. When the jet flap is absent $(C_J = 0)$ we obtain the problem solved in section 2.3, whereas when the approaching stream is uniform

$(a_n = 0)$, we obtain the problem solved in ref. 7. The present problem appears to be a rather difficult one. The method of solution of the simpler problem of ref. 7 with some modifications may still be applicable·but this needs to be investigated.

3. RESULTS AND DISCUSSION

The effect of proximity of a surface of discontinuity on the list of an airfoil is shown in figure 7. Results from reference 4 are included in this figure for comparison. Glauert (ref. 4) represented the airfoil by a single vortex distribution. This the only reason for Glauert's results not agreeing with the present ones. For large values of h/c, as can be seen from the figure, both methods give the same results. This trend is to be expected. At lower values of h/c, the present results are more accurate. For values of h/c very close to zero, the results are, however, not reliable.

Figures 8 and 9 illustrate the effect of a uniform jet on the lift of a flat plate airfoil. Figure 8 shows that as the ratio of the jet width to the airfoil chord (h/c) increases, the lift ratio (L/L_0) increases and reaches a value of 1.0 asymptotically. The lift of the airfoil depends on the location of the airfoil in the jet. Figure 9 shows that the lift is maximum when the airfoil is on the centerline of the jet and decreases as the airfoil is moved away from the centerline. This is obviously a result of linearization.

Changes in the lift and pitching moment due to a nonuniform stream and wall effects were reported by Ting and Liu (ref. 5). Figure 10 shows the results for the example case of reference 5, obtained by the present analysis and compared with those of reference 5. The two results should have been identical. The small differences that can be noticed in this figure are due to a small error that had crept in the results of reference 5. When this was corrected, their results were identical with the present ones.

The effect of four surfaces of discontinuity on the lift and pitching moment of a flate plate airfoil is illustrated in figure 11. The airfoil is assumed to be located on the axis of the central stream. The widths of the middle three streams are the same. Even though the velocity profile is assumed to be symmetric in this example, the computer program developed to solve this problem is quite general and accepts different values for the velocity in the streams adjacent to the middle stream increases, the lift

(and the pitching moment) ratio of the airfoil increases approaching the value of 1.0; this increase can be looked upon as the result of an increase in effective jet width.

The effect of a nonuniform stream of a smooth velocity profile is given in figures 12, 13, and 14. Figure 12 shows the effect on the lift of a flat plate airfoil due to a jet of a Gaussian velocity profile in a uniform stream. In figure 12a, a uniform approach stream corresponds to $a = 0$, and hence the value L/L_0 at this point is unity. For values of 'a' greater than zero, the stream has a jet like velocity porfile and the low velocity streams above and below the airfoil cause the lift ratio to drop. For values of 'a' less than zero (but greater than -1.0) the stream has a wake like profile. In this case the high velocity stream in the neighborhood of the airfoil causes an increase in the airfoil lift ratio.

Figure 12b shows the effect of the spread of the jet on the airfoil lift ratio. For small values of the parameter (d/c) the airfoil lift ratio is small. As (d/c) increases, the lift ratio increases and reaches the value of 1.0 asymptotically as (d/c) tends to infinity.

Lifting characteristics of a Joukowski airfoil in a nonuniform stream have been studied by Chow et al. (ref. 1) by solving Euler's equations. Some of their results are given in figures 13 and 14 for comparison with the results of the present analysis. It may be recalled that the present analysis is a linear potential flow analysis.

As the first example following reference 1, the upstream nonuniform velocity profile is represented by

$$U(y) = U_\infty \left[1 + a \exp\left(-(y-y_s)^2/d^2 \right) \right] \tag{33}$$

where 'a' is the ratio between maximum excess velocity and the velocity U_∞ (at $y = \infty$). Parameter d represents the spread of the velocity nonuniformity and y_s represents the vertical location of the jet centerline with respect to the airfoil. Figure 13 illustrates the effect of varying the location of the airfoil on the lift of a thin Joukowski airfoil (camber = 5%) at zero angle of attack for three different values of the jet spread (d/c). In all these cases the present results show that the lift is maximum

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when the airfoil is located on the centerline of the jet. The numerical results of Euler's equations included in figure 13 for comparison show that the maximum lift occurs when the airfoil is slightly below the jet centerline. Generally there is no satisfactory agreement between the two results in figure 13.

As the second example, the lifting characteristics of the Joukowski airfoil in a shear layer between two parallel streams is studied. For analysis, the upstream velocity profile is chosen to be

$$U(y) = U_0 \left[1 + a \tanh \left((y - y_s)/d \right) \right] \tag{34}$$

The mean velocity U_0 is used as the velocity scale. The parameter d represents the spread of the shear layer. As before, y_s denotes the vertical location of the airfoil relative to the upstream profile. Figure 14 shows the lift coefficient of the Joukowski airfoil placed in such a stream obtained by the present analysis together with the numerical results of Euler's equation from reference 1. The results show the expected trend, but the agreement between the two results is not satisfactory.

There could be several reasons for the poor agreement between the present results and those of the numerical solution of Euler's equation. The present analysis is a linearized potential flow theory. Linearization may have introduced some errors. Yet another reason could be due to possible errors in the numerical results of reference 1. For example, in reference 1, a value of 1.315 has been quoted for cc_l for the Joukowski airfoil ($c = 1.808$) at zero angle of attack in uniform stream. This implies a c_l of 0.727. But the linearized theory gives a c_l of only 0.625. Considering these differences, the lack of good agreement in results in figures 13 and 14 is not surprising.

When a small perturbation approximation is introduced in the vorticity transport equation, we arrive at the following linear partial differential equation for the perturbation velocity components.

$$U \left(u_y - v_x \right)_x + v \left(U_y \right)_y = 0 \tag{35}$$

where $U = U(y)$ is the undisturbed nonuniform velocity and $u(x,y)$ and

v(x,y) are the perturbation velocity components. The subscripts x and y stand for partial differentiation with respect to x and y respectively. Since U = U(y) is known, the equation is a linear p.d.e. with variable coefficients. The linearized boundary condition on the airfoil is as before v = Uy', where y' is the slope of the mean camber line.

The perturbation velocity components u and v satisfy the equation $(U_y - v_x) = 0$. Hence, it is evident that the present potential flow solution of linearized Euler's equation (35) only if U_{yy} is small.

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4. CONCLUDING REMARKS

A linearized potential flow theory has been applied to the problem of a thin airfoil in the middle of five streams with four surfaces of discontinuity. The theory is then extended to solve the problem of a thin airfoil in a nonuniform stream having a smooth velocity profile. Results have been obtained for two examples - one for a stream having a Gaussian velocity profile and the other having a hyperbolic tangent velocity profile. These results have been compared with more sophisticated numerical results of Euler's equation. Some differences are noticed between these two results, and possible reasons for this poor agreement are indicated.

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Figure 1. Airfoil Below a Surface of Discontinuity

Figure 2. Vortex System for an Airfoil Below A Surface of
Discontinuity

$$\alpha = \frac{U_0^2 - U_1^2}{U_0^2 + U_1^2} \qquad \beta = \frac{U_0^2 - U_{-1}^2}{U_0^2 + U_{-1}^2}$$

Jet system

Part of the vortex system
for the middle stream

Figure 3. Airfoil in the Middle of Three Parallel Streams

U_2

U_1

U_0 h Γ

U_{-1}

U_{-2}

U_{-3}

U_n

nK_{n+2}

nK_{n+1} $2h$

nK_n

nK_{n-1}

nK_{n-2}

nK_{n-3}

Jet system   Part of the vortex system
                                 for the nth jet

Figure 4. Airfoil in an Infinite Series of Jets

Figure 5. Airfoil in the Middle of Five Parallel Streams

Figure 6.   Airfoil in a Stream of Smooth Velocity Profile

Figure 7. Effect of Proximity of a Surface of Discontinuity
on the Airfoil Lift

Figure 8. Effect of Jet Width on the Airfoil Lift

Figure 9. Effect of Offset from Jet Centerline on the Airfoil Lift

Figure 10. Changes in Airfoil Lift and Pitching Moment Due to Nonuniform Stream and Wall Effects

Figure 11. Effect of Variation of U_1/U_0 on the Airfoil Lift

Figure 12a. Effect of Nonuniform Stream on the Airfoil
Lift (Variation with the jet excess velocity
factor a)

Figure 12b.   Effect of Nonuniform Stream on the Airfoil Lift
              (Variation with the jet spread parameter d/c)

Figure 13. Lift Variation of a Joukowski Airfoil in a Stream
of Gaussian Velocity Profile

Figure 14. Lift Variation of a Joukowski Airfoil in a Shear Layer

DERIVATION OF THE RECURRENCE RELATION FOR THE STRENGTH OF IMAGE VORTICES

Consider the relations (16a) to (16d)

$$\beta_2 \; I_{s+2} = J_{s+2} - \alpha_2 \; J_{1-s} \tag{16a}$$

$$\beta_1 \; J_{s+1} = \Gamma_{s+1} - \alpha_1 \; \Gamma_{-s} \tag{16b}$$

$$\beta_0 \; \Gamma_s = K_s - \alpha_0 \; K_{-s-1} \tag{16c}$$

$$\beta_{-1} \; K_{s-1} = L_{s-1} - \alpha_{-1} \; L_{-s-2} \tag{16d}$$

with $\qquad I_s = 0$, for $s > 2$

$$J_1 = 0, \; \Gamma_0 = \Gamma, \; K_{-1} = 0.$$

and $\qquad L_s = 0$, for $s < -2$ $\tag{A1}$

We can rewrite (16b) as

$$\Gamma_{s+1} = \beta_1 \; J_{s+1} + \alpha_1 \; \Gamma_{-s} \tag{A2}$$

The relation (16a) with s replaced by $(s-1)$ gives

$$\beta_2 \; I_{s+1} = J_{s+1} - \alpha_2 \; J_{2-s}$$

Since $I_{s+1} = 0$ for $s > 1$, this can be written as

$$0 = J_{s+1} - \alpha_2 \; J_{2-s} \quad \text{for } s > 1$$

or $\quad J_{s+1} = \alpha_2 \; J_{2-s}$ $\tag{A3}$

Using (A3) in (A2) we obtain

$$\Gamma_{s+1} = \alpha_2 \beta_1 J_{2-s} + \alpha_1 \Gamma_{-s} \quad \text{for } s \geqslant 1 \qquad (A4)$$

which can be rewritten with s replaced by $(1-s)$ as

$$\Gamma_{2-s} = \beta_1 J_{2-s} + \alpha_1 \Gamma_{s-1} \qquad (A5)$$

Combining (A5) and (A4) we obtain

$$\Gamma_{s+1} = \alpha_2 (\Gamma_{2-s} - \alpha_1 \Gamma_{s-1}) + \alpha_1 \Gamma_{-s} \qquad (A6)$$

or $\quad \Gamma_s = \alpha_1 \Gamma_{-(s-1)} + \alpha_2 \Gamma_{-(s-3)} - \alpha_1 \alpha_2 \Gamma_{s-2}$

Next since $L_s = 0$ for $s \leqslant -2$, the relation (16d) reduces to

$$\beta_{-1} K_{s-1} = -\alpha_{-1} L_{-s-2} \quad \text{for } s \leqslant -1 \qquad (A7)$$

Also replacing s by $-(s+1)$ in (16d), we obtain

$$\beta_{-1} K_{-(s+2)} = L_{-(s+2)} - \alpha_{-1} L_{s-1} \qquad (A8)$$

Using (A7) in (A8) and noting $L_{(s-1)} = 0$ for $s \leqslant -1$, we get

$$K_{s-1} = -\alpha_{-1} K_{-(s+2)} \quad \text{for } s \leqslant -1 \qquad (A9)$$

or $\quad K_s = -\alpha_{-1} K_{-(s+3)} \quad \text{for } s \leqslant -2.$

Next replacing s by $-(s+1)$ in (16c), we obtain

$$\beta_0 \Gamma_{-(s+1)} = K_{-(s+1)} - \alpha_0 K_s \qquad (A10)$$

If this relation is used in (16c), then we find

$$\alpha_0 \Gamma_{-(s+1)} = -\Gamma_s + \beta_0 K_s \qquad (A11)$$

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which can also be written with s replaced by $-(s+3)$ as follows:

$$\alpha_0 \; \Gamma_{s+2} = - \Gamma_{-(s+3)} + \beta_0 \; K_{-(s+3)} \tag{A12}$$

On using (A11) and (A12) in (A9), we arrive at

$$\Gamma_s = - \alpha_{-1} \; \Gamma_{-(s+3)} - \alpha_0 \; \Gamma_{-(s+1)} - \alpha_0 \; \alpha_{-1} \; \Gamma_{s+2} \tag{A13}$$

(A6) and (A13) are the required recurrence relations for Γs for $s > 2$ and $s < -2$, respectively. It can be easily shown with $\Gamma_0 = \Gamma$ that $\Gamma_1 = \alpha_0 \Gamma$, $\Gamma_{-1} = \alpha_{-1} \Gamma$. Other values of Γ's can be determined using the above recurrence relations.