NASA Contractor Report 172176

ICASE

NASA-CR-172176
19830026428

THE LINEAR REGULATOR PROBLEM FOR PARABOLIC SYSTEMS

H. T. Banks
and
K. Kunisch

Contract Nos. NAS1-15810, NAS1-16394
July 1983

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

The Linear Regulator Problem for Parabolic Systems

H. T. BANKS*
Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, RI 02912
and
Department of Mathematics
Southern Methodist University
Dallas, Texas 75275

and

K. KUNISCH†
Institut für Mathematik
Technische Universität Graz
A-8010 Graz, Austria
and
Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, RI 02912

ABSTRACT

We present an approximation framework for computation (in finite dimensional spaces) of Riccati operators that can be guaranteed to converge to the Riccati operator in feedback controls for abstract evolution systems in a Hilbert space. It is shown how these results may be used in the linear optimal regulator problem for a large class of parabolic systems.

* Research supported in part by NSF grant MCS-8205335, by AFOSR contract 81-0198, and ARO contract ARO-DAAG-29-79-C-0161. Parts of the research were carried out while both authors were visitors at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, Va., which is operated under NASA contracts No. NAS1-15810 and No. NAS1-16394.

† This author gratefully acknowledges support from the Max Kade Foundation. Research supported in part by the Fonds zur Förderung der Wissenschaftlichen Forschung, AUSTRIA, No. P4534, and by the Steiermärkischen Wissenschafts - und Forschungsförderungsfonds.

1. Introduction

In this note we consider feedback controls for parabolic partial differential equations and the related Riccati operator theory when an infinite horizon integral quadratic cost functional is optimized. A general convergence framework for approximation ideas which can be used in computational techniques is developed in the context of the regulator problem theory pursued by Gibson in several recent investigations [8, 9, 10]. To illustrate our ideas we shall consider a specific model problem: The infinite horizon regulator problem for the parabolic control system

$$(1.1) \qquad \frac{\partial y}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial y}{\partial x_i} + cy + Bu(t),$$

for $t > 0$, $x \in \Omega \subset R^n$, with Dirichlet boundary conditions $y|_{\partial\Omega} = 0$ and known initial data $y|_{t=0} = \phi$. Consideration of this model problem is motivated by our desire to develop efficient computational schemes for optimal control problems in connection with the insect dispersal investigations detailed in [1, 2]. These problems (involving parabolic partial differential equations) will entail distributed controls (e.g., spraying of pesticides over a region with frequency and intensity of spraying constituting important control variables). We expect the theoretical results presented in this paper to form a sound foundation for the development in the near future of computational procedures for feedback controls in such problems.

In section 2, we state carefully a convergence theory for approximate Riccati operators that is essentially a modification and refinement of the theory presented by Gibson in [8]. (In an appendix, we indicate details as to how our framework follows from the results of Gibson.) We then in section 3 state precisely our control problem for the system (1.1) and show that under reasonable

assumptions (which imply a certain "preservation of exponential stability under approximation" condition) the abstract framework of section 2 can be used to guarantee convergence of approximate solutions in the event the basic approximation scheme enjoys rather fundamental convergence properties. These are sufficiently relaxed to allow a generous number of practical schemes (modal, splines of several types and orders) to fall within our treatment.

A concluding section contains remarks on the potential usefulness of the results in this presentation.

2. Approximation of an Abstract Linear Regulator Problem

In this section we summarize approximation results for an abstract linear optimal regulator problem that we shall subsequently employ in our treatment of parabolic systems. The results given here involve a minor but important modification of the abstract theory developed by Gibson in [8]. Specifically our presentation is formulated so as to facilitate approximation of the regulator problem by a sequence of finite dimensional state space problems, each defined on a subspace of the state space of the original problem. Gibson's presentation [8] requires the approximating problems each be defined on the entire original state space and as we shall explain below, this can lead to some tedious technical considerations. Our modified framework of this section really follows directly from that of Gibson, but we shall defer to an appendix a detailed explanation of this aspect of our considerations.

We suppose throughout that H and U are Hilbert spaces, that $A : \mathrm{dom}\, A \subset H \to H$ is the infinitesimal generator of a strongly continuous or C_0 semigroup $T(t)$ on H and that $B \in \mathcal{L}(U,H)$. We consider a control system in H given by

$$\begin{aligned} &\dot{y}(t) = Ay(t) + Bu(t), \qquad t > 0 , \\ &y(0) = y_0 , \end{aligned} \tag{2.1}$$

and an associated performance measure

$$J(y_0,u) = \int_0^\infty \{\langle Dy(t),y(t)\rangle + \langle Qu(t),u(t)\rangle\}dt \tag{2.2}$$

where $D \in \mathcal{L}(H)$, $Q \in \mathcal{L}(U)$ are selfadjoint and satisfy $D \geq 0$, $Q > 0$. Our fundamental abstract linear optimal regulator problem can then be stated as

(R) Minimize $J(y_0,u)$ over $u \in L^2(0,\infty;U)$ subject to $y = y(\cdot;u)$ satisfying (2.1).

We shall say that a function $u \in L^2(0,\infty;U)$ is an admissible control for the initial state $y_0 \in H$ if $J(y_0,u)$ is finite. As usual, a certain algebraic Riccati equation will play a fundamental role in our analysis and an operator $\Pi \in \mathcal{L}(H)$ is called a solution of the algebraic Riccati equation (A.R.E) if Π maps dom A into dom A^* and satisfies on H the equation

$$(2.3) \qquad A^*\Pi + \Pi A - \Pi BQ^{-1}B^*\Pi + D = 0.$$

Here A^* is the Hilbert space adjoint of A and we recall [4, p. 51] that it is the generator of the C_0 semigroup $T(t)^*$ which is adjoint to $T(t)$. We note that if Π satisfies (2.3) on dom A then (2.3) can be taken as an equation on H since $\Pi BQ^{-1}B^*\Pi - D$ is bounded so that $A^*\Pi + \Pi A$ has a bounded extension to all of H.

The following result is taken from [8, Theorems 4.11, 4.6].

Theorem 2.1. Let A, B, Q, D be as given above. Then there exists a nonnegative selfadjoint solution Π of the algebraic Riccati equation (2.3) if and only if for each $y_0 \in H$, there exists an admissible control. If this latter holds, then the unique optimal control and corresponding trajectory for (R) are given by

$$(2.4) \qquad \bar{u}(t) = - Q^{-1}B^*\Pi_\infty \bar{y}(t)$$

$$(2.5) \qquad \bar{y}(t) = S(t)y_0 \ ,$$

where Π_∞ is the minimal nonnegative selfadjoint solution of the A.R.E. (2.3) and $S(t)$ is the C_0 semigroup generated by $A - BQ^{-1}B^*\Pi_\infty$. If $|y(t;u)| \to 0$ as $t \to \infty$ for any admissible control (this is guaranteed for example by the condition $D > 0$), then Π_∞ is the unique nonnegative selfadjoint solution of the A.R.E. If $D > 0$, then we also have that $S(t)$ is uniformly exponentially stable.

In this theorem the term minimal for a selfadjoint operator is in reference to the usual ordering of selfadjoint nonnegative operators on a Hilbert space. We note that the minimal solution Π_∞ of (2.3) can be obtained as the limit of a sequence of Riccati operators for associated finite interval regulator problems (see [8, Theorems 4.10, 4.11]) in a manner analogous to the usual procedure for finite dimensional state space regulator problems [13].

We next formulate a sequence of approximate regulator problems and present a convergence result for the corresponding Riccati operators. Let H^N, $N = 1,2,\ldots$, be a sequence of finite dimensional closed linear subspaces of H and $P^N : H \to H^N$ be the canonical orthogonal projections. Assume that $T^N(t)$ is a sequence of C_0 semigroups on H^N with infinitesimal generators $A^N \in \mathcal{L}(H^N)$. Given operators $B^N \in \mathcal{L}(U,H^N)$ and $D^N \in \mathcal{L}(H^N)$ we consider the family of regulator problems:

(R^N) Minimize $J^N(y^N(0),u)$ over $u \in L^2(0,\infty;U)$,

where

$$\begin{aligned} \dot{y}^N(t) &= A^N y^N(t) + B^N u(t), \qquad t > 0 , \\ y^N(0) &= y_0^N \equiv P^N y_0 , \end{aligned} \tag{2.6}$$

and

$$J^N(y^N(0),u) = \int_0^\infty \{\langle D^N y^N(t), y^N(t)\rangle + \langle Qu(t),u(t)\rangle\}dt. \tag{2.7}$$

We note that since $B^N : U \to H^N$, the trajectories of (2.6) evolve in H^N and consequently (R^N) is a linear regulator problem in the finite dimensional state space H^N so that finite dimensional control theory is applicable here. We shall need several assumptions in a convergence statement regarding solutions of (R^N) and (R).

(H1): For each $y_0^N \in H^N$ there exists an admissible control $u^N \in L^2(0,\infty;U)$ for (R^N) and any admissible control for (2.6), (2.7) drives the state of (2.6) to zero asymptotically.

(H2): (i) For each $z \in H$, we have $T^N(t)P^N z \to T(t)z$ with the convergence uniform in t on bounded subsets of $[0,\infty)$.

(ii) For each $z \in H$, we have $T^N(t)^* P^N z \to T(t)^* z$ with the convergence uniform in t on bounded subsets of $[0,\infty)$.

(iii) For each $v \in U$, $B^N v \to Bv$ and for each $z \in H$, $B^{N*} z \to B^* z$.

(iv) For each $z \in H$, $D^N P^N z \to Dz$.

We remark that (H2)(i) implies in particular (take $t = 0$) that $P^N z \to z$ for each $z \in H$ and in this sense we have the subspaces H^N approximate H.

If assumption (H1) holds, then the optimal control $\bar{u}^N$ for (R^N) is given in feedback form by

$$\bar{u}^N(t) = - Q^{-1} B^{N*} \Pi^N \bar{y}^N(t) \tag{2.8}$$

where $\Pi^N \in \mathcal{L}(H^N)$ is the unique nonnegative selfadjoint solution of the algebraic Riccati equation on H^N

$$A^{N*}\Pi^N + \Pi^N A^N - \Pi^N B^N Q^{-1} B^{N*} \Pi^N + D^N = 0, \tag{2.9}$$

and $\bar{y}^N$ is the corresponding solution of (2.6) with $u = \bar{u}^N$. Moreover

$$J^N(y_0^N, \bar{u}^N) = \langle \Pi^N y_0^N, y_0^N \rangle .$$

We also have the following fundamental convergence results.

Theorem 2.2. Suppose (H1), (H2) hold, $Q > 0$, $D \geq 0$ and $D^N \geq 0$ and let Π^N denote the unique nonnegative selfadjoint Riccati operators on H^N for the problems (R^N). Further assume that a unique nonnegative selfadjoint Riccati operator on H for the problem (R) exists. Let $S(t)$ and $S^N(t)$ be the semigroups generated by $A - BQ^{-1}B^*\Pi$ and $A^N - B^NQ^{-1}B^{N*}\Pi^N$ on H and H^N, respectively. If there are positive constants M_1, M_2 and ω independent of N and t such that

$$|S^N(t)|_{H^N} \leq M_1 e^{-\omega t} \quad \text{for} \quad t \geq 0, \quad N = 1,2,\ldots, \tag{2.10}$$

and

$$|\Pi^N|_{H^N} \leq M_2 , \tag{2.11}$$

then

$$\Pi^N P^N z \to \Pi z \quad \text{for every } z \in H, \tag{2.12}$$

$$S^N(t)P^N z \to S(t)z \quad \text{for every } z \in H, \tag{2.13}$$

where the convergence is uniform in t on bounded subsets of $[0,\infty)$, and

$$|S(t)| \leq M_1 e^{-\omega t} \quad \text{for} \quad t \geq 0. \tag{2.14}$$

We present a proof of Theorem 2.2 in the Appendix. Meanwhile we remark that under the hypotheses of this theorem, $\Pi^N P^N$ is an extension of $\Pi^N \in \mathcal{L}(H^N)$ to all of H. If D^N, A^N are replaced by $D^N P^N$, $A^N P^N$, respectively and (2.9) is considered as an equation on H, then $\Pi^N P^N$ is its unique minimal nonnegative selfadjoint solution.

Theorem 2.2 is essentially contained in [8]. The main difference between the theorem here and the result in [8] is, as stated earlier, that here each

of the finite dimensional state problems $(\mathcal{R}^N)$ is defined in the subspace H^N only, whereas in [8], Gibson requires that the approximate regulator problems be defined on the entire space H. This causes some unnecessary technical difficulties: First note that if $D > 0$ and $D^N = P^N D$ (as an operator in H^N), then $D^N > 0$ on H^N. But $D^N = 0$ on $H^{N\perp}$. This difficulty can be circumvented by considering instead $\hat{D}^N = P^N D + I - P^N$ as an operator on H -- see [9, p. 698].

To explain a second disadvantage to the formulation of $(\mathcal{R}^N)$ on all of H, let us assume that $|T^N(t)|_{H^N} \leq Me^{-\alpha t}$ for positive constants M and α. This allows one to infer existence of Riccati operators Π^N on H^N; however, if the semigroups $T^N(t)$ are extended to H by taking $T^N(t)P^N + I - P^N$, then these extensions are not uniformly exponentially stable and the existence of feedback solutions to $(\mathcal{R}^N)$ on H is not guaranteed. However there is a more subtle difficulty regarding verification of the analogue of (2.10) on H (e.g., see Theorem 4.3, condition (5.17) of [8]) if the approximate problems are defined on H. Even if one has the condition $|T^N(t)|_{H^N} \leq Me^{-\alpha t}$ with $T^N(t)$ extended to H as mentioned above, the feedback operators $S^N(t)$ on H satisfy $S^N(t)z = z$ for $z \in H^{N\perp}$ and hence it is not possible to satisfy directly the stability requirement (2.10) on H. In [9] this difficulty is handled by taking $T^N(t)z = e^{-t}z$ for $z \in H^{N\perp}$. But then $T(t)$ and $T^N(t)$ are essentially unrelated on $H^{N\perp}$.

In the next two sections we shall see that the version of approximation results given in our Theorem 2.2 lends itself to easy verification for certain classes of approximation schemes for parabolic systems.

3. Convergence of Approximate Riccati Operators for Parabolic Systems

We use the framework summarized in the previous section to treat the optimization of integral quadratic cost functionals for parabolic systems of the form given in (1.1). We shall follow the notation introduced in section 2 and for our state space H we choose $H^0(\Omega)$ with Ω a bounded domain in R^n possessing a piecewise C^1 boundary $\partial\Omega$. Unless otherwise indicated, all of the function spaces below are to be understood as spaces of functions with domain Ω and range R^1.

For B, D, and Q as given in defining problem $(\mathcal{R})$ of section 2, with $D > 0$ and $Q > 0$, we consider the regulator problem $(\mathcal{R})$ with the system (2.1) for the state $y(t) = y(t,\cdot)$ in $H = H^0(\Omega)$ taken as the parabolic system

$$y_t = \sum_{i,j=1}^{n} D^i(a_{ij}D^j y) + \sum_{i=1}^{n} b_i D^i y + cy + Bu, \quad t > 0,$$

$$y(0,\cdot) = \phi, \quad y(t,\cdot)\Big|_{\partial\Omega} = 0, \tag{3.1}$$

where $u \in L^2(0,\infty;U)$ and $D^i = \dfrac{\partial}{\partial x_i}$ denotes differentiation with respect to the i^{th} spatial variable x_i.

We make the following standing assumptions on the coefficients in (3.1):

There exist positive constants γ and μ such that

$$\gamma \sum_i \xi_i^2 \le \sum_{i,j} a_{ij}\xi_i\xi_j \le \mu \sum_i \xi_i^2$$

for every $\xi \in R^n$; $a_{ij} = a_{ji}$, and $a_{ij} \in L^\infty(\Omega)$, $b_i \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$ for every $i,j = 1,\ldots,n$.

Throughout our discussions the concept of a solution of (3.1) will be that of a weak solution (i.e., in the sense of distributional derivatives).

We introduce the sesquilinear form $\sigma: H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ defined by

$$\sigma(z,v) = \int_\Omega \Big\{ \sum_{i,j=1}^{n} a_{ij} D^j z D^i \bar{v} - \Big(\sum_{i=1}^{n} b_i D^i z + \tilde{c} z \Big) \bar{v} \Big\} dx$$

where $\tilde{c}(x) = c(x) - k$, with $k = k(\Omega,\gamma) > 0$ determined so that the inequality

$$\text{Re}\, \sigma(z,z) \geq c_1 |z|_1^2 \,, \quad z \in H_0^1(\Omega), \tag{3.2}$$

holds for some positive constant c_1 independent of z (see [14, p.144]). Here and throughout $|\cdot|_1$ and $|\cdot|$ denote the $H^1(\Omega)$ and $H^0(\Omega)$ norms respectively. Furthermore, to allow use of the theory of sectorial operators and sesquilinear forms in discussing the spectra of various operators, we assume in defining σ that the functions in $H_0^1(\Omega)$ are complex valued. For the sesquilinear form σ it can be shown (see [14, p.143]) that there is also a constant $c_2 = c_2(\Omega,\mu)$ so that

$$|\sigma(z,v)| \leq c_2 |z|_1 |v|_1 \tag{3.3}$$

for all $z,v \in H_0^1(\Omega)$. Furthermore, it follows from the bounds (3.2), (3.3) and well known results on sesquilinear forms (see e.g., [12, p.101]) that there exist operators A_k, A_k^* in $H^0(\Omega)$ such that

$$\sigma(z,v) = \langle -A_k z, v \rangle \quad \text{for} \quad z \in \text{dom}\, A_k, v \in H_0^1(\Omega), \tag{3.4}$$

and

$$\overline{\sigma(z,v)} = \langle -A_k^* v, z \rangle \quad \text{for} \quad v \in \text{dom}\, A_k^*, \; z \in H_0^1(\Omega). \tag{3.5}$$

In addition dom A_k and dom A_k^* are dense in $H_0^1(\Omega)$,

and we have A_k dom $A_k = H^0(\Omega)$, A_k^* dom $A_k^* = H^0(\Omega)$.

From (3.2), (3.4) and (3.5), we find that

$$\text{(3.6)} \qquad \begin{aligned} \mathrm{Re}\langle A_k z, z\rangle &\leq -c_1 |z|_1^2 \quad , \quad z \in \mathrm{dom}\, A_k, \\ \mathrm{Re}\langle A_k^* z, z\rangle &\leq -c_1 |z|_1^2 \quad , \quad z \in \mathrm{dom}\, A_k^*. \end{aligned}$$

In view of (3.6) and the range statements above for A_k and A_k^*, we may invoke standard results from linear semigroup theory [15, p.16, thm. 4.5] to assert that A_k and A_k^* are the infinitesimal generators of linear C_0-semigroups $T_k(t)$ and $T_k^*(t)$, respectively. (As we have noted above, in fact $T_k^*(t) = T_k(t)^*$.)

We note that the solution semigroup $T(t)$ for (3.1) is given by

$$\text{(3.7)} \qquad T(t) = e^{kt} T_k(t)$$

with the infinitesimal generator A of $T(t)$ given by $A = A_k + kI$ and $\mathrm{dom}\, A = \mathrm{dom}\, A_k$. Similarly, we have

$$\text{(3.8)} \qquad T(t)^* = e^{kt} T_k^*(t)$$

with the infinitesimal generator A^* of $T(t)^*$ given by $A^* = A_k^* + kI$ and $\mathrm{dom}\, A^* = \mathrm{dom}\, A_k^*$. We also have $|T(t)| \leq e^{(k-c_1)t}$ for $t \geq 0$.

Turning next to approximations for $(\mathcal{R})$, we suppose we have a sequence of finite dimensional (real) subspaces $H^N \subset H_0^1(\Omega)$, $N = 1,2,\ldots$, which satisfy the approximation condition:

(C1): For each $z \in H_0^1(\Omega)$, there exists an element $\tilde{z}^N$ in H^N such that $|z - \tilde{z}^N|_1 \leq \varepsilon(N)$, where $\varepsilon(N) \to 0$ as $N \to \infty$.

We remark that this condition is fulfilled in the event H^N is chosen

as in many classes of finite element approximation schemes [5, Chap.III, 3.2]. In particular, (C1) holds for the case where Ω is a rectangle in R^2 and the H^N are the usual linear spans of tensor products of standard one dimensional piecewise linear splines [16] with mesh size approaching zero as $N \to \infty$.

Proceeding in standard fashion, we observe that the restriction of σ to $H^N \times H^N$ defines, in a unique manner, bounded linear operators A_k^N, A_k^{*N} on H^N such that

$$\sigma(z,v) = \langle -A_k^N z, v\rangle \quad , \quad z,v \in H^N, \tag{3.9}$$

and

$$\sigma(z,v) = \langle -A_k^{*N} v, z\rangle \quad , \quad z,v \in H^N. \tag{3.10}$$

Here $A_k^{*N} = A_k^{N*}$. We let $A^N = A_k^N + kI$, $A^{N*} = A_k^{N*} + kI$ with domains H^N and note that A^N, A^{N*} generate C_0 semigroups $T^N(t)$, $T^N(t)^*$ on H^N, with $T^N(t)^*$ the adjoint of $T^N(t)$. For the finite dimensional approximating problems (R^N) we choose

$$B^N = P^N B, \quad D^N = P^N D \tag{3.11}$$

where $P^N: H^0(\Omega) \to H^N$ is, as in section 2, the canonical orthogonal projection. We have thus specified all of the needed entities for the problem (R^N) of section 2. As we noted previously the trajectories of this problem evolve in H^N and hence it is a finite dimensional regulator problem for which computational techniques are readily available (assuming of course that one has made a thoughtful decision in defining the H^N).

We turn to a verification of (H2) of section 2 for the approximations at hand. Since it is a trivial matter to see that (C1) implies that

(3.12) $$P^N z \to z, \quad \text{as} \quad N \to \infty, \quad \text{for} \quad z \in H^0(\Omega),$$

the conditions (H2)-(iii), (H2)-(iv) follow at once from (3.11). We next argue that

(3.13) $$T_k^N(t)P^N z \to T_k(t)z$$

(3.14) $$T_k^N(t)^* P^N z \to T_k(t)^* z,$$

for $z \in H^0(\Omega)$, with the convergence uniform in t on bounded subsets of $[0,\infty)$. This taken with (3.7), (3.8) will imply conditions (H2)-(i), (H2)-(ii). First we note that from (3.2), (3.9), (3.10) and the fact that $H^N \subset H_0^1(\Omega)$, we have

(3.15) $$\mathrm{Re}\langle A_k^N z, z\rangle \leq -c_1 |z|_1^2$$

(3.16) $$\mathrm{Re}\langle A_k^{N*} z, z\rangle \leq -c_1 |z|_1^2$$

for all $z \in H^N$. Moreover, we shall demonstrate that the following convergence statements hold:

(3.17) $$(I-A_k^N)^{-1} P^N z \to (I-A_k)^{-1} z, \quad z \in H^0(\Omega),$$

(3.18) $$(I-A_k^{N*})^{-1} P^N z \to (I-A_k^*)^{-1} z, \quad z \in H^0(\Omega).$$

We then may use the Trotter-Kato theorem (see, e.g., the version given in [18]) to conclude that (3.15)-(3.18) imply at once the statements (3.13), (3.14).

Thus we turn to establish (3.17) and (3.18). We shall employ a result given in [7, p.756, Lemma 3.3], which we state without proof here.

Lemma 3.1. There exists a positive constant δ_1 and a constant θ_1 in $(0,\pi/2)$ such that

$$|\lambda||z|^2 + |z|_1^2 \le \delta_1\left|\lambda|z|^2 - \sigma(z,z)\right| \tag{3.19}$$

for all $z \in H_0^1(\Omega)$ and $\lambda \in \left\{\xi \in \mathbb{C}: \theta_1 \le |\arg \xi| \le \pi\right\}$.

We use this to show that (3.17) holds. For $z \in H^0(\Omega)$, define $w = (I-A_k)^{-1}z$ and $w^N = (I-A_k^N)^{-1}P^N z$. Then we have for all $z^N \in H^N$

$$\begin{aligned} &<w,z^N> + \sigma(w,z^N) = <z,z^N>, \text{ and} \\ &<w^N,z^N> + \sigma(w^N,z^N) = <z,z^N>. \end{aligned}$$

Consequently, defining $e^N = w - w^N$, we find

$$<e^N,z^N> + \sigma(e^N,z^N) = 0$$

for all $z^N \in H^N$. Taking $\lambda = -1$ and $z = e^N$ in (3.19) - note that $e^N \in H_0^1(\Omega)$ - we obtain using this last equation

$$\begin{aligned} |e^N|^2 + |e^N|_1^2 &\le \delta_1\left|-|e^N|^2 - \sigma(e^N,e^N)\right| \\ &= \delta_1|-<e^N,e^N+z^N> - \sigma(e^N,e^N+z^N)| \end{aligned}$$

for all $z^N \in H^N$. Let $z^N = w^N - \tilde{w}^N$, where $\tilde{w}^N$ is an approximation for w chosen according to (C1). (Here we again use the fact that $w \in \text{dom } A_k \subset H_0^1(\Omega)$.) We thus obtain the estimate

$$\begin{aligned} |e^N|^2 + |e^N|_1^2 &\le \delta_1|<e^N,w-\tilde{w}^N> + \sigma(e^N,w-\tilde{w}^N)| \\ &\le c_2\varepsilon(N)\delta_1\left\{|e^N| + |e^N|_1\right\}, \end{aligned}$$

where we have, without loss of generality, assumed that $c_2 \ge 1$. This last

estimate implies $e^N \to 0$ in $H^1(\Omega)$ and, in particular, (3.17) holds.

Turning to (3.18), we recall that $\overline{\sigma(z,v)} = \langle -A_k^* v, z\rangle$ and define $\tau: H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ by $\tau(z,v) = \overline{\sigma(z,v)}$. Then τ satisfies the same inequalities (3.2), (3.3) as σ. We may therefore verify (3.18) by referring to the analysis for (3.17). We summarize our discussions to this point.

Lemma 3.2. Let (C1) hold. Then (H2) obtains with B^N, D^N, $T^N(t)$, and $T^N(t)^*$ defined as in and just above (3.11).

To use Theorems 2.1, 2.2 of section 2, we shall make the following stabilizability hypothesis.

(C2): The pair (A,B) is exponentially stabilizable, i.e., there exists a bounded linear operator $K: H^0(\Omega) \to U$ such that the semigroup $T_s(t)$ generated by $A + BK$ satisfies $|T_s(t)| \leq M_1 e^{-\omega_1 t}$ for some positive constants M_1 and ω_1.

For a discussion of (C2), we refer the reader to [17] and the references given there.

To make use of the theory of section 2, we need to verify that a certain preservation of exponential stabilizability under approximation condition holds for our problem. This condition can be stated as:

(POES): Suppose that condition (C2) holds. Then there exists an integer N_0 such that for all $N \geq N_0$ the pairs $(A^N, P^N B)$ are uniformly exponentially stabilizable by the operator K of (C2), i.e., there exist positive constants (independent of N) M_s and ω_s such that the semigroups $T_s^N(t)$ generated by $A^N + P^N BK$ satisfy $|T_s^N(t)| \leq M_s e^{-\omega_s t}$ for all $N \geq N_0$ and $t \geq 0$.

Before returning to the theoretical results of section 2, we argue that the class of approximations for our system (3.1) does indeed satisfy the preservation of stabilizability condition (POES).

Lemma 3.3. Let (C1), (C2) hold. Then the approximations defined through (3.9), (3.10), (3.11) yield systems that satisfy (POES).

Proof. We define a sesquilinear form $\sigma_B: H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ by $\sigma_B(z,v) = \sigma(z,v) + \langle -BKz, v\rangle + \langle \tilde{k}z, v\rangle$ where $\tilde{k}$ is chosen so that

$$\text{Re}\,\sigma_B(z,z) \geq c_3|z|_1^2 \quad , \quad z \in H_0^1(\Omega) \tag{3.20}$$

and

$$|\sigma_B(z,v)| \leq c_4|z|_1|v|_1 \quad , \; z,v \in H_0^1(\Omega), \tag{3.21}$$

for some positive constants c_3, c_4 (recall (3.2), (3.3)). Then arguments similar to those in [7, p.756, Lemmas 3.2, 3.3] can be used to establish that the numerical range of σ_B is contained in a right sector $S_{\theta,\gamma} = \left\{\lambda \in \mathbb{C} : |\arg(\lambda-\gamma)| \leq \theta\right\}$ where $0 < \theta < \pi/2$, γ real.

We next consider the restriction of σ_B to $H^N \times H^N$ and, in a manner already discussed, this gives rise to bounded linear operators $\hat{A}_B^N$, $\hat{A}_B^{N*}$ on H^N such that $\hat{A}_B^N = A^N + P^N BK - \hat{k}$ where $\hat{k} = k + \tilde{k}$ (see the definition of k in σ, $\tilde{k}$ in σ_B). Indeed $\sigma_B(z,v) = \langle -\hat{A}_B^N z, v\rangle$, $\overline{\sigma_B(z,v)} = \langle -\hat{A}_B^{N*} v, z\rangle$ for $z,v \in H^N$, with $\hat{A}_B^{N*}$ the adjoint of $\hat{A}_B^N$. Furthermore, the numerical range of $\hat{A}_B^N$ (and $\hat{A}_B^{N*}$) is contained in the left sector $S_{\theta,\gamma}^- = \left\{\lambda: -\lambda \in S_{\theta,\gamma}\right\}$, uniformly in N. Thus the numerical range and hence the spectrum (see [1], p.280]) of $A_B^N = A^N + P^N BK = \hat{A}_B^N + \hat{k}$ are contained in a left sector $\hat{S} = S_{\theta,\gamma}^- + \hat{k}$, uniformly in N. It follows that the set of all eigenvalues λ of A_B^N with $\text{Re}\lambda > -\delta$ is bounded, uniformly in N,

for any fixed δ.

Using arguments similar to those behind (3.17), (3.18) - see Lemma 3.1 and the proof of Lemma 3.2 - it is easily shown that for some $\zeta \in R^1$, $\zeta > 0$ sufficiently large, we have ζ in all the resolvent sets $\rho(A_B^N)$, $\rho(A+BK)$, $N = 1,2,\ldots,$ and

$$(\zeta - A_B^{N*})^{-1} P^N z \to (\zeta - (A+BK)^*)^{-1} z \quad \text{for } z \in H^0(\Omega). \tag{3.22}$$

To complete our proof, let us assume that (POES) does not hold even though (C2) does. We argue a contradiction. If (POES) does not hold, then there exists a sequence N_j with $N_j \to \infty$ and λ^{N_j} an eigenvalue of $A_B^{N_j}$ satisfying $\mathrm{Re}\,\lambda^{N_j} > -1/j$. From our findings on the spectrum of A_B^N, $N = 1,2,\ldots,$ we know there exists a limit point $\hat{\lambda}$ of $\{\lambda^{N_j}\}$ with $\mathrm{Re}\,\hat{\lambda} \geq 0$, $\hat{\lambda} \in \hat{S}$. We shall argue that $\hat{\lambda}$ is an eigenvalue of $A + BK$, which is a contradiction since (C2) implies $\mathrm{Re}\,\lambda \leq -\omega_1$ for λ in the spectrum of $A + BK$ (see [4, p.32]).

For convenience, we relabel and drop the subsequential notation, assuming henceforth that $\lambda^N \to \hat{\lambda}$, with λ^N an eigenvalue of A_B^N. Let ϕ^N be an eigenvector with $|\phi^N| = 1$ and $A_B^N \phi^N = \lambda^N \phi^N$ for all N sufficiently large. Then we have

$$\lambda^N \phi^N - \zeta \phi^N = A_B^N \phi^N - \zeta \phi^N$$

and hence

$$(A_B^N - \zeta)^{-1} (\lambda^N \phi^N - \zeta \phi^N) = \phi^N.$$

Let χ in dom A^* be arbitrary and put $\psi = (A+BK-\zeta)^* \chi$. Then

$$\langle \lambda^N \phi^N - \zeta \phi^N,\ ((A_B^N - \zeta)^{-1})^* P^N \psi \rangle = \langle \phi^N, P^N \psi \rangle. \tag{3.23}$$

Using (3.2), one can readily show that the set $\{\phi^N\}$ is a bounded set in $H_0^1(\Omega)$. Consequently, there exists a subsequence, again denoted by $\{\phi^N\}$, converging strongly in $H^0(\Omega)$ to some nontrivial $w \in H^0(\Omega)$. Thus from (3.22), (3.23) and (3.12) - a result of (C1) - we have

$$<(\hat{\lambda}-\zeta)w,((A+BK-\zeta)^{-1})^*\psi> = <w,\psi>$$

or

$$<(\hat{\lambda}-\zeta)w,\chi> = <w,(A+BK-\zeta)^*\chi>$$

for all χ in dom A^*. Thus $w \in$ dom A and

$$<(\hat{\lambda}-\zeta)w,\chi> = <(A+BK-\zeta)w,\chi>$$

or

$$<(A+BK-\hat{\lambda})w,\chi> = 0$$

for all χ in dom A^*. It follows that w is an eigenvector corresponding to the eigenvalue $\hat{\lambda}$ for $A + BK$ and this yields the desired contradiction and completes the proof of Lemma 3.3.

We return finally to a discussion of the convergence theory of section 2 as it is applied to the specific parabolic system control problem that is the focus of the present section. We assume (C1) and (C2) hold. Then (POES) along with Theorem 2.1 yields the existence of nonnegative selfadjoint Riccati operators Π and Π^N (for N sufficiently large) associated with the problems (R) and (R^N) in $H^0(\Omega)$ and H^N, respectively. Since $D > 0$ and $D^N = P^N D > 0$ on H^N, these Riccati operators are unique and furthermore (H1) obtains.

Turning to Theorem 2.2, we first verify that (2.11) holds.

Recall that

$$|\Pi^N|_{H^N} = \sup\{\langle \Pi^N z, z\rangle \mid z \in H^N, |z| = 1\} = \sup_{|z|=1} J(z, \bar{u}^N) \tag{3.24}$$

where $\bar{u}^N$ is the optimal feedback control (2.8) of (R^N). Define, for $z^N \in H^N$ with $|z^N| = 1$, the control $u_s^N(t) = KT_s^N(t)z^N$ where $T_s^N(t)$ is the semigroup defined in (POES). Then

$$J(z^N, \bar{u}^N) \leq \int_0^\infty \langle D^N T_s^N(t)z^N, T_s^N(t)z^N\rangle dt + \int_0^\infty \langle QKT_s^N(t)z^N, KT_s^N(t)z^N\rangle dt$$

$$\leq \left\{|D| + |Q||K|^2\right\} M_s^2 \omega_s^{-2} \equiv M_2$$

so that from (3.24) we may infer (2.11). To establish (2.10), we first note that $|S^N(t)|_{H^N} \leq K_1 e^{\beta t}$ for some constants K_1 and β independent of N. This follows from (3.15), (3.16), (2.11) and the fact that $S^N(t)$ is generated by $A^N - P^N B Q^{-1} B^* P^N \Pi^N$. Moreover we have

$$\int_0^\infty \langle DS^N(t)z^N, S^N(t)z^N\rangle dt \leq \langle \Pi^N z^N, z^N\rangle \leq M_2 |z^N|^2.$$

Since $D > 0$, a theorem of Datko (see [6], [8, p.540, Thm. 2.2]) implies existence of positive constants M_1 and ω, independent of N, such that

$$|S^N(t)|_{H^N} \leq M_1 e^{-\omega t}.$$

Hence (2.10) of Theorem 2.2 holds.

Using the convergence results of (2.12), (2.13) it is easy to argue that the optimal feedback controls for (R^N) converge to that of (R). We summarize our findings for the regulator problems for (3.1) in the following theorem.

Theorem 3.1. Assume that the subspace approximation condition (C1) holds for $H^N \subset H_0^1(\Omega)$, that the stabilizability condition (C2) holds for (3.1), and that $Q > 0$, $D > 0$. Then there exist unique Riccati operators Π and Π^N associated with the regulator problems (R) and (R^N) on $H^0(\Omega)$ and H^N for (3.1) and

$$\Pi^N P^N z \to \Pi z \quad \text{for} \quad z \in H^0(\Omega),$$
$$S^N(t) P^N z \to S(t) z \quad \text{for} \quad z \in H^0(\Omega),$$

and

$$\bar{u}^N(t) \to \bar{u}(t),$$

with these last two statements holding uniformly in t on compact subsets of $[0,\infty)$. Here $S^N(t)$ and $S(t)$ are the semigroups generated by $A^N - P^N B Q^{-1} B^* P^N \Pi^N$ and $A - BQ^{-1}B^*\Pi$, and $\bar{u}^N$ and $\bar{u}$ are the optimal feedback controls for (R^N) and (R), respectively. Moreover, $|S(t)| \leq M_1 e^{-\omega t}$ with $\omega > 0$.

4. Concluding Remarks

The conclusions in Theorem 3.1, especially (3.26) and (3.27), are important since they reveal that the finite dimensional control laws when employed in the systems that we can compute (i.e., the approximate systems) allow us to anticipate what might happen qualitatively if we used the infinite dimensional feedback controls in the original distributed system. However of equal importance are findings (which are simple corollaries to the results of section 3) that indicate that use of the approximate (readily computed and usually easily implemented) controls in the actual distributed system can be expected to produce satisfactory performance. More precisely, let $U^N = Q^{-1}B^*\Pi^N P^N$ and consider the sequence $\bar{A}^N = A - BU^N$ of operators in $H^0(\Omega)$. Then the operators $\bar{A}^N$ generate semigroups $\bar{S}^N(t)$ which are uniformly exponentially stable and $\bar{S}^N(t)z \to S(t)z$, uniformly on compact sets in $[0,\infty)$, $z \in H^0(\Omega)$, provided, of course, that the assumptions of Theorem 3.1 are fulfilled. The uniform exponential stability can be established using arguments similar to those in the proof of Lemma 3.3. The significance of results for such finite dimensional feedback into the original distributed system was noted by Gibson in [9, p. 699].

We note that the techniques described in this paper are not restricted to parabolic equations of the form (1.1) with distributed control. Indeed as can be seen from the arguments in section 3, the essential property required for application of these ideas is that the differential equation operator in (2.1) be sectorial or, more precisely, that the systems (including feedback) generate sesquilinear forms with numerical range in some sector (e.g., see the arguments behind Lemmas 3.1, 3.3). Indeed, even though our treatment here is concerned with the practically important (in view of the applications mentioned in section 1) case of disbributed controls, we recognize that there

are important applications where boundary control problems for parabolic equations are of primary interest. In some of these applications our treatment and techniques are readily used. (We note that the only restriction on B is that it be bounded and some boundary control problems are readily transformed to the form (2.1)). Furthermore, certain control problems for higher order equations can also successfully be treated with the ideas presented in this paper.

Finally, we note that our approximation approach involves almost no restrictions on the subspaces H^N so that we again can treat a large variety of problems. For example, we specifically do not require that H^N be contained in dom A (or dom A_k) about which we may have only partial information in some cases. Thus we may readily employ linear spline approximations with second order operators in the framework of our results. Based on our previous efforts with spline based approximations in parameter estimation [1, 2] and control problems [3], we are confident that optimism concerning use of splines in the present framework is justified.

Appendix

We give here a proof for Theorem 2.2 using in a fundamental way some of the results of Gibson. As we have already mentioned, Theorem 2.2 in its present form is not given in [8] and in fact his arguments for an analogous result appear to contain some technical inaccuracies which we shall attempt to avoid.

To make our arguments, we need to consider regulator problems on the finite intervals $[s,t_f]$, $-\infty < s < t_f$, with a weighting operator G for the final state $y(t_f)$. We assume throughout that A generates the C_0 semigroup $T(t)$ on H, that D, Q, G are selfadjoint with $D \geq 0$, $Q > 0$, $G \geq 0$, and $B \in \mathcal{L}(U,H)$. The finite interval problems are given by:

$$(R,t_f) \qquad \text{Minimize } J(s,y(s),u) = \langle Gy(t_f),y(t_f)\rangle + \int_s^{t_f} \{\langle Dy(t),y(t)\rangle + \langle Qu(t),u(t)\rangle\}dt$$

$$\text{subject to } y(t) = T(t-s)y(s) + \int_s^t T(t-\sigma)Bu(\sigma)d\sigma \quad \text{for } s \leq t \leq t_f .$$

Under our assumptions a unique nonnegative selfadjoint Riccati operator Π_s can be associated with (R,t_f). That is, Π_s is the unique nonnegative selfadjoint solution of the integral Riccati equation for $z \in H$

$$\Pi_s(t)z = T(t_f-t)^*GT(t_f-t)z + \int_t^{t_f} T(\eta-t)^*[D - \Pi_s(\eta)BQ^{-1}B^*\Pi_s(\eta)]T(\eta-t)z\,d\eta$$

with $\Pi_s(\xi) \in \mathcal{L}(H)$ for $s \leq \xi \leq t_f$, (see [8, Theorems 3.1, 3.2 and equation (3.28)]). We then have the following limit results.

Theorem A.1. Assume that the unique nonnegative selfadjoint solution Π of the A.R.E. (2.3) exists. Let Π_s be the unique Riccati operator function associated with the problem $(R,0)$. If $\lim_{t\to\infty} |S(t)z| = 0$ for all $z \in H$ where

where $S(t)$ is generated by $A - BQ^{-1}B^*\Pi$, then

(A.1) $\lim_{s \to -\infty} \Pi_s(s)z = \Pi z$ for all $z \in H$.

If moreover $G \geq \Pi$ and there exist positive constants M and β such that

(A.2) $|S(t)| \leq Me^{-\beta t}$, $t \geq 0$,

then

(A.3) $\Pi \leq \Pi_s(s) \leq \Pi + M^2 e^{2\beta s}|G|$ for $s \leq 0$.

Proof. If Π is the unique nonnegative, selfadjoint solution of (2.3), then by the calculations in [8, p. 557-558], it is also the unique solution of the first integral Riccati equation of [8] on the infinite interval and corresponds to the operator P_∞ of that paper. Theorem A.1 then follows directly from Theorem 4.10 of [8].

We note that if in addition to the above hypotheses we have $D > 0$, then (A.2) is satisfied (see Theorem 4.8 of [8]).

We next recall an approximation result for the finite horizon regulator problem $(\mathcal{R}, t_f)$ in H. Let (H2) hold with operators as in $(\mathcal{R}^N)$ given; in addition assume $G^N \in \mathcal{L}(H^N)$, $G^N \geq 0$, are given. To consider a related finite horizon problem in H, we define $\tilde{G}^N = G^N P^N$ and $\tilde{D}^N = D^N P^N$ on H. Consider for $-\infty < s < t_f$ and $y(s) \in H$ given the problem:

$(\mathcal{R}^N, t_f)$ Minimize $J^N(s, y^N(s), u) = \langle \tilde{G}^N y^N(t_f), y^N(t_f) \rangle$

$$+ \int_s^{t_f} \{\langle \tilde{D}^N y^N(t), y^N(t) \rangle + \langle Qu(t), u(t) \rangle\} dt$$

subject to $y^N(t) = T^N(t-s)P^N y(s) + \int_s^t T^N(t-\sigma)B^N u(\sigma)d\sigma$ for $s \leq t \leq t_f$.

The problem $(\mathcal{R}^N,t_f)$ is considered as a problem in H even though we note that $y^N(t) \in H^N$ for each t so that $\tilde{D}^N y^N(t) = D^N y^N(t)$ and $\tilde{G}^N y^N(t_f) = G^N y^N(t_f)$. We denote the unique nonnegative selfadjoint Riccati operator function associated with $(\mathcal{R}^N,0)$ by Π_s^N (see Theorem 3.2 of [8]). The following is a consequence of Theorem 5.1 of [8].

Theorem A.2. Let (H2) hold and assume that $G^N P^N z \to Gz$ for $z \in H$. Then for $s < 0$ we have

$$\bar{u}^N \to \bar{u} \qquad \text{uniformly on } [s,0] ,$$

$$\bar{y}^N \to \bar{y} \qquad \text{uniformly on } [s,0] ,$$

$$\Pi_s^N(\xi)z \to \Pi_s(\xi)z \quad \text{for } z \in H, \quad \text{uniformly in } \xi \text{ on } [s,0].$$

Here $\bar{u}^N$, $\bar{u}$, $\bar{y}^N$, $\bar{y}$ denote optimal controls and trajectories of the problems $(\mathcal{R}^N,0)$ and $(\mathcal{R},0)$, respectively.

With these preliminaries, we are now prepared to prove Theorem 2.2.

Proof of Theorem 2.2.

Denote by Π_s and Π_s^N, $s \leq 0$, the Riccati operator functions associated with $(\mathcal{R},0)$ and $(\mathcal{R}^N,0)$ in H where we take $G = M_2 I$, $G^N = M_2 P^N$ with M_2 the constant in inequality (2.11). From Theorem A.1 applied to each of the problems $(\mathcal{R}^N,0)$ on H^N with (2.10) (and hence (A.2) with $M = M_1$, $\beta = \omega$) holding we conclude that for $s < 0$ one has on H^N

$$\Pi^N \leq \Pi_s^N(s) \leq \Pi^N + M_1^2 e^{2\omega s} M_2 .$$

This implies that on H we have for $s < 0$ and each N

(A.4) $$\Pi^N P^N \le \Pi_s^N(s) P^N \le \Pi^N P^N + M_1^2 e^{2\omega s} M_2 .$$

Since $\Pi^N P^N$ is selfadjoint in H, we conclude from (A.4) and (A.1) that for each $\varepsilon > 0$ and $z \in H$, there exists $\zeta = \zeta(z,\varepsilon)$ in $(-\infty,0)$ such that

(A.5) $$|\Pi_\zeta^N(\zeta) P^N - \Pi^N P^N| < \varepsilon \qquad \text{for every } N = 1,2,\ldots,$$

and

(A.6) $$|\Pi z - \Pi_\zeta(\zeta) z| < \varepsilon .$$

Therefore we have

$$|\Pi^N P^N z - \Pi z| \le |\Pi^N P^N z - \Pi_\zeta^N(\zeta) P^N z| + |\Pi_\zeta^N(\zeta) P^N z - \Pi_\zeta^N(\zeta) z|$$

(A.7) $$+ |\Pi_\zeta^N(\zeta) z - \Pi_\zeta(\zeta) z| + |\Pi_\zeta(\zeta) z - \Pi z|$$

$$\le \varepsilon |z| + |\Pi_\zeta^N(\zeta)| \, |P^N z - z| + |\Pi_\zeta^N(\zeta) z - \Pi_\zeta(\zeta) z| + \varepsilon .$$

But by Theorem A.2 and the uniform boundedness principle we have $|\Pi_\zeta^N(\zeta)|$ uniformly bounded in N and $\Pi_\zeta^N(\zeta) z \to \Pi_\zeta(\zeta) z$. Finally from (H2)(ii) we have $P^N z \to z$ and thus (A.7) implies $\Pi^N P^N z \to \Pi z$ for every $z \in H$. Hence (2.12) is established.

From (H2)(iii) and (2.11) it follows that $|B^N Q^{-1} B^{N*} \Pi^N|_{H^N}$ is uniformly bounded and moreover $B^N Q^{-1} B^{N*} \Pi^N P^N z \to B Q^{-1} B^* \Pi z$ for each $z \in H$. Therefore (2.13) follows from use of the variation of parameters representations for $\bar{y}^N(t) = S^N(t) z$ and $\bar{y}(t) = S(t) z$ and the Gronwall inequality along with (2.8), (2.10) and (H2)(i). Finally (2.14) is a consequence of (2.13) and (2.10).

References

[1] H. T. Banks and P. Kareiva, Parameter estimation techniques for transport equations with application to population dispersal and tissue bulk flow models, LCDS Tech. Rep. 82-13, Brown University, July 1982; J. Math. Biology, to appear.

[2] H. T. Banks, P. L. Daniel, and P. Kareiva, Estimation of temporally and spatially varying coefficients in models for insect dispersal, LCDS Tech. Rep. 83-14, Brown University, Providence, June, 1983.

[3] H. T. Banks, I. G. Rosen, and K. Ito, A spline based technique for computing Riccati operators and feedback controls in regulator problems for delay equations, ICASE Rep. 82-31, Sept. 1982; SIAM J. Comp. and Stat. Sci., submitted.

[4] P. L. Butzer and H. Berens, Semigroups of Operators and Approximation, Springer, Berlin-Herdelberg-New York, 1967.

[5] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North Holland, 1978.

[6] R. Datko, Uniform asymptotic stability of evolutionary processes in a Banach space, SIAM J. Math. Anal. 3 (1972), 428-445.

[7] H. Fujita and A. Mizutani, On the finite element method for parabolic equations, I: approximation of holomorphic semi-groups, J. Math. Soc. Japan, 28 (1976), 749-771.

[8] J. S. Gibson, The Riccati integral equations for optimal control problems on Hilbert spaces, SIAM J. Control and Optimization, 17 (1979), 537-565.

[9] J. S. Gibson, An analysis of optimal model regulation: convergence and stability, SIAM J. Control and Optimization, 19 (1981), 686-707.

[10] J. S. Gibson, Linear-quadratic optimal control of hereditary differential systems: infinite dimensional Riccati equations and numerical approximations, SIAM J. Control and Optimization, 21 (1983), 95-139.

[11] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1966.

[12] S. G. Krein, Linear Differential Equations in Banach Spaces, Transl. Math. Mono. 29, American Math Sociecy, Providence, 1971.

[13] H. Kwakernaak and R. Sivan Linear Optimal Control Systems, Wiley, New York, 1972.

[14] O. A. Ladyzhenskaya and N. N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968.

[15] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Lecture Notes, 10, University of Maryland, College Park, 1974.

[16] M. H. Schultz, Spline Analysis, Prentice-Hall, Englewood Cliffs, 1973.

[17] R. Triggiani, On the stabilizability problem in Banach space, J. Math. Anal. Appl. 52 (1975), 383-403.

[18] T. Ushijima, On the finite element approximation of parabolic equations--Consistency, boundedness and convergence, Mem. Numer. Math. 2 (1975), 21-33.

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-172176		

4. Title and Subtitle	5. Report Date
The Linear Regulator Problem for Parabolic Systems	July 1983
	6. Performing Organization Code
7. Author(s)	8. Performing Organization Report No.
H. T. Banks and K. Kunisch	83-34
	10. Work Unit No.
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langely Research Center Hampton, VA 23665	11. Contract or Grant No. NAS1-15810 & NAS1-16394
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546	13. Type of Report and Period Covered Contractor report
	14. Sponsoring Agency Code

15. Supplementary Notes Additional support: NSF grant MCS-8205335, AFOSR contract 81-0198, ARO contract ARO-DAAG-29-79-C-0161, Fonds zur Forderung der Wissenschaft lichen Forschung, AUSTRIA, No. P4535, and Steiermarkischen Wissenschafts - und Forschungsforderungsfonds. Technical Monitor: Robert H. Tolson, Final report

16. Abstract

We present an approximation framework for computation (in finite dimensional spaces) of Riccati operators that can be guaranteed to converge to the Riccati operator in feedback controls for abstract evolution systems in a Hilbert space. It is shown how these results may be used in the linear optimal regulator problem for a large class of parabolic systems.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
parabolic systems feedback controls approximation schemes	66 Systems Analysis Unclassified-Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	30	A03

End of Document