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I - Ellipsoidal Region

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1. INTRODUCTION

Volume integrals associated with the integration of inhomogeneous Helmholtz equation are of practical interest in determining physical quantities in acoustic, electromagnetic and elastic fields. The inhomogeneous scaler Helmholtz equation takes the following form:

$$\nabla^2\Phi + \alpha^2\Phi = -4\pi\rho(\underset{\sim}{r}) \tag{1}$$

where $\rho(\underset{\sim}{r})$ is the source distribution or density function, ∇^2 and α are the Laplacian and wavenumber, respectively. It is well known [1,2] that a particular solution to Eq. (1) is:

$$\Phi(\underset{\sim}{r}) = \iiint_{\Omega} \rho(\underset{\sim}{r}') \, R^{-1} \exp i\alpha R \, dV' \tag{2}$$

$$R = |\underset{\sim}{r} - \underset{\sim}{r}'|$$

in which $(4\pi R)^{-1} \exp i\alpha R$ is the free space Green's function, and Ω is the region where the source is distributed. The source distribution function $\rho(\underset{\sim}{r})$ can in general be either expanded or approximated in a polynomial form and hence $\rho(\underset{\sim}{r}')$ is normally written as

$$\rho(\underset{\sim}{r}') = (x')^{\lambda} \, (y')^{\mu} \, (z')^{\nu} \tag{3}$$

where λ, μ, ν are integers. The integration of the vector Helmholtz equation is analogous, [2].

The reduction of time harmonic fields of frequency ω in the acoustic and electromagnetic fields to that of integrating the inhomogeneous Helmholtz equation over a given volume can be found in many standard texts [3,4]. A formulation that leads to the required form of volume integration, Eqs. (2,3) such that the elastic fields can be determined has recently been given in [5-7]. Using the dynamic version of the Betti-Rayleigh reciprocal theorem, an integral representation of the displacement field u_i in an elastic medium containing an inhomogeneity can be given in terms of the eigenstrains $\varepsilon^{*(1)}_{ij}$ and eigenforce π^*_j as:

$$u_m(\underset{\sim}{r}') = -\iiint_\Omega C_{jkrs}\, g_{jm,k'}(\underset{\sim}{r},\underset{\sim}{r}')\, \varepsilon^{*(1)}_{rs}(\underset{\sim}{r})\, dV - \iiint_\Omega g_{jm}(\underset{\sim}{r},\underset{\sim}{r}')\, \pi^*_j(\underset{\sim}{r})\, dV \tag{4}$$

where g_{jm} are the spatial part of the free space Green's tensor function. For a linear isotropic elastic medium,

$$g_{jm}(\underset{\sim}{r}-\underset{\sim}{r}') = \frac{1}{4\pi\rho_o\omega^2}\left\{ \beta^2\, \delta_{jm}\, \frac{\exp i\beta R}{R} - \left[\frac{\exp i\alpha R}{R} - \frac{\exp i\beta R}{R}\right]_{,jm} \right\} \tag{5}$$

in which ρ_o is mass density, α and β are wavenumbers for longitudinal and shear waves, respectively. Expanding the eigenstrains and eigenforces in a polynomial of position vector $\underset{\sim}{r}$ yields:

$$\pi^*_j(\underset{\sim}{r}) = A_j + A_{jk}\, x_k + A_{jk\ell}\, x_k x_\ell + \dots \tag{6a}$$

$$\varepsilon^*_{ij}(\underset{\sim}{r}) = B_{ij} + B_{ijk}\, x_k + B_{ijk\ell}\, x_k x_\ell + \dots \tag{6b}$$

and substituting it and (5) in (4), the displacement field is found to be

$$u_m(\underset{\sim}{r}) = f_{mj}(\underset{\sim}{r})\, A_j + f_{mjk}(\underset{\sim}{r})\, A_{jk} + \dots + F_{mij}(\underset{\sim}{r})\, B_{ij} + F_{mijk}(\underset{\sim}{r})\, B_{ijk} + \dots \tag{7}$$

where

$$4\pi\rho_o\omega^2 f_{mj}(r) = -\beta^2 \phi \delta_{mj} + \psi,_{mj} - \phi,_{mj} \tag{8a}$$

$$4\pi\rho_o\omega^2 f_{mjk}(r) = -\beta^2 \phi_k \delta_{mj} + \psi_{k,mj} - \phi_{k,mj} \tag{8b}$$

$$4\pi\rho_o\omega^2 F_{mij}(r) = -[\lambda\alpha^2 \psi,_m \delta_{ij} + 2\mu \beta^2 \phi,_i \delta_{mj} - 2\mu \psi,_{mij} + 2\mu \phi,_{mij}] \tag{8c}$$

$$4\pi\rho_o\omega^2 F_{mijk}(r) = -[\lambda \alpha^2 \psi_{k,m} \delta_{ij} + 2\mu \beta^2 \phi_{k,i} \delta_{mj} - 2\mu \psi_{k,mij} + 2\mu \phi_{k,mij}] \tag{8d}$$

Here, λ, μ are Lamé's constants, and

$$\psi(r) = \iiint_\Omega R^{-1} \exp(i\alpha R)\, dV', \tag{9a}$$

$$\psi_k(r) = \iiint_\Omega x'_k R^{-1} \exp(i\alpha R)\, dV', \tag{9b}$$

$$\psi_{kl\ldots s} = \iiint_\Omega x'_k x'_\ell \ldots x'_s R^{-1} \exp(i\alpha R)\, dV', \tag{9c}$$

$$\phi(r) = \iiint_\Omega R^{-1} \exp(i\beta R)\, dV', \tag{9d}$$

$$\phi_k(r) = \iiint_\Omega x'_k R^{-1} \exp(i\beta R)\, dV', \tag{9e}$$

...

$$\phi_{kl\ldots s} = \iiint_\Omega x'_k x'_\ell \ldots x'_s R^{-1} \exp(i\beta R)\, dV'. \tag{9f}$$

This paper presents results for the volume integrals over a region that is either an ellipsoid, a finite cylinder or a rectangular parallelpipe with semi-axes a_1, a_2 and a_3, Fig. 1. The integrals in (9) are subsequently referred to as the Φ-integrals and they are obtained in series form by expanding R^{-1} exp $(i\alpha R)$ in appropriate Taylor series expansions for regions $r > r'$ and $r < r'$, and by using the multinomial theorem with also the assistance of the classical result of Dyson [8] in the case of an ellipsoid. Certain derivatives of the Φ-integrals that are of interest are also presented.

Professors C. P. Yang and C. Saltzer of the Department of Physics and Mathematics at The Ohio State University, respectively, participated in many helpful discussions.

2. SERIES REPRESENTATION OF THE Φ-INTEGRALS

Let R^{-1} exp $i\alpha R$ be expanded in a Taylor Series expansion for $\bar{r}'$ as

$$R^{-1} \exp i\alpha R = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[x_i' \frac{\partial}{\partial x_i'}\right]^n \left[r^{-1} \exp i\alpha r\right], \qquad (10)$$

for $r > r'$

and in a Taylor Series expansion for $\bar{r}$ as

$$R^{-1} \exp i\alpha R = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(x_i \frac{\partial}{\partial x_i'}\right)^n \left[(r')^{-1} \exp i\alpha r'\right], \qquad (11)$$

for $r < r'$

in which the summation convention is observed and $i = 1,2,3$.

Employing the multinomial theorem as suggested in Ref. [1], the Φ-integrals can be explicitly written as triple sums:

$$\Phi_>(\underset{\sim}{r}) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \sum_{k=0}^{n-\ell} \frac{(-1)^n}{\ell!\ K!\ (n-\ell-k)!} \cdot \frac{\partial^n}{\partial x^\ell \partial y^k \partial z^{n-\ell-k}} \left\{\frac{\exp i\alpha r}{r}\right\} \cdot$$

$$\cdot \iiint_\Omega (x')^\ell (y')^k (z')^{n-\ell-k} \rho(x',y',z')\, dV' ,$$

for $r > r'$ (12)

and

$$\Phi_<(\underset{\sim}{r}) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \sum_{k=0}^{n-\ell} \frac{(-1)^n}{\ell!\ k!\ (n-\ell-k)!} \cdot x^{\ell}\ y^{k}\ z^{n-\ell-k}\ \cdot$$

$$\cdot \iiint_{\Omega} \rho(x',y',z') \frac{\partial^n}{\partial x'^{\ell} \partial y'^{k} \partial z'^{n-\ell-k}} \left\{ \frac{\exp\ i\alpha r'}{r'} \right\} dV' \quad ,$$

$$\text{for } r < r' \tag{13}$$

The Taylor series representations given in Eq. (10) and Eq. (11) converge for the region $r > r'$ and $r < r'$, respectively. The integral $\Phi_>(\underset{\sim}{r})$ in Eq. (12) is normally used to evaluate physical quantities measured at large distance from the region Ω. The apparent singularities present in Eq. (13) appear as $\ell n\ \varepsilon$, ε^{-1}, ε^{-2},... where ε is a small positive number. These singularities disappear, however, if ε is taken to be the radius of a sphere centered around the origin. In evaluating $\Phi_<(\underset{\sim}{r})$ for an ellipsoid, care must be taken in determining the contribution to the integral from the lower limit ε. A further note on this is given at the end of Section 3.

3. INTEGRATION OVER AN ELLIPSOIDAL REGION

The integrals in (12, 13) are of either one of the following forms:

$$\Phi^O = \iiint_{\Omega} (x')^p\ (y')^q\ (z')^s\ dV' \quad , \tag{14}$$

$$\Phi^S = \iiint_{\Omega} \rho(x',y',z') \frac{\partial^n}{\partial x'^{\ell} \partial y'^{k} \partial z'^{n-\ell-k}} \left\{ \frac{\mathrm{Sin}\ \alpha r'}{r'} \right\} dV' \tag{15}$$

$$\Phi^C = \iiint_{\Omega} \rho(x',y',z') \frac{\partial^n}{\partial x'^{\ell} \partial y'^{k} \partial z'^{n-\ell-k}} \left\{ \frac{\mathrm{Cos}\ \alpha r'}{r'} \right\} dV' \tag{16}$$

These integrals can be further evaluated as follows:

(a) $$\phi^0 = \iiint_\Omega (x')^f \, (y')^g \, (z')^h \, dV'$$

$$= \begin{cases} \dfrac{a_1^{f+1} \, a_2^{g+1} \, a_3^{h+1}}{(2m+3)} \, \dfrac{4\pi}{(2m+1)} \, \dfrac{R(f/2)R(g/2)R(h/2)}{R(m)} \\ 0 \quad \textit{If any one of the superscript power f,g, or h is odd.} \end{cases} \tag{17}$$

where a_1, a_2, a_3 are the axes of the ellipsoid, and

$$2m = f + g + h$$

$$R(m) = \frac{(2m)!}{m!} \tag{18}$$

This result was first obtained by Moschovidis [9].

(b) $n = 0$, ϕ^S:

$$\phi^S = \iiint_\Omega \rho(x',y',z') \cdot \frac{\sin \alpha r'}{r'} \, dV'$$

$$= \iiint_\Omega \rho(x',y',z') \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\alpha^{2m-1}}{(2m-1)!} (r')^{2m-2} \, dV'$$

$$= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\alpha^{2m-1}}{(2m-1)!} S_{m,p} \tag{19}$$

where

$$S_{m,p} = \iiint_\Omega (x')^\lambda \, (y')^\mu \, (z')^\nu \, (x'^2+y'^2+z'^2)^{m-1} \, dV'$$

$$= \frac{a_1^{\lambda+1} a_2^{\mu+1} a_3^{\nu+1}}{(\lambda+\mu+\nu+2m+1)} \, \frac{4\pi}{(\lambda+\mu+\nu+2m-1)}$$

$$\sum_{m_1,m_2,m_3}^{\infty} \frac{(m-1)!}{m_1!m_2!m_3!} \frac{a_1^{2m_1} a_2^{2m_2} a_3^{2m_3} (2m_1+\lambda)!\ (2m_2+\mu)!(2m_3+\nu)!\ [2(m-1)+\lambda+\mu+\nu]/2!}{(2m_1+\lambda)/2!\ (2m_2+\mu)/2!\ (2m_3+\nu)/2!\ [2(m-1)+\lambda+\mu+\nu]!}$$

$$m_1+m_2+m_3 = m-1 \quad , \text{ if } \lambda,\mu,\nu \text{ all even,} \tag{19a}$$

$$= 0, \text{ if } \lambda,\mu, \text{ or } \nu \text{ odd.} \tag{19b}$$

in which the multinomial formula

$$(r')^{2m} = (x'^2 + y'^2 + z'^2)^m$$

$$= \sum_{m_1,m_2,m_3} \frac{m!}{m_1!\ m_2!\ m_3!} (x')^{2m_1} (y')^{2m_2} (z')^{2m_3} \tag{20}$$

is used. In (20), the sums are taken over all non-negative integers m_1, m_2 and m_3 for which $m_1+m_2+m_3 = m$.

(c) $n = 0 \quad , \quad \Phi^C$

$$\Phi^C = \iiint_\Omega \rho(x',y',z') \frac{\cos \alpha r'}{r'} dV' \tag{21}$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} \iiint_\Omega (x')^\lambda (y')^\mu (z')^\nu \cdot \frac{(r')^{2m}}{r'} dV'$$

Using the multinomial formula and letting the integral in (21) be denoted by

$$C_{m,p} = \iiint_\Omega (x')^\lambda (y')^\mu (z')^\nu (r')^{2m-1} dV'$$

$$= \sum_{m_1,m_2,m_3,} \frac{m!}{m_1!m_2!m_3!} \iiint_\Omega \frac{(x')^{2m_1+\lambda}(y')^{2m_2+\mu}(z')^{2m_3+\nu}}{r'} dV' \tag{22}$$

The volume integral in Eq. (22) may be viewed as the potential of variable densities observed at the origin, $\underset{\sim}{r}=0$. Applying the results on volume integration over an ellipsoid given by Dyson [8], Eq. (22) can be written as

$$C_{m,p} = \sum_{m_1, m_2, m_3,} \pi\, a_1 a_2 a_3 \cdot \frac{(m)!}{m_1!\; m_2!\; m_3!}\; a_1^{2m_1+\lambda} a_2^{2m_2+\mu} a_3^{2m_3+\nu}$$

$$\cdot \int_0^\infty \frac{\psi^{m+p}}{2^{2(m+p)}\,(m+p)!\,(m+p+1)'}\; \delta^{m+p} \left[\left(\frac{a_1 x}{a_1^2+\psi}\right)^{2m_1+\lambda}, \left(\frac{a_2 y}{a_2^2+\psi}\right)^{2m_2+\mu},\right.$$

$$\left., \left(\frac{a_3 z}{a_3^2+\psi}\right)^{2m_3+\nu}\right] \cdot \frac{d\psi}{Q} \quad , \text{ if } \lambda,\mu,\nu, \text{ all even} \qquad (23.a)$$

$$= 0 \qquad , \text{ if } \lambda,\mu,\nu, \text{ is odd} \qquad (23.b)$$

where $2p = (\lambda+\mu+\nu)$

$$Q^2 = (a_1^2 + \psi)\;(a_2^2 + \psi)\;(a_3^2 + \psi)$$

$$\delta = \frac{a_1^2+\psi}{a_1^2}\,\frac{d^2}{dx^2} + \frac{a_2^2+\psi}{a_2^2}\,\frac{d^2}{dy^2} + \frac{a_3^2+\psi}{a_3^2}\,\frac{d^2}{dz^2}$$

$$\delta^2 = \delta \cdot \delta$$

$$\vdots$$

$$\delta^\ell = \sum_{\ell_1, \ell_2, \ell_3} \frac{\ell!}{\ell_1!\,\ell_2!\,\ell_3!} \left(\frac{a_1^2+\psi}{a_1^2}\right)^{\ell_1} \left(\frac{a_2^2+\psi}{a_2^2}\right)^{\ell_2} \left(\frac{a_3^2+\psi}{a_3^2}\right)^{\ell_3} \cdot$$

$$\cdot \frac{d^{2\ell}}{dx'^{2\ell_1}\; dy'^{2\ell_2}\; dz'^{2\ell_3}} \quad , \qquad (24)$$

in which the sums are taken over non-negative values of ℓ_1, ℓ_2, ℓ_3 for which $\ell_1 + \ell_2 + \ell_3 = \ell$. Using the definition of δ^ℓ, (24), and noting that

$$F = (x')^{2m_1+\lambda}(y')^{2m_2+\mu}(z')^{2m_3+\nu} = \left(\frac{a_1x'}{a_1^2+\psi}\right)^{2m_1+\lambda}\left(\frac{a_2y'}{a_2^2+\psi}\right)^{2m_2+\mu} \cdot \left(\frac{a_3z'}{a_3^2+\psi}\right)^{2m_3+\nu}$$

it can be easily shown that

$$\delta^{m+p}\,F\left(\frac{a_1x'}{a_1^2+\psi},\ \frac{a_2y'}{a_2^2+\psi},\ \frac{a_3z'}{a_3^2+\psi}\right)$$

$$= \frac{(m+p)!}{(m_1+\lambda/2)!\,(m_2+\mu/2)!\,(m_3+\nu/2)!}\ (2m_1+\lambda)!\,(2m_2+\mu)!\,(2m_3+\nu)! \cdot$$

$$\cdot\left(\frac{1}{a_1^2+\psi}\right)^{m_1+\lambda/2} \cdot \left(\frac{1}{a_2^2+\psi}\right)^{m_2+\mu/2} \cdot \left(\frac{1}{a_3^2+\psi}\right)^{m_3+\nu/2} \tag{25}$$

Finally, for n = 0

$$\phi^C = \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!}\ C_{m,p} \tag{26}$$

where

$$C_{m,p} = \sum_{m_1,m_2,m_3,} \frac{\pi m!\,(2m_1+\lambda)!\,(2m_2+\mu)!\,(2m_3+\nu)!}{m_1!\,m_2!\,m_3!\,(m_1+\lambda/2)!\,(m_2+\mu/2)!\,(m_3+\nu/2)!}$$

$$\cdot \frac{a_1^{2m_1+\lambda+1} \; a_2^{2m_2+\mu+1} \; a_3^{2m_3+\nu+1}}{2^{2m+2p} \; (m+p+1)!} \cdot$$

$$\cdot \int_0^\infty \frac{\psi^{m+p} \; d\psi}{(a_1^2+\psi)^{m_1+\lambda/2} \; (a_2^2+\psi)^{m_2+\mu/2} \; (a_3^2+\psi)^{m_3+\nu/2} Q} \cdot \tag{27}$$

(d) $n \neq 0, \quad \Phi^S$:

$$\Phi^S = \iiint_\Omega \rho(x',y',z') \frac{\partial^n}{\partial x'^{\ell} \partial y'^{k} \partial z'^{n-\ell-k}} \quad \frac{\sin \alpha r'}{r'} dV'$$

$$= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\alpha^{2m-1}}{(2m-1)!} \; S_{m,p}^n \tag{28}$$

where

$$S_{m,p}^n = \iiint_\Omega (x')^\lambda (y')^\mu (z')^\nu \cdot \frac{\partial^n}{\partial x'^{\ell} \partial y'^{k} \partial y'^{n-\ell-k}} \cdot (r')^{2m-2} \; dV'$$

$$= \sum_{m_1, m_2, m_3} \frac{(m-1)! \; (2m_1)! \; (2m_2)! \; (2m_3)!}{\ell! \; k! \; (n-\ell-k)!} \cdot$$

$$\cdot \iiint_\Omega (x')^{\lambda+2m_1-\ell} \; (y')^{\mu+2m_2-k} \; (z')^{\nu+2m_3-n+\ell+k} \; dV' \tag{29}$$

in which the multinomial formula is used and m_1, m_2, m_3 are summed over all integers greater than and equal to unity and $m_1+m_2+m_3$ are summed over all integers greater than and equal to unity and $m_1+m_2+m_3 = (m-1)$. The integral in (29) can be obtained by using the formula given in (17), and is easily shown to be

$$S^n_{m,p} = \sum_{m_1,m_2,m_3} \frac{(m-1)!\ (2m_1)!\ (2m_2)!\ (2m_3)!}{\ell!\ k!\ (n-\ell-k)!}\ 4\pi$$

$$\cdot \frac{a_1^{\lambda+2m_1-\ell+1}\ a_2^{\mu+2m_2-k+1}\ a_3^{\nu+2m_3-n+\ell+k+1}}{(2p+2m+1-n)\ (2p+2m-n-1)} \cdot$$

$$\frac{R(\frac{1}{2}(\lambda+2m_1-\ell+1))R(\frac{1}{2}(\mu+2m_2-k-1))R(\frac{1}{2}(\nu+2m_3-n+\ell+k+1))}{R(\frac{1}{2}(p+m-1-n/2))},$$

if $(\lambda-\ell)$, $(\mu-k)$, $(\nu-n+\ell+k)$ all even (30.a)

$= 0$ if $(\lambda-\ell)$, $(\mu-k)$ or $(\nu-n+\ell+k)$ is odd (30.b)

where

$$2p = \lambda+\mu+\nu$$

(e) $n \neq 0$, ϕ^C

$$\phi^C = \iiint_\Omega \rho(x',y',z')\ \frac{\partial^n}{\partial x'^\ell \partial y'^k \partial z'^{n-\ell-k}}\ \frac{\cos \alpha r'}{r'}\ dV'$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} \cdot C^n_{m,p} \tag{31}$$

where

$$C^n_{m,p} = \iiint_\Omega x'^\lambda y'^\mu z'^\nu \frac{\partial^n}{\partial x'^\ell \partial y'^k \partial z'^{n-\ell-k}}\ (r')^{2m-1}\ dV' \tag{32}$$

When $n \neq 0$, it is not as easy to find a compact form for these integrals. For the determination of the elastodynamic fields of an ellipsoidal inhomogeneity as formulated in [6,7] it is sufficient to determine $\Phi_{<}(r)$ for a finite number of n's in determining the B_{ij}, B_{ijk}, and A_j, A_{jk}, ... in [6,7]. For example, if it is necessary to determine the eigenstrains ε^*_{ij} up to a second order distribution, it is then sufficient to find Φ^C for $1 \leq n \leq 6$.

The integral $\Phi_{<}(r)$ in (13) can be replaced for n = k by

$$\Phi_{<}(\underset{\sim}{r}) = \frac{1}{2!} x_p x_q \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} C^{(k)}_{m,p} \tag{33}$$

where

$$C^{(k)}_{m,p} = \iiint_{\Omega} (x')^{\lambda} (y')^{\mu} (z')^{\nu} \cdot \frac{\partial^k (r')^{2m-1}}{\partial x'_p \, \partial x'_q \cdots \partial x'_u} \cdot dV' \tag{34}$$

The substitutions of the derivatives of $(r')^{2m-1}$ in Eq. (34), lead to integrals that can be easily evaluated by using Eqs. (23,24,25), for the cases $m \geq 1$, $k = 1, 3$ and $m \geq 2$, $k = 4$, etc. Special attention must be given to the cases $m = 0$, $k = 2,3$, and $m = 0,1$ $k = 4$.

Using the notations given in Ref.[10] and noting that

$$dV' = d\underset{\sim}{x}' = dr'\, dS = dr' \cdot r'^2\, d\omega , \tag{35}$$

we obtain

$$r'(\ell) = \left(\frac{1}{g}\right)^{1/2} \tag{36}$$

where

$$g = \ell_i^2/a_i^2 \quad , \quad \ell_i = x_i'/r' \quad . \tag{37}$$

and f = 0, e = 1 due to the fact that here we consider the source point being situated at the origin, i.e. $\underset{\sim}{r} = 0$. Volume integrals associated with Eq. (32), $1 \leq n \leq 4$, can be written into surface integrals by using Eq. (35). and finally reduced to simple integrals through the work of Routh [11], e.g.

$$\iiint_{\Omega} \rho \cdot (r')^{-3}\, dV' = \iint \rho \cdot (r')^{-3} \cdot dr' \cdot r'^2 \cdot d\omega$$

$$= \iiint \rho \cdot (r')^{-1}\, dr'\, d\omega$$

If $\rho = 1$, [11, p. 901],

$$\iiint (r')^{-3}\, dV' = \iint_{\Sigma} \{\ell n\; r'(\ell_i) - \ell n\; \varepsilon\}\, d\omega$$

$$= - \iint_{\Sigma} \{\tfrac{1}{2}\, \ell n\;\; g + \ell n\; \varepsilon\}\, d\omega$$

$$= A_1 \tag{38}$$

$$\iiint_{\Omega} (r')^{-5}\, dV' = -\frac{1}{2}\iint_{\Sigma} \{[r'(\ell_i)]^{-2} - \varepsilon^{-2}\}\, d\omega$$

$$= -\frac{1}{2}\iint_{\Sigma} g\, d\omega + A_2$$

$$= -\frac{2\pi}{3} \cdot \frac{1}{a_i a_i} + A_2 \tag{39}$$

The surface integral of the type $\iint_{\Sigma} \ell_1^m\, \ell_2^n\, \ell_3^k\, g^{-1}\, d\omega$ can be reduced to simple integrals as well by using the work of Routh [7] in the same manner as listed in Ref. [6] and therefore will not be repeated here. The constants

The constants A_1 and A_2 are equal to $4\pi(\ell_n a - \ell_n \varepsilon)$ and $+(2\pi/3)(\varepsilon^{-2})$, respectively for a sphere of radius a, where ε is a small positive number. The coefficient of these types of terms, $\ell n\ \varepsilon$, ε^{-1}, ε^{-2}, ..., in the Φ-integral can be shown to be identically zero in a straight forward manner if Ω is a sphere. When Ω is an ellipsoid, the lower limit of integration should be taken from the surface of a small sphere with radius ε, (38,39). The contribution to the Φ-integral from the lower limit can therefore be identified as zero.

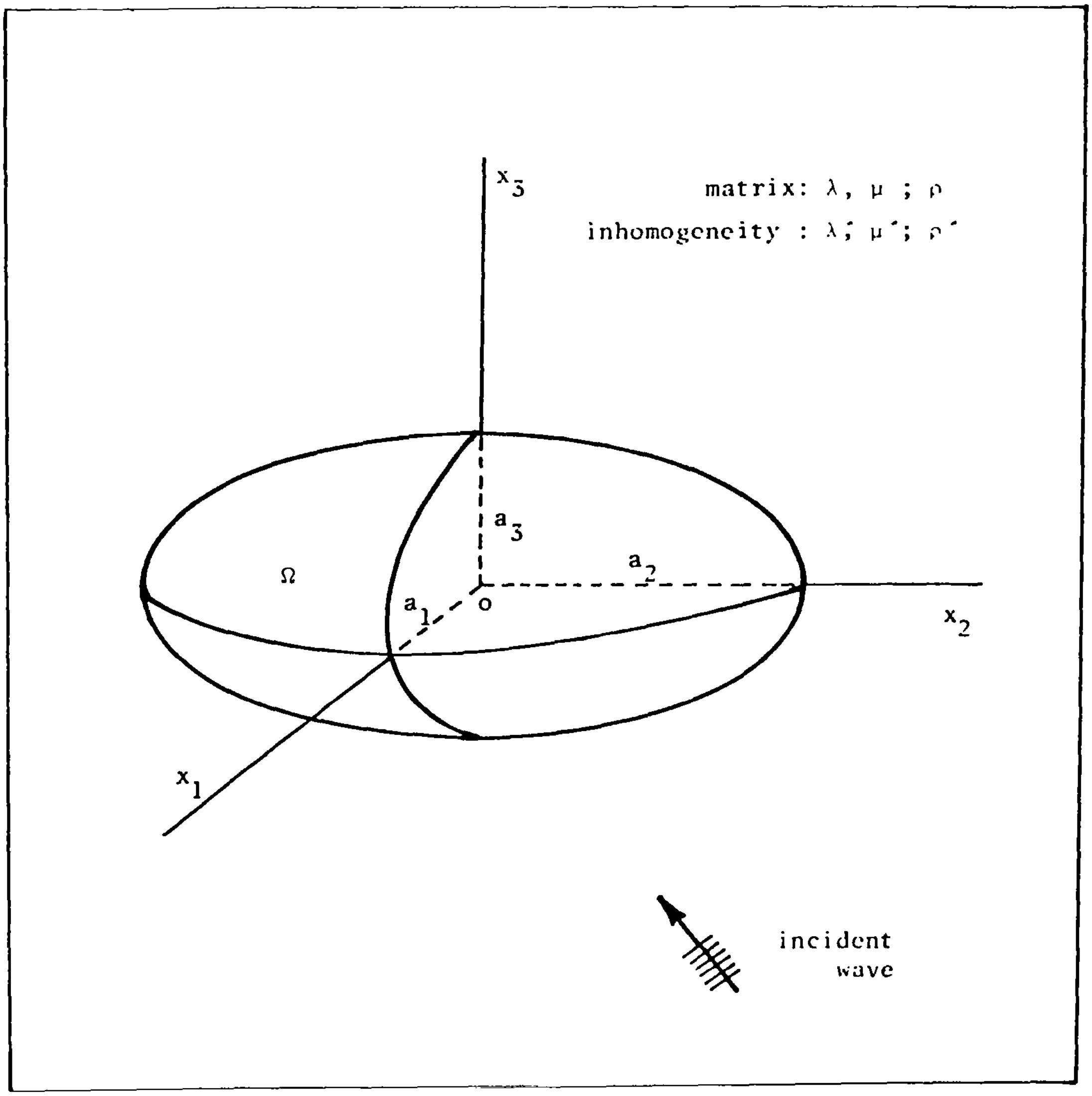

Fig. 1 An ellipsoidal region of integration.

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16. Abstract

Problems of wave phenomena in fields of acoustics, electromagnetics and elasticity are often reduced to an integration of the inhomogenous Helmholtz equation. Results are presented for volume integrals associated with the Helmholtz operator, $\nabla^2 + \alpha^2$, for the cases of an ellipsoidal region, a finite cylindrical region, and a region of rectangular parallelepiped. By using appropriate Taylor series expansions and multinomial theorem, these volume integrals are obtained in series form for regions r > r' and r < r', where r and r' are distances from the origin to the point of observation and source, respectively. Derivatives of these integrals are easily evaluated. When the wave number approaches zero, the results reduce directly to the potentials of variable densities.

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