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A Solution Procedure for Behavior of Thick Plates on a Nonlinear Foundation and Postbuckling Behavior of Long Plates

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SUMMARY

Approximate solutions are presented in this paper for three nonlinear orthotropic plate problems. The problems are to determine the deformations and forces in: (1) a thick plate attached to a pad having nonlinear material properties which, in turn, is attached to a substructure which is then deformed; (2) a long plate loaded in inplane longitudinal compression beyond its buckling load; and (3) a long plate loaded in inplane shear beyond its buckling load. For all three problems, the two-dimensional plate equations are reduced to one-dimensional equations in the y-direction by using a one-dimensional trigonometric approximation in the x-direction. Each problem uses different trigonometric terms. Solutions are obtained using an existing algorithm for simultaneous, first-order, nonlinear, ordinary differential equations subject to two-point boundary conditions. For each problem, reasons are discussed for choosing the trigonometric terms, ordinary differential equations are derived to determine the variable coefficients of the trigonometric terms, and sample results are presented. This solution procedure provides a quick, easy way to solve some nonlinear plate problems.

INTRODUCTION

Linear equations are often inadequate in the analysis of plates that appear in space vehicle structures and aircraft structures. Some problems of plate analysis involve stretching and bending coupling so that the strains depend nonlinearly on the deformations. Other problems may involve nonlinear material properties, such as those due to plasticity or nonhomogeneous materials. Strength or limits in deformation may be considered in the design of such plates. Useful estimates of the deformations can usually be found from approximate solutions of the nonlinear equations of equilibrium. Meaningful strain distributions, however, require much more accurate solutions of the nonlinear equations.

Many methods are available for solving nonlinear plate problems. A fairly complete survey, with examples of methods used before the high-speed computer was developed, is contained in reference 1. Analytical solutions of the differential equations are sometimes possible through special devices or inverse procedures. Direct procedures include iterative methods, such as Newton's method, and methods which make use of trial and error. For problems involving a parameter of small value, a regular perturbation method may be employed; for problems involving a boundary layer, methods of asymptotic integration may be employed. Weighted residual techniques are also available which include the energy method, the Galerkin method, collocation, and the method of least squares. Finite-element and finite-difference methods are often used now with the high-speed computer. These available methods have certain limitations, and solution to many nonlinear plate problems are not readily obtained by these methods.

In the present paper, simplifying assumptions are used in a derivation from basic relations for plate problems to replace the nonlinear, partial (two-dimensional) differential equations of plate theory with nonlinear, ordinary (one-dimensional) differential equations. The derivation employs the principle of virtual work in conjunction with the assumption that the displacements may be represented by the first few terms of a Fourier series. Solution of the ordinary differential equations, subject to the boundary conditions which arise naturally in the derivation, may be obtained by using the algorithm described in the next paragraph.

An algorithm based on Newton's method has been developed by Lentini and Pereyra (ref. 2) to solve a system of simultaneous first-order, nonlinear, ordinary differential equations subject to two-point boundary conditions. The system of equations is of the form

$$\bar{y}' = \bar{F}(x,\bar{y})$$

where $\bar{y}$ is the vector of dependent variables, x is the independent variable defined in the interval (a,b), and $\bar{F}$ is a vector, the components of which may be nonlinear functions of x and $\bar{y}$. The boundary conditions of the problem are specified by

$$\bar{g}[\bar{y}(a),\bar{y}(b)] = 0$$

(The components of $\bar{g}$ identify each of the boundary conditions.) This algorithm uses finite differences with deferred corrections, and adaptive mesh spacings are automatically produced to detect and resolve mild boundary layers.

Approximate solutions for three nonlinear plate problems are presented in this report. They are as follows: (1) a thick plate attached to a foundation having nonlinear material properties, which in turn is attached to a substructure, which is then deformed; (2) a long plate loaded in inplane compression beyond its buckling load; and (3) a long plate loaded in inplane shear beyond its buckling load. The first problem is important in designing thermal protection systems for space transportation vehicles and analyzing bonded joints. The second and third problems are important in structural design to take advantage of the stress-carrying ability of supported plates beyond the buckling load. The purposes of this paper are (1) to indicate the reasons for choosing the trigonometric Fourier series terms that are used, (2) to present the derivation of the differential equations that determine the coefficients of the Fourier series terms, and (3) to present sample results for each of the three problems.

SYMBOLS

$\left.\begin{matrix} A_{11}, A_{12}, A_{22}, \\ A_{44}, A_{55}, A_{66} \end{matrix}\right\}$ plate extensional stiffnesses

$A_{33p}, A_{44p}, A_{55p}$ pad extensional stiffnesses

a,b dimensions of rectangular plate parallel to x- and y-axes, respectively

C_x, C_y, C_z constant coefficients of cubic parts of stress-strain laws for pad

$D_{11}, D_{12}, D_{22}, D_{66}$	plate bending stiffnesses
E, G	Young's modulus and shear modulus for beam
E_p, G_p	Young's modulus and shear modulus for pad
L	length of beam
M_x, M_y, M_{xy}	bending moments in plate or beam
N_x, N_y, N_{xy}	inplate stress resultants in plate or beam
N_{xav}	average N_x in plate
N_{xyav}	average N_{xy} in plate
N_{zp}	normal stress resultant in pad
Q_x, Q_y	transverse shear stress resultants in plate
Q_{xp}, Q_{yp}	transverse shear stress resultants in pad
T	thickness of pad
t	thickness of plate or beam
U, V, W	deformation of substructure in x-, y-, and z-directions
u, v, w	displacements in x-, y-, and z-directions
$u_1, u_2, v_0, v_1, v_2, w_0, w_1, w_2, \beta_{x1}, \beta_{x2}, \beta_{y0}, \beta_{y1}, \beta_{y2}$	arbitrary functions of y in equations (14), (15), and (30) for u, v, w, β_x, and β_y
$u_0, u_s, u_c, v_0, v_s, v_c, w_s, w_c$	arbitrary functions of y in equations (41) for u, v, and w
$\bar{u}_0, \bar{u}_{00}$	arbitary constants in expressions (14)
$\underline{u}$	total axial displacement of beam
$\bar{u}_{cn}$	applied compressive displacement
$\bar{u}_{sh}$	applied shearing displacement
$\underline{w}$	total deflection normal to beam
x, y, z	plate or beam coordinates

β_x, β_y rotations in x- and y-directions

$\left.\begin{array}{l}\varepsilon_x, \varepsilon_y, \\ \gamma_{xy}, \gamma_{xz}, \gamma_{yz}\end{array}\right\}$ neutral surface strains

$\varepsilon_{zp}, \gamma_{xzp}, \gamma_{yzp}$ strains in pad

$\kappa_x, \kappa_y, \kappa_{xy}$ curvatures

λ half-wavelength in x-direction

Prime (') indicates differentiation with respect to y. Subscripts x and y which follow a comma indicate partial differential of the principal symbol with respect to the subscript.

APPLICATIONS

Derivations of ordinary differential equations are presented in this section for (1) a thick plate on a nonlinear foundation, (2) the postbuckling behavior of long plates in compression, and (3) the postbuckling behavior of long plates in shear. The choices of trigonometric Fourier series terms used for each plate problem, and the rationale for choosing these terms, are discussed. To clarify the derivation of the differential equations for the aforementioned problems, the governing equations for a simpler problem - the deep beam on a nonlinear foundation - is derived first. The deep-beam problem is the one-dimensional counterpart of the two-dimensional thick-plate problem and shows in a much simpler problem the steps involved in deriving the differential equations of equilibrium from the principle of virtual work.

Deep Beam on Nonlinear Foundation

As indicated in figure 1, the beam is attached to the top of a pad, which is attached to a substructure that may be deformed in the x- and z-directions (U and W). The effects of transverse shearing are included, in addition to extension and bending. The pad resists transverse extension and shearing and has nonlinear material properties. The problem is to determine the strains in the beam and in the pad due to deformation of the substructure. Generally, the beam is considered to be much stiffer than the pad.

Let u be the x-displacement of the neutral surface, w the deflection of the neutral surface, and β_x the rotation. The neutral surface strain is then

$$\varepsilon_x = u' \tag{1}$$

the transverse shearing strain is

$$\gamma_{xz} = \beta_x + w' \tag{2}$$

and the curvature of the neutral surface is

$$\kappa_x = \beta'_x \tag{3}$$

The bottom of the beam thickness t displaces $u - (t/2)\beta_x$; thus, the rotation of the pad of thickness T is $[u - (t/2)\beta_x - U]/T$. The average deflection of the pad is $(w + W)/2$; the transverse shearing strain in the pad is then

$$\frac{u - (t/2)\beta_x - U}{T} + \frac{w' + W'}{2} \tag{4}$$

the sum of its rotation and slope. The beam is assumed to be much stiffer than the pad; accordingly, the slope of the beam w' is neglected in comparison with the substructure slope W'. Therefore, the transverse shearing strain in the pad is

$$\gamma_{xzp} = \frac{u - (t/2)\beta_x - U}{T} + \frac{W'}{2} \tag{5}$$

The extension of the pad $w - W$ over the thickness T determines the transverse extensional strain in the pad as

$$\varepsilon_{zp} = \frac{w - W}{T} \tag{6}$$

The virtual work of the system for a beam of unit width and of length L is

$$\delta\Pi = \int_0^L (N_x\,\delta\varepsilon_x + M_x\,\delta\kappa_x + Q_x\,\delta\gamma_{xz} + N_{zp}\,\delta\varepsilon_{zp} + Q_{xp}\,\delta\gamma_{xzp})\,dx \tag{7}$$

where the coefficients of the virtual strains are force resultants. Substitution for the strains and integration by parts and rearrangement results in

$$\begin{aligned}\delta\Pi = \int_0^L \Bigg[&\left(-N'_x + \frac{1}{T}\,Q_{xp}\right)\delta u + \left(Q_x - M'_x - \frac{t}{2T}\,Q_{xp}\right)\delta\beta_x \\ &+ \left(-Q'_x + \frac{1}{T}\,N_{zp}\right)\delta w\Bigg]dx + (N_x\,\delta u + M_x\,\delta\beta_x + Q_x\,\delta w)_0^L\end{aligned} \tag{8}$$

According to the principle of virtual work, the virtual and actual displacements must satisfy the same boundary conditions, but the virtual displacements are arbitrary in the interior region. For equilibrium, the virtual work is an extremum;

$\delta\Pi = 0$. Therefore, equation (8) for virtual work can be replaced by the following differential equations and boundary condition terms:

$$\left.\begin{aligned}
&-N_x' + \frac{1}{T} Q_{xp} = 0 \\
&Q_x - M_x' - \frac{t}{2T} Q_{xp} = 0 \\
&-Q_x' + \frac{1}{T} N_{zp} = 0 \\
&(N_x \, \delta u)_0^L = 0 \qquad (M_x \, \delta\beta_x)_0^L = 0 \qquad (Q_x \, \delta w)_0^L = 0
\end{aligned}\right\} \tag{9}$$

Products of forces and virtual displacements appear in the boundary terms. The edges of the beam are free; therefore, $N_x = M_x = Q_x = 0$ at $x = 0$ and L. For other problems, the displacements can be taken to be equal to zero or combinations of forces and displacements (such that the above expressions are satisfied) can be set equal to zero.

A linear stress-strain law is assumed for the beam as follows:

$$\left.\begin{aligned}
N_x &= Et\varepsilon_x = Etu' \\
M_x &= \frac{Et^3}{12} \kappa_x = \frac{Et^3}{12} \beta_x' \\
Q_x &= Gt\gamma_{xz} = Gt(\beta_x + w')
\end{aligned}\right\} \tag{10}$$

For the pad, a cubic nonlinear stress-strain law is assumed as follows (see fig. 1):

$$\left.\begin{aligned}
N_{zp} &= E_p T \left(\varepsilon_{zp} + C_z \varepsilon_{zp}^3\right) \\
Q_{xp} &= G_p T \left(\gamma_{xzp} + C_x \gamma_{xzp}^3\right)
\end{aligned}\right\} \tag{11}$$

The equations expressing the stress-strain law assumed for the beam equations (10) and the equilibrium equations (9), together with the stress-strain law for the pad equations (5), (6), and (11) can be identified as six simultaneous first-order, ordinary, nonlinear differential equations with unknowns u, β_x, w, N_x, M_x, and Q_x with boundary conditions. This system of equations may be readily solved by the algorithm discussed in the Introduction. The six equations in the desired form are as follows:

$$u' = \frac{N_x}{Et} \tag{12a}$$

$$\beta'_x = \frac{12M_x}{Et^3} \tag{12b}$$

$$w' = \frac{Q_x}{Gt} - \beta_x \tag{12c}$$

$$N'_x = G_p\left\{\frac{u - (t/2)\beta_x - U}{T} + \frac{W'}{2} + C_x\left[\frac{u - (t/2)\beta_x - U}{T} + \frac{W'}{2}\right]^3\right\} \tag{12d}$$

$$M'_x = Q_x - \frac{G_p t}{2}\left\{\frac{u - (t/2)\beta_x - U}{T} + \frac{W'}{2} + C_x\left[\frac{u - (t/2)\beta_x - U}{T} + \frac{W'}{2}\right]^3\right\} \tag{12e}$$

$$Q'_x = E_p\left[\frac{w - W}{T} + C_z\left(\frac{w - W}{T}\right)^3\right] \tag{12f}$$

Thick Plate on Nonlinear Foundation

For a thick plate on a pad with nonlinear material properties and substructure, the idealization to a deep beam discussed in the preceding section would not be satisfactory if deformations across both the length and width of the plate are important. This derivation is the two-dimensional extension of the beam problem (see fig. 2 for the coordinate system), and the substructure is permitted to deform V in the y-direction as well as the U and W in the x- and z-directions, respectively, for the beam. However, the U,V,W deformations considered are limited to

$$\left.\begin{aligned} U &= \bar{U}_{00} + \bar{U}_0 x + U_1(y)\sin\frac{\pi x}{a} + U_2(y)\sin\frac{2\pi x}{a} \\ V &= V_0(y) + V_1(y)\cos\frac{\pi x}{a} + V_2(y)\cos\frac{2\pi x}{a} \\ W &= W_0(y) + W_1(y)\cos\frac{\pi x}{a} + W_2(y)\cos\frac{2\pi x}{a} \end{aligned}\right\} \tag{13}$$

where the barred quantities are constants, and the indicated functions of y can be chosen as desired. As for the deep beam, transverse shearing is permitted in the thick plate, and the plate is considered to be much stiffer than the pad. The pad has nonlinear material properties and resists transverse extension and shearing in both the x- and y-directions. To make use of the algorithm described in the Introduction, it is necessary to make approximations and derive ordinary differential equations to replace the partial differential equations of plate theory.

In some linear problems solved by the use of Fourier series, the loading terms are expanded in Fourier series, and the solution is in the form of a Fourier series which is exact term by term. For various approximations, the series may be limited to the first few terms of the series. A similar approximate method is used here for a nonlinear problem. The substructure deformations represented by equations (13) can be considered to be loading terms, and, after identifying the first few terms of the series for the solution, no further terms are considered (and no further approximations are made).

Trigonometric terms of the same form as those considered for U, V, and W in equations (13) are assumed for the plate displacements u, v, and w in the x-, y-, and z-directions as follows:

$$\left.\begin{aligned} u &= \bar{u}_{00} + \bar{u}_0 x + u_1(y) \sin\frac{\pi x}{a} + u_2(y) \sin\frac{2\pi x}{a} \\ v &= v_0(y) + v_1(y) \cos\frac{\pi x}{a} + v_2(y) \cos\frac{2\pi x}{a} \\ w &= w_0(y) + w_1(y) \cos\frac{\pi x}{a} + w_2(y) \cos\frac{2\pi x}{a} \end{aligned}\right\} \tag{14}$$

Similar trigonometric terms are assumed for the plate rotations β_x and β_y as follows:

$$\left.\begin{aligned} \beta_x &= \beta_{x1}(y) \sin\frac{\pi x}{a} + \beta_{x2}(y) \sin\frac{2\pi x}{a} \\ \beta_y &= \beta_{y0}(y) + \beta_{y1}(y) \cos\frac{\pi x}{a} + \beta_{y2} \cos\frac{2\pi x}{a} \end{aligned}\right\} \tag{15}$$

The neutral surface strains and curvatures in the plate are

$$\left.\begin{aligned} &\varepsilon_x = u_{,x} && \gamma_{xy} = u_{,y} + v_{,x} && \kappa_x = \beta_{x,x} \\ &\varepsilon_y = v_{,y} && \gamma_{xz} = \beta_x + w_{,x} && \kappa_y = \beta_{y,y} \\ & && \gamma_{yz} = \beta_y + w_{,y} && \kappa_{xy} = \beta_{x,y} + \beta_{y,x} \end{aligned}\right\} \tag{16}$$

The commma before a subscript denotes partial differential with respect to that subscript. The strains in the pad are given as

$$\gamma_{xzp} = \frac{u - (t/2)\beta_x - U}{T} + \frac{W_{,x}}{2} \tag{17a}$$

$$\gamma_{yzp} = \frac{v - (t/2)\beta_y - V}{T} + \frac{W_{,y}}{2} \tag{17b}$$

$$\varepsilon_{zp} = \frac{w - W}{T} \tag{17c}$$

By substitution from equations (14) and (15), the strains and the curvatures are

$$\varepsilon_x = \bar{u}_0 + u_1 \frac{\pi}{a} \cos \frac{\pi x}{a} + u_2 \frac{2\pi}{a} \cos \frac{2\pi x}{a} \tag{18a}$$

$$\varepsilon_y = v_0' + v_1' \cos \frac{\pi x}{a} + v_2' \cos \frac{2\pi x}{a} \tag{18b}$$

$$\gamma_{xy} = \left(u_1' - v_1 \frac{\pi}{a}\right) \sin \frac{\pi x}{a} + \left(u_2' - v_2 \frac{2\pi}{a}\right) \sin \frac{2\pi x}{a} \tag{18c}$$

$$\gamma_{xz} = \left(\beta_{x1} - w_1 \frac{\pi}{a}\right) \sin \frac{\pi x}{a} + \left(\beta_{x2} - w_2 \frac{2\pi}{a}\right) \sin \frac{2\pi x}{a} \tag{18d}$$

$$\gamma_{yz} = \beta_{y0} + w_0' + \left(\beta_{y1} + w_1'\right) \cos \frac{\pi x}{a} + \left(\beta_{y2} + w_2'\right) \cos \frac{2\pi x}{a} \tag{18e}$$

$$\kappa_x = \beta_{x1} \frac{\pi}{a} \cos \frac{\pi x}{a} + \beta_{x2} \frac{2\pi}{a} \cos \frac{2\pi x}{a} \tag{18f}$$

$$\kappa_y = \beta_{y0}' + \beta_{y1}' \cos \frac{\pi x}{a} + \beta_{y2}' \cos \frac{2\pi x}{a} \tag{18g}$$

$$\kappa_{xy} = \left(\beta_{x1}' - \beta_{y1} \frac{\pi}{a}\right) \sin \frac{\pi x}{a} + \left(\beta_{x2}' - \beta_{y2} \frac{2\pi}{a}\right) \sin \frac{2\pi x}{a} \tag{18h}$$

$$\gamma_{xzp} = \frac{1}{T}\left[\left(\bar{U}_{00} - \bar{u}_{00}\right) + \left(\bar{U}_0 - \bar{u}_0\right)x\right] + \left(\frac{u_1 - \frac{t}{2}\beta_{x1} - U_1}{T} - \frac{W_1}{2}\frac{\pi}{a}\right)\sin\frac{\pi x}{a}$$

$$+ \left(\frac{u_2 - \frac{t}{2}\beta_{x2} - U_2}{T} - \frac{W_2}{2}\frac{2\pi}{a}\right)\sin\frac{2\pi x}{a} \quad (18i)$$

$$\gamma_{yzp} = \frac{v_0 - \frac{t}{2}\beta_{y0} - V_0}{T} + \frac{W_0'}{2} + \left(\frac{v_1 - \frac{t}{2}\beta_{y1} - V_1}{T} + \frac{W_1'}{2}\right)\cos\frac{\pi x}{a}$$

$$+ \left(\frac{v_2 - \frac{t}{2}\beta_{y2} - V_2}{T} + \frac{W_2'}{2}\right)\cos\frac{2\pi x}{a} \quad (18j)$$

$$\varepsilon_{zp} = \frac{w_0 - W_0}{T} + \frac{w_1 - W_1}{T}\cos\frac{\pi x}{a} + \frac{w_2 - W_2}{T}\cos\frac{2\pi x}{a} \quad (18k)$$

With no loss in generality, the constants appearing in u can be set equal to the corresponding constants appearing in U as follows:

$$\bar{u}_{00} = \bar{U}_{00}$$

$$\bar{u}_0 = \bar{U}_0$$

The stress-strain laws for the plate are

$$\left.\begin{aligned} N_x &= A_{11}\varepsilon_x + A_{12}\varepsilon_y & Q_x &= A_{44}\gamma_{xz} & M_x &= D_{11}\kappa_x + D_{12}\kappa_y \\ N_y &= A_{22}\varepsilon_y + A_{12}\varepsilon_x & Q_y &= A_{55}\gamma_{yz} & M_y &= D_{22}\kappa_y + D_{12}\kappa_x \\ N_{xy} &= A_{66}\gamma_{xy} & & & M_{xy} &= D_{66}\kappa_{xy} \end{aligned}\right\} \quad (19)$$

The stress-strain laws for the pad are

$$\left.\begin{aligned} Q_{xp} &= A_{44p}(\gamma_{xzp} + C_x\gamma_{xzp}^3) \\ Q_{yp} &= A_{55p}(\gamma_{yzp} + C_y\gamma_{yzp}^3) \\ N_{zp} &= A_{33p}(\varepsilon_{zp} + C_z\varepsilon_{zp}^3) \end{aligned}\right\} \quad (20)$$

The form of the force and moment resultants for the plate in terms of the trigonometric terms in the x-direction with coefficients, functions of y, is similar to the form of the strain for the plate:

$$\left.\begin{aligned}
N_x &= N_{x0}(y) + N_{x1}(y)\cos\frac{\pi x}{a} + N_{x2}(y)\cos\frac{2\pi x}{a} \\
N_y &= N_{y0}(y) + N_{y1}(y)\cos\frac{\pi x}{a} + N_{y2}(y)\cos\frac{2\pi x}{a} \\
N_{xy} &= N_{xy1}(y)\sin\frac{\pi x}{a} + N_{xy2}(y)\sin\frac{2\pi x}{a} \\
M_x &= M_{x0}(y) + M_{x1}(y)\cos\frac{\pi x}{a} + M_{x2}(y)\cos\frac{2\pi x}{a} \\
M_y &= M_{y0}(y) + M_{y1}(y)\cos\frac{\pi x}{a} + M_{y2}(y)\cos\frac{2\pi x}{a} \\
M_{xy} &= M_{xy1}(y)\sin\frac{\pi x}{a} + M_{xy2}(y)\sin\frac{2\pi x}{a} \\
Q_x &= Q_{x1}(y)\sin\frac{\pi x}{a} + Q_{x2}(y)\sin\frac{2\pi x}{a} \\
Q_y &= Q_{y0}(y) + Q_{y1}(y)\cos\frac{\pi x}{a} + Q_{y2}(y)\cos\frac{2\pi x}{a}
\end{aligned}\right\} \qquad (21)$$

However, the form of the force resultants for the pad include terms

$$\left.\begin{aligned}
Q_{xp} &= Q_{xp1}(y)\sin\frac{\pi x}{a} + Q_{xp2}(y)\sin\frac{2\pi x}{a} + \ldots + Q_{xp6}(y)\sin\frac{6\pi x}{a} \\
Q_{yp} &= Q_{yp0}(y) + Q_{yp1}(y)\cos\frac{\pi x}{a} + Q_{yp2}(y)\cos\frac{2\pi x}{a} + \ldots + Q_{yp6}(y)\cos\frac{6\pi x}{a} \\
N_{zp} &= N_{zp0}(y) + N_{zp1}(y)\cos\frac{\pi x}{a} + N_{zp2}(y)\cos\frac{2\pi x}{a} + \ldots + N_{zp6}(y)\cos\frac{6\pi x}{a}
\end{aligned}\right\} \qquad (22)$$

The first few of these coefficients are

$$
\left.\begin{aligned}
Q_{xp1} &= A_{44p}\left[\gamma_{xzp1} - \frac{3}{2} C_x\left(\frac{1}{2}\gamma_{xzp1}^3 + \gamma_{xzp1}\gamma_{xzp2}^2\right)\right] \\
Q_{xp2} &= A_{44p}\left[\gamma_{xzp2} - \frac{3}{2} C_x\left(\gamma_{xzp1}^2\gamma_{xzp2} + \frac{1}{2}\gamma_{xzp2}^3\right)\right] \\
Q_{yp0} &= A_{55p}\left[\gamma_{yzp0} + C_y\left(\gamma_{yzp0}^3 + \frac{3}{2}\gamma_{yzp0}\gamma_{yzp1}^2 + \frac{3}{2}\gamma_{yzp0}\gamma_{yzp2}^2 + \frac{3}{4}\gamma_{yzp1}^2\gamma_{yzp2}\right)\right] \\
Q_{yp1} &= A_{55p}\left[\gamma_{yzp1} + 3C_y\left(\gamma_{yzp0}^2\gamma_{yzp1} + \gamma_{yzp0}\gamma_{yzp1}\gamma_{yzp2} + \frac{1}{4}\gamma_{yzp1}^3 + \frac{1}{4}\gamma_{yzp1}\gamma_{yzp2}^2\right)\right] \\
Q_{yp2} &= A_{55p}\left[\gamma_{yzp2} + 3C_y\left(\gamma_{yzp0}^2\gamma_{yzp2} + \frac{1}{2}\gamma_{yzp0}\gamma_{yzp1}^2 + \frac{1}{4}\gamma_{yzp2}^3\right)\right] \\
N_{zp0} &= A_{33p}\left[\varepsilon_{z0} + C_z\left(\varepsilon_{z0}^3 + \frac{3}{2}\varepsilon_{z0}\varepsilon_{z1}^2 + \frac{3}{2}\varepsilon_{z0}\varepsilon_{z2}^2 + \frac{3}{4}\varepsilon_{z1}^2\varepsilon_{z2}\right)\right] \\
N_{zp1} &= A_{33p}\left[\varepsilon_{z1} + 3C_z\left(\varepsilon_{z0}^2\varepsilon_{z1} + \varepsilon_{z0}\varepsilon_{z1}\varepsilon_{z2} + \frac{1}{4}\varepsilon_{z1}^3 + \frac{1}{4}\varepsilon_{z1}\varepsilon_{z2}^2\right)\right] \\
N_{zp2} &= A_{33p}\left[\varepsilon_{z2} + 3C_z\left(\varepsilon_{z0}^2\varepsilon_{z2} + \frac{1}{2}\varepsilon_{z0}\varepsilon_{z1}^2 + \frac{1}{4}\varepsilon_{z2}^3\right)\right]
\end{aligned}\right\} \quad (23)
$$

where

$$
\gamma_{xzp1} = \left(u_1 - \frac{t}{2}\beta_{x1} - U_1\right)\Big/T - \frac{W_1}{2}\frac{\pi}{a}
$$

All the other pad strain coefficients are similar. (See eqs. (18).)

The virtual work of the system for a plate of width b and length a is

$$\delta\Pi = \int_0^b \int_0^a \Big(N_x \, \delta\varepsilon_x + N_y \, \delta\varepsilon_y + N_{xy} \, \delta\gamma_{xy} + Q_x \, \delta\gamma_{xz} + Q_y \, \delta\gamma_{yz}$$

$$+ M_x \, \delta\kappa_x + M_y \, \delta\kappa_y + M_{xy} \, \delta\kappa_{xy}$$

$$+ Q_{xp} \, \delta\gamma_{xzp} + Q_{yp} \, \delta\gamma_{yzp} + N_{zp} \, \delta\varepsilon_{zp} \Big) \, dx \, dy \tag{24}$$

Substituting for the force and moment resultants from equations (21) and equations (22), substituting for the strains from equations (18), and integrating over x results in

$$\delta\Pi = \frac{a}{2} \int_0^b \Big[N_{x1} \, \delta u_1 \frac{\pi}{a} + N_{x2} \, \delta u_2 \frac{2\pi}{a} + 2N_{y0} \, \delta v'_0 + N_{y1} \, \delta v'_1 + N_{y2} \, \delta v'_2$$

$$+ N_{xy1}\left(\delta u'_1 - \delta v_1 \frac{\pi}{a}\right) + N_{xy2}\left(\delta u'_2 - \delta v_2 \frac{2\pi}{a}\right) + Q_{x1}\left(\delta\beta_{x1} - \delta w_1 \frac{\pi}{a}\right)$$

$$+ Q_{x2}\left(\delta\beta_{x2} - \delta w_2 \frac{2\pi}{a}\right) + 2Q_{y0}(\delta\beta_{y0} + \delta w'_0) + Q_{y1}(\delta\beta_{y1} + \delta w'_1)$$

$$+ Q_{y2}(\delta\beta_{y2} + \delta w'_2) + M_{x1} \, \delta\beta_{x1} \frac{\pi}{a} + M_{x2} \, \delta\beta_{x2} \frac{2\pi}{a} + 2M_{y0} \, \delta\beta'_{y0}$$

$$+ M_{y1} \, \delta\beta'_{y1} + M_{y2} \, \delta\beta'_{y2} + M_{xy1}\left(\delta\beta'_{x1} - \delta\beta_{y1} \frac{\pi}{a}\right) + M_{xy2}\Big(\delta\beta'_{x2}$$

$$- \beta_{y2} \frac{2\pi}{a}\Big) + Q_{xp1} \frac{1}{T}\left(\delta u_1 - \frac{t}{2} \, \delta\beta_{x1}\right) + Q_{xp2} \frac{1}{T}\left(\delta u_2 - \frac{t}{2} \, \delta\beta_{x2}\right)$$

$$+ 2Q_{yp0} \frac{1}{T}\left(\delta v_0 - \frac{t}{2} \, \delta\beta_{y0}\right) + Q_{yp1} \frac{1}{T}\left(\delta v_1 - \frac{t}{2} \, \delta\beta_{y1}\right)$$

$$+ Q_{yp2} \frac{1}{T}\left(\delta v_2 - \frac{t}{2} \, \delta\beta_{y2}\right) + 2N_{zp0} \frac{1}{T} \, \delta w_0 + N_{zp1} \frac{1}{T} \, \delta w_1$$

$$+ N_{zp2} \frac{1}{T} \, \delta w_2 \Big] \, dy \tag{25}$$

Because of the integration properties of products of orthogonal functions, only the first few coefficients for the forces in the pad appear. Integration by parts results in

$$\delta\Pi = \frac{a}{2}\int_0^b \Bigg[\left(-N'_{xy1} + N_{x1}\frac{\pi}{a} + \frac{Q_{xp1}}{T}\right)\delta u_1 + \left(-N'_{xy2} + N_{x2}\frac{2\pi}{a} + \frac{Q_{xp2}}{T}\right)\delta u_2 + \left(-2N'_{y0} + \frac{2Q_{yp0}}{T}\right)\delta v_0 + \left(-N'_{y1} - N_{xy1}\frac{\pi}{a} + \frac{Q_{yp1}}{T}\right)\delta v_1$$

$$+ \left(-N'_{y2} - N_{xy2}\frac{2\pi}{a} + \frac{Q_{yp2}}{T}\right)\delta v_2 + \left(-2Q'_{y0} + \frac{2N_{zp0}}{T}\right)\delta w_0$$

$$+ \left(-Q'_{y1} - Q_{x1}\frac{\pi}{a} + \frac{N_{zp1}}{T}\right)\delta w_1 + \left(-Q'_{y2} - Q_{x2}\frac{2\pi}{a} + \frac{N_{zp2}}{T}\right)\delta w_2$$

$$+ \left(-M'_{xy1} + M_{x1}\frac{\pi}{a} + Q_{x1} - \frac{Q_{xp1}t}{2T}\right)\delta\beta_{x1} + \left(-M'_{xy2} + M_{x2}\frac{2\pi}{a} + Q_{x2} - \frac{Q_{xp2}t}{2T}\right)\delta\beta_{x2} + \left(-2M'_{y0} + 2Q_{y0} - \frac{Q_{yp0}t}{T}\right)\delta\beta_{y0} + \left(-M'_{y1} - M_{xy1}\frac{\pi}{a}\right.$$

$$\left. + Q_{y1} - \frac{Q_{yp1}t}{2T}\right)\delta\beta_{y1} + \left(-M'_{y2} - M_{xy2}\frac{2\pi}{a} + Q_{y2} - \frac{Q_{yp2}t}{2T}\right)\delta\beta_{y2}\Bigg] dy$$

$$+ \frac{a}{2}\Big(N_{xy1}\,\delta u_1 + N_{xy2}\,\delta u_2 + 2N_{y0}\,\delta v_0 + N_{y1}\,\delta v_1 + N_{y2}\,\delta v_2 + 2Q_{y0}\,\delta w_0$$

$$+ Q_{y1}\,\delta w_1 + Q_{y2}\,\delta w_2 + M_{xy1}\,\delta\beta_{x1} + M_{xy2}\,\delta\beta_{x2} + 2M_{y0}\,\delta\beta_{y0} + M_{y1}\,\delta\beta_{y1}$$

$$+ M_{y2}\,\delta\beta_{y2}\Big)_0^b$$

Thus, the principle of virtual work requires satisfaction of the following differential equations and choice of boundary conditions:

$$N'_{xy1} = \frac{\pi}{a}N_{x1} + \frac{Q_{xp1}}{T} \tag{26a}$$

$$N'_{xy2} = \frac{2\pi}{a}N_{x2} + \frac{Q_{xp2}}{T} \tag{26b}$$

$$N'_{y0} = \frac{Q_{yp0}}{T} \tag{26c}$$

$$N'_{y1} = \frac{\pi}{a} N_{xy1} + \frac{Q_{yp1}}{T} \tag{26d}$$

$$N'_{y2} = -\frac{2\pi}{a} N_{xy2} + \frac{Q_{yp2}}{T} \tag{26e}$$

$$M'_{xy1} = \frac{\pi}{a} M_{x1} + Q_{x1} - \frac{Q_{xp1}t}{2T} \tag{26f}$$

$$M'_{xy2} = \frac{2\pi}{a} M_{x2} + Q_{x2} - \frac{Q_{xp2}t}{2T} \tag{26g}$$

$$M'_{y0} = Q_{y0} - \frac{Q_{yp0}t}{2T} \tag{26h}$$

$$M'_{y1} = -\frac{\pi}{a} M_{xy1} + Q_{y1} - \frac{Q_{yp1}t}{2T} \tag{26i}$$

$$M'_{y2} = -\frac{2\pi}{a} M_{xy2} + Q_{y2} - \frac{Q_{yp2}t}{2T} \tag{26j}$$

$$Q'_{y0} = \frac{N_{zp0}}{T} \tag{26k}$$

$$Q'_{y1} = -Q_{x1} \frac{\pi}{a} + \frac{N_{zp1}}{T} \tag{26l}$$

$$Q'_{y2} = -Q_{x2} \frac{2\pi}{a} + \frac{N_{zp2}}{T} \tag{26m}$$

$$N_{xy1} \ \delta u_1 \Big|_0^b = 0 \qquad N_{xy2} \ \delta u_2 \Big|_0^b = 0 \tag{26n}$$

$$N_{y0} \ \delta v_0 \Big|_0^b = 0 \qquad N_{y1} \ \delta v_1 \Big|_0^b = 0 \qquad N_{y2} \ \delta v_2 \Big|_0^b = 0 \tag{26o}$$

$$M_{xy1} \ \delta\beta_{x1} \Big|_0^b = 0 \qquad M_{xy2} \ \delta\beta_{x2} \Big|_0^b = 0 \tag{26p}$$

$$M_{y0}\ \delta\beta_{y0}\Big|_0^b = 0 \qquad M_{y1}\ \delta\beta_{y1}\Big|_0^b = 0 \qquad M_{y2}\ \delta\beta_{y2}\Big|_0^b = 0 \tag{26q}$$

$$Q_{y0}\ \delta w_0\Big|_0^b = 0 \qquad Q_{y1}\ \delta w_1\Big|_0^b = 0 \qquad Q_{y2}\ \delta w_2\Big|_0^b = 0 \tag{26r}$$

The boundary condition assumed for the results presented in this paper is that the edges are free. Therefore, the stress and moment resultants are zero; so at $y = 0$ and b

$$N_{xy1} = N_{xy2} = N_{y0} = N_{y1} = N_{y2} = Q_{y0} = Q_{y1} = Q_{y2} = M_{xy1} = M_{xy2} = 0$$

$$M_{y0} = M_{y1} = M_{y2} = 0$$

The system of first-order, ordinary differential equations to be solved for this problem are presented in terms of the following 26 unknowns:

$$u_1,\ u_2,\ v_0,\ v_1,\ v_2,\ w_0,\ w_1,\ w_2,\ \beta_{x1},\ \beta_{x2},\ \beta_{y0},\ \beta_{y1},\ \beta_{y2}$$

$$N_{xy1},\ N_{xy2},\ N_{y0},\ N_{y1},\ N_{y2},\ Q_{y0},\ Q_{y1},\ Q_{y2},\ M_{xy1},\ M_{xy2},\ M_{y0},\ M_{y1},\ M_{y2}$$

Equations (26), which were obtained from the virtual work, present 13 of the differential equations used. The remaining 13 equations were obtained from the strain-displacement relations (eqs. (18)), the stress-strain relations (eqs. (19)) for the plate, and equations (21). These remaining equations are as follows:

$$u_1' = \frac{\pi}{a} v_1 + \frac{N_{xy1}}{A_{66}} \tag{27a}$$

$$u_2' = \frac{2\pi}{a} v_2 + \frac{N_{xy2}}{A_{66}} \tag{27b}$$

$$v_o' = \frac{N_{y0}}{A_{22}} - \left(\frac{A_{12}}{A_{22}}\right) \bar{u}_0 \tag{27c}$$

$$v_1' = \frac{N_{y1}}{A_{22}} - \left(\frac{A_{12}}{A_{22}}\right)\frac{\pi}{a}\, u_1 \qquad (27d)$$

$$v_2' = \frac{N_{y2}}{A_{22}} - \left(\frac{A_{12}}{A_{22}}\right)\frac{2\pi}{a}\, u_2 \qquad (27e)$$

$$w_0' = \frac{Q_{y0}}{A_{55}} - \beta_{y0} \qquad (27f)$$

$$w_1' = \frac{Q_{y1}}{A_{55}} - \beta_{y1} \qquad (27g)$$

$$w_2' = \frac{Q_{y2}}{A_{55}} - \beta_{y2} \qquad (27h)$$

$$\beta_{x1}' = \frac{M_{xy1}}{D_{66}} + \frac{\pi}{a}\,\beta_{y1} \qquad (27i)$$

$$\beta_{x2}' = \frac{M_{xy2}}{D_{66}} + \frac{2\pi}{a}\,\beta_{y2} \qquad (27j)$$

$$\beta_{y0}' = \frac{M_{y0}}{D_{22}} \qquad (27k)$$

$$\beta_{y1}' = \frac{M_{y1}}{D_{22}} - \left(\frac{D_{12}}{D_{22}}\right)\frac{\pi}{a}\,\beta_{x1} \qquad (27l)$$

$$\beta_{y2}' = \frac{M_{y2}}{D_{22}} - \left(\frac{D_{12}}{D_{22}}\right)\frac{2\pi}{a}\,\beta_{x2} \qquad (27m)$$

In addition, the stress-strain laws for the plate directly result in

$$\left.\begin{aligned} Q_{x1} &= A_{44}\left(\beta_{x1} - \frac{\pi}{a} w_1\right) \\ Q_{x2} &= A_{44}\left(\beta_{x2} - \frac{2\pi}{a} w_2\right) \end{aligned}\right\} \quad (28)$$

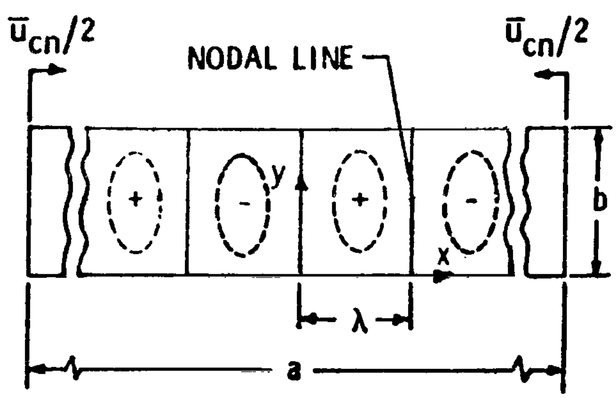
and, after some elementary manipulation, result in

$$\left.\begin{aligned} N_{x1} &= \left(A_{11} - \frac{A_{12}^2}{A_{22}}\right)\frac{\pi}{a} u_1 + \left(\frac{A_{12}}{A_{22}}\right) N_{y1} \\ N_{x2} &= \left(A_{11} - \frac{A_{12}^2}{A_{22}}\right)\frac{2\pi}{a} u_2 + \left(\frac{A_{12}}{A_{22}}\right) N_{y2} \\ M_{x1} &= \left(D_{11} - \frac{D_{12}^2}{D_{22}}\right)\frac{\pi}{a} \beta_{x1} + \left(\frac{D_{12}}{D_{22}}\right) M_{y1} \\ M_{x2} &= \left(D_{11} - \frac{D_{12}^2}{D_{22}}\right)\frac{2\pi}{a} \beta_{x2} + \left(\frac{D_{12}}{D_{22}}\right) M_{y2} \end{aligned}\right\} \quad (29)$$

Sample results for this problem are presented subsequently.

Long Plate in Compression Loaded Beyond Buckling

The analysis of the postbuckling behavior of long, rectangular orthotropic plates in longitudinal compression is presented in this section. The plate has a length a and a width b, and the long edges are supported. The following sketch of a buckled plate identifies the periodic deflection w of half-wavelength λ:



At buckling, the equations for w are satisfied by

$$w_1(y) \sin \frac{\pi x}{\lambda}$$

and the equations for u and v are satisfied by $-\bar{u}_{cn}\left(\frac{x}{a} - \frac{1}{2}\right)$ and $\bar{v}_0\left(\frac{y}{b} - \frac{1}{2}\right)$, respectively, where $\bar{u}_{cn}$ and $\bar{v}_0$ are constants. The study of the postbuckling behavior of a plate presented in reference 3, which solves for the unknowns in the order of their importance (perturbation method), indicates that the next group of terms to be considered are

$$u_2(y) \sin \frac{2\pi x}{\lambda}$$

$$v_2(y) \cos \frac{2\pi x}{\lambda}$$

The next term to be considered is

$$w_3(y) \sin \frac{3\pi x}{\lambda}$$

However, for up to three or four times the buckling load, the coefficient w_3 was found to be very small compared with w_1 for a long plate (ref. 3). Accordingly, the assumed displacements are

$$\left.\begin{aligned} u &= -\bar{u}_{cn}\left(\frac{x}{a} - \frac{1}{2}\right) + u_2(y) \sin \frac{2\pi x}{\lambda} \\ v &= v_0(y) + v_2(y) \cos \frac{2\pi x}{\lambda} \\ w &= w_1(y) \sin \frac{\pi x}{\lambda} \end{aligned}\right\} \tag{30}$$

Thus, the deflection w is assumed to be sinusoidally periodic with half-wavelength λ. The deflection w is exact at buckling, the displacements u and v are sinusoidally periodic with half-wavelength $\lambda/2$, and u has an extra term which is linear in the x-direction associated with the constant $\bar{u}_{cn}$ which is specified. No further approximations are made.

A Lévy-type solution is developed for the postbuckling behavior of long plates in compression. The neutral surface strains and curvatures as given by Von Kármán nonlinear plate theory are

$$\begin{aligned} \varepsilon_x &= u_{,x} + \frac{1}{2} w_{,x}^2 \\ &= -\frac{\bar{u}_{cn}}{a} + u_2 \frac{2\pi}{\lambda} \cos \frac{2\pi x}{\lambda} + \frac{1}{4}\left(\frac{\pi}{\lambda}\right)^2 w_1^{\,2}\left(1 + \cos \frac{2\pi x}{\lambda}\right) \end{aligned} \tag{31a}$$

$$\varepsilon_y = v_{,y} + \frac{1}{2} w_{1y}^2$$

$$= v_0' + v_2' \cos\frac{2\pi x}{\lambda} + \frac{1}{4} w_1'^2\left(1 - \cos\frac{2\pi x}{\lambda}\right) \tag{31b}$$

$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y}$$

$$= \left(u_2' - \frac{2\pi}{\lambda} v_2 + \frac{1}{2} w_1 w_1' \frac{\pi}{\lambda}\right) \sin\frac{2\pi x}{\lambda} \tag{31c}$$

$$\kappa_x = -w_{,xx} = \left(\frac{\pi}{\lambda}\right)^2 w_1 \sin\frac{\pi x}{\lambda} \tag{31d}$$

$$\kappa_y = -w_{,yy} = -w_1'' \sin\frac{\pi x}{\lambda} \tag{31e}$$

$$\kappa_{xy} = -2w_{,xy} = -2w_1' \frac{\pi}{a} \cos\frac{\pi x}{\lambda} \tag{31f}$$

From the stress-strain law for an orthotropic plate, the form of the stress resultants can be identified as follows:

$$\left.\begin{aligned}
N_x &= A_{11}\varepsilon_x + A_{12}\varepsilon_y = N_{x0}(y) + N_{x2}(y)\cos\frac{2\pi x}{\lambda} \\
N_y &= A_{22}\varepsilon_y + A_{12}\varepsilon_x = N_{y0}(y) + N_{y2}(y)\cos\frac{2\pi x}{\lambda} \\
N_{xy} &= A_{66}\gamma_{xy} = N_{xy2}(y)\sin\frac{2\pi x}{\lambda} \\
M_x &= D_{11}\kappa_x + D_{12}\kappa_y = M_{x1}\sin\frac{\pi x}{\lambda} \\
M_y &= D_{22}\kappa_y + D_{12}\kappa_x = M_{y1}\sin\frac{\pi x}{\lambda} \\
M_{xy} &= D_{66}\kappa_{xy} = M_{xy1}(y)\cos\frac{\pi x}{\lambda}
\end{aligned}\right\} \tag{32}$$

where

$$N_{x0} = A_{11}\left[-\frac{\bar{u}_{cn}}{a} + \frac{1}{4}\left(\frac{\pi}{\lambda}\right)^2 w_1^2\right] + A_{12}\left(v_0' + \frac{1}{4} w_1'^2\right) \tag{33a}$$

$$N_{x2} = A_{11}\left[\frac{2\pi}{\lambda} u_2 + \frac{1}{4}\left(\frac{\pi}{\lambda}\right)^2 w_1^{\,2}\right] + A_{12}\left(v_2' - \frac{1}{4} w_1'^{\,2}\right) \tag{33b}$$

$$N_{y0} = A_{22}\left(v_0' + \frac{1}{4} w_1'^{\,2}\right) + A_{12}\left[-\frac{\bar{u}_{cn}}{a} + \frac{1}{4}\left(\frac{\pi}{\lambda}\right)^2 w_1^{\,2}\right] \tag{33c}$$

$$N_{y2} = A_{22}\left(v_2' - \frac{1}{4} w_1'^{\,2}\right) + A_{12}\left[\frac{2\pi}{\lambda} u_2 + \frac{1}{4}\left(\frac{\pi}{\lambda}\right)^2 w_1^{\,2}\right] \tag{33d}$$

$$N_{xy2} = A_{66}\left(u_2' - \frac{2\pi}{\lambda} v_2 + \frac{1}{2}\frac{\pi}{\lambda} w_1 w_1'\right) \tag{33e}$$

$$M_{x1} = D_{11}\left(\frac{\pi}{\lambda}\right)^2 w_1 - D_{12} w_1'' \tag{33f}$$

$$M_{y1} = -D_{22} w_1'' + D_{12}\left(\frac{\pi}{\lambda}\right)^2 w_1 \tag{33g}$$

$$M_{xy1} = -2D_{66}\frac{\pi}{\lambda} w_1' \tag{33h}$$

The virtual work of the system is

$$\delta\Pi = \int_0^b \int_0^\lambda (N_x\ \delta\varepsilon_x + N_y\ \delta\varepsilon_y + N_{xy}\ \delta\gamma_{xy} + M_x\ \delta\kappa_x + M_y\ \delta\kappa_y + M_{xy}\ \delta\kappa_{xy})\ dx\ dy \tag{34}$$

Substitution of equations (31) and (32) into equation (34) and integration over x results in

$$\begin{aligned}\delta\Pi = \frac{\lambda}{2}\int_0^b \Bigg\{ & N_{x0}\left(\frac{\pi}{\lambda}\right)^2 w_1\ \delta w_1 + N_{x2}\left[\frac{2\pi}{\lambda}\ \delta u_2 + \frac{1}{2}\left(\frac{\pi}{\lambda}\right)^2 w_1\ \delta w_1\right] \\ & + 2N_{y0}\left(\delta v_0' + \frac{1}{2} w_1'\ \delta w_1'\right) + N_{y2}\left(\delta v_2' - \frac{1}{2} w_1'\ \delta w_1'\right) \\ & + N_{xy2}\left[\delta u_2' - \frac{2\pi}{\lambda}\ \delta v_2 + \frac{1}{2}\frac{\pi}{\lambda}\left(w_1\ \delta w_1' + w_1'\ \delta w_1\right)\right] \\ & + M_{x1}\left(\frac{\pi}{\lambda}\right)^2 \delta w_1 - M_{y1}\ \delta w_1'' - 2M_{xy1}\frac{\pi}{\lambda}\ \delta w_1' \Bigg\}\ dy\end{aligned} \tag{35}$$

Integration of equation (35) by parts results in

$$\delta\Pi = \frac{\lambda}{2}\int_0^b \Bigg\{\left(-N'_{xy2} + \frac{2\pi}{\lambda}N_{x2}\right)\delta u_2 - 2N'_{y0}\,\delta v_0$$

$$- \left(N'_{y2} + \frac{2\pi}{\lambda}N_{xy2}\right)\delta v_2$$

$$+ \left[V'_y + \left(\frac{\pi}{\lambda}\right)^2 M_{x1} + \frac{1}{2}\left(\frac{\pi}{\lambda}\right)^2 (2N_{x0} + N_{x2})\,w_1\right.$$

$$\left. + \frac{1}{2}\,\frac{\pi}{\lambda}\,N_{xy2}\beta\right]\delta w_1\Bigg\}\,dy$$

$$+ \frac{\lambda}{2}\Big[N_{xy2}\,\delta u_2 + 2N_{y0}\,\delta v_0 + N_{y2}\,\delta v_2 - M_{y1}\,\delta\beta - V_y\,\delta w_1\Big]_0^b \tag{36}$$

where, by definition,

$$\left.\begin{aligned} V_y &= -M'_{y1} + 2M_{xy1}\frac{\pi}{\lambda} - \frac{1}{2}(2N_{y0} - N_{y2})\beta - \frac{1}{2}\,\frac{\pi}{\lambda}\,N_{xy2}w_1 \\ \beta &= w'_1 \end{aligned}\right\} \tag{37}$$

Thus, the principle of virtual work requires satisfaction of the following differential equations and choice of boundary conditions:

$$\left.\begin{aligned} N'_{xy2} &= \frac{2\pi}{\lambda}N_{x2} \\ N'_{y0} &= 0 \\ N'_{y2} &= -\frac{2\pi}{\lambda}N_{xy2} \\ V'_y &= -\left(\frac{\pi}{\lambda}\right)^2 M_{x1} - \frac{1}{2}\left(\frac{\pi}{\lambda}\right)^2(2N_{x0} + N_{x2})w_1 - \frac{1}{2}\,\frac{\pi}{\lambda}\,N_{xy2}\beta \end{aligned}\right\} \tag{38}$$

$$N_{xy2}u_2\Big|_0^b = 0 \tag{39a}$$

$$N_{y0}v_0\Big|_0^b = 0 \tag{39b}$$

$$N_{y2}v_2\Big|_0^b = 0 \tag{39c}$$

$$M_{y1}\beta\Big|_0^b = 0 \tag{39d}$$

$$V_y w_1\Big|_0^b = 0 \tag{39e}$$

Since the variations must satisfy the same boundary conditions as the actual displacements, the variation signs were omitted in the boundary terms in equations (39). For the y boundaries, the simply supported, straightedge boundary condition requires that at $y = 0$ and b

$$w_1 = N_{y0} = u_2 = v_2 = M_{y1} = 0$$

The clamped, straightedge boundary condition requires that at $y = 0$ and b

$$w_1 = N_{y0} = u_2 = v_2 = \beta = 0$$

Thus, $N_{y0} = 0$ throughout the region for both boundary conditions considered. The simultaneous first-order differential equations to be solved can be written in terms of the unknowns

$$u_2,\ v_2,\ w_1,\ \beta,\ N_{xy2},\ N_{y2},\ M_{y1},\ V_y$$

with v_0 determined by integration after the other unknowns are determined. The following differential equations for these unknowns have been determined by manipulation of equations already stated:

$$u_2' = \frac{N_{xy2}}{A_{66}} + \frac{2\pi}{\lambda} v_2 - \frac{1}{2}\frac{\pi}{\lambda} w_1\beta \tag{40a}$$

$$v_2' = \frac{N_{y2}}{A_{22}} + \frac{1}{4}\beta^2 - \frac{A_{12}}{A_{22}}\left[u_2 \frac{2\pi}{\lambda} + \frac{1}{4}\left(\frac{\pi}{\lambda}\right)^2 w_1^2\right] \tag{40b}$$

$$\beta' = -\frac{M_{y1}}{D_{22}} + \frac{D_{11}}{D_{22}}\left(\frac{\pi}{\lambda}\right)^2 w_1 \tag{40c}$$

$$w_1' = \beta \tag{40d}$$

$$N_{xy2}' = \frac{2\pi}{\lambda}\left\{\frac{A_{12}}{A_{22}} N_{y2} + \left(A_{11} - \frac{A_{12}^2}{A_{22}}\right)\left[u_2 \frac{2\pi}{\lambda} + \frac{1}{4}\left(\frac{\pi}{\lambda}\right)^2 w_1^2\right]\right\} \tag{40e}$$

$$N'_{y2} = -\frac{2\pi}{\lambda} N_{xy2} \tag{40f}$$

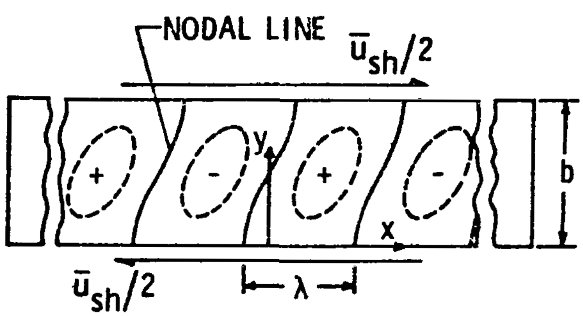
$$M'_{y1} = -V_y - 4D_{66}\left(\frac{\pi}{\lambda}\right)^2 \beta + \frac{1}{2} N_{y2}\beta - \frac{1}{2}\frac{\pi}{\lambda} N_{xy2}w_1 \tag{40g}$$

$$\begin{aligned} V'_y = &-\left(\frac{\pi}{\lambda}\right)^2\left[\frac{D_{12}}{D_{22}} M_{y1} + \left(D_{11} - \frac{D_{12}^2}{D_{22}}\right)\left(\frac{\pi}{\lambda}\right)^2 w_1\right] - \frac{1}{2}\frac{\pi}{\lambda} N_{xy2}\beta \\ &- \frac{1}{2}\left(\frac{\pi}{\lambda}\right)^2\left\{\frac{A_{12}}{A_{22}} N_{y2} + \left(A_{11} - \frac{A_{12}^2}{A_{22}}\right)\left[-\frac{2\bar{u}_{cn}}{a} + \frac{2\pi}{\lambda} u_2 + \frac{3}{4}\left(\frac{\pi}{\lambda}\right)^2 w_1^{\;2}\right]\right\} w_1 \end{aligned} \tag{40h}$$

Equations (40) are the eight ordinary differential equations to be solved for this problem. Sample results for this case are presented subsequently.

Long Plate in Shear Beyond Buckling Load

Analysis of the postbuckling behavior of long, rectangular orthotropic plates in inplane shear is presented in this section. The plate has a width b, and the long edges are supported. The following sketch of a buckled plate identifies the skewed periodic deflection w of half-wavelength λ:



At buckling (see ref. 4), the equations for w are satisfied by

$$w_s(y) \sin\frac{\pi x}{\lambda} + w_c(y) \cos\frac{\pi x}{\lambda}$$

The subscripts s and c refer to sine and cosine. The equations for u and v are satisfied by $\bar{u}_{sh}\left(\frac{y}{b} - \frac{1}{2}\right)$ and 0, respectively. The choice of trigonometric terms for the shear problem is not as straightforward as for the compression problem. Guided by experimental and theoretical results for shear which indicate that the buckle mode does not change much in the initial postbuckling range (as with compression), the postbuckling mode in shear is given by the buckling mode and a few elementary terms beyond. Accordingly, the assumed displacements for this problem are

$$\left.\begin{aligned}
u &= u_0(y) + u_s(y)\sin\frac{2\pi x}{\lambda} + u_c(y)\cos\frac{2\pi x}{\lambda}\\
v &= v_0(y) + v_s(y)\sin\frac{2\pi x}{\lambda} + v_c(y)\cos\frac{2\pi x}{\lambda}\\
w &= w_s(y)\sin\frac{\pi x}{\lambda} + w_c(y)\cos\frac{\pi x}{\lambda}
\end{aligned}\right\} \tag{41}$$

No further assumptions are made. Following steps similar to those for compression loading, the equations of equilibrium as obtained from virtual work are as follows:

$$\left.\begin{aligned}
N'_{xy0} &= 0\\
N'_{xys} &= \frac{2\pi}{\lambda} N_{xc}\\
N'_{xyc} &= -\frac{2\pi}{\lambda} N_{xs}\\
N'_{y0} &= 0\\
N'_{ys} &= \frac{2\pi}{\lambda} N_{xyc}\\
N'_{yc} &= -\frac{2\pi}{\lambda} N_{xys}\\
V'_{ys} &= -\left[\left(\frac{\pi}{\lambda}\right)^2 M_{xs} + \left(\frac{\pi}{\lambda}\right)^2\left(N_{x0}w_s - \frac{1}{2}N_{xs}w_c + \frac{1}{2}N_{xc}w_s\right) + \frac{\pi}{\lambda}\left(N_{xy0}w'_c\right.\right.\\
&\qquad \left.\left. + \frac{1}{2}N_{xys}w'_s + \frac{1}{2}N_{xyc}w'_c\right)\right]\\
V'_{yc} &= -\left[\left(\frac{\pi}{\lambda}\right)^2 M_{xc} + \left(\frac{\pi}{\lambda}\right)^2\left(N_{x0}w_c - \frac{1}{2}N_{xs}w_s - \frac{1}{2}N_{xc}w_c\right) - \frac{\pi}{\lambda}\left(N_{xy0}w'_s\right.\right.\\
&\qquad \left.\left. + \frac{1}{2}N_{xys}w'_c - \frac{1}{2}N_{xyc}w'_s\right)\right]
\end{aligned}\right\} \tag{42}$$

Differential equations obtained from the stress-strain law are as follows:

$$u'_0 = -\frac{1}{2}\frac{\pi}{\lambda}\left(w_s\beta_c - w_c\beta_s\right) + \frac{N_{xy0}}{A_{66}} \tag{43a}$$

$$u_s' = \frac{2\pi}{\lambda} v_c - \frac{1}{2} \frac{\pi}{\lambda}\left(w_s\beta_s - w_c\beta_c\right) + \frac{N_{xys}}{A_{66}} \tag{43b}$$

$$u_c' = -\frac{2\pi}{\lambda} v_s - \frac{1}{2} \frac{\pi}{\lambda}\left(w_s\beta_c + w_c\beta_s\right) + \frac{N_{xyc}}{A_{66}} \tag{43c}$$

$$v_0' = -\frac{1}{4}\left(\beta_s{}^2 + \beta_c{}^2\right) - \frac{A_{12}}{A_{22}} \frac{1}{4}\left(\frac{\pi}{\lambda}\right)^2\left(w_s{}^2 + w_c{}^2\right) + \frac{N_{y0}}{A_{22}} \tag{43d}$$

$$v_s' = -\frac{1}{2} \beta_s\beta_c + \left(\frac{A_{12}}{A_{22}}\right)\left[u_c \frac{2\pi}{\lambda} + \frac{1}{2}\left(\frac{\pi}{\lambda}\right)^2 w_s w_c\right] + \frac{N_{ys}}{A_{22}} \tag{43e}$$

$$v_c' = \frac{1}{4}\left(\beta_s{}^2 - \beta_c{}^2\right) - \left(\frac{A_{12}}{A_{22}}\right)\left[u_s \frac{2\pi}{\lambda} + \frac{1}{4}\left(\frac{\pi}{\lambda}\right)^2 \left(w_s{}^2 - w_c{}^2\right)\right] + \frac{N_{yc}}{A_{22}} \tag{43f}$$

$$\beta_s' = \frac{D_{12}}{D_{22}}\left(\frac{\pi}{\lambda}\right)^2 w_s - \frac{M_{ys}}{D_{22}} \tag{43g}$$

$$\beta_c' = \left(\frac{D_{12}}{D_{22}}\right)\left(\frac{\pi}{\lambda}\right)^2 w_c - \frac{M_{yc}}{D_{22}} \tag{43h}$$

By definition,

$$\left.\begin{aligned}
w_s' &= \beta_s \\
w_c' &= \beta_c \\
M_{ys}' &= V_{ys} - \frac{1}{2}\left(2N_{y0}\beta_s + N_{ys}\beta_c - N_{yc}\beta_s\right) - \frac{1}{2} \frac{\pi}{\lambda}\Big(-2N_{xy0}w_c + N_{xys}w_s \\
&\qquad + N_{xyc}w_c\Big) + \frac{2\pi}{\lambda} M_{xyc} \\
M_{yc}' &= V_{yc} - \frac{1}{2}\left(2N_{y0}\beta_c + N_{ys}\beta_s + N_{yc}\beta_c\right) - \frac{1}{2} \frac{\pi}{\lambda}\Big(-2N_{xy0}w_s - N_{xys}w_c \\
&\qquad + N_{xyc}w_s\Big) - \frac{2\pi}{\lambda} M_{xys}
\end{aligned}\right\} \tag{44}$$

Additional equations from the stress-strain law are as follows:

$$
\left.
\begin{aligned}
N_{x0} &= \left(A_{11} - \frac{A_{12}^2}{A_{22}}\right)\frac{1}{4}\left(\frac{\pi}{\lambda}\right)^2\left(w_s^{\,2} + w_c^{\,2}\right) + \left(\frac{A_{12}}{A_{22}}\right)N_{y0} \\
N_{xs} &= -\left(A_{11} - \frac{A_{12}^2}{A_{22}}\right)\left[u_c\,\frac{2\pi}{\lambda} + \frac{1}{2}\left(\frac{\pi}{\lambda}\right)^2 w_s w_c\right] + \left(\frac{A_{12}}{A_{22}}\right)N_{ys} \\
N_{xc} &= \left(A_{11} - \frac{A_{12}^2}{A_{22}}\right)\left[u_s\,\frac{2\pi}{\lambda} + \frac{1}{4}\left(\frac{\pi}{\lambda}\right)^2\left(w_s^{\,2} - w_c^{\,2}\right)\right] + \left(\frac{A_{12}}{A_{22}}\right)N_{yc} \\
M_{xs} &= \left(D_{11} - \frac{D_{12}^2}{D_{22}}\right)\left(\frac{\pi}{\lambda}\right)^2 w_s + \left(\frac{D_{12}}{D_{22}}\right)M_{ys} \\
M_{xc} &= \left(D_{11} - \frac{D_{12}^2}{D_{22}}\right)\left(\frac{\pi}{\lambda}\right)^2 w_c + \left(\frac{D_{12}}{D_{22}}\right)M_{yc} \\
M_{xys} &= \frac{2\pi}{\lambda}\,D_{66}\beta_c \\
M_{xyc} &= -\,\frac{2\pi}{\lambda}\,D_{66}\beta_s
\end{aligned}
\right\} \qquad (45)
$$

Equations (42), (43), and (44) (except for the fourth equation of (42) and the fourth equation of (43)) are the 18 ordinary differential equations to be solved for this problem. The fourth equation of (42) shows that N_{y0} is a constant or zero and the fourth equation of (43) permits a solution for v_0 when N_{y0} has been determined and when the rest of the equations have been solved. The applied shearing displacement $\bar{u}_{sh}$ enters the problem through boundary condition on u_0. Sample results for this case are presented subsequently.

SAMPLE RESULTS AND DISCUSSION

Sample results are presented for each of the three plate problems. The solution to each of the problems appears to be more efficient by this method than any other method, including finite elements and finite differences.

Thick Plate on Nonlinear Foundation

The thick plate considered has dimensions of 5 in. by 5 in. by 2 in., with extensional and flexural properties as follows:

$$A_{11} = A_{22} = 51\ 674 \text{ lb/in.} \qquad D_{11} = D_{22} = 17\ 225 \text{ in-lb}$$

$$A_{12} = 9301 \text{ lb/in.} \qquad D_{12} = 3100 \text{ in-lb}$$

$$A_{44} = A_{55} = A_{66} = 21\ 000 \text{ lb/in.} \qquad D_{66} = 7000 \text{ in-lb}$$

The pad is 0.16 in. thick with properties in the initial (linear) range given by: transverse extension modulus $E = 500$ psi and transverse shear modulus $G_1 = G_2 = 12.5$ psi. Therefore,

$$A_{33p} = 80 \text{ lb/in.}$$

$$A_{44p} = A_{55p} = 2 \text{ lb/in.}$$

The substructure is deformed such that

$$U = -0.0001\left(\frac{a}{2} - x\right)$$

$$V = 0.0003\left(\frac{b}{2} - y\right)$$

$$W = 0.12 \sin \frac{\pi y}{b} + 0.03 \cos \frac{\pi y}{b} \cos \frac{\pi x}{a} + 0.01 \sin \frac{2\pi y}{b} \cos \frac{2\pi x}{a}$$

where U, V, and W are measured in inches.

The solution was obtained by the present method for the thick plate on the pad and deformed substructure. The values of the moment M_y and stress resultant N_y at the center of the plate for various values of the pad cubic stiffness coefficients are shown in figure 3. When the coefficients are zero, the solution of the equations is exact. The moment doubles in the range shown as the cubic stiffness coefficients increase. The stress resultant increases by a factor of about 9 for the same range.

Long Plate in Compression Loaded Beyond Buckling

Equations of equilibrium have been derived in the section entitled "Applications" which may be solved directly by using the algorithm described in reference 2. For a given value of the applied compressive displacement $\bar{u}_{cn}$, and for prescribed values of the dimensions, the material properties, and the half-wavelength λ, the equations may be solved and the average load intensity N_{xav} may be determined as follows:

$$N_{xav} = -\frac{1}{2b\lambda}\int_0^b \int_0^{2\lambda} N_x \, dx \, dy$$

For finite plates, equilibrium paths can be determined in the postbuckling range which may be associated with buckling modes or changes in mode. In addition, from examination of the energy, these paths can be identified as being stable or unstable. For an infinitely long plate, consideration may be restricted to wavelength, and, by comparison with results for finite plates, the wavelength of the stable path corresponds to the wavelength that gives minimum energy. An example of a path which is not of interest beyond buckling is one that gives zero deflections. The load of interest for postbuckling studies is the one that corresponds to an applied displacement that is in the postbuckling range and on the equilibrium path corresponding to the wavelength of interest. For this problem, these conditions are satisfied if (1) the applied displacement is larger than its buckling value, (2) the results give nonzero deflections, and (3) various half-wavelengths are tried until a minimum energy is obtained.

Characteristic curves are plotted in figure 4 for the compression of long, isotropic plates and long ±45° laminated composite plates with quasi-isotropic lay up. The average stress intensity coefficient is plotted as a function of the applied displacement coefficient for the long edges held straight (inplane) and simply supported or clamped. The isotropic curves apply to isotropic metal or composites with an isotropic lay up. The ±45° laminate curves apply to graphite-epoxy filamentary material with properties given by the following dimensionless quantities:

$$\frac{D_{12} + 2D_{66}}{\sqrt{D_{11}D_{22}}} = 2.28$$

$$\frac{A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}}{2A_{66}\sqrt{A_{11}A_{22}}} = -0.431$$

For the isotropic plate, both these quantities are unity, and for both the isotropic and ±45° laminate,

$$\frac{A_{22}D_{11}}{A_{11}D_{22}} = 1$$

The slope of the load-displacement curve is a measure of the overall plate stiffness. As shown in figure 4, this curve is a straight line with a slope equal to one prior to buckling. After buckling, this line changes slope according to the boundary conditions and according to $\frac{D_{12} + 2D_{66}}{\sqrt{D_{11}D_{22}}}$ and $\frac{A_{22}D_{11}}{A_{11}D_{22}}$. (See ref. 5.) The ±45° laminate has a slightly lower postbuckling slope than the isotropic and, therefore, has a slightly lower postbuckling stiffness.

The only assumptions made in this analysis are that Von Kármán theory applies and that the deflection w and the displacements u and v are sinusoidally periodic with half-wavelengths λ and $\lambda/2$, respectively. Comparisons with experiment show that Von Kármán theory is satisfactory. The deflection pattern is exact at buckling, and, by comparison with other work, this method should be accurate to three times the buckling load or higher. The results given herein for simply supported edges are the same as those presented in reference 6. The results given herein for clamped edges have not been presented previously. The curves for the clamped edges are similar to the curves for the simply supported edges, except for higher slopes in the postbuckling range. This difference indicates that clamped plates are stiffer in the postbuckling range.

Long Plate in Shear Loaded Beyond Buckling

For the plate loaded in shear, equations of equilibrium have been derived in the section entitled "Applications" which may be solved directly by using the algorithm described in reference 2. For a given value of the applied shearing displacement $\bar{u}_{sh}$, and for prescribed values of the dimensions, the material properties, and the half-wavelength λ, the equations may be solved and the average shear load intensity may be determined as follows:

$$N_{xyav} = \frac{1}{2b\lambda}\int_0^b \int_0^{2\lambda} N_{xy}\ dx\ dy$$

As with the long plate loaded in compression, to get a postbuckling solution, (1) the applied displacement must be larger than its buckling value, (2) the results must be on the equilibrium path where the deflections are nonzero, and (3) various wavelengths must be tried until the wavelength is obtained which corresponds to minimum energy. Characteristic curves are plotted in figure 5 for the shearing of long isotropic plates and long ±45° laminated composite plates with quasi-isotropic lay up. The average stress intensity coefficient is plotted as a function of the applied displacement coefficient for the long edges held straight (inplane) and simply supported, or clamped. In addition to the parameters mentioned for compression (see ref. 6), the shear case depends on the value of $\frac{A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}}{2A_{66}\sqrt{A_{11}A_{22}}}$. Comparing the postbuckling slopes shown in figures 4 and 5 shows that the stiffness of plates buckled in shear is higher than the stiffness of plates buckled in compression.

Again, the only assumptions made in this analysis are that Von Kármán theory applies and that the deflection w and the displacements u and v are sinusoidally periodic with half-wavelengths λ and $\lambda/2$, respectively. Comparisons with experiment show that Von Kármán theory is satisfactory for shear loading. The deflection pattern and the deformation pattern are skewed, unlike the compression case, and the deflection pattern is exact at buckling. The method should also be accurate to at least three times the buckling load. The results given herein for simply supported edges are the same as those presented in reference 6. The results given herein for clamped edges have not been presented previously. As with the plates loaded in compression, the curves for clamped edges are similar to the curves for simply supported edges, except for higher slopes in the postbuckling range. The higher slopes indicate that clamped plates are stiffer in the postbuckling range.

CONCLUDING REMARKS

A solution procedure is presented in this paper for three nonlinear plate problems. This procedure is demonstrated by deriving one-dimensional equations in the y-direction using a one-dimensional trigonometric approximation in the x-direction for three different (two-dimensional) plate problems and by presenting sample results for these problems. For a plate on a nonlinear foundation, the solution is a nonlinear extension of a Fourier series solution of a linear differential equation in which the loading terms are also expanded in a Fourier series to get an exact solution, term by term. The postbuckling problems are solved by a nonlinear extension of the (linear) Lévy-type solution for long plates. The trigonometric terms are exact for the linear range of stiffness for the plate on the foundation. The trigonometric terms are exact for initial buckling in the postbuckling problems. The sample-problem results for the plate on a nonlinear foundation show that with an increase in nonlinear stiffness of the foundation, the neutral-surface direct stress resultant is affected much more than the corresponding moment. Postbuckling plate problems studied include isotropic plates and a ±45° filamentary composite laminated plate. The ±45° laminated plate is not as stiff as the isotropic plate in the postbuckling range for the compression or shear loadings. There is not much difference for compression, but they differ considerably for shear. For both plates and both loadings, the postbuckling stiffness is higher for clamped edges than it is for simply supported edges.

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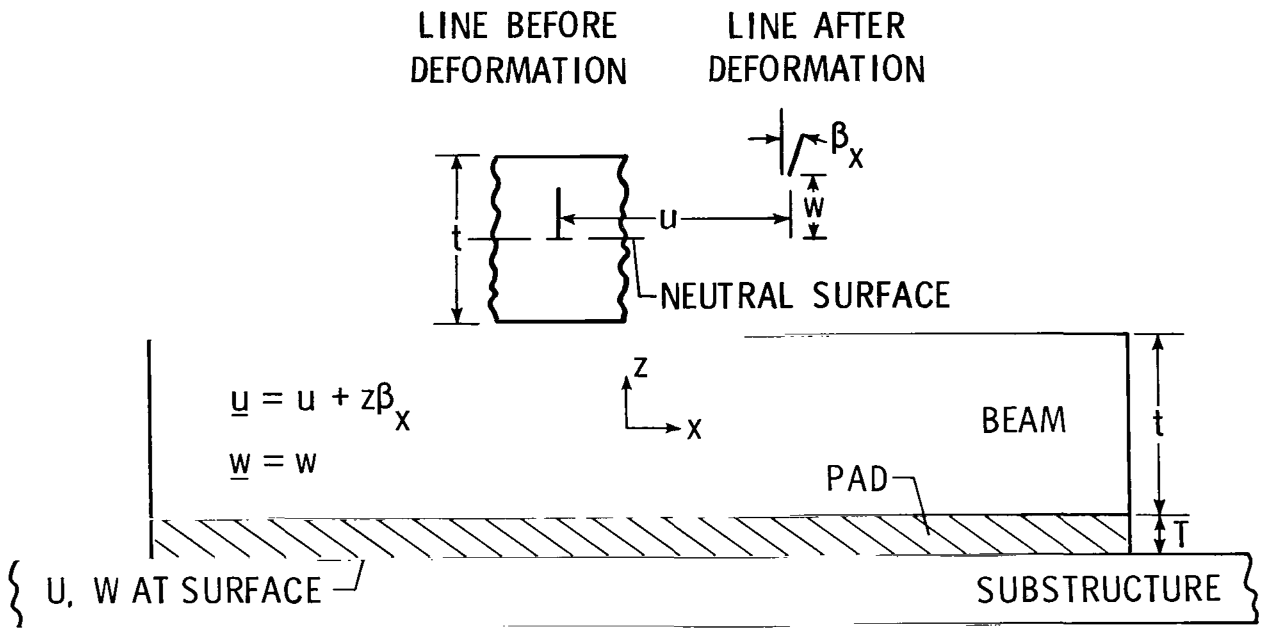

STRAINS IN BEAM

$$\varepsilon_x = u'$$
$$\kappa_x = \beta_x'$$
$$\gamma_{xz} = \beta_x + w'$$

STRAINS IN PAD

$$\varepsilon_{zp} = (w - W)/T$$
$$\gamma_{xzp} = \left(u - \frac{t}{2}\beta_x - U\right)\Big/T + W'/2$$

STRESS-STRAIN LAW FOR PAD

$$N_{zp} = E_p T\left(\varepsilon_{zp} + C_z \varepsilon_{zp}^3\right)$$
$$Q_{xp} = G_p T\left(\gamma_{xzp} + C_x \gamma_{xzp}^3\right)$$

Figure 1.- Analytical model of deep beam attached to a substructure by a pad with nonlinear properties.

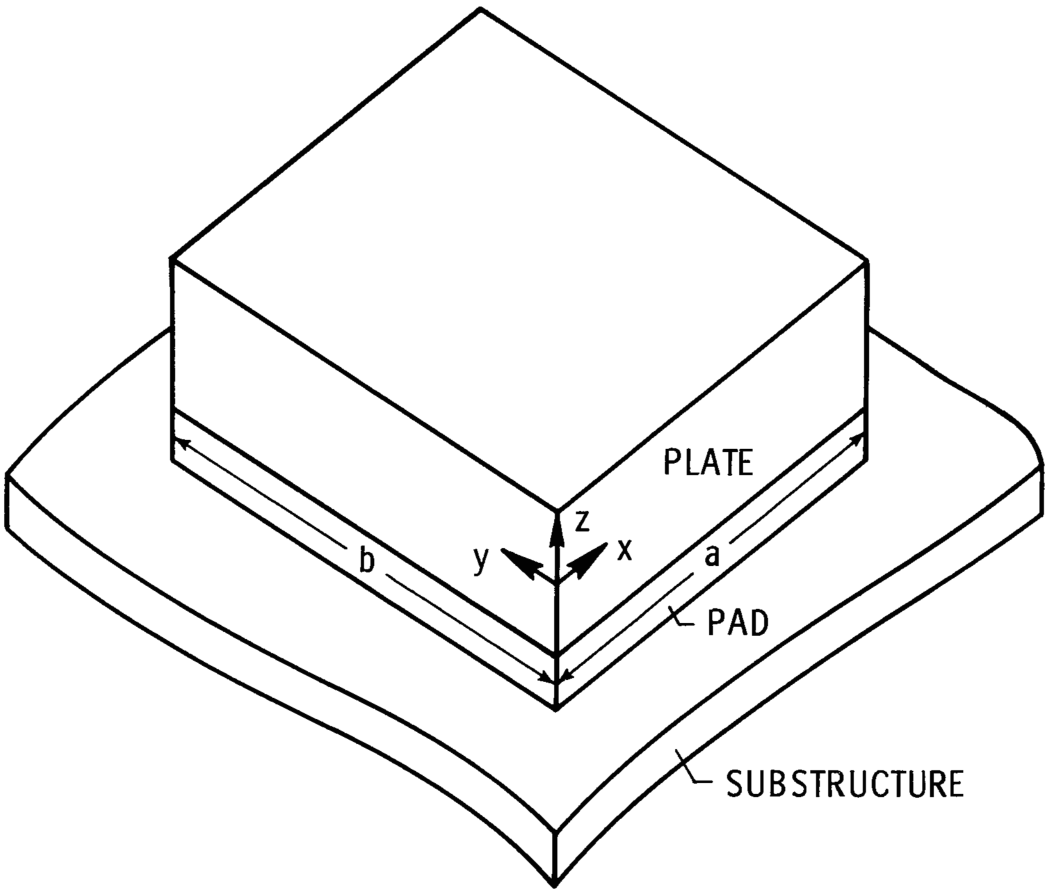

Figure 2.- Coordinates for a thick plate attached to a substructure by a pad with nonlinear properties.

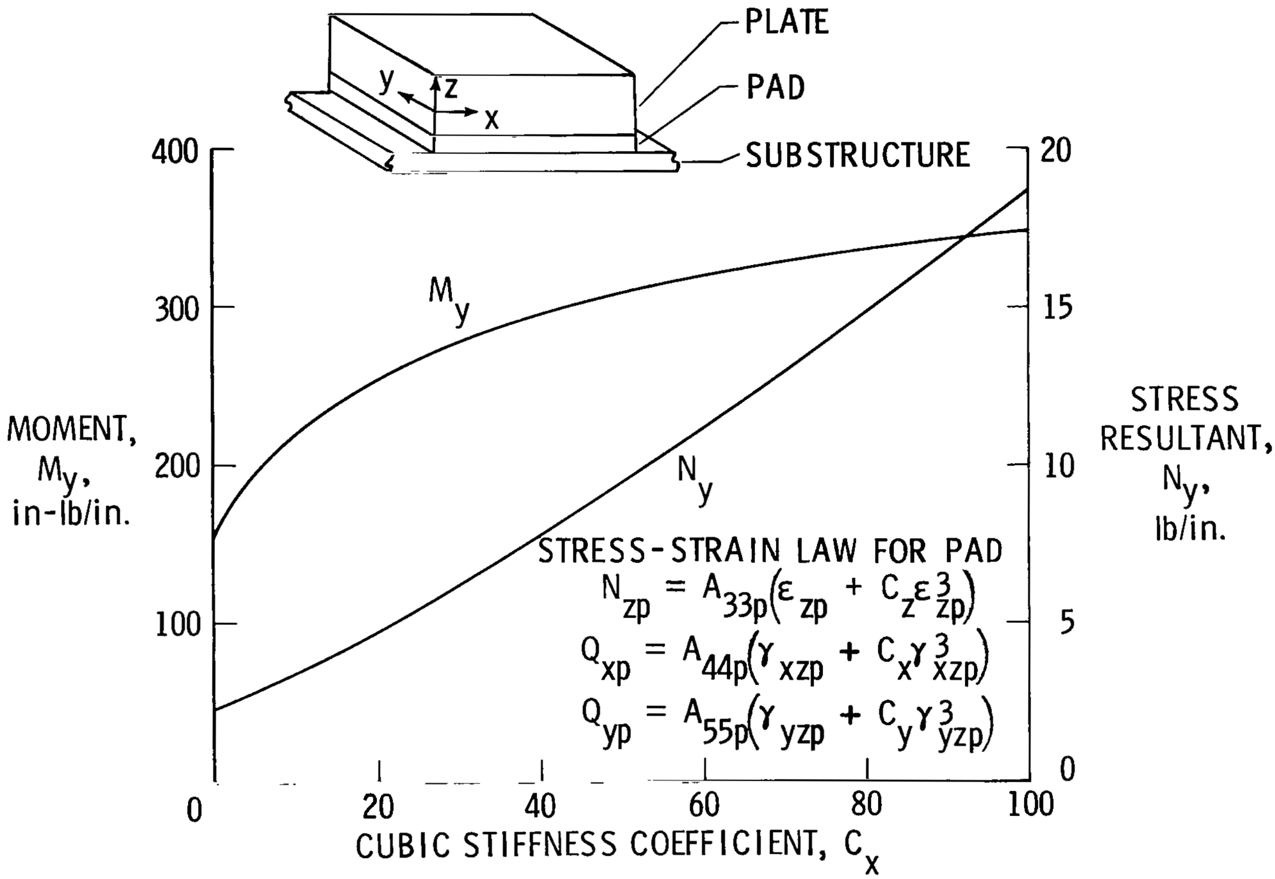

Figure 3.- Moment and stress resultant at center of plate with increase in nonlinear stiffness of pad. $C_y = C_x$; $C_z = 1.5C_x$.

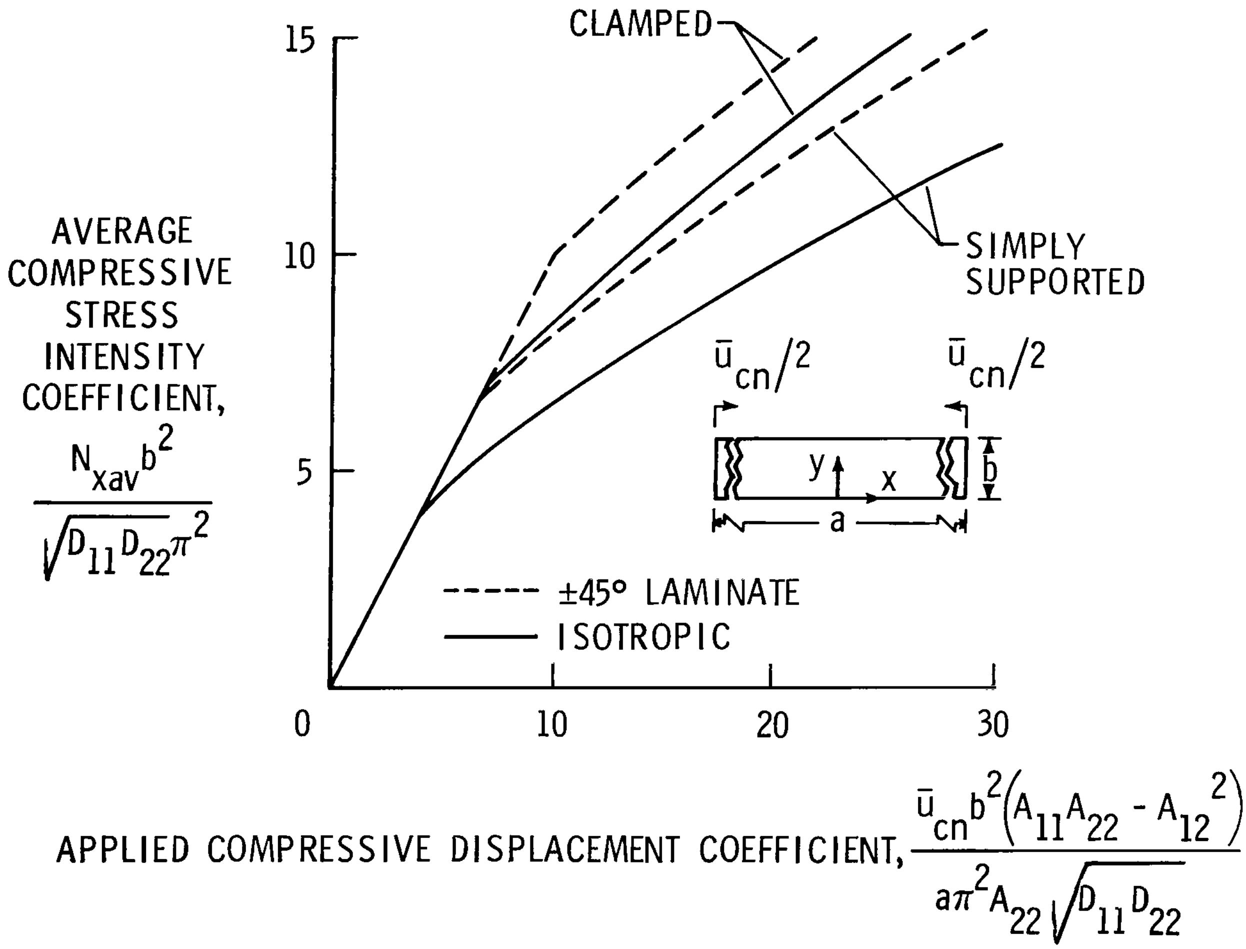

Figure 4.- Characteristic curves for plates in compression.

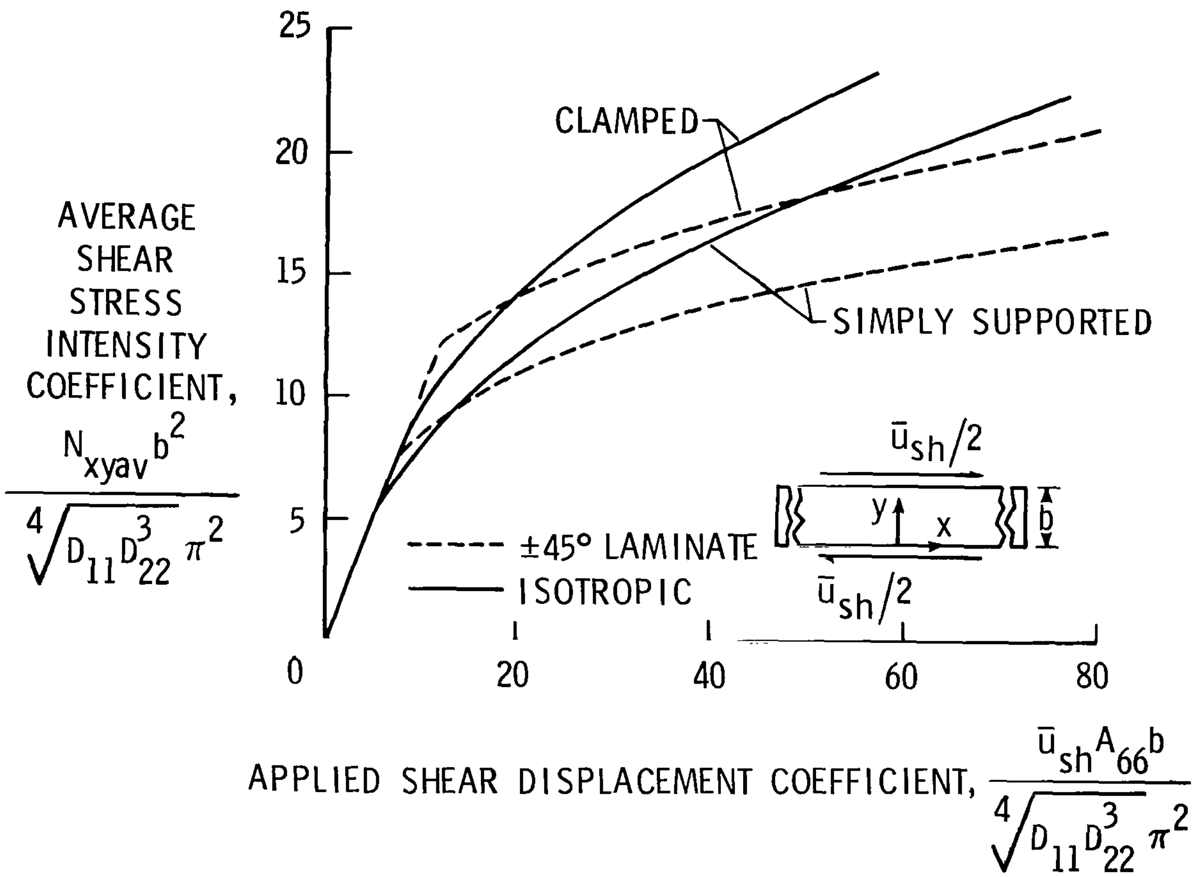

Figure 5.- Characteristic curves for plates in shear.

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16. Abstract

Approximate solutions are presented in this paper for three nonlinear orthotropic plate problems: (1) a thick plate attached to a pad having nonlinear material properties which, in turn, is attached to a substructure which is then deformed; (2) a long plate loaded in inplane longitudinal compression beyond its buckling load; and (3) a long plate loaded in inplane shear beyond its buckling load. For all three problems, the two-dimensional plate equations are reduced to one-dimensional equations in the y-direction by using a one-dimensional trigonometric approximation in the x-direction. Each problem uses different trigonometric terms. Solutions are obtained using an existing algorithm for simultaneous, first-order, nonlinear, ordinary differential equations subject to two-point boundary conditions. For each problem, reasons are discussed for choosing the trigonometric terms, ordinary differential equations are derived to determine the variable coefficients of the trigonometric terms, and sample results are presented.

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