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MULTIDIMENSIONAL EXPLICIT DIFFERENCE SCHEMES FOR HYPERBOLIC CONSERVATION LAWS

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MULTIDIMENSIONAL EXPLICIT DIFFERENCE SCHEMES FOR HYPERBOLIC CONSERVATION LAWS

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ABSTRACT

First- and second-order explicit difference schemes are derived for a three-dimensional hyperbolic system of conservation laws, without recourse to dimensional factorization. All schemes are upwind biased and optimally stable.

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1. INTRODUCTION

For the solution of initial-value problems governed by hyperbolic conservation equations some fine numerical techniques are available. Most methods are based on a one-dimensional scheme

$$u^{n+1} = L_x(\Delta t)\ u^n \ , \tag{1}$$

integrating the system of conservation laws

$$u_t + \left[f(u)\right]_x = 0 \tag{2}$$

from time t^n to $t^{n+1} = t^n + \Delta t$. A multi-dimensional system like

$$u_t + \left[f(u)\right]_x + \left[g(u)\right]_y + \left[h(u)\right]_z = 0 \tag{3}$$

may always be approximated by a sequence of one-dimensional steps; most commonly used is

$$u^{n+1} = L_z(\Delta t)\ L_y(\Delta t)\ L_x(\Delta t)\ u^n \ , \tag{4.1}$$

$$u^{n+2} = L_x(\Delta t)\ L_y(\Delta t)\ L_z(\Delta t)\ u^{n+1} \ , \tag{4.2}$$

with second-order accuracy in time at every other time-level [1].

The convenience of such a factorization is twofold. Firstly, in developing multi-dimensional methods one may concentrate on one-dimensional operators; secondly, multi-dimensional codes reduce to a sequence of one-dimensional sweeps.

A display of the power of the above approach is found in the review paper by Woodward and Colella [2] on the numerical simulation of two-dimensional, strongly compressible flow.

Considering the success of dimensional factorization we may ask ourselves if there is any point in designing genuinely multi-dimensional methods, i.e. methods that can not be implemented as a sequence of one-dimensional operators. The answer still is "yes", but may very well tend to "no" if computers will continue to grow, in speed and capacity, at the current pace. The same, however, may have been said, more than 20 years ago, on the matter of developing second-order methods. In both cases the increase in complexity of the methods is meant to pay off via an increase in efficiency.

Genuinely multi-dimensional schemes are most efficient in modeling essentially multi-dimensional phenomena in a relatively small region, such as push-pull flow or flow around a sharp corner. If such regions have a strong influence on the overall solution, then the use of a multi-dimensional scheme is recommendable.

In the further sections of this paper I shall indicate how to construct second-order difference schemes in two or three space dimensions. First-order schemes were published earlier [3]; their construction is briefly reviewed.

2. A FIRST-ORDER SCHEME

In order to derive a first-order scheme for Eq.(3), consider the scalar linear equation

$$u_t + au_x + bu_y + cu_z = 0, \quad a > 0 \ , \ b > 0 \ , \ c > 0 \ , \tag{5}$$

in combination with a piecewise uniform initial-value distribution

$$u^n(x,y,z) = u^n_{ijk}, \quad x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}, \ y_{j-\frac{1}{2}} < y < y_{j+\frac{1}{2}}, \ z_{k-\frac{1}{2}} < z < z_{k+\frac{1}{2}}, \tag{6}$$

with $x_{i\pm\frac{1}{2}} = x_i \pm \frac{1}{2}\Delta x$, $y_{j\pm\frac{1}{2}} = y_j \pm \frac{1}{2}\Delta y$, $z_{k\pm\frac{1}{2}} = z_k \pm \frac{1}{2}\Delta z$.

The exact solution of (5) at time t^{n+1} in terms of the initial values at t^n is

$$u^{n+1}(x,y,z) = u^n(x-a\Delta t, y-b\Delta t, z-c\Delta t). \tag{7}$$

Inserting the initial values (6) into (7) yields, for sufficiently small Δt, the following difference scheme:

$$u^{n+1} = [1 - \sigma_x\Delta_x\{1 - \tfrac{1}{2}\sigma_y\Delta_y(1 - \frac{1}{3}\sigma_z\Delta_z) - \tfrac{1}{2}\sigma_z\Delta_z(1 - \frac{1}{3}\sigma_y\Delta_y)\}$$

$$- \sigma_y\Delta_y\{1 - \tfrac{1}{2}\sigma_z\Delta_z(1 - \frac{1}{3}\sigma_x\Delta_x) - \tfrac{1}{2}\sigma_x\Delta_x(1 - \frac{1}{3}\sigma_z\Delta_z)\}$$

$$- \sigma_z\Delta_z\{1 - \tfrac{1}{2}\sigma_x\Delta_x(1 - \frac{1}{3}\sigma_y\Delta_y) - \tfrac{1}{2}\sigma_y\Delta_y(1 - \frac{1}{3}\sigma_x\Delta_x)\}]u^n,$$

$$\sigma_x = a\Delta t/\Delta x < 1 , \quad \sigma_y = b\Delta t/\Delta y < 1 , \quad \sigma_z = c\Delta t/\Delta z < 1 ; \tag{8}$$

here Δ_x, Δ_y, Δ_z denote backward (upwind) differencing in the x-, y-, z-direction.

The terms in the above formula have been especially arranged in order to facilitate their interpretation. The terms in parentheses are one-dimensional updates over a time-step 1/3 Δt, with the first-order upwind scheme. The braced terms are two-dimensional updates over a time-step ½ Δt, and represent the values of u at cell interfaces, averaged both in space and in time. The full operator, between brackets, shows the three-dimensional upwind nature of the scheme.

Godunov [4] has indicated how to transform upwind differencing for a scalar linear convection equation into a method for a nonlinear hyperbolic system of the form (2). For scheme (8) his recipe implies replacement of $\sigma_x\Delta_x u$ by

$$\frac{\Delta t}{\Delta x}\left[(\Delta_x f)^+ + (\nabla_x f)^-\right] \tag{9.1}$$

or

$$\frac{\Delta t}{\Delta x}\left[\Delta_x(f^+) + \nabla_x(f^-)\right] , \tag{9.2}$$

where ∇ denotes forward differencing and the superscripts + and - indicate the splitting of a flux-difference or a flux in parts associated with forward and backward signals, respectively (see [5] and [6]). The expressions $\sigma_y\Delta_y u$ and $\sigma_x\Delta_x u$ are replaced analogously.

Note that (8) can be factorized exactly as indicated in (4):

$$u^{n+1}_{ijk} = (1-\sigma_z\Delta_z)(1-\sigma_y\Delta_y)(1-\sigma_x\Delta_x)\ u^n_{ijk} . \tag{10}$$

While this property in general is lost when extending the scheme to a linear or nonlinear hyperbolic system, it nevertheless shows that dimensional splitting basically is a sound idea.

3. SECOND-ORDER SCHEMES

Second-order schemes for Eq. (3) may be derived, again, by extending schemes for Eq. (5). In order to achieve second-order accuracy in time and space we must use a piecewise linear initial-value distribution (see [7]), i.e.,

$$u^n(x,y,z) = u^n_{ijk} + \frac{x-x_i}{\Delta x}\,\delta_x u^n_{ijk} + \frac{y-y_i}{\Delta y}\,\delta_y u^n_{ijk} + \frac{z-z_k}{\Delta z}\,\delta_z u^n_{ijk},$$

$$x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}} \ , \quad y_{j-\frac{1}{2}} < y < y_{j+\frac{1}{2}} \ , \quad z_{k-\frac{1}{2}} < z < z_{k+\frac{1}{2}} \ ; \qquad (11)$$

here $\delta_x u$, $\delta_y u$ and $\delta_z u$ represent locally averaged differences of u, e.g. $\delta_x = \frac{1}{2}(\Delta_x + \nabla_x)$.

Combining (11) with the exact solution (7) of (5) reveals that the second-order scheme contains terms up to $O[(\Delta t)^2]$ when restricted to one dimension, up to $O[(\Delta t)^3]$ in two dimensions, and up to $O[(\Delta t)^4]$ in three dimensions. The two- and three-dimensional schemes therefore can not be equivalent to products of steps with the one-dimensional scheme.

For simplicity, consider first the two-dimensional scheme; with the terms arranged as in (8) it reads

$$u^{n+1}_{ijk} = [1 - \sigma_x\Delta_x\{1 + \tfrac{1}{2}(1 - \sigma_x)\delta_x - \tfrac{1}{2}\sigma_y\Delta_y[1 + (\tfrac{1}{2} - \tfrac{2}{3}\,\sigma_x)\delta_x + (\tfrac{1}{2} - \tfrac{1}{3}\,\sigma_y)\delta_y]\}$$

$$- \sigma_y\Delta_y\{1 + \tfrac{1}{2}(1 - \sigma_y)\delta_y - \tfrac{1}{2}\sigma_x\Delta_x[1 + (\tfrac{1}{2} - \tfrac{2}{3}\,\sigma_y)\delta_y + (\tfrac{1}{2} - \tfrac{1}{3}\,\sigma_x)\delta_x]\}]u^n_{ijk},$$

$$\sigma_x < 1 \ , \quad \sigma_y < 1 \ , \ \sigma_z < 1. \qquad (12)$$

When changing this scalar scheme into a scheme for a nonlinear system, we must replace $\sigma_x\delta_x u^n_{ijk}$ by

$$\frac{\Delta t}{\Delta x}\ \{\ f[(1+\tfrac{1}{2}\delta_x)u^n_{ijk}] - f[(1-\tfrac{1}{2}\delta_x)u^n_{ijk}]\ \}\ , \qquad (13)$$

a central difference of fluxes inside of volume (ijk). The resulting method is unattractive because of the many intermediate steps needed for a full update; moreover, it is not clear that all steps are relevant for a non-diagonalizable hyperbolic system.

A slight simplification results when the third-order terms on the right-hand side of (12) are taken together, yielding

$$\sigma_x\sigma_y\Delta_x\Delta_y[\tfrac{1}{2}(1-\sigma_x)\delta_x + \tfrac{1}{2}(1-\sigma_y)\delta_y], \qquad (14)$$

and then redistributed in a more convenient fashion over the expressions in braces, e.g. with weight $\frac{1}{2}$ in each expression. The most drastic simplification results by omitting <u>all</u> third-order terms. This does not appear to affect the stability condition of the scheme. Its short-wave stability is still dictated by the first-order scheme (8), while its long-wave stability is slightly improved (dropping (14) causes the long waves to slow down).

The three-dimensional scalar linear version of this simplemost scheme is

$$u_{ijk}^{n+1} = \Big[1 - \sigma_x\Delta_x\{1 + \tfrac{1}{2}(1 - \sigma_x)\delta_x - \tfrac{1}{2}\sigma_y\Delta_y(1 - \tfrac{1}{3}\,\sigma_z\Delta_z) - \tfrac{1}{2}\sigma_z\Delta_z(1 - \tfrac{1}{3}\,\sigma_y\Delta_y)\}$$

$$- \sigma_y\Delta_y\{1 + \tfrac{1}{2}(1 - \sigma_y)\delta_y - \tfrac{1}{2}\sigma_z\Delta_z(1 - \tfrac{1}{3}\,\sigma_x\Delta_x) - \tfrac{1}{2}\sigma_x\Delta_x(1 - \tfrac{1}{3}\,\sigma_z\Delta_z)\}$$

$$- \sigma_z\Delta_z\{1 + \tfrac{1}{2}(1 - \sigma_z)\delta_z - \tfrac{1}{2}\sigma_x\Delta_x(1 - \tfrac{1}{3}\,\sigma_y\Delta_y) - \tfrac{1}{2}\sigma_y\Delta_y(1 - \tfrac{1}{3}\,\sigma_x\Delta_x)\}\Big]u_{ijk}^{n},$$

$$\sigma_x < 1\ ,\quad \sigma_y < 1\ ,\ \sigma_z < 1. \qquad (15)$$

It turns out that a two-dimensional version of this scheme for the Euler equations was derived independently by P. Collela (private communication, 1983) and has already been applied successfully to an aerodynamics flow problem in a curvilinear grid [8].

4. CONCLUSIONS

In the preceding sections it has been shown that fully three-dimensional difference schemes for hyperbolic systems of conservation laws can be constructed on the basis of a simple convection principle.

The resulting schemes are explicit, upwind biased and stable under a combination of one-dimensional Courant-Friedrichs-Lewy conditions. The latter property they share with dimensionally factorized schemes.

A two-dimensional second-order scheme of this kind has been put into practice by Eidelman, Colella and Shreeve [8]. A numerical comparison between factorized and non-factorized schemes remains to be made.

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16. Abstract

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