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CR-170934

Final Report

Contract NAS8-34750

NASA Marshall Space Flight Center



'AGCE Related Studies of Baroclinic Flows in Spherical Geometry'

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OF BAROCLINIC FLOWS IN SPHERICAL GEOMETRY  
Final Report (Clarkson Coll. of Technology)  
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by

J. M. Hyun

Principal Investigator

Mechanical and Industrial Engineering Department

Clarkson College of Technology

Potsdam, New York

November 1983

## Summary

Steady-state, axisymmetric motions of a Boussinesq fluid contained in rotating spherical annulus are considered. The motions are driven by latitudinally-varying temperature gradients at the shells. Linearized formulations for a narrow gap are derived and the flow field is divided into the Ekman layers and the geostrophic interior. The Ekman layer flows are consistent with the known results for cylindrical geometries. Within the framework of rather restrictive assumptions, the interior flows are solved by a series of associated Legendre polynomials. These solutions show qualitative features valid at mid-latitudes.

## 1. Introduction

The dynamics of rotating and stratified fluids in a container is generally influenced by the presence of Coriolis forces acting on the fluid motion in planes perpendicular to the rotation axis. Laboratory models have been developed to study the flows to simulate the geophysical fluid systems. Historically, cylindrical geometries have been widely used because of its maximal simplicity and much qualitative information has been made available from such model studies.

One of the most important ingredients in the modeling efforts of realistic geophysical fluid systems is the spherically radial gravity. Recently, there is work on developing a spherical laboratory experiment for Spacelab flights, the Atmospheric General Circulation Experiment (AGCE) [see Ref. 1]. Significant features of AGCE are that a radial body force, simulating the spherical gravity, is imposed via an electric field across rotating spherical shells. The fluid contained in the spherical gap is stably stratified in the radial direction, and the motion is driven by the latitudinal temperature gradients on the boundaries.

In an effort to obtain qualitative characteristics for such experimental configurations, we study the steady, axisymmetric motions of a thin layer of fluid between two rotating spherical shells. As was pointed out in Ref. 2, the axisymmetric flows would provide the needed basic-states for stability analysis.

We shall follow the analytical techniques of Ref. 3, and attempt to give an analytical solution within the framework of the specified assumptions. The main thrust is in the examination of the geostrophic interior flows at mid-latitudes.

## 2. The formulation

Consider a Boussinesq fluid occupying a spherical annulus of thickness  $D$ . The radii of the spherical shells, which are rotating at angular velocity  $\Omega$  about the polar axis, are  $R$  and  $R + D$ , where  $\delta \equiv D/R \ll 1$  is assumed. The coordinate  $\theta$  measures latitude,  $r$  is the radial distance from the sphere's center. The eastward, northward, and radially outward velocity components are  $U, V, W$ , respectively.

We introduce the nondimensionalization anticipating that in the main body of fluid the flows are in geostrophic and hydrostatic balance:

$$\begin{aligned} r &= R + \delta R Z', \\ (U, V) &= \bar{U} [U'(\theta, Z'), V'(\theta, Z')], \quad W = \delta \bar{U} W'(\theta, Z'), \\ T &= T_0 + \Delta T_V Z' + (2\Omega R \bar{U} / g \alpha D) T'(\theta, Z'), \\ \rho &= \rho_0 [1 - \alpha (T - T_0)], \\ P &= P_0 - \rho_0 g D Z' + \alpha \Delta T_V \rho_0 g D^2 \frac{1}{2} Z'^2 + 2\Omega R \bar{U} \rho_0 P'(\theta, Z'), \end{aligned}$$

for which the usual notation is employed (Ref. 2).

The equations of motion for linearized, steady-state,

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axisymmetric motion on a rotating spherical coordinate are (the primes have been removed from the nondimensional quantities):

$$-V \sin\theta + \delta W \cos\theta = \frac{E_H}{2} \left[ \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} \cos\theta \frac{\partial V}{\partial\theta} - \frac{V}{\cos^2\theta} \right] + \frac{E_V}{2} \left[ \frac{\partial^2 U}{\partial Z^2} + 2\delta \frac{\partial U}{\partial Z} \right]. \quad (1)$$

$$U \sin\theta = -\frac{\partial P}{\partial\theta} + \frac{E_H}{2} \left[ \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} \cos\theta \frac{\partial V}{\partial\theta} - \frac{V}{\cos^2\theta} \right] + \frac{E_V}{2} \left[ \frac{\partial^2 V}{\partial Z^2} + 2\delta \frac{\partial V}{\partial Z} \right]. \quad (2)$$

$$-\delta U \cos\theta = -\frac{\partial P}{\partial Z} + T + \frac{\delta^2 E_H}{2} \left[ \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} \cos\theta \frac{\partial W}{\partial\theta} - \frac{W}{\cos^2\theta} \right] + \delta^2 \frac{E_V}{2} \left[ \frac{\partial^2 W}{\partial Z^2} + 2\delta \frac{\partial W}{\partial Z} \right]. \quad (3)$$

$$WS = \frac{E_H}{2\sigma_H} \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} \cos\theta \frac{\partial T}{\partial\theta} + \frac{E_V}{2\sigma_V} \left[ \frac{\partial^2 T}{\partial Z^2} + 2\delta \frac{\partial T}{\partial Z} \right]. \quad (4)$$

$$\frac{\partial W}{\partial Z} + 2\delta W + \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} (\cos\theta V) = 0. \quad (5)$$

The relevant nondimensional parameters are

$$E_H = \nu_H / \Omega R^2, \quad E_V = \nu_V / K_H, \quad \delta = D/R, \\ S = \alpha g \Delta T_V D / \Omega^2 R^2, \quad \sigma_H = \nu_H / K_H, \quad \sigma_V = \nu_V / K_V.$$

where  $\nu_V, K_V; \nu_H, K_H$  are the mixing coefficients for momentum and heat in the vertical and horizontal directions, respectively. These are introduced to develop a model of greater geophysical relevance.

The boundary conditions are

$$(u, v, w) = (U_T, 0, 0) \quad \text{on} \quad z = 1. \quad (6)$$

$$(u, v, w) = (U_L, 0, 0) \quad \text{on} \quad z = 0. \quad (7)$$

$$T = T_T \quad \text{on} \quad z = 1. \quad (8)$$

$$T = T_L \quad \text{on} \quad z = 0. \quad (9)$$

(6) and (7) are to be used when there is a mechanical forcing of the flow due to a differential rotation of the shells. For a purely thermally-driven flow,  $U_L = U_T = 0$ .

Now, the strategy is to divide the flow field for mid-latitudes into the viscously-controlled Ekman layers on the shells and the geostrophic interior. Ref. 2 scales the variables as

$$U = U_I(\theta, Z) + \tilde{U}_T(\theta, \zeta_1) + \tilde{U}_L(\theta, \zeta_2). \quad (10)$$

$$V = E_V V_I(\theta, Z) + \tilde{V}_T(\theta, \zeta_1) + \tilde{V}_L(\theta, \zeta_2). \quad (11)$$

$$W = E_V^{1/2} W_I(\theta, Z) + E_V^{1/2} \tilde{W}_T(\theta, \zeta_1) + E_V^{1/2} \tilde{W}_L(\theta, \zeta_2). \quad (12)$$

$$P = P_I(\theta, Z) + \delta E_V^{1/2} \tilde{P}_T(\theta, \zeta_1) + \delta E_V^{1/2} \tilde{P}_L(\theta, \zeta_2). \quad (13)$$

$$T = T_I(\theta, Z) + E_V^{1/2} \sigma_V S \tilde{T}_T(\theta, \zeta_1) + E_V^{1/2} \sigma_V S \tilde{T}_L(\theta, \zeta_2). \quad (14)$$

where the subscript I denotes the interior flow and the tilde represents the Ekman layer corrections.  $\zeta_1$  and  $\zeta_2$  are the  $E_V^{-1/2}$ -stretched Z coordinates in the Ekman layers.

The solutions for the Ekman layer velocities are well-known (e.g., Ref. 4). The boundary condition for the interior flow is

$$W_I(\theta, 0) = - \frac{\text{sgn} \theta}{2 \cos \theta} \frac{\partial}{\partial \theta} \frac{\cos \theta [U_I(\theta, 0) - U_L]}{|\sin \theta|^{1/2}} \quad (15)$$

and a similar one for  $W_I(\theta, 1)$ .

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The correction to the temperature in the Ekman layer is  $O(\sigma_v S E_v^{1/2})$ ; thus, to  $O(E_v^{1/2})$ , the temperature boundary conditions for the interior flows are

$$T_I(\theta, 1) = T_T, \quad T_I(\theta, 0) = T_L. \tag{16}$$

3. The geostrophic interior

At mid-latitudes  $\theta \sim 0(1)$ , the interior flow equations are

$$U_I \sin\theta = - \frac{\partial P_I}{\partial \theta}. \tag{17}$$

$$-V_I \sin\theta = - \delta E_v^{-1/2} W_I \cos\theta + \frac{1}{2} \frac{\partial^2 U_I}{\partial Z^2} + \frac{1}{2} \left[ \frac{E_H}{E_v} \left( \frac{1}{\cos\theta} \frac{\partial}{\partial \theta} \cos\theta \frac{\partial U_I}{\partial \theta} - \frac{U_I}{\cos^2\theta} \right) \right]. \tag{18}$$

$$T_I = \frac{\partial P_I}{\partial Z} - \delta U_I \cos\theta. \tag{19}$$

$$\frac{\partial W_I}{\partial Z} + \frac{E_v^{1/2}}{\cos\theta} \frac{\partial}{\partial \theta} (\cos\theta V_I) + 2 \delta W_I = 0. \tag{20}$$

$$2W_I \sigma_v S / E_v^{1/2} = \frac{\partial^2 T_I}{\partial Z^2} + \frac{E_H}{E_v} \frac{\sigma_v}{\sigma_H} \frac{1}{\cos\theta} \frac{\partial}{\partial \theta} \cos\theta \frac{\partial T_I}{\partial \theta} + 2\delta \frac{\partial T_I}{\partial Z}. \tag{21}$$

We note that for the actual AGCE,  $\delta E_v^{-1/2} \ll 1$  may not be valid. From (20), to  $O(E_v^{1/2})$  and to  $O(\delta)$ ,

$$W_I(\theta) = - \frac{\text{sgn}\theta}{4 \cos\theta} \frac{\partial}{\partial \theta} \frac{\cos\theta}{|\sin\theta|^2} \frac{\partial}{\partial \theta} \int_{z=0}^{z=1} T_I(\theta, z) dz. \tag{22}$$



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Following the techniques of Ref. 3, an integral equation involving  $\partial T_I / \partial \theta$  is formed. The solutions of that equation may be found in a series of associated Legendre functions of order 1 :

$$\begin{aligned} \frac{\partial T_I}{\partial \theta} = & - \frac{\lambda(\theta)}{1+\lambda(\theta)} (U_T - U_L) \sin \theta \\ & + \sum_{n=1}^{\infty} [a_n \sinh \ell_n z + b_n \cosh \ell_n z] P_n^{(1)}(\sin \theta) \\ & - \frac{\lambda(\theta)}{1+\lambda(\theta)} \sum_{n=1}^{\infty} \left\{ a_n \left[ \frac{\cosh \ell_n}{\ell_n} - \frac{1}{\ell_n} \right] + b_n \left[ \frac{\sinh \ell_n}{\ell_n} \right] \right\} P_n^{(1)}(\sin \theta), \end{aligned} \quad (23)$$

where

$$\lambda(\theta) = \sigma_V S / 2\gamma^2 |\sin \theta|^{\frac{3}{2}} E_V^{\frac{1}{2}},$$

$$\ell_n^2 = n(n+1)\gamma^2,$$

$$\gamma^2 = E_H \sigma_V / E_V \sigma_H.$$

The coefficients  $a_n$  and  $b_n$  are found through

$$\begin{aligned} A_n = b_n - \left( \frac{2n+1}{2} \right) \left( \frac{1}{n(n+1)} \right) \left[ a_n \left( \frac{\cosh \ell_n}{\ell_n} - \frac{1}{\ell_n} \right) \right. \\ \left. + b_n \frac{\sinh \ell_n}{\ell_n} \right] \sum_{m=1}^{\infty} q_{nm}, \end{aligned} \quad (24)$$

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$$B_n = [a_n \sinh \lambda_n + b_n \cosh \lambda_n] - \left(\frac{2n+1}{2}\right) \left(\frac{1}{n(n+1)}\right) \left[ a_n \left(\frac{\cosh \lambda_n}{\lambda_n} - \frac{1}{\lambda_n}\right) + b_n \frac{\sinh \lambda_n}{\lambda_n} \right] \sum_{m=1}^{\infty} q_{nm}, \quad (25)$$

where

$$q_{nm} = \int_{-\pi/2}^{\pi/2} \frac{\lambda(\theta)}{1+\lambda(\theta)} P_n^{(1)}(\sin\theta) P_m^{(1)}(\sin\theta) \cos\theta \, d\theta, \quad (26)$$

The boundary conditions provide  $A_n$  and  $B_n$ ,

$$A_n = \frac{2n+1}{2} \frac{1}{n(n+1)} \int_{-\pi/2}^{\pi/2} \left(\frac{\partial T_L}{\partial \theta}\right) P_n^{(1)}(\sin\theta) \cos\theta \, d\theta, \quad (27)$$

$$B_n = \frac{2N+1}{2} \frac{1}{n(n+1)} \int_{-\pi/2}^{\pi/2} \left(\frac{\partial T_T}{\partial \theta}\right) P_n^{(1)}(\sin\theta) \cos\theta \, d\theta. \quad (28)$$

From the thermal wind relation

$$-\sin\theta \frac{\partial U_I}{\partial z} = \frac{\partial T_I}{\partial \theta}.$$

and noting that

$$U_I(\theta, 1) + U_I(\theta, 0) = U_L + U_T.$$

we have

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$$\begin{aligned}
 U_I &= \frac{\lambda(\theta)}{1+\lambda(\theta)} (U_T - U_L) Z \\
 &- \sum_1^{\infty} \left[ \frac{a_n}{\ell_n} \cosh \ell_n Z + \frac{b_n}{\ell_n} \sinh \ell_n Z \right] \frac{P_n^{(1)}(\sin\theta)}{\sin\theta} \\
 &+ \sum_1^{\infty} \left[ a_n \left( \frac{\cosh \ell_n}{\ell_n} - \frac{1}{\ell_n} \right) + b_n \left( \frac{\sinh \ell_n}{\ell_n} \right) \right] \frac{\lambda(\theta)}{1+\lambda(\theta)} \frac{P_n^{(1)}(\sin\theta)}{\sin\theta} Z.
 \end{aligned} \tag{29}$$

Then, from (20), to  $O(\delta)$ , we have

$$W_I(\theta, Z) = W^{(0)}(\theta) [(1 + \delta) - 2 \delta Z]. \tag{30}$$

where  $W^{(0)}(\theta)$  is given in (22).

From (18), we obtain v

$$\begin{aligned}
 V_I &= \delta E_V^{-1/2} W \cos\theta / \sin\theta \\
 &+ \frac{1}{2} \sum_1^{\infty} \left[ a_n \ell_n \cosh \ell_n Z + b_n \ell_n \sinh \ell_n Z \right] \frac{P_n^{(1)}(\sin\theta)}{\sin^2\theta} \\
 &- \frac{1}{2} \frac{1}{\sin\theta} \frac{E_H}{E_V} \left( \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} \cos\theta \frac{\partial U_I}{\partial\theta} - \frac{U_I}{\cos^2\theta} \right)
 \end{aligned} \tag{31}$$

Now, it is apparent that (29) and (31) are valid only in the mid-latitudes  $\theta \sim O(1)$  so that  $\sin\theta \sim O(1)$ .

#### 4. Remarks

The above analysis for the interior flows at mid-latitudes

rely heavily on rather severe scaling assumptions. Since the boundary conditions for the interior flows were based on the presence of the Ekman suction, the analysis is applicable to rapidly-rotating fluids. The narrow-gap assumption, which is essential for deriving the simplified equations of motion, is appropriate for realistic atmospheric flows but less accurate for laboratory experiments. The interior solutions presented lose validity as the equator is approached. The horizontal component of the rotation vector vanishes at the equator, and, therefore, the Ekman layer fades away. The interior flows themselves should satisfy the boundary conditions on the shells, and the viscously-controlled horizontal velocities near the equator have to be accounted for. To remove the singularity at the equator, an equatorial boundary layer is called for. Analytical treatments of this equatorial boundary layer appear to be very difficult and the attempts so far have been inconclusive. The results of the interior flows for  $\theta \sim 0$  (1) show features qualitatively consistent with other studies. A full-length paper is being prepared to report on the findings.

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Item C., Attachment A of Contract Requirements for Quarterly Reporting:

- (1) Total cumulative costs incurred as of this report date: \$28,765.80
- (2) Estimate of cost to complete contract: \$0.0
- (3) Estimated percentage of physical completion of contract: 100%
- (4) N/A

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## Summary

Steady-state, axisymmetric motions of a Boussinesq fluid contained in rotating spherical annulus are considered. The motions are driven by latitudinally-varying temperature gradients at the shells. Linearized formulations for a narrow gap are derived and the flow field is divided into the Ekman layers and the geostrophic interior. The Ekman layer flows are consistent with the known results for cylindrical geometries. Within the framework of rather restrictive assumptions, the interior flows are solved by a series of associated Legendre polynomials. These solutions show qualitative features valid at mid-latitudes.

## 1. Introduction

The dynamics of rotating and stratified fluids in a container is generally influenced by the presence of Coriolis forces acting on the fluid motion in planes perpendicular to the rotation axis. Laboratory models have been developed to study the flows to simulate the geophysical fluid systems. Historically, cylindrical geometries have been widely used because of its maximal simplicity and much qualitative information has been made available from such model studies.

One of the most important ingredients in the modeling efforts of realistic geophysical fluid systems is the spherically radial gravity. Recently, there is work on developing a spherical laboratory experiment for Spacelab flights, the Atmospheric General Circulation Experiment (AGCE) [see Ref. 1]. Significant features of AGCE are that a radial body force, simulating the spherical gravity, is imposed via an electric field across rotating spherical shells. The fluid contained in the spherical gap is stably stratified in the radial direction, and the motion is driven by the latitudinal temperature gradients on the boundaries.

In an effort to obtain qualitative characteristics for such experimental configurations, we study the steady, axisymmetric motions of a thin layer of fluid between two rotating spherical shells. As was pointed out in Ref. 2, the axisymmetric flows would provide the needed basic-states for stability analysis.



We shall follow the analytical techniques of Ref. 3, and attempt to give an analytical solution within the framework of the specified assumptions. The main thrust is in the examination of the geostrophic interior flows at mid-latitudes.

## 2. The formulation

Consider a Boussinesq fluid occupying a spherical annulus of thickness  $D$ . The radii of the spherical shells, which are rotating at angular velocity  $\Omega$  about the polar axis, are  $R$  and  $R + D$ , where  $\delta \equiv D/R \ll 1$  is assumed. The coordinate  $\theta$  measures latitude,  $r$  is the radial distance from the sphere's center. The eastward, northward, and radially outward velocity components are  $U, V, W$ , respectively.

We introduce the nondimensionalization anticipating that in the main body of fluid the flows are in geostrophic and hydrostatic balance:

$$\begin{aligned}
 r &= R + \delta R Z', \\
 (U, V) &= \bar{U} [U'(\theta, Z'), V'(\theta, Z')], \quad W = \delta \bar{U} W'(\theta, Z'), \\
 T &= T_0 + \Delta T_V Z' + (2\Omega R \bar{U} / g \alpha D) T'(\theta, Z'), \\
 \rho &= \rho_0 [1 - \alpha (T - T_0)], \\
 P &= P_0 - \rho_0 g D Z' + \alpha \Delta T_V \rho_0 g D^2 \frac{1}{2} Z'^2 + 2\Omega R \bar{U} \rho_0 P'(\theta, Z'),
 \end{aligned}$$

for which the usual notation is employed (Ref. 2).

The equations of motion for linearized, steady-state,

axisymmetric motion on a rotating spherical coordinate are (the primes have been removed from the nondimensional quantities):

$$-V \sin\theta + \delta W \cos\theta = \frac{E_H}{2} \left[ \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} \cos\theta \frac{\partial V}{\partial\theta} - \frac{V}{\cos^2\theta} \right] + \frac{E_V}{2} \left[ \frac{\partial^2 U}{\partial Z^2} + 2\delta \frac{\partial U}{\partial Z} \right]. \quad (1)$$

$$U \sin\theta = -\frac{\partial P}{\partial\theta} + \frac{E_H}{2} \left[ \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} \cos\theta \frac{\partial V}{\partial\theta} - \frac{V}{\cos^2\theta} \right] + \frac{E_V}{2} \left[ \frac{\partial^2 V}{\partial Z^2} + 2\delta \frac{\partial V}{\partial Z} \right]. \quad (2)$$

$$-\delta U \cos\theta = -\frac{\partial P}{\partial Z} + T + \frac{\delta^2 E_H}{2} \left[ \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} \cos\theta \frac{\partial W}{\partial\theta} - \frac{W}{\cos^2\theta} \right] + \delta^2 \frac{E_V}{2} \left[ \frac{\partial^2 W}{\partial Z^2} + 2\delta \frac{\partial W}{\partial Z} \right]. \quad (3)$$

$$WS = \frac{E_H}{2\sigma_H} \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} \cos\theta \frac{\partial T}{\partial\theta} + \frac{E_V}{2\sigma_V} \left[ \frac{\partial^2 T}{\partial Z^2} + 2\delta \frac{\partial T}{\partial Z} \right]. \quad (4)$$

$$\frac{\partial W}{\partial Z} + 2\delta W + \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} (\cos\theta V) = 0. \quad (5)$$

The relevant nondimensional parameters are

$$E_H = v_H / \Omega R^2, \quad E_V = v_V / K_H, \quad \delta = D/R, \\ S = \alpha g \Delta T_V D / \Omega^2 R^2, \quad \sigma_H = v_H / K_H, \quad \sigma_V = v_V / K_V.$$

where  $v_V, K_V; v_H, K_H$  are the mixing coefficients for momentum and heat in the vertical and horizontal directions, respectively. These are introduced to develop a model of greater geophysical relevance.

The boundary conditions are

$$(u, v, w) = (U_T, 0, 0) \quad \text{on } z = 1. \quad (6)$$

$$(u, v, w) = (U_L, 0, 0) \quad \text{on } z = 0. \quad (7)$$

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$$T = T_T \quad \text{on} \quad z = 1. \quad (8)$$

$$T = T_L \quad \text{on} \quad z = 0. \quad (9)$$

(6) and (7) are to be used when there is a mechanical forcing of the flow due to a differential rotation of the shells. For a purely thermally-driven flow,  $U_L = U_T = 0$ .

Now, the strategy is to divide the flow field for mid-latitudes into the viscously-controlled Ekman layers on the shells and the geostrophic interior. Ref. 2 scales the variables as

$$U = U_I(\theta, Z) + \tilde{U}_T(\theta, \zeta_1) + \tilde{U}_L(\theta, \zeta_2). \quad (10)$$

$$V = E_V V_I(\theta, Z) + \tilde{V}_T(\theta, \zeta_1) + \tilde{V}_L(\theta, \zeta_2). \quad (11)$$

$$W = E_V^{1/2} W_I(\theta, Z) + E_V^{1/2} \tilde{W}_T(\theta, \zeta_1) + E_V^{1/2} \tilde{W}_L(\theta, \zeta_2). \quad (12)$$

$$P = P_I(\theta, Z) + \delta E_V^{1/2} \tilde{P}_T(\theta, \zeta_1) + \delta E_V^{1/2} \tilde{P}_L(\theta, \zeta_2). \quad (13)$$

$$T = T_I(\theta, Z) + E_V^{1/2} \sigma_V S \tilde{T}_T(\theta, \zeta_1) + E_V^{1/2} \sigma_V S \tilde{T}_L(\theta, \zeta_2). \quad (14)$$

where the subscript I denotes the interior flow and the tilde represents the Ekman layer corrections.  $\zeta_1$  and  $\zeta_2$  are the  $E_V^{-1/2}$ -stretched Z coordinates in the Ekman layers.

The solutions for the Ekman layer velocities are well-known (e.g., Ref. 4). The boundary condition for the interior flow is

$$W_I(\theta, 0) = - \frac{\text{sgn}\theta}{2\cos\theta} \frac{\partial}{\partial\theta} \frac{\cos\theta [U_I(\theta, 0) - U_L]}{|\sin\theta|^{1/2}} \quad (15)$$

and a similar one for  $W_I(\theta, 1)$ .

The correction to the temperature in the Ekman layer is  $O(\sigma_v S E_v^{1/2})$ ; thus, to  $O(E_v^{1/2})$ , the temperature boundary conditions for the interior flows are

$$T_I(\theta, 1) = T_T, \quad T_I(\theta, 0) = T_L. \quad (16)$$

### 3. The geostrophic interior

At mid-latitudes  $\theta \sim 0$  (1), the interior flow equations are

$$U_I \sin\theta = - \frac{\partial P_I}{\partial \theta}. \quad (17)$$

$$\begin{aligned} -V_I \sin\theta = & -\delta E_v^{-1/2} W_I \cos\theta + \frac{1}{2} \frac{\partial^2 U_I}{\partial Z^2} \\ & + \frac{1}{2} \left[ \frac{E_H}{E_v} \left( \frac{1}{\cos\theta} \frac{\partial}{\partial \theta} \cos\theta \frac{\partial U_I}{\partial \theta} - \frac{U_I}{\cos^2\theta} \right) \right]. \end{aligned} \quad (18)$$

$$T_I = \frac{\partial P_I}{\partial Z} - \delta U_I \cos\theta. \quad (19)$$

$$\frac{\partial W_I}{\partial Z} + \frac{E_v^{1/2}}{\cos\theta} \frac{\partial}{\partial \theta} (\cos\theta V_I) + 2\delta W_I = 0. \quad (20)$$

$$2W_I \sigma_v S / E_v^{1/2} = \frac{\partial^2 T_I}{\partial Z^2} + \frac{E_H}{E_v} \frac{\sigma_v}{\sigma_H} \frac{1}{\cos\theta} \frac{\partial}{\partial \theta} \cos\theta \frac{\partial T_I}{\partial \theta} + 2\delta \frac{\partial T_I}{\partial Z}. \quad (21)$$

We note that for the actual AGCE,  $\delta E_v^{-1/2} \ll 1$  may not be valid. From (20), to  $O(E_v^{1/2})$  and to  $O(\delta)$ ,

$$W_I(\theta) = - \frac{\operatorname{sgn}\theta}{4 \cos\theta} \frac{\partial}{\partial \theta} \frac{\cos\theta}{|\sin\theta|^{3/2}} \frac{\partial}{\partial \theta} \int_{z=0}^{z=1} T_I(\theta, Z) dz. \quad (22)$$

Following the techniques of Ref. 3, an integral equation involving  $\partial T_I / \partial \theta$  is formed. The solutions of that equation may be found in a series of associated Legendre functions of order 1 :

$$\begin{aligned} \frac{\partial T_I}{\partial \theta} = & - \frac{\lambda(\theta)}{1+\lambda(\theta)} (U_T - U_L) \sin \theta \\ & + \sum_{n=1}^{\infty} [a_n \sinh \ell_n z + b_n \cosh \ell_n z] P_n^{(1)}(\sin \theta) \\ & - \frac{\lambda(\theta)}{1+\lambda(\theta)} \sum_{n=1}^{\infty} \left\{ a_n \left[ \frac{\cosh \ell_n}{\ell_n} - \frac{1}{\ell_n} \right] + b_n \left[ \frac{\sinh \ell_n}{\ell_n} \right] \right\} P_n^{(1)}(\sin \theta), \end{aligned} \quad (23)$$

where

$$\lambda(\theta) = \sigma_V S / 2\gamma^2 |\sin \theta|^{\frac{3}{2}} E_V^{\frac{1}{2}},$$

$$\ell_n^2 = n(n+1)\gamma^2,$$

$$\gamma^2 = E_H \sigma_V / E_V \sigma_H.$$

The coefficients  $a_n$  and  $b_n$  are found through

$$\begin{aligned} A_n = b_n - \left( \frac{2n+1}{2} \right) \left( \frac{1}{n(n+1)} \right) \left[ a_n \left( \frac{\cosh \ell_n}{\ell_n} - \frac{1}{\ell_n} \right) \right. \\ \left. + b_n \frac{\sinh \ell_n}{\ell_n} \right] \sum_{m=1}^{\infty} q_{nm}, \end{aligned} \quad (24)$$

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$$B_n = [a_n \sinh \lambda_n + b_n \cosh \lambda_n] - \left(\frac{2n+1}{2}\right) \left(\frac{1}{n(n+1)}\right) \left[ a_n \left(\frac{\cosh \lambda_n}{\lambda_n} - \frac{1}{\lambda_n}\right) + b_n \frac{\sinh \lambda_n}{\lambda_n} \right] \sum_{m=1}^{\infty} q_{nm}, \quad (25)$$

where

$$q_{nm} = \int_{-\pi/2}^{\pi/2} \frac{\lambda(\theta)}{1+\lambda(\theta)} P_n^{(1)}(\sin\theta) P_m^{(1)}(\sin\theta) \cos\theta \, d\theta, \quad (26)$$

The boundary conditions provide  $A_n$  and  $B_n$  :

$$A_n = \frac{2n+1}{2} \frac{1}{n(n+1)} \int_{-\pi/2}^{\pi/2} \left(\frac{\partial T_L}{\partial \theta}\right) P_n^{(1)}(\sin\theta) \cos\theta \, d\theta, \quad (27)$$

$$B_n = \frac{2n+1}{2} \frac{1}{n(n+1)} \int_{-\pi/2}^{\pi/2} \left(\frac{2T_T}{\partial \theta}\right) P_n^{(1)}(\sin\theta) \cos\theta \, d\theta. \quad (28)$$

From the thermal wind relation

$$-\sin\theta \, \partial U_I / \partial Z = \partial T_I / \partial \theta.$$

and noting that

$$U_I(\theta, 1) + U_I(\theta, 0) = U_L + U_T.$$

we have

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$$\begin{aligned}
 U_I &= \frac{\lambda(\theta)}{1+\lambda(\theta)} (U_T - U_L) Z \\
 &- \sum_1^{\infty} \left[ \frac{a_n}{\ell_n} \cosh \ell_n Z + \frac{b_n}{\ell_n} \sinh \ell_n Z \right] \frac{P_n^{(1)}(\sin\theta)}{\sin\theta} \\
 &+ \sum_1^{\infty} \left[ a_n \left( \frac{\cosh \ell_n}{\ell_n} - \frac{1}{\ell_n} \right) + b_n \left( \frac{\sinh \ell_n}{\ell_n} \right) \right] \frac{\lambda(\theta)}{1+\lambda(\theta)} \frac{P_n^{(1)}(\sin\theta)}{\sin\theta} Z.
 \end{aligned} \tag{29}$$

Then, from (20), to  $O(\delta)$ , we have

$$W_I(\theta, Z) = W^{(0)}(\theta) [(1 + \delta) - 2 \delta Z]. \tag{30}$$

where  $W^{(0)}(\theta)$  is given in (22).

From (18), we obtain

$$\begin{aligned}
 V_I &= \delta E_V^{-1/2} W \cos\theta / \sin\theta \\
 &+ \frac{1}{2} \sum_1^{\infty} \left[ a_n \ell_n \cosh \ell_n Z + b_n \ell_n \sinh \ell_n Z \right] \frac{P_n^{(1)}(\sin\theta)}{\sin^2\theta} \\
 &- \frac{1}{2} \frac{1}{\sin\theta} \frac{E_H}{E_V} \left( \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} \cos\theta \frac{\partial U_I}{\partial\theta} - \frac{U_I}{\cos^2\theta} \right)
 \end{aligned} \tag{31}$$

Now, it is apparent that (29) and (31) are valid only in the mid-latitudes  $\theta \sim O(1)$  so that  $\sin\theta \sim O(1)$ .

#### 4. Remarks

The above analysis for the interior flows at mid-latitudes

rely heavily on rather severe scaling assumptions. Since the boundary conditions for the interior flows were based on the presence of the Ekman suction, the analysis is applicable to rapidly-rotating fluids. The narrow-gap assumption, which is essential for deriving the simplified equations of motion, is appropriate for realistic atmospheric flows but less accurate for laboratory experiments. The interior solutions presented lose validity as the equator is approached. The horizontal component of the rotation vector vanishes at the equator, and, therefore, the Ekman layer fades away. The interior flows themselves should satisfy the boundary conditions on the shells, and the viscously-controlled horizontal velocities near the equator have to be accounted for. To remove the singularity at the equator, an equatorial boundary layer is called for. Analytical treatments of this equatorial boundary layer appear to be very difficult and the attempts so far have been inconclusive. The results of the interior flows for  $\theta \sim 0$  (1) show features qualitatively consistent with other studies. A full-length paper is being prepared to report on the findings.



## References

1. Fowles, W. W. and G. H. Fichtl, 1977: Geophysical fluid flow model experiments in spherical geometry. Proc. Third NASA Weather and Climate Program Science Review. NASA Conf. Publ. 2029, Paper No. 32, 177-181.
2. Miller, T. L. and R. L. Gall, 1983: Thermally-driven flow in a rotating spherical shell: Axisymmetric states. J. Atmos. Sci., 40, 856-867.
3. Pedlosky, J., 1969: Axially symmetric motion of a stratified, rotating fluid in a spherical annulus of narrow gap. J. Fluid Mech., 36, 401-415.

### Item C., Attachment A of Contract Requirements for Quarterly Reporting:

- (1) Total cumulative costs incurred as of this report date: \$28,765.80
- (2) Estimate of cost to complete contract: \$0.0
- (3) Estimated percentage of physical completion of contract: 100%
- (4) N/A