DEPARTMENT OF ELECTRICAL ENGINEERING
SCHOOL OF ENGINEERING
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA

INVESTIGATION OF ANALYTICAL METHODS FOR EFFICIENT PARTITIONING OF ON-BOARD PROCESSING FUNCTIONS FOR REMOTE SENSING APPLICATIONS

By

David Livingston

and

John W. Stoughton, Principal Investigator

Final Report
For the period September 5, 1980 to September 5, 1981

Prepared for
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia

Under
Research Grant NAS1-15648
Task Authorization No. 43
James A. Dorst, Technical Monitor
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INTRODUCTION

This final report details the results developed to fulfill the requirements for Task NAS1-15648-43 [1]. It is divided into four sections:

Section I Risk Decomposition

Section II Universal Algebras

Section III Unary Functions

Section IV Current Topics

Section I generalizes the results reported in the previous report and presents some new examples based on ring theory. Section II further generalizes the use of lattice techniques to any system with algebraic structure. The third section deals with a specific algebra called unary functions. Areas which are currently under consideration are briefly described in the final section. It is recommended that the previous report be reviewed since the functions for this report are described therein.

SECTION I. RING DECOMPOSITION

The decomposition of multiplication and addition processes may be obtained simultaneously if the algebraic structure under consideration has the form of a ring. As in group decomposition, it is necessary to obtain the lattice of substructures. The substructures of rings are called ideals and are defined as follows:

Let $\{R, +, \cdot\}$ be a ring where R is a set which is closed under addition and multiplication. $\{I, +, \cdot\}$ is an ideal if and only if:

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1) $I \leq R$,

2) $\{I, +\}$ is a subgroup of $\{R, +\}$, and

3) For all $a \in I$ and all $b \in R$, $a \cdot b = c$, where $c \in I$.

The set of all ideals belonging to a ring form a lattice under the partial ordering of inclusion. As in the case of groups, equivalence classes or partitions may be obtained by finding the additive cosets of each ideal. A lattice of the resulting partitions may be constructed from which the decomposition structure and element representations can be obtained.

Example 1. Integers Modulo Eight

The addition and multiplication tables are:

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

and

•	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

From these tables the following ideals may be found:

1) The ring itself

2)

+	0	2	4	6
0	0	2	4	6
2	2	4	6	0
4	4	6	0	2
6	6	0	2	4

•	0	2	4	6
0	0	0	0	0
2	0	4	0	4
4	0	0	0	0
6	0	4	0	4

3)

+	0	4
0	0	4
4	4	0

•	0	4
0	0	0
4	0	0

4)

+	0
0	0

•	0
0	0

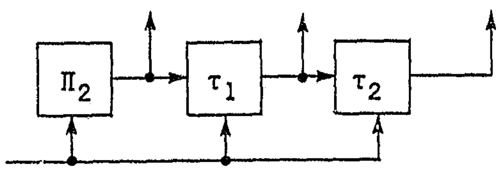
The partitions associated with each ideal and its corresponding lattice are:

1)	$\{\overline{0, 1, 3, 4, 5, 6, 7}\}$	$= \Pi_1$	Π_1	
2)	$\{\overline{0, 2, 4, 6}; \overline{3, 5, 7}\}$	$= \Pi_2$	Π_2	τ_1
3)	$\{\overline{0, 4}; \overline{1, 5}; \overline{2, 6}; \overline{3, 7}\}$	$= \Pi_3$	Π_3	τ_2
4)	$\{\overline{0}; \overline{1}; \overline{2}; \overline{3}; \overline{4}; \overline{5}; \overline{6}; \overline{7}\}$	$= \Pi\phi$	$\Pi\phi$	

Two partitions τ_1 and τ_2 may be created such that $\Pi_2 \cdot \tau_1 \cdot \tau_2 + \Pi_3 \cdot \tau_2 = \Pi\phi$.

$\tau_1 = \{\overline{0, 1, 4, 5}; \overline{2, 3, 6, 7}\}$ and $\tau_2 = \{\overline{0, 1, 2, 3}; \overline{4, 5, 6, 7}\}$.

By choosing τ_1 and τ_2 as such, a pipeline or serial structure is obtained:



This decomposition structure may be used for both the addition and multiplication processes differing only in the realization of each block. Element representations may be chosen such that they are the equivalent binary representations. Given the decomposition and representations, the ring structure may be realized in hardware using standard procedures.

Polynomial arithmetic can also be decomposed, as in the proceeding example, if a ring structure is evident.

Example 2. Ring of Polynomials Modulo $x^2 + 2$ over Ground Field Modulo 3

The addition and multiplication tables are:

+	0	1	2	x	2x	x+1	x+2	2x+2	2x+1
0	0	1	2	x	2x	x+1	x+2	2x+2	2x+1
1	1	2	0	x+1	2x+1	x+2	x	2x	2x+2
2	2	0	1	x+2	2x+2	x	x+1	2x+1	2x
x	x	x+1	x+2	2x	0	2x+1	2x+2	2	1
2x	2x	2x+1	2x+2	0	x	1	2	x+2	x+1
x+1	x+1	x+2	x	2x+1	1	2x+2	2x	0	2
x+2	x+2	x	x+1	2x+2	2	2x	2x+1	1	0
2x+2	2x+2	2x	2x+1	2	x+2	0	1	x+1	x
2x+1	2x+1	2x+2	2x	1	x+1	2	0	x	x+2

and

•	0	1	2	x	2x	x+1	x+2	2x+2	2x+1
0	0	0	0	0	0	0	0	0	0
1	0	1	2	x	2x	x+1	x+2	2x+2	2x+1
2	0	2	1	2x	x	2x+2	2x+1	x+1	x+2
x	0	x	2x	1	2	x+1	2x+1	2x+2	x+2
2X	0	2x	x	2	1	2x+2	x+2	x+1	2x+1
x+1	0	x+1	2x+2	x+1	2x+2	2x+2	0	x+1	0
x+2	0	x+2	2x+1	2x+1	x+2	0	x+2	0	2x+1
2x+2	0	2x+2	x+1	2x+2	x+1	x+1	0	2x+2	0
2x+1	0	2x+1	x+2	x+2	2x+1	0	2x+1	0	x+2

The ideals are:

1) The ring itself

2)

+	0	x+1	2x+2
0	0	x+1	2x+2
x+1	x+1	2x+2	0
2x+2	2x+2	0	x+1

•	0	x+1	2x+2
0	0	0	0
x+1	0	2x+2	x+1
2x+2	0	x+1	2x+2

3)

+	0	x+2	2x+1
0	0	x+2	2x+1
x+2	x+2	2x+1	0
2x+1	2x+1	0	x+2

•	0	x+2	2x+1
0	0	0	0
x+2	0	x+2	2x+1
2x+1	0	2x+1	x+2

4)

+	0
0	0

•	0
0	0

Note that the two non-trivial ideals are generated by the linear factors of the modulus, $x^2 + 2$.

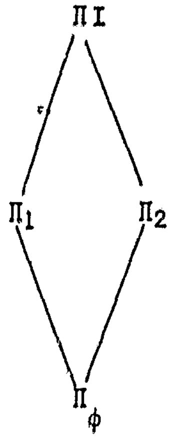
The partitions and associated lattice corresponding to the ideals are:

1) $\{\overline{0,\ 1,\ 2,\ x,\ 2x,\ x+1,\ x+2,\ 2x+1,\ 2x+2}\} = \Pi I$

2) $\{\overline{0,\ x+1,\ 2x+2};\ \overline{1,\ 2x,\ x+2};\ \overline{2,\ x,\ 2x+1}\} = \Pi_1$

3) $\{\overline{0,\ x+2,\ 2x+1};\ \overline{1,x,\ 2x+2};\ \overline{2,\ 2x,\ x+1}\} = \Pi_2$

4) $\{\overline{0};\ \overline{1};\ \overline{2};\ \overline{2x};\ \overline{x+1};\ \overline{x+2};\ \overline{2x+1};\ \overline{2x+2};\ \overline{2x+1};\ \overline{2x+2}\} = \Pi_\phi$



This lattice indicates a parallel decomposition. The element representations may be made as follows:

$$\Pi_1 = \{\overset{0}{\overline{0,\ x+1,\ 2x+2}};\ \overset{1}{\overline{1,\ 2x,\ x+2}};\ \overset{2}{\overline{2,\ x,\ 2x+1}}\}$$

$$\Pi_2 = \{\overset{0}{\overline{0,\ x+2,\ 2x+1}};\ \overset{1}{\overline{1,\ x,\ 2x+2}};\ \overset{2}{\overline{2,\ 2x,\ x+1}}\}$$

It is important to note the implication of this representation. Polynomial arithmetic can be performed using simple integer arithmetic. As an example, consider $|(2x)\cdot(2x+2)|_{x^2+2}$.

$$\begin{aligned} |(2x)\cdot(2x+2)|_{x^2+2} &= |\cdot\ 2x2x + 2x\cdot 2|_{x^2+2} \\ &= |4x^2 + 4x|_{x^2+2} \\ &= |x^2 + x|_{x^2+2} \\ &= x+1. \end{aligned}$$

This result may also be obtained by using the representations

$$|(2x).\ (2x+2)|_{x^2+2})| \rightarrow (1,2)\cdot(0,1) = (1\cdot 0, 2\cdot 1) = (0,2)$$

$$(0,2) \rightarrow x+1.$$

In other terms, multiplication of polynomials modulo another polynomial can be considered a cyclic convolution. Interpreting the results of the lattice analysis, a cyclic convolution may be performed by a term-by-term multiplication of two integer sequences representing the polynomials. This is equivalent to performing a cyclic convolution of two sequences by transforming, multiplying and inverse transforming the sequences using a method such as the finite Fourier transform. It is interesting to note that this result was obtained quite systematically using lattice analysis.

SECTION II. UNIVERSAL ALGEBRAS

It has been shown that Group [1] and ring decompositions may be obtained in a systematic fashion using lattice analysis. This method may be generalized to any algebra belonging to a class following the rules of universal algebra [2].

An algebra may be defined as a pair $\{S,F\}$, where S is a nonempty set of elements and F is a set of operations which map a Cartesian power of S into S, that is,

$$F_a: \; S^{n(a)} \rightarrow S, \text{ for all } F_a \varepsilon F.$$

Groups, rings, and lattices are all examples of algebras. A field is not an algebra under this definition since the inverse function is not defined for zero.

A subalgebra may be defined as a pair $\{T,F,\}$, where T<S and F_a: $T^{n(a)} \rightarrow T$ for all $F_a \varepsilon F$. That is, the set T is closed under all F_a and T is a subset of S. The importance of subalgebras to decomposition is evident by the following theorem and corollary from Birkhoff [2].

<u>Theorem 1</u>: Any union of subalgebras of an algebra is a subalgebra, and an algebra is a subalgebra of itself.

Corollary 1: The subalgebras of an algebra form a complete lattice.

Corollary 1 provides the vehicle by which arithmetic systems may be decomposed in a systematic fashion. The structure of the system is first examined to determine if it forms an algebra. All subalgebras are then found and a lattice of the subalgebras is constructed. By augmenting each subalgebra with its cosets a lattice of partitions is formed from which the possible decompositions are obtained.

SECTION III. UNARY FUNCTIONS

Unary functions are of particular interest, especially with the availability of new memory technologies which allow the implementation of unary functions by look-up tables stored in read only memories. If the range of a unary function is a subset of its domain, then the function forms an algebra and readily lends itself to possible decompositions. The value of decomposing a unary function, in terms of look-up tables, is a reduction of the total memory required to implement the table.

Example: x+1 Modulo 12

x	x+1
0	1
1	2
2	3
3	4
4	5
5	6

x	x+1
6	7
7	8
8	9
9	10
10	11
11	0

This table forms an algebra, but there are no apparent subalgebra, but there are no apparent subalgebras for the function x+1. If the table is treated as a sequential machine, the partitions may be readily obtained.

Partitions:

$\Pi_I = \{\overline{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}\}$

$\Pi_1 = \{\overline{0, 2, 4, 6, 8, 10}; \overline{1, 3, 5, 7, 9, 11}\}$

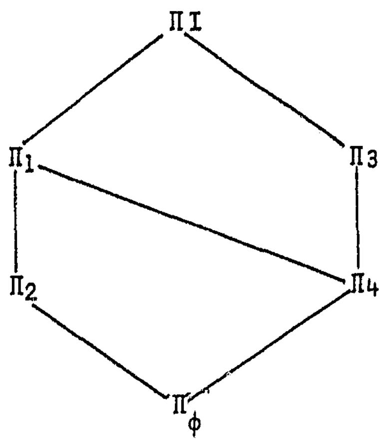
$\Pi_2 = \{\overline{0, 4, 8}; \overline{1, 5, 9}; \overline{2, 6, 10}; \overline{3, 7, 11}\}$

$\Pi_3 = \{\overline{0, 3, 6, 9}; \overline{1, 4, 7, 10}; \overline{2, 5, 8, 11}\}$

$\Pi_4 = \{\overline{0, 6}; \overline{1, 7}; \overline{2, 8}; \overline{3, 9}; \overline{4, 10}; \overline{5, 11}\}$

The $\Pi_\phi = \{\overline{0}; \overline{1}; \overline{2}; \overline{3}; \overline{4}; \overline{5}; \overline{6}; \overline{7}; \overline{8}; \overline{9}; \overline{10}; \overline{11}\}$

The Lattice of Partitions:



$\Pi_3 \cdot \Pi_2 = \Pi_\phi$, suggesting a parallel decomposition.

The element representations may be made:

$$\Pi_2 = \{\overset{0}{\overline{0, 4, 8}}; \overset{1}{\overline{1, 5, 9}}; \overset{2}{\overline{2, 6, 10}}; \overset{3}{\overline{3, 7, 11}}\}$$

$$\Pi_3 = \{\overset{0}{\overline{0, 3, 6, 9}}; \overset{1}{\overline{1, 4, 7, 10}}; \overset{3}{\overline{2, 5, 8, 11}}\}.$$

Thus the original function (table) may be decomposed into two functions (tables):

Y	F(Y)
0	1
1	2
2	3
3	0

Z	F(Z)
0	1
1	2
2	0

Notice the total number of memory locations needed is 4 + 3 = 7 as opposed to 12. In this decomposition, the transformations are systematic:

$Y = |\, x \,|_4$ and $Z = |\, x \,|_3$.

If the transformations are not systematic the decomposition may be more difficult to implement than the original function, unless it is a part of a larger system using the decomposed representations. This is the biggest disadvantage to using this method for unary functions.

SECTION IV. CURRENT TOPICS

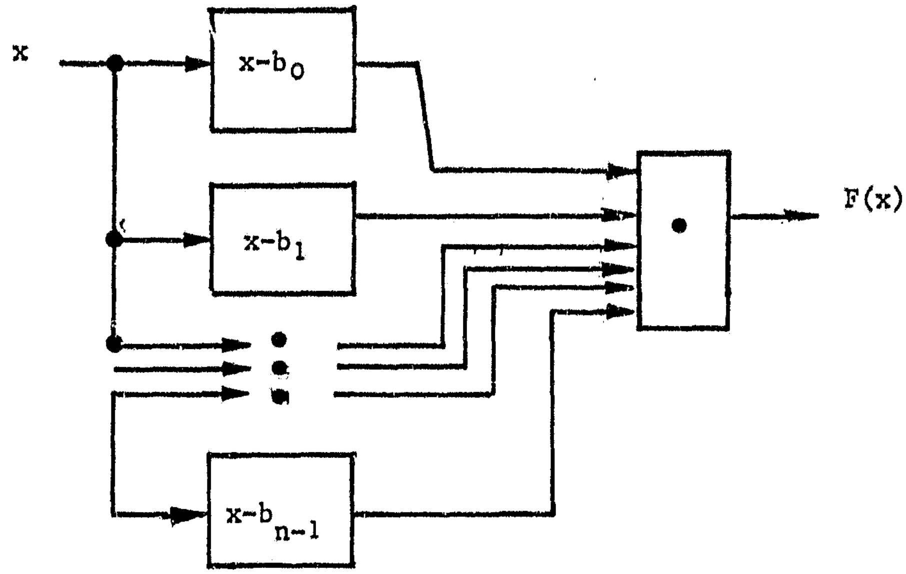
Polynomial evaluations may be represented in various decompositions including serial and parallel. A simple example of a parallel decomposition is the representation of the polynomial as a product of its linear factors. That is, a polynomial may be factored:

$$an\text{-}1\ x^{n-1} + \ldots + a_2x^2 + a_1x + a_o =$$

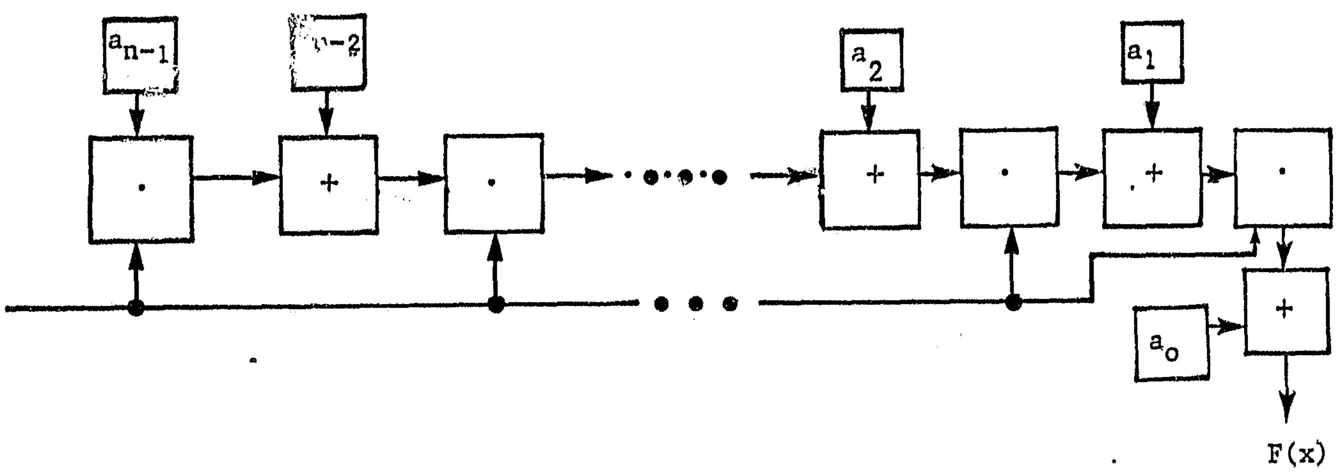
$$(\ldots(an\text{-}1\ x + an\text{-}2)\ x + \ldots a_2)\ x + a_o\ .$$

The structures of the two decompositions may be represented in block form.

Parallel



Serial



As of yet, a correspondence between these decompositions and a lattice from which they may be obtained has not been found. Such a correspondence will help generalize the lattice method.

Decompositions of matrix products are very desirable from the standpoint of decreasing computational complexity. There exist decompositions of matrix products which reduce the number of element multiplications. For example, the product of two 2 x 2 matrices can be performed using six multiplications rather than the usual eight. Matrix operations are highly algebraic in structure and should lend themselves to many possible decompositions.

Current research involves using lattice analysis to obtain beneficial decompositions. There is a major problem encountered using these techniques. Match algebras are very large. For example, a 2 x 2 matrix over a field of sixteen elements has 65536 representations. This size problem may be overcome by generating the technique to large problems by examining smaller problems in detail and using computer assistance in handling the large information sets.

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2. Birkhoff, Garrett, Lattice Theory, American Mathematical Society Colloquium Publications Vol. 25., American Mathematical Society, Providence, Rhode Island, 1967.