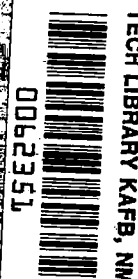


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Volume Integrals Associated With the Inhomogeneous Helmholtz Equation



II - Cylindrical Region; Rectangular Region

W. F. Zhong and L. S. Fu

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Volume Integrals Associated With the Inhomogeneous Helmholtz Equation

II - Cylindrical Region; Rectangular Region

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Prepared for
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I. INTEGRATION OVER A FINITE CYLINDRICAL REGION, Fig. 1

The integrals, Eqs. (12,13) [1], see also [2], are of either one of the following forms:

$$\phi^0 = \iiint_{\Omega} (x')^p (y')^q (z')^s dv' \quad (1)$$

$$\phi^S = \iiint_{\Omega} \rho(x', y', z') \frac{\partial^n}{\partial x'^{\ell} \partial y'^k \partial z'^{n-\ell-k}} \left\{ \frac{\sin \alpha r'}{r'} \right\} dv' \quad (2)$$

$$\phi^C = \iiint_{\Omega} \rho(x', y', z') \frac{\partial^n}{\partial x'^{\ell} \partial y'^k \partial z'^{n-\ell-k}} \left\{ \frac{\cos \alpha \gamma'}{\gamma'} \right\} dv' \quad (3)$$

Letting $x'=x'$, $y'=\zeta \cos \theta$, $z'=\zeta \sin \theta$ and $dv'=dx'dy'dz'=\zeta d\zeta d\theta dx'$, these integrals can be further evaluated as follows: [3]

(a)

$$\begin{aligned} \phi^0 &= \iiint_{\Omega} (x')^p (y')^q (z')^s dv' \\ &= \frac{(q-1)!!(s-1)!!(2\pi)}{(q+s)!!} \left(\frac{a^{2+q+s}}{2+q+s} \right) \left(\frac{2 \ell^{1+p}}{1+p} \right) \quad \text{if } p, q, s \text{ all even} \end{aligned} \quad (4a)$$

$$= 0 \quad \text{if any one of } p, q, s \text{ is odd.} \quad (4b)$$

where according to the definition of factorial,

$$(q-1)!! = \frac{(q-1)!}{2^{\left(\frac{q}{2}-1\right)} \left(\frac{q}{2}-1\right)!} = 1 \cdot 3 \cdot 5 \dots (q-1), \quad q \text{ even} \quad (5a)$$

$$(q+s)!! = 2^{\left(\frac{q+s}{2}\right)} \left(\frac{q+s}{2}\right)! = 2 \cdot 4 \cdot 6 \dots s(s+2) \dots (s+q), \quad q, s \text{ even} \quad (5b)$$

and a is the radius of the cylinder, ℓ the length.

(b) $n=0$, ϕ^S :

$$\begin{aligned} \phi^S &= \iiint_{\Omega} (x')^p (y')^q (z')^s \frac{\sin \alpha \gamma'}{\gamma'} dv' \\ &= \iiint_{\Omega} (x')^p (y')^q (z')^s \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m+1}}{(2m+1)!} (r')^{2m} dv' \end{aligned}$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m+1}}{(2m+1)!} S_{m,p} \quad (6)$$

where

$$S_{m,p} = \iiint_{\Omega} (x')^p (y')^q (z')^s (r')^{2m} dv' \quad (7)$$

Using the multinomial formula, [4]

$$(r')^{2m} = (x'^2 + y'^2 + z'^2)^m = \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} (x')^{2m_1} (y')^{2m_2} (z')^{2m_3} \quad (8)$$

the integral

$$\begin{aligned} S_{m,p} &= \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} \iiint_{\Omega} (x')^{2m_1+p} (y')^{2m_2+q} (z')^{2m_3+s} dv' \\ &= \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} \frac{(\mu-1)!! (\nu-1)!! (2\pi)}{(\mu+\nu)!!} \left(\frac{a}{2+\mu+\nu}\right) \left(\frac{2\ell}{1+\lambda}\right) \\ &\quad m=0, 1, 2, \dots, \text{ if } \lambda, \mu, \nu \text{ all even} \\ &= 0 \quad \text{if any one of } \lambda, \mu, \nu \text{ is odd} \end{aligned} \quad (9)$$

where $(\mu-1)!!$, $(\nu-1)!!$, $(\mu+\nu)!!$ are defined as the same as (5a) and (5b),

and,

$$\begin{aligned} \lambda &= 2m_1 + p \\ \mu &= 2m_2 + q \\ \nu &= 2m_3 + s \\ m &= m_1 + m_2 + m_3 \end{aligned} \quad (10)$$

(c) $n=0$, ϕ^c :

$$\begin{aligned} \phi^c &= \iiint_{\Omega} (x')^p (y')^q (x')^s \frac{\cos \alpha r'}{r'} dv' \\ &= \iiint_{\Omega} (x')^p (y')^q (x')^s \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} \frac{(r')^{2m}}{r'} dv' \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} C_{m,p} \end{aligned} \quad (11)$$

where

$$C_{m,p} = \iiint_{\Omega} (x')^p (y')^q (z')^s \frac{(r')^{2m}}{r'} dv' \\ = \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} \iiint_{\Omega} (x')^{\lambda} (y')^{\mu} (z')^{\nu} \frac{dv'}{r'} \quad (12)$$

$$\lambda = 2m_1 + p$$

$$\mu = 2m_2 + q$$

$$\nu = 2m_3 + s$$

$$m_1 + m_2 + m_3 = m \quad (13)$$

The integral on the right hand side of Eq.(12) can be evaluated as follows:

$$\iiint_{\Omega} (x')^{\lambda} (y')^{\mu} (z')^{\nu} \frac{dv'}{r'} = \\ = \frac{(\mu-1)!!(\nu-1)!!}{(\mu+\nu)!!} (2\pi) \cdot I \quad \text{if } \mu, \nu \text{ all even} \quad (14a)$$

$$= 0 \quad \text{if any one of } \mu, \nu \text{ is odd.} \quad (14b)$$

In the above expression

$$I = \frac{1}{k+1} \sum_{t=1}^{\frac{1}{2}(k-1)} (-1)^{t+1} \frac{(k+1)(k-1)\dots(k-2t+3)}{k(k-2)\dots(k-2t+2)} (a)^{k-2t+1} \int_{-\ell}^{\ell} (x')^{\lambda+2t-2} (a^2+x'^2)^{1/2} dx' \\ + (-1)^{\frac{1}{2}(k-1)} \frac{(k-1)!!}{k!!} \int_{-\ell}^{\ell} (x')^{\lambda+k-1} (a^2+x'^2)^{1/2} {}_{-x'}^{\lambda+k} dx' \quad (15)$$

where

$$k = 1 + \mu + \nu \quad (16)$$

is odd, and

$$(k-1)!! = 2^{\frac{(k-1)}{2}} \cdot \left(\frac{k-1}{2}\right)! = 2 \cdot 4 \dots (k-1) \quad (17a)$$

$$(k)!! = \frac{k!}{2^{\frac{(k-1)}{2}} \left(\frac{k-1}{2}\right)!} = 1 \cdot 3 \dots (k) \quad (17b)$$

The integrals

$$\begin{aligned}
 & \int_{-l}^l (x')^{\lambda+2t-2} (a^2+x'^2)^{1/2} dx' \\
 &= \frac{2}{k_1+1} \sum_{t_1=1}^{\frac{k_1}{2}} (-1)^{t_1+1} \frac{(k+1)(k_1-1)\dots(k_1-2t_1+3)}{(k_1+2)k_1\dots(k_1-2t_1+4)} (a)^{2t_1-2} \cdot (l)^{k_1-2t_1+1} \cdot (a^2+l^2)^{3/2} \\
 &+ (-1)^{\frac{k_1}{2}} \frac{(k_1-1)!!}{(k_1+2)(k_1)!!} 2a^{k_1} (l\sqrt{a^2+l^2} + a^2 \operatorname{sh}^{-1} \frac{l}{a}) \quad \text{if } \lambda, \mu, \nu \text{ are even} \\
 &= 0 \quad \text{if any one of } \lambda, \mu, \nu \text{ is odd.} \quad (18)
 \end{aligned}$$

where

$$k_1 = \lambda + 2t - 2 \quad (19)$$

is even, and

$$(k_1-1)!! = \frac{(k_1-1)!}{2^{\frac{k_1}{2}-1} \cdot (\frac{k_1}{2}-1)!} = 1 \cdot 3 \cdot 5 \dots (k_1-1) \quad (20a)$$

$$(k_1)!! = 2^{\frac{k_1}{2}} \cdot (\frac{k_1}{2})! = 2 \cdot 4 \dots (k_1) \quad k_1 \text{ is even} \quad (20b)$$

The integral

$$\begin{aligned}
 & \int_{-l}^l (x')^{\lambda+k-1} (a^2+x'^2)^{1/2} dx' \\
 &= \int_{-l}^l (x')^{k_2} (a^2+x'^2)^{1/2} dx' \\
 &= \frac{2}{k_2+1} \sum_{t_2=1}^{\frac{k_2}{2}} (-1)^{t_2+1} \frac{(k_2+1)(k_2-1)\dots(k_2-2t_2+3)}{(k_2+2)k_2\dots(k_2-2t_2+4)} (a)^{2t_2-2} \cdot (l)^{k_2-2t_2+1} \cdot (a^2+l^2)^{3/2} \\
 &= (-1)^{\frac{k_2}{2}} \frac{(k_2-1)!!}{(k_2+2) \cdot (k_2)!!} \cdot 2a^{k_2} (l\sqrt{a^2+l^2} + a^2 \operatorname{sh}^{-1} \frac{l}{a}) \quad \text{if } \lambda, \mu, \nu \text{ are even} \quad (21a)
 \end{aligned}$$

$$= 0 \quad \text{if any one of } \lambda, \mu, \nu \text{ is odd.} \quad (21b)$$

where

$$k_2 = \lambda + k - 1 = \lambda + \mu + \nu \quad (22)$$

The integral

$$\int_{-\ell}^{\ell} (x')^{\lambda+k} dx'$$

$$= \frac{2\ell^{\lambda+k+1}}{\lambda+k+1}$$

$$\text{if } \lambda, \mu, \nu \text{ are even} \quad (23a)$$

$$= 0$$

$$\text{if any one of } \lambda, \mu, \nu \text{ is odd.} \quad (23b)$$

II. INTEGRATION OVER A RECTANGULAR PARALLELEPIPED, Fig. 2

In the case of a rectangular parallelepiped, the integrals in (14), (15), (16) can be evaluated as follows:

(a)

$$\begin{aligned}\phi^0 &= \iiint_{\Omega} (x')^p (y')^q (z')^s dv' \\ &= \frac{8}{(p+1)(q+1)(s+1)} (a)^{q+1} (b)^{s+1} (\ell)^{p+1} \quad \text{if } p, q, s \text{ all are even} \quad (24a)\end{aligned}$$

$$= 0 \quad \text{if any one of } p, q, s \text{ is odd.} \quad (24b)$$

where ℓ, a, b is length of the rectangular parallelepiped toward x', y', z' -direction, respectively.

(b) $n=0, \phi^S$

$$\begin{aligned}\phi^S &= \iiint_{\Omega} (x')^p (y')^q (z')^s \frac{\sin \alpha r'}{r'} dv' \\ &= \iiint_{\Omega} (x')^p (y')^q (z')^s \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m+1}}{(2m+1)!} (r')^{2m} \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m+1}}{(2m+1)!} S_{m,p} \quad (25)\end{aligned}$$

and

$$\begin{aligned}S_{m,p} &= \iiint_{\Omega} (x')^p (y')^q (z')^s (r')^{2m} dv' \\ &= \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} \iiint_{\Omega} (x')^{2m_1+p} (y')^{2m_2+q} (z')^{2m_3+s} dv' \\ &= \sum_{m_1, m_2, m_3} \frac{8m!}{m_1! m_2! m_3!} \frac{(a)^{\mu+1} (b)^{\nu+1} (\ell)^{\lambda+1}}{(\lambda+1)(\mu+1)(\nu+1)} \\ & \quad m=0, 1, 2, \dots \quad \text{if } \lambda, \mu, \nu \text{ all are even} \quad (26a)\end{aligned}$$

$$= 0 \quad \text{if any one of } \lambda, \mu, \nu \text{ is odd.} \quad (26b)$$

where

$$\begin{aligned}
m &= m_1 + m_2 + m_3 \\
\lambda &= 2m_1 + p \\
\mu &= 2m_2 + q \\
\nu &= 2m_3 + s
\end{aligned} \tag{27}$$

(c) $n=0$, ϕ^c .

$$\begin{aligned}
\phi^c &= \iiint_{\Omega} (x')^p (y')^q (z')^s \frac{\cos \alpha r'}{r'} dv' \\
&= \iiint_{\Omega} (x')^p (y')^q (z')^s \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} \frac{(r')^{2m}}{r'} \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m}}{(2m)!} C_{m,p}
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
C_{m,p} &= \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} \iiint_{\Omega} (x')^{\lambda} (y')^{\mu} (z')^{\nu} \frac{dv'}{r'} \\
&= \sum_{m_1, m_2, m_3} \frac{m!}{m_1! m_2! m_3!} \int_{-l}^l (x')^{\lambda} \left[\int_{-a}^a (y')^{\mu} \left(\int_{-b}^b \frac{(z')^{\nu}}{\sqrt{\alpha^2 + z'^2}} dz' \right) dy' \right] dx'
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
\lambda &= 2m_1 + p \\
\mu &= 2m_2 + q \\
\nu &= 2m_3 + s \\
\alpha^2 &= x'^2 + y'^2 \\
m_1 + m_2 + m_3 &= m
\end{aligned} \tag{30}$$

The integrals in (34) can be evaluated as follows:

$$\begin{aligned}
(i) & \int_{-b}^b \frac{(z')^{\nu}}{\sqrt{\alpha^2 + z'^2}} dz' \\
&= \Pi_{\nu} (b)^{\nu-2t+1} (x'^2 + y'^2)^{\nu/2} \sqrt{b^2 + x'^2 + y'^2} + (-1)^{\nu/2} \frac{2(\nu-1)!!}{(\nu)!!} (x'^2 + y'^2)^{\nu/2} \operatorname{sh}^{-1} \frac{b}{\sqrt{x'^2 + y'^2}}
\end{aligned}$$

$$=0 \quad \begin{array}{l} \text{if } \nu \text{ is even} \\ \text{if } \nu \text{ is odd.} \end{array} \quad \begin{array}{l} (31a) \\ (31b) \end{array}$$

where Π_ν is defined as an operator as follows:

$$\Pi_\nu = \frac{2}{\nu+1} \sum_{t=1}^{\nu/2} (-1)^{t+1} \frac{(\nu+1)(\nu-1)\dots(\nu-2t+3)}{\nu(\nu-2)(\nu-4)\dots(\nu-2t+2)} \quad (32a)$$

$$(\nu-1)!! = \frac{(\nu-1)!}{2^{\frac{\nu}{2}-1} (\frac{\nu}{2}-1)!} = 1 \cdot 3 \dots (\nu-1) \quad (32b)$$

$$(\nu)!! = 2^{\frac{\nu}{2}} (\frac{\nu}{2})! = 2 \cdot 4 \dots \nu, \quad \nu \text{ is even} \quad (32c)$$

(ii)

$$\begin{aligned} & \int_{-a}^a \int_{-b}^b (y')^\mu (z')^\nu \frac{dy' dz'}{\sqrt{x'^2 + y'^2 + z'^2}} \\ &= \Pi_\nu \sum_{n_1, n_2} (-1)^{\nu/2} \frac{n!}{n_1! n_2!} (b)^{\nu-2t+1} (x')^{2n_1} I_1(x') \\ &+ \sum_{i_1, i_2} (-1)^{\nu/2} \frac{i!}{i_1! i_2!} \frac{2(\nu-1)!!}{(\nu)!!} (b) (x')^{2i_1} I_2(x') \\ &- \sum_{i_1, i_2} (-1)^{\nu/2} \frac{i!}{i_1! i_2!} \frac{(\nu-1)!!}{3(\nu)!!} (b)^3 (x')^{2i_1} I_3(x') \\ &+ \sum_{i_1, i_2} \sum_{j=2}^{\infty} (-1)^{\nu/2+j} \frac{i!}{i_1! i_2!} \frac{2(\nu-1)!!}{(\nu)!!} \frac{(2j)!}{2^{2j} (j!)^2 (2j+1)} (b)^{2j+1} (x')^{2i_1} I_4(x') \quad (33) \end{aligned}$$

where

$$\begin{aligned} I_1(x') &= \int_{-a}^a (y')^{2n_2+\mu} \sqrt{b^2+x'^2+y'^2} dy' \\ &= \Pi_\xi (a)^{\xi-2t_1+1} (b^2+x'^2)^{t_1-1} \sqrt{(a^2+b^2+x'^2)^3} + (-1)^{\xi/2} \frac{2(\xi-1)!!}{(\xi+2)!!} (b^2+x'^2)^{\xi/2} \\ &\cdot (a\sqrt{a^2+b^2+x'^2} + (b^2+x'^2) \operatorname{sh}^{-1} \frac{a}{\sqrt{b^2+x'^2}}) \end{aligned}$$

$$\text{if } \mu \text{ is even} \quad (34a)$$

$$= 0 \quad \text{if } \mu \text{ is odd.} \quad (34b)$$

In the above expression

$$\Pi_{\xi} = \frac{2}{\nu+1} \sum_{t_1=1}^{\xi/2} (-1)^{t_1+1} \frac{(\xi+1)(\xi-1)\dots(\xi+2t_1+3)}{(\xi+2)\xi\dots(\xi-2t_1+4)} \quad (35)$$

$$(\xi-1)!! = 1 \cdot 3 \dots (\xi-1) \quad \xi \text{ is even} \quad (36a)$$

$$(\xi+2)!! = 2 \cdot 4 \dots (\xi+2) \quad \xi \text{ is even} \quad (36b)$$

$$\xi = 2n_2 + \mu \quad (37)$$

The integral

$$I_2(x') = \int_{-a}^a (y')^{2i_2+\mu} \frac{dy'}{\sqrt{x'^2+y'^2}}$$

$$= \Pi_{\xi_1}(a) \xi_1^{-2t_3+1} (x',2)^{t_3-1} \sqrt{a^2+x'^2} + (-1)^{\xi_1/2} \frac{2(\xi_1-1)!!}{(\xi_1)!!} (x',2)^{\xi_1/2} \text{sh}^{-1} \frac{a}{x'} \quad (38a)$$

$$\text{if } \mu \text{ is even} \quad (38a)$$

$$= 0 \quad \text{if } \mu \text{ is odd.} \quad (38b)$$

where Π_{ξ_1} is defined as the same as (32a), only if replace ν with ξ_1 and

$$\xi_1 = 2i_2 + \mu \quad (39a)$$

$$(\xi_1-1)!! = 1 \cdot 3 \dots (\xi_1-1) \quad \xi_1 \text{ is even} \quad (39b)$$

$$(\xi_1)!! = 2 \cdot 4 \dots (\xi_1) \quad \xi_1 \text{ is even} \quad (39c)$$

The integral

$$I_3(x') = \int_{-a}^a \frac{(y')^{\xi_1}}{\sqrt{(x'^2+y'^2)^3}} dy'$$

$$= \Pi_{\xi_1}^{\prime}(a) \xi_1^{-2t_4+1} (x',2)^{t_4-1} \sqrt{a^2+x'^2}$$

$$+ (-1)^{\frac{\xi_1-2}{2}} \frac{2(\xi_1-1)!!}{(\xi_1-2)!!} (x',2)^{\frac{\xi_1}{2}-1} \left(\text{sh}^{-1} \frac{a}{x'} - \frac{a}{\sqrt{a^2+x'^2}} \right)$$

$$\text{if } \mu \text{ is even} \quad (40a)$$

$$= 0 \quad \text{if } \mu \text{ is odd.} \quad (40b)$$

where

$$\Pi_{\xi_1}^I = \frac{2}{\xi_1+1} \sum_{t_4=1}^{\xi_1/2-1} (-1)^{t_4+1} \frac{(\xi_1+1)(\xi_1-1)\dots(\xi_1-2t_4+3)}{(\xi_1-2)(\xi_1-4)\dots(\xi_1-2t_4)} \quad (41)$$

The integral

$$\begin{aligned} I_4(x') &= \int_{-a}^a \frac{(y')^{\xi_1}}{\sqrt{(x'^2+y'^2)^{2j+1}}} dy' \\ &= \Pi_{\xi_1}^{II} (a)^{\xi_1-2t_2+1} (x'^2)^{t_2-1} / \sqrt{(a^2+x'^2)^{2j-1}} \\ &\quad + \sum_{t_0=1}^{j-1} (-1)^{\xi_1/2} \frac{2(\xi_1-1)!! (2)^{t_0}}{(\xi_1-2j)(\xi_1-2j-2)\dots(2-2j)} \frac{(j-1)(j-2)\dots(j-t_0)}{(2j-1)(2j-3)\dots(2j-2t_0+1)} \\ &\quad \cdot (a)(x'^2)^{\frac{\xi_1}{2}-t_0-1} \frac{(a^2+x'^2)^{t_0}}{\sqrt{(a^2+x'^2)^{2j-1}}} \\ &\quad + (-1)^{\xi_1/2} \frac{2(\xi_1-1)!!}{(\xi_1-2j)(\xi_1-2j-2)\dots(2-2j)(2j-1)} (a)(x'^2)^{\frac{\xi_1}{2}-1} \cdot \frac{1}{\sqrt{(a^2+x'^2)^{2j-1}}} \end{aligned}$$

$$\text{if } \mu \text{ is even} \quad (42a)$$

$$= 0 \quad \text{if } \mu \text{ is odd} \quad (42b)$$

where

$$\Pi_{\xi_1}^{III} = \frac{2}{\xi_1+1} \sum_{t_0=1}^{\xi_1/2} (-1)^{t_0+1} \frac{(\xi_1+1)(\xi_1-1)\dots(\xi_1-2t_0+3)}{(\xi_1-2j)(\xi_1-2j-2)\dots(\xi_1-2j-2t_0+2)} \quad (43)$$

(iii)

$$\begin{aligned}
& \iiint_{\Omega} (x')^{\lambda} (y')^{\mu} (z')^{\nu} \frac{dv'}{\sqrt{x'^2 + y'^2 + z'^2}} \\
&= \pi_{\nu} \pi_{\xi} \sum_{n_1, n_2} (-1)^{\nu/2} \frac{n!}{n_1! n_2!} (a)^{\xi - 2t_1 + 1} \cdot (b)^{\nu - 2t + 1} \cdot I_1 \\
&+ (-1)^{\xi/2} \pi_{\nu} \frac{2(\xi - 1)!!}{(\xi + 2)!!} (b)^{\nu - 2t + 1} \cdot I_2 \\
&+ \pi_{\xi_1} \sum_{i_1, i_2} (-1)^{\nu/2} \frac{i!}{i_1! i_2!} \frac{2(\nu - 1)!!}{(\nu)!!} (a)^{\xi_1 - 2t_2 + 1} \cdot (b) \cdot I_3 \\
&+ \sum_{i_1, i_2} (-1)^{\frac{\nu + \xi_1}{2}} \frac{i!}{i_1! i_2!} \frac{2(\nu - 1)!!}{(\nu)!!} \frac{2(\xi_1 - 1)!!}{(\xi_1)!!} (b) \cdot I_4 \\
&- \pi_{\xi_1} \sum_{i_1, i_2} (-1)^{\nu/2} \frac{i!}{i_1! i_2!} \frac{(\nu - 1)!!}{3(\nu)!!} (a)^{\xi_1 - 2t_4 + 1} (b)^3 \cdot I_5 \\
&- \sum_{i_1, i_2} (-1)^{\frac{\xi_1 + \nu - 1}{2}} \frac{i!}{i_1! i_2!} \frac{(\nu - 1)!!}{3(\nu)!!} \frac{2(\xi_1 - 1)!!}{(\xi_1 - 2)!!} (b)^3 \cdot I_6 \\
&+ \pi_{\xi_1} \sum_{i_1, i_2} \sum_{j=2}^{\infty} (-1)^{j + \frac{\nu}{2}} \frac{i!}{i_1! i_2!} \frac{2(\nu - 1)!!}{(\nu)!!} \frac{(2j)!}{2^{2j} (j!)^2 (2j+1)} (a)^{\xi_1 - 2t_2 + 1} \cdot (b)^{2j+1} \cdot I_7 \\
&+ \sum_{i_1, i_2} \sum_{j=2}^{j-1} \sum_{t_0=1}^{t_0=1} (-1)^{j + \frac{\nu}{2} + \frac{\xi_1}{2}} \frac{i!}{i_1! i_2!} \frac{2(\nu - 1)!!}{(\nu)!!} \frac{2(\xi_1 - 1)!!}{(\xi_1 - 2j)(\xi_1 - 2j - 2) \dots (2 - 2j)} \\
&\cdot \frac{(2j)!}{2^{2j} (j!)^2 (2j+1)} \frac{(j-1)(j-2) \dots (j-t_0)}{(2j-1)(2j-3) \dots (2j-2t_0+1)} (a)(b)^{2j+1} \cdot I_8 \\
&+ \sum_{i_1, i_2} \sum_{j=2}^{\infty} (-1)^{j + \frac{\nu}{2} + \frac{\xi_1}{2}} \frac{i!}{i_1! i_2!} \frac{2(\nu - 1)!!}{(\nu)!!} \frac{(2j)!}{2^{2j} (j!)^2 (2j+1)} \\
&\cdot \frac{2(\xi_1 - 1)!!}{(\xi_1 - 2j)(\xi_1 - 2j - 2) \dots (2 - 2j)(2j-1)} (a)(b)^{2j+1} \cdot I_9
\end{aligned} \tag{44}$$

The integrals in (44) can be evaluated as in Appendix.

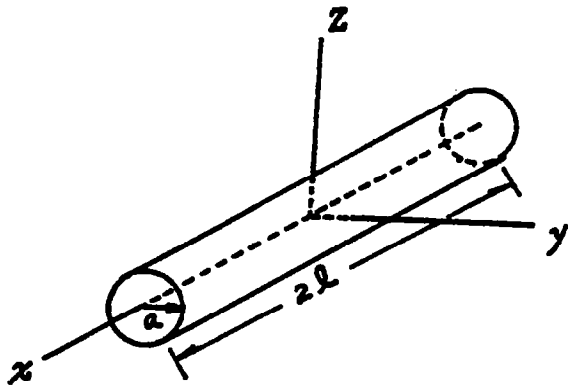


FIG. 1

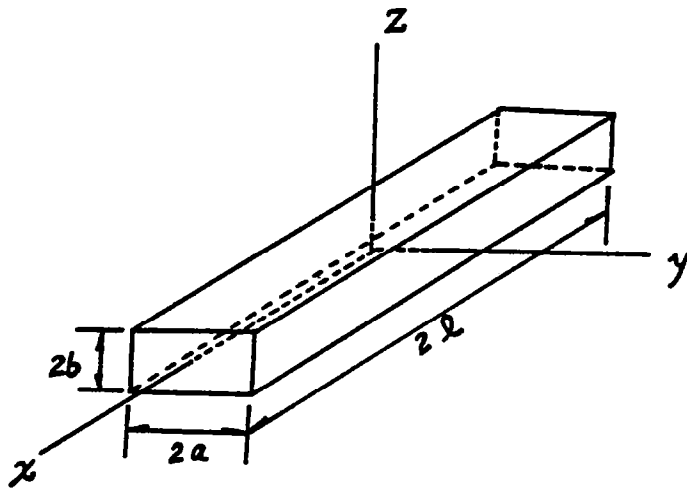


FIG. 2

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APPENDIX

The I-Integrals and π -Functions

(a) The I-integrals

$$\begin{aligned}
 (i) \quad I_1 &= \int_{-\ell}^{\ell} (x')^{\lambda+2\eta} (b^2+x'^2)^{t_1-1} \frac{dx'}{\sqrt{(a^2+b^2+x'^2)^3}} \\
 &= \pi_{\eta, \Gamma_1, \Gamma_2} \sum \frac{\Gamma!}{\Gamma_1! \Gamma_2!} (b)^{2\Gamma_1} (a^2+b^2)^{t_3-1} \cdot (\ell)^{\eta-2t_3+1} \frac{1}{\sqrt{(a^2+b^2+\ell^2)^5}} \\
 &\quad + \sum_{\Gamma_1, \Gamma_2} (-1)^{\eta/2} \frac{\Gamma!}{\Gamma_1! \Gamma_2!} \frac{8(\eta-1)!!}{(\eta+4)!!} (b)^{2\Gamma_1} (a^2+b^2)^{\eta/2} \left[\frac{1}{2} \ell \cdot \sqrt{(a^2+b^2+\ell^2)^3} + \right. \\
 &\quad \left. + \frac{3}{4}(a^2+b^2) (\ell \sqrt{a^2+b^2+\ell^2} + (a^2+b^2) \operatorname{sh}^{-1} \frac{\ell}{\sqrt{a^2+b^2}}) \right]
 \end{aligned}$$

if λ is even

= 0

if λ is odd

in which the π -function π_{η} is defined in (b) of the Appendix.

$$(ii) \quad I_2 = I_2' + I_2''$$

$$\begin{aligned}
 I_2' &= \int_{-\ell}^{\ell} a(x')^{\lambda+2\eta_1} (b^2+x'^2)^{\xi/2} \frac{dx'}{\sqrt{a^2+b^2+x'^2}} \\
 &= \pi_{\eta_1, \theta_1, \theta_2} \sum \frac{\theta!}{\theta_1! \theta_2!} (a)(b)^{2\theta} (a^2+b^2)^{t_4-1} \cdot (\ell)^{\eta_1-2t_4+1} \frac{1}{\sqrt{(a^2+b^2+\ell^2)^3}} \\
 &\quad + \sum_{\theta_1, \theta_2} (-1)^{\eta_1/2} \frac{\theta!}{\theta_1! \theta_2!} \frac{2(\eta_1-1)!!}{(\eta_1+2)!!} (a)(b)^{2\theta} (a^2+b^2)^{\eta_1/2} \\
 &\quad \cdot (\ell \sqrt{a^2+b^2+\ell^2} + (a^2+b^2) \operatorname{sh}^{-1} \frac{\ell}{\sqrt{a^2+b^2}})
 \end{aligned}$$

if λ is even

= 0

if λ is odd.

$$\begin{aligned}
I_2'' &= \int_{-\ell}^{\ell} (x')^{\lambda+2n_1} (b^2+x'^2)^{3\xi/2} \text{sh}^{-1} \frac{a}{\sqrt{b^2+x'^2}} dx' \\
&= 2 \sum_{e_1, e_2} \frac{e!}{e_1! e_2!} \frac{a(b)^{2\theta_1} \ell^{\eta_3+1}}{\eta_3+1} - \frac{1}{3} \sum_{e_1! e_2!} \frac{e!}{e_1! e_2!} \left(\sum_{t_6=1}^{\eta_3/2} (-1)^{t_6+1} \right. \\
&\quad \cdot \frac{(b^2)^{t_6-1} \ell^{\eta_3-2t_6+1}}{\eta_3-2t_6+1} + (-1)^{\eta_3/2} (b)^{\eta_3-1} \cdot t_g^{-1} \frac{\ell}{b} (a)^3 (b)^{2e_1} + \\
&\quad + \sum_{e_1, e_2} \sum_{j_1=2}^{\infty} (-1)^{j_1} \frac{(2j_1)!}{2^{2j_1} j_1! (2j_1+1)} \frac{e!}{e_1! e_2!} (a)^{2j_1+1} \\
&\quad \cdot \left(\Pi_{\eta_3} \frac{(b^2)^{t_7-1} \ell^{\eta_3-2t_7+1}}{(b^2+\ell^2)^{j_1-1}} + (-1)^{\eta_3/2} \frac{(\eta_3-1)!!}{(\eta_3-2j_1+1)(\eta_3-2j_1-1)\dots(3-2j_1)} \right. \\
&\quad \cdot (b)^{\eta_3} (\Pi_{\eta_3}'(\ell) + (-1)^{j_1-1} \frac{2}{b} t_g^{-1} \frac{\ell}{b}) (b)^{2e_1} \\
&\qquad\qquad\qquad \text{if } \lambda \text{ is even} \\
&= 0 \qquad\qquad\qquad \text{if } \lambda \text{ is odd.}
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \quad I_3 &= \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+2t_3-2} \frac{a}{\sqrt{a^2+x'^2}} dx' \\
&= \Pi_{\eta_2} (a^2)^{t_5-1} \ell^{\eta_2-2t_5+1} \frac{a}{\sqrt{(a^2+\ell^2)^3}} + (-1)^{\eta_2/2} \frac{2(\eta_2-1)!!}{(\eta_2+2)!!} (a^2)^{\eta_2/2} \\
&\quad \cdot (\ell \sqrt{a^2+\ell^2} + a^2 \text{sh}^{-1} \frac{\ell}{a}) \qquad \text{if } \lambda \text{ is even} \\
&= 0 \qquad\qquad\qquad \text{if } \lambda \text{ is odd.}
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad I_4 &= \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+\xi_1} \text{sh}^{-1} \frac{a}{x'} dx' \\
&= \frac{a}{\eta_9+1} \left[\Pi_{\eta_9} (a^2)^{t_{15}-1} \ell^{\eta_9-2t_{15}+1} \frac{a}{\sqrt{a^2+\ell^2}} + (-1)^{\eta_9/2} \frac{2(\eta_9-1)!!}{(\eta_9)!!} (a)^{\eta_9} \text{sh}^{-1} \frac{\ell}{a} \right] \\
&\qquad\qquad\qquad \text{if } \lambda \text{ is even} \\
&= 0 \qquad\qquad\qquad \text{if } \lambda \text{ is odd.}
\end{aligned}$$

$$\begin{aligned}
(v) \quad I_5 &= \int_{-l}^l (x')^{\lambda+2i_1+2t_4-2} \frac{dx'}{\sqrt{a^2+x'^2}} \\
&= \Pi_{\eta_8} (a)^{t_{14}-1} \cdot (\ell)^{\eta_8-2t_{14}-1} \cdot \sqrt{a^2+\ell^2} + (-1)^{\eta_8/2} \frac{2(\eta_8-1)!!}{(\eta_8)!!} (a^2)^{\eta_8/2} \text{sh}^{-1} \frac{\ell}{a} \\
& \qquad \qquad \qquad \text{if } \lambda \text{ is even} \\
&= 0 \qquad \qquad \qquad \text{if } \lambda \text{ is odd.}
\end{aligned}$$

(vi)

$$I_6 = I'_6 - I''_6$$

$$\begin{aligned}
I'_6 &= \int_{-l}^l (x')^{\lambda+2i_1+\xi_1-2} \text{sh}^{-1} \frac{a}{x'} dx' \\
&= \frac{a}{\eta_{10}+1} \left[\Pi_{\eta_{10}} (a^2)^{t_{16}-1} \cdot (\ell)^{\eta_{16}-2t_{16}+1} \cdot \sqrt{a^2+\ell^2} + (-1)^{\eta_{10}/2} \frac{2(\eta_{10}-1)!!}{(\eta_{10})!!} \cdot \right. \\
& \quad \left. \cdot (a)^{\eta_{10} \text{sh}^{-1} \frac{\ell}{a}} \right] \qquad \qquad \qquad \text{if } \lambda \text{ is even} \\
&= 0 \qquad \qquad \qquad \text{if } \lambda \text{ is odd.}
\end{aligned}$$

$$\begin{aligned}
I''_6 &= \int_{-l}^l (x')^{\lambda+2i_1+\xi_1-2} \frac{a}{\sqrt{a^2+x'^2}} dx' \\
&= \Pi_{\eta_5} (a)^{t_9} \cdot (\ell)^{\eta_5-2t_9-1} \cdot \sqrt{a^2+\ell^2} + (-1)^{\eta_5/2} \frac{2(\eta_5-1)!!}{(\eta_5)!!} (a)^{\eta_5/2} \text{sh}^{-1} \frac{\ell}{a} \\
& \qquad \qquad \qquad \text{if } \lambda \text{ is even} \\
&= 0 \qquad \qquad \qquad \text{if } \lambda \text{ is odd.}
\end{aligned}$$

$$\begin{aligned}
(vii) \quad I_7 &= \int_{-l}^l (x')^{\lambda+2i_1+2t_2-2} \frac{dx'}{\sqrt{(a^2+x'^2)^{2j-1}}} \\
&= \Pi_{\eta_6} (a^2)^{t_{10}-1} \frac{(\ell)^{\eta_6-2t_{10}+1}}{\sqrt{(a^2+\ell^2)^{2j-3}}} + (-1)^{\eta_6/2} \frac{2(\eta_6-1)!!}{(\eta_6-2j-2)(\eta_6-2j-4)\dots(-2j)}
\end{aligned}$$

$$\frac{a^{\eta_{6\ell}}}{(2j-3)a^2\sqrt{(a^2+\ell^2)}2^{j-3}} \left(1 + \sum_{k_{11}=1}^{j-2} \frac{8^{k_{11}}(j-2)(j-3)\dots(j-k_{11}-1)}{(2j-5)(2j-7)\dots(2j-2k_{11}-3)} \frac{(a^2+\ell^2)^{k_{11}}}{(4a^2)^{k_{11}}}\right)$$

if λ is even

= 0

if λ is odd.

$$(viii) \quad I_8 = \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+\xi_1-2t_0-2} \frac{(a^2+x'^2)^{t_0} dx'}{\sqrt{(a^2+x'^2)}2^{j-1}}$$

$$= \Pi_{\eta_7} \sum_{\phi_1, \phi_2} \frac{\phi_1!}{\phi_1! \phi_2!} \frac{(a^2)^{t_{12}-1+\phi_1} \binom{\eta_7-2t_{12}+1}{\ell}}{\sqrt{(a^2+\ell^2)}2^{j-3}} + \sum_{\phi_1, \phi_2} \frac{\phi_1!}{\phi_1! \phi_2!}$$

$$\cdot (-1)^{\eta_7/2} \frac{2(\eta_7-1)!!}{(\eta_7-2j-2)(\eta_7-2j-4)\dots(-2j)}$$

$$\frac{(a^2)^{\eta_7/2+\phi_1} \ell}{(2j-3)a^2\sqrt{(a^2+\ell^2)}2^{j-3}} \left(1 + \sum_{k_{13}=1}^{j-2} \frac{8^{k_{13}}(j-2)(j-3)\dots(j-k_{13}-1)(a^2+\ell^2)^{k_{13}}}{(2j-5)(2j-7)\dots(2j-2k_{13}-3)(4a^2)^{k_{13}}}\right)$$

if λ is even

= 0

if λ is odd.

$$(ix) \quad I_9 = \int_{-\ell}^{\ell} (x')^{\lambda+2i_1+\xi_1-2} \frac{dx'}{\sqrt{(a^2+x'^2)}2^{j-1}}$$

$$= \Pi_{\eta_8} \frac{(a^2)^{t_{12}-1} \binom{\eta_8-2t_{12}+1}{\ell}}{\sqrt{(a^2+\ell^2)}2^{j-1}} + (-1)^{\eta_8/2} \frac{2(\eta_8-1)!!}{(\eta_8-2j-2)(\eta_8-2j-4)\dots(-2j)}$$

$$\frac{a^{\eta_{8\ell}}}{(2j-3)a^2\sqrt{(a^2+\ell^2)}2^{j-3}} \left(1 + \sum_{k_{12}=1}^{j-2} \frac{8^{k_{12}}(j-2)(j-3)\dots(j-k_{12}-1)(a^2+\ell^2)^{k_{12}}}{(2j-5)(2j-7)\dots(2j-2k_{12}-3)(4a^2)^{k_{12}}}\right)$$

if λ is even

= 0

if λ is odd.

(b) The π -Functions

The π -functions in the I-integrals listed in (a) of this Appendix are defined as follows:

$$(i) \quad \Pi_{\eta} = \frac{2}{\eta+1} \sum_{t_3=1}^{\eta/2} (-1)^{t_3} \frac{(\eta+1)(\eta-1)\dots(\eta-2t_3+3)}{(\eta+4)(\eta+2)\dots(\eta-2t_3+6)}$$

$$\eta = \lambda + 2n_2 + 2\Gamma_2 \quad (\text{is even})$$

$$\Gamma_1 + \Gamma_2 = \Gamma = t_1 - 1$$

$$(ii) \quad \Pi_{\eta_1} \text{ as the same as (35), if replace } \xi \text{ with } \eta_1$$

$$\eta_1 = \lambda + 2n_1 + 2\theta_2 \quad (\text{is even})$$

$$\theta_1 + \theta_2 = \theta = \xi/2$$

(iii)

$$\Pi_{\eta_3} = \frac{2}{\eta_3+1} \sum_{t_7=1}^{\eta_3/2} (-1)^{t_7} \frac{t_7+1}{(n_3-2j_1+1)(n_3-2j_1-1)\dots(n_3-2j_1-2t_7+3)}$$

$$\Pi'_{\eta_3} = \frac{4}{3-2(j_1+1)} \sum_{t_8=1}^{j_1-1} (-1)^{t_8} \cdot (2)^{t_8-1} \cdot \frac{[3-2(j_1+1)][3-2j_1]\dots[3-2(j_1-t_8+2)]}{(j_1-1)(j_1-2)\dots(j_1-t_8)}$$

$$\eta_3 = \lambda + 2n_1 + 2e_2 \quad (\text{is even})$$

$$e_1 + e_2 = e = \frac{3}{2} \xi$$

$$(iv) \quad \Pi_{\eta_2} \text{ as the same as (35), if replace } \xi \text{ with } \eta_2$$

$$\eta_2 = \lambda + 2i_1 + 2t_3 - 2 \quad (\text{is even})$$

$$(v) \quad \Pi_{\eta_9} \text{ as the same as (32a), if replace } \nu \text{ with } \eta_9$$

$$\eta_9 = \lambda + 2i_1 + \xi_1 \quad (\text{is even})$$

(vi) Π_{η_8} as the same as (32a), if replace ν with η_8

$$\eta_8 = \lambda + 2i_1 + 2t_4 - 2 \quad (\text{is even})$$

(vii) $\Pi_{\eta_{10}}$ as the same as (32a), if replace ν with η_{10}

$$\eta_{10} = \lambda + 2i_1 + \xi_1 - 2 \quad (\text{is even})$$

(viii) Π_{η_5} as the same as (32a), if replace ν with η_5

$$\eta_5 = \lambda + 2i_1 + \xi_1 - 2 \quad (\text{is even})$$

$$(ix) \quad \Pi_{\eta_6} = \frac{2}{\eta_6+1} \sum_{t_{10}=1}^{\eta_6/2} (-1)^{t_{10}+1} \frac{(\eta_6+1)(\eta_6-1)\dots(\eta_6-2t_{10}+3)}{(\eta_6-2j-2)(\eta_6-2j-4)\dots(\eta_6-2j-2t_{10})}$$

$$\eta_6 = \lambda + 2i_1 + 2t_2 - 2 \quad (\text{is even})$$

(x) Π_{η_7} as the same as Π_{η_6} , if replace η_6 with η_7

$$\eta_7 = \lambda + 2i_1 + \xi_1 + 2\phi_2 - 2t_0 - 2 \quad (\text{is even})$$

$$\phi_1 + \phi_2 = \phi = t_0$$

(xi) Π_{η_8} as the same as Π_{η_6} , if replace η_6 with η_8

$$\eta_8 = \lambda + 2i_1 + \xi_1 - 2 \quad (\text{is even})$$

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