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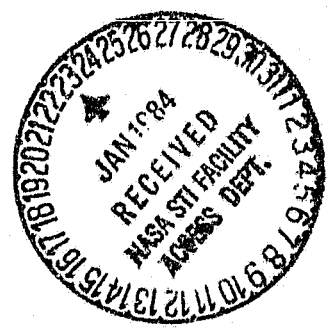
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DIGITAL MULTI-SHAKER MODAL TESTING

by

Mark Blair
Roy R. Craig, Jr.

DIGITAL MULTI-SHAKER MODAL TESTING

A Report to

NASA Marshall Space Flight Center

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by

Mark Blair[*]

Roy R. Craig, Jr.[†]

ASE-EM Department
The University of Texas at Austin
Austin, Texas   78712

[*] Graduate Student, ASE
[†] Professor, ASE-EM

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Introduction

In the design of a structure it is important to know how the structure will respond to forces that are expected to be present in its operating environment. Much time is spent developing analytical models that predict responses, and when the structure is finally built it needs to be physically tested before being placed into operation. This modal testing has become very important in the last few decades with the structures becoming more complex and being placed in more extreme operating environments.

In Chapter 1 of this thesis, a review of several modal testing techniques will be made, along with brief discussions of their advantages and limitations. In Chapter 2, a new technique will be presented which overcomes many of the previous limitations. Several simulated experiments are included in Chapter 3 to verify the validity and accuracy of the new method. Finally, in Chapter 4 conclusions are drawn from the simulation studies and recommendations for further work are presented.

The complete computer code configured for the simulation study is presented in an Appendix.

Acknowledgement

Chapter 1

1.1   Overview

The need for modal testing (sometimes referred to as modal analysis) of structures is well known and documented. The problem is that there exists no fool-proof and totally efficient method to extract modal coefficients from structures. Over the past 40 years several different types of analysis have been developed and practiced with varying degrees of success. The analyses range from time-domain to frequency-domain, and from single-excitation to multiple-excitation. To employ each of these methods properly, it is necessary to examine them to find their respective advantages and disadvantages.

1.2   Multi-Shaker Sine Dwell Testing

The oldest and most widely used method is the Multi-Shaker Sine Dwell (MSSD) test, first described by Lewis and Wrisley (1). This method uses shakers attached at several points on the structure. The shakers are tuned in amplitude and phase simultaneously at a given frequency such that the acceleration outputs are 90 degrees out of phase with the corresponding force inputs (this is indicated by a circle on a Lissajous plot). The amplitudes and phases are then recorded giving the mode shape of the structure at that given frequency. The damping can also be obtained by simultaneously shutting off all the applied forces and analyzing the resultant free-decay rate.

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The advantages of this method are that a very accurate mode shape can be obtained due to the high signal-to-noise ratio, and input levels can be varied to permit identification of pseudo-modal parameters for non-linear structures. The disadvantages, though, usually outweigh these benefits. To measure the mode shapes accurately, accessibility to several shakers is required, and in some instances the attachment of a shaker at a given location is physically impossible. A second drawback is the time involved in stepping through the frequency range one mode at a time and tuning and re-tuning each shaker to obtain a 90 degree phase lag (or lead).

A revision of the MSSD method by Anderson (2) speeds up the tuning procedure. In this method, interest is placed on maximizing the quadrature part of the output to input relationship. Several closely-spaced modes are tuned in conjunction with one another with the number of shakers used being dependent upon the number of modes being tuned at a time. Though the time required by this method is less, it is still an iterative procedure, which, in itself, makes this method time consuming.

1.3   Time-Domain Analysis

As far as time-domain analyses go, there are basically two types, the Least-Squares Curve-Fitting Time-Domain (LSTD) Method (3) and the eigenvalue Ibrahim Time Domain (ITD) Method (4). The LSTD method is an older and more widely known method than ITD, though ITD is more sophisticated and has fewer limitations.

The LSTD method uses the time history of a free-decay response and upon an initial approximation of the modal parameters, it performs an iterative solution routine, based on the Newton-Raphson procedure, to

produce accurate estimates. The problems with this method are the computer time involved for the iterations (up to 50), the need for user-supplied initial approximations, and the reported high inaccuracies after the first two or three modes (3).

A revision of the LSTD method has recently been developed by Smith (3). In this revision a new type of Newton-Raphson iterative procedure is used. This method is also semi-automatic, with the computer analyzing an FFT of the free-decay response to gather the initial natural frequencies (the damping estimates are still user-supplied). This revision, though, also has difficulty in separating closely-coupled modes. If two peaks cannot be resolved in the initial FFT, then one of the two modes will be omitted from the analysis. Also, to properly identify two closely-spaced modes, the time history must include one or two beat frequency cycles, thus causing a significant increase in the length of the required data record.

The ITD method uses time histories of free-decay responses and time-delayed free-decay responses of several degrees-of-freedom (DOF) to set up an eigenvalue problem that solves for the modal coefficients. To solve for more modes than there are measured DOF, ITD creates pseudo-DOF by using time-delayed responses based on the original responses. This also provides the means for determining the validity of each acquired mode by a comparison of mode shape coefficients.

The ITD method has proven to be an effective tool in modal analysis. There is a problem, though, in the use of this method: it is fairly complicated. For each test there are seven analysis parameters to choose. Also, when there is a low signal-to-noise ratio, ITD produces some systematic errors that must be compensated for by several additional analyses (5).

1.4   Frequency-Domain Analysis

Frequency-domain analysis encompasses several different styles of curve-fitting or eigenvalue analysis. The common features among all the methods are their similarity of excitation and their use of curve-fitting of frequency response functions (FRFs). The structure is commonly excited by one shaker with an input signal in the form of a swept sine wave or some type of random noise. Sometimes, though, the structure may be excited by a single impulse, usually administered by a hammer with a force cell attached.

FRFs are usually acquired by a single-input, single-output procedure. The system is either excited by a hammer or a single point random (SPR) shaker set-up (6). By dividing the cross-spectrum of the input and output by the auto-spectrum of the input an accurate estimate of the FRF can be obtained.

Another procedure for gathering transfer functions has been described by Allemang (7). In this method the structure is excited simultaneously by several random-noise-driven shakers. The excitations and responses are measured and then manipulated.

$$\{\ddot{X}\} = [H] \{F\}$$
$$[GXF] = [H] \, \llcorner GFF]$$
$$[H] = [GXF] [GFF]^{-1}$$

where,

$[GXF]$ = m x n frequency dependent matrix
$[GXF]$ = Input-output cross-spectrum matrix
$[GFF]$ = n x n frequency dependent matrix
$[GFF]$ = Input cross-spectrum matrix
$GFF(i,j)$ = Cross-spectrum of input i with respect to input j

This method has the advantage over SPR in that several columns of the transfer matrix can be computed at one time. This method has the disadvantage, though, of a high computational 'cost,' since the [GFF] matrix must be inverted at each frequency.

The simplest curve-fitting technique is the 'peak-picking' method. A plot of imaginary response vs. frequency is used to find the points of maximum magnitude in the imaginary component of the response. If the structure can be represented as a single DOF (SDOF) system in the vicinity of the natural frequency, the point of maximum magnitude corresponds to the natural frequency, and its value is the mode shape component. A half-power scheme can be employed to determine the damping value.

If another mode is close to the chosen mode, errors in the modal parameters will become significant. One way to compensate for this is the SDOF circle-fit technique. With this method a plot of imaginary vs. real values of the FRF is analyzed. When a natural frequency occurs, a circle will be formed in the Argand plane. If two or more modes interact, the circles will not be complete; one circle will be only partially formed when the next circle starts (8). An estimate for each complete circle can be made, and from this the amplitude of the mode shape (diameter of the circle), natural frequency (mid-point on the circle), and the damping (the circle is bisected and the corresponding magnitudes and frequencies are used) can be obtained. This procedure was originally suggested by Kennedy and Pancu (9).

This SDOF circle-fitting method will also break down if the modes become too closely spaced such that the individual circles interfere significantly with each other.

One method that can help produce better estimates is a routine described by Richardson and Kniskern (10). With this method FRFs are taken from the structure using various reference exciter locations, so that components from more than one column of the transfer matrix, H, are included. After initial analysis, the natural frequencies and residues are input to an algorithm that weights and averages the residues to produce modal vectors.

The problem with this method is that it is not self-sufficient nor analytically rigorous. Since a preliminary analysis must be performed beforehand, the time factor increases significantly. The added accuracy is not always sufficient to warrant the use of this algorithm either. A numerical example cited in the paper by Richardson and Kniskern showed only a slight improvement as a result of using this method.

There are, though, several curve-fitting routines that can properly handle multiple DOF (MDOF) situations. Among them is one called the Least-Squares Complex Exponential (LSCE) technique (11). In this method the FRFs are obtained and then transformed back into the time domain, forming the unit impulse response functions. The analytical form of the response function can be written

$$X(t_k) = \sum_{r=1}^{2N} A_r U_r^k \tag{1-1}$$

where,

$$A_r \quad = \quad \text{residue of mode } r$$

$$A_{r+1} \quad = \quad \text{complex conjugate of } A_r \ (r=1,3,5,\dots)$$

$$U_r \quad = \quad e^{s_r T}$$

$$s_r = -\xi_r w_r + j w_r (1-\xi_r^2)^{1/2}$$

s_{r+1} = complex conjugate of s_r $(r=1,3,5,\ldots)$

T = value of time subinterval

k = time point index $(k=0,1,2,\ldots,2n)$

$2n$ = number of time steps

N = number of modes

If $N = n$, then there is a polynomial of order $2N$ that must satisfy the equation

$$U^{2N} + a_1 U^{2N-1} + a_2 U^{2N-1} + \ldots + a_{2N-1} U_1 + a_{2N} = 0 \qquad (1-2)$$

whose roots are $U_1, U_2, \ldots, U_{2N}$.

Pre-multiplying Eq. (1) by a_{2N-k} and summing yields

$$\sum_{k=0}^{2N} a_{2N-k} X(t_k) = \sum_{r=1}^{2N} A_r \left(\sum_{k=0}^{2N} a_{2N-k} U_r^k \right)$$

The following equality holds

$$\sum_{k=0}^{2N} a_{2N-k} U_r^k = 0$$

since U_r is a root of Eq. (1-2).

Thus

$$\sum_{k=0}^{2N} a_{2N-k} X(t_k) = 0 \ , \qquad a_0 = 1 \qquad (1-3)$$

Equation (1-3) can be expanded into a series of equations by beginning each successive equation with a different response time, i.e.

$$\sum_{i=0}^{m} \left(\sum_{k=0}^{2N} a_{2N-k} X(t_{k+i}) = 0 \right) \ , \qquad m > 2N \qquad (1-4)$$

Since a_0 is known, the other a's may be obtained from the equation

$$\sum_{i=0}^{m} \left(\sum_{k=0}^{2N-1} a_{2N-k} \, X(t_{t+i}) = -X(t_{2N+i}) \right) \qquad (1\text{-}5)$$

Once the a's are known they can be substituted back into Eq. (1-2) and a polynomial root solver can be used to find the roots, U_r. The values for W_r and ξ_r can be found directly from U_r. The residues, A_r, for each U_r can be found by creating 2N equations in the form of Eq. (1-1), each with a different initial time.

Though this method was just described for one unit impulse response record, it can be easily extended to several records. Since the a's are system parameters, they should remain constant for each response record. Therefore, the summation on i can be extended to include any number of response records. This also means that the excitation location can be changed so that more than one column of the H matrix can be used simultaneously in the analysis.

To establish whether the estimate for N, the number of modes, is sufficiently large, a simple procedure is used involving a series of progressively larger estimates for the number of modes. Equation (1-5) is first rewritten in matrix form

$$Aa = B \qquad (1\text{-}6)$$

and the least-squares error is found by computing

$$R = Aa - B \qquad (1\text{-}7)$$

and the sum square error E is

$$E = R^T R \qquad (1\text{-}8)$$

A plot of the error vs. the number of modes assumed will show a significant drop-off when N is sufficiently large.

The LSCE method has been incorporated into commercial software packages and has demonstrated great success in extracting modal parameters. LSCE does have a problem, though. It must transform frequency-domain data back to the time domain. When this additional transform is performed, information can be lost or contaminated.

A second method which uses FRFs from SPR testing and which can use multiple columns of the transfer matrix, H , is an eigenvalue solver developed by Coppolino, called the Simultaneous Frequency Domain (SFD) Method (12). The idea of 'simultaneous' is that single-input FRFs from several response locations and reference locations can be used simultaneously in the analysis. In this method the matrix equation of motion for a structure may be transformed to the frequency domain and written as

$$[M] \{\ddot{X}(w)\} + [B] \{\dot{X}(w)\} + [K] \{X(w)\} = \{D\} f(w) \qquad (1\text{-}9)$$

where

$$
\begin{array}{lll}
[M] & = & \text{mass matrix} \\
[B] & = & \text{damping matrix} \\
[K] & = & \text{stiffness matrix} \\
\{X(w)\} & = & \text{array of displacements of points on the structure} \\
\{D\} & = & \text{external force distribution array} \\
f(w) & = & \text{external force amplitude} \\
f(w) & = & 1, \text{ since the transfer function for a unit impulse is } 1.
\end{array}
$$

Pre-multiplying Eq. (1-9) by $[M]^{-1}$ and solving for $\{\ddot{X}(w)\}$ gives

$$\{\ddot{X}(w)\} = -[[M]^{-1}[B] \mid [M]^{-1}[K] \mid [M]^{-1}[D]] \begin{Bmatrix} \dot{X}(w) \\ X(w) \\ -1 \end{Bmatrix} \qquad (1\text{-}10)$$

The response arrays can be expanded into matrices such that each column corresponds to a frequency point. A least-squares solution can then be obtained for the quantities $[M]^{-1}[B]$ and $[M]^{-1}[K]$. Once these two dynamic matrices are known, it is a simple procedure to use them in an eigenvalue problem

$$\lambda \begin{Bmatrix} \dot{X} \\ X \end{Bmatrix} + \begin{bmatrix} [M]^{-1}[B] & [M]^{-1}[K] \\ -[I] & [0] \end{bmatrix} \begin{Bmatrix} \dot{X} \\ X \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \cdot \qquad (1\text{-}11)$$

to solve for the frequencies and modes.

The only disadvantage of SFD compared to the LSCE single-input MDOF parameter solver is that the order of the matrices, thus the number of modes obtained at a time, is directly dependent upon how many response locations are available for the analysis.

A third method that utilizes multiple rows and columns is the force appropriation, or Asher, method (13-15). The Asher method is based on the same idea as the MSSD test; that at a natural frequency an appropriate force vector will produce responses 90 degrees out of phase with the excitation. Assuming that the applied forces are real, this means

$$\{X\} = [H]\{F\} = ([H_R] + i[H_I])\{F\} \qquad (1\text{-}12)$$

For the response to be in quadrature with the excitation

$$[H_R]\{F\} = 0 \qquad (1\text{-}13)$$

Thus, the condition for natural frequencies is

$$\text{Det } [H_R (w_r)] = 0 \qquad r = 1,2,\ldots,n \tag{1-14}$$

Once the frequencies have been found from Eq. (1-14), a solution for the required force vector, F_r, can be obtained by using Eq. (1-13). Finally, the mode shape can be obtained from

$$\{X(w_r)\} = i[H_I (w_r)] \{F\} \tag{1-15}$$

The Asher method is usually accurate, but the condition of Eq. (1-13) is only a necessary, but not sufficient, requirement for a natural mode. If the order of the determinant is much less than the order of the system, then artificial or 'spurious' modes can appear.

The Asher method requires that the number of excitation and response points be equal, whereas the number of response points usually far exceeds the number of excitation points used in a test. To avoid this Asher method limitation, another method has been developed which uses non-square matrices, and thus permits more response points. This method, termed the minimum coincident response (MCR) method (15-16), starts with basically the same equations

$$\{X\}_{m \times 1} = ([H_R]_{m \times p} + i[H_I]) \{F\}_{p \times 1} \qquad p \ll n \qquad m < n \tag{1-16}$$

$$\{X_R\} = [H_R] \{F\} \tag{1-17}$$

Instead of setting Eq. (1-17) to zero, a coincident response error norm given by

$$\varepsilon = \{X_R\}^T \{X_R\}$$

is minimized. The force vector F is appropriated such that ε is minimized subject to an amplitude constraint on X_{iI}, quadrature

response at the ith response point. This leads to a force vector

$$\{F\} = \frac{1}{\{H_{iI}\}\ ([H_R]^T\ [H_R])^{-1}\ \{H_{iI}\}^T}\ ([H_R]^T\ [H_R])^{-1}\ \{H_{iI}\}^T \quad (1\text{-}18)$$

where H_{iI} is the ith row of the imaginary part of the transfer matrix. The mode shape can then be obtained from

$$\{X_I\} = i[H_I\ (w_r)]\ \{F\} \quad (1\text{-}19)$$

The MCR method has been shown to give favorable results (15), although it can still produce 'spurious' modes. This problem can be diagnosed by various methods. One is to check the slope of the ε vs. frequency curve. Shallow slopes seem to indicate 'spurious' modes, while steep slopes correlate with actual modes (15). Another difficulty with the MCR method is that, while the Asher method interpolates to find zero crossings, MCR searches for minima.

Chapter 2

2.1 Development of a New Multi-Shaker Algorithm

For several years, since the advent of digital computers and the development of the Fast Fourier Transform (FFT), considerable time has gone into performing and perfecting single point random (SPR) excitation structural dynamics measurement techniques. The SPR method has proven to be reasonably fast and simple to perform. There are, though, some inherent problems with an SPR test. Two of the chief problems are energy dissipation within the structure and the possibility of locating a shaker near a node line.

A technique to avoid these problems is to apply forces at multiple locations. The method to be developed in this chapter is an expansion of the eigenvalue SFD method described by Coppolino (12). Since the standard modal analysis methods are applicable to single-shaker testing only, this alternative approach involves a reformulation of the equations of motion to accommodate the use of multiple forces.

The Multi-Shaker Modal Analysis (MSMA) method presented here assumes that a linear structure can be accurately described over a limited frequency range by a finite number of degrees of freedom (DOF). This assumption yields the standard frequency-domain matrix equation relating the output accelerations at given points on the structure to the input forces

$$\{\ddot{X}(w)\} = [H(w)] \{F(w)\} \qquad (2\text{-}1)$$

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where,

$$\{\ddot{X}(w)\} \;=\; \text{output accelerations} \;,\;\; (n \times 1)$$

$$\{F(w)\} \;=\; \text{input forces} \;,\;\; (n \times 1)$$

$$[H(w)] \;=\; \text{complex frequency response function} \;,\;\; (n \times n)$$

$$[H(w)] \;=\; [\emptyset]\,[/A/]\,[\emptyset]^T$$

$$\lceil\emptyset\rfloor \;=\; \text{modal matrix} \;,\;\; (n \times n)$$

$$[/A/] \;=\; \text{diagonal modal admittance matrix} \;,\;\; (n \times n)$$

$$[/A/] \;=\; \frac{-w^2}{((w_r^2 - w^2) + i(2\xi_r\, ww_r))} \qquad r=i,n$$

$$w_r \;=\; \text{natural circular frequency of mode } r$$

$$\xi_r \;=\; \text{critical damping ratio of mode } r$$

$$w \;=\; \text{circular frequency}$$

The force vector is rewritten in a form that includes only the DOF where forces are applied. A distribution matrix relates this new force vector, $\{f(w)\}$, to the original force vector, $\{F(w)\}$.

$$\{F(w)\} \;=\; [D]\,\{f(w)\} \tag{2-2}$$

For example, if a 4DOF system has exciters at points 1 and 3, the equation would look like

$$\{F(w)\} \;=\; \begin{Bmatrix} f_1 \\ 0 \\ f_3 \\ 0 \end{Bmatrix} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} f_1 \\ f_3 \end{Bmatrix}$$

So, Eq. (2-1) is now

$$\{\ddot{X}(w)\} \;=\; [\emptyset]\,[/A/]\,[\emptyset]^T\,[D]\,\{f(w)\} \tag{2-3}$$

It is assumed that there are n points where the responses are measured, and if it is assumed that the structure can be accurately described over the frequency range of interest by m DOF ($m < n$), then Eq. (2-3) can be partitioned into sets of 'independent' DOF and 'dependent' DOF equations. Thus,

$$\begin{Bmatrix} \ddot{X}_i(w) \\ \ddot{X}_d(w) \end{Bmatrix} = \begin{bmatrix} [\emptyset_i] \\ [\emptyset_d] \end{bmatrix} [/A(w)/] \begin{bmatrix} [\emptyset_i]^T & [\emptyset_d]^T \end{bmatrix} \begin{bmatrix} [D_i] \\ [D_d] \end{bmatrix} \{f(w)\} \quad (2\text{-}4)$$

or

$$\begin{Bmatrix} \ddot{X}_i(w) \\ \ddot{X}_d(w) \end{Bmatrix} = \begin{bmatrix} [I] \\ [\emptyset_d][\emptyset_i]^{-1} \end{bmatrix} [\emptyset_i] [/A(w)/] [\emptyset_i]^T [D_{id}] \{f(w)\} \quad (2\text{-}5)$$

where

$$[D_{id}] = [D_i] + [\emptyset_i]^{-T} [\emptyset_D]^T [D_d]$$

From Eq. (2-5) a relationship between the independent and dependent DOF accelerations is

$$\{\ddot{X}_d(w)\} = [\emptyset_d] [\emptyset_i]^{-1} \{\ddot{X}_i(w)\} \quad (2\text{-}6a)$$

This equation will be employed later to determine the modal components of the dependent DOF for the determined natural frequencies. The equation relating the independent accelerations to the forces is

$$\{\ddot{X}_i\} = [\emptyset_i] [/A(w)/] [\emptyset_i]^T [D_{id}] \{f(w)\} \quad (2\text{-}6b)$$

Pre-multiplying both sides of Eq. (2-6b) by $([\emptyset_i] [/A(w)/] [\emptyset_i]^T)^{-1}$ and expanding, yields

$$([\emptyset_i] \ [/A(w)/] \ [\emptyset_i]^T)^{-1} \left\{\ddot{X}_i\right\} = [D_{id}] \left\{f(w)\right\}$$

$$[\emptyset_i]^{-T} \ [/(1/A(w))/] \ [\emptyset_i]^{-1} \left\{\ddot{X}_i\right\} = [D_{id}] \left\{f(w)\right\}$$

$$[\emptyset_i]^{-T} \ [/(1 - i2\xi_r \ (w_r/w) - (w_r/w)^2)/] \ [\emptyset_i]^{-1} \left\{\ddot{X}_i\right\} = [D_{id}] \left\{f(w)\right\}$$

$$[\emptyset_i]^{-T} \ [\emptyset_i]^{-1} \left\{\ddot{X}_i\right\} + [\emptyset_i]^{-T} \ [/2\xi_r \ w_r/] \ [\emptyset_i]^{-1} \ (-i/w) \left\{\ddot{X}_i\right\}$$
$$+ \ [\emptyset_i]^{-T} \ [/w_r^2/] \ [\emptyset_i]^{-1} \ (-1/w^2) \left\{\ddot{X}_i\right\} = [D_{id}] \left\{f(w)\right\}$$

or, in summary,

$$[M_i] \left\{\ddot{X}_i\right\} + [C_i] \left\{\dot{X}_i\right\} + [K_i] \left\{X_i\right\} = [D_{id}] \left\{f(w)\right\} \tag{2-7}$$

where

$$[M_i] = [\emptyset_i]^{-T} \ [\emptyset_i]^{-1}$$

$$[C_i] = [\emptyset_i]^{-T} \ [/2\xi_r \ w_r/] \ [\emptyset_i]^{-1}$$

$$[K_i] = [\emptyset_i]^{-T} \ [/w_r^2/] \ [\emptyset_i]^{-1}$$

$$\left\{\dot{X}_i\right\} = (-i/w) \left\{\ddot{X}_i\right\}$$

$$\left\{X_i\right\} = (-1/w^2) \left\{\ddot{X}_i\right\}$$

Least-squares averaging may be employed to estimate the matrices $[M_i]$, $[C_i]$, and $[K_i]$. Therefore, the accelerations, velocities, displacements and forces in Eq. (2-7) are expanded into matrices, where each column corresponds to a discrete frequency point. That is,

$$[\ddot{X}_i] = [\ddot{X}_i(w_1) \mid \ddot{X}_i(w_2) \mid ... \mid \ddot{X}_i(w_k)] \tag{2-8}$$

where $k > m$. Multiplication of Eq. (2-7) by $[M_i]^{-1}$ yields

$$-[\ddot{X}_i] = \left[[\tilde{C}_i] \ [\tilde{K}_k] \ [\tilde{D}_{id}]\right] \begin{bmatrix} [\dot{X}_i] \\ [X_i] \\ [f(w)] \end{bmatrix} \tag{2-9}$$

where,

$$[\tilde{C}_i] = [M_i]^{-1} [C_i] \qquad (2\text{-}10a)$$

$$[\tilde{K}_i] = [M_i]^{-1} [K_i] \qquad (2\text{-}10b)$$

$$[\tilde{D}_{id}] = -[M_i]^{-1} [D_{id}] \qquad (2\text{-}10c)$$

Since $[\tilde{C}_i]$, $[\tilde{K}_i]$, and $[\tilde{D}_{id}]$ are real matrices, Eq. (2-9) can be rewritten for the real and imaginary parts of the responses and inputs.

$$-\left[[\ddot{X}_{i_{Re}}] \; [\ddot{X}_{i_{Im}}]\right] = \left[[\tilde{C}_i] \; [\tilde{K}_i] \; [\tilde{D}_{id}]\right] \begin{bmatrix} [\dot{X}_{i_{Re}}] & [\dot{X}_{i_{Im}}] \\ [X_{i_{Re}}] & [X_{i_{Im}}] \\ [f_{Re}] & [f_{Im}] \end{bmatrix} \qquad (2\text{-}11)$$

Since this equation is over-determined (k > m), a least-squares solution is used. Equation (2-11) may be rewritten symbolically

$$[Y] = [T] [Z] \qquad (2\text{-}12)$$

The least-squares solution for $[T]$ is

$$[T] = [Y] [Z]^T \left[[Z] [Z]^T\right]^{-1} \qquad (2\text{-}13)$$

The $[\tilde{C}_i]$ and $[\tilde{K}_i]$ that were solved for in Eq. (2-13) can now be used to determine the eigenvalues and eigenvectors for the system in the following manner. From Eq. (2-9) the free vibration equation

$$\ddot{X}_i + [\tilde{C}_i] \dot{X}_i + [\tilde{K}_i] X_i = 0 \qquad (2\text{-}14)$$

may be written. Since the matrices $[\tilde{C}_i]$ and $[\tilde{K}_i]$ are constant, X and its derivatives in Eq. (2-14) may be written in the time-domain, rather than in the frequency-domain as originally expressed in Eq. (2-7). Thus, let

$$X_i = Xe^{\lambda t} \tag{2-15a}$$

$$\dot{X}_i = \lambda X_i \tag{2-15b}$$

$$\ddot{X}_i = \lambda \dot{X}_i \tag{2-15c}$$

From Eq. (2-15b)

$$\lambda X_i - \dot{X}_i = 0 \tag{2-16d}$$

Thus, combining Eqs. (2-14) and (2-15) yields

$$\lambda \begin{Bmatrix} \dot{X}_i \\ X_i \end{Bmatrix} + \begin{bmatrix} [\tilde{C}_i] & [\tilde{K}_i] \\ [-I] & [0] \end{bmatrix} \begin{Bmatrix} \dot{X}_i \\ X_i \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2-17}$$

which is a standard linear eigenvalue problem. The eigenvectors will produce mode shape components for the 'independent' DOF only. To include the 'dependent' DOF, Eq. (2-6a) is expanded to matrix form for the k frequency points and then manipulated, via the least-squares technique, to produce a relationship between the 'dependent' and 'independent' modal coefficients.

$$[\ddot{X}_d] = [\emptyset_d] [\emptyset_i]^{-1} [\ddot{X}_i]$$
$$[\emptyset_d] = [\ddot{X}_d] [\ddot{X}_i]^T [[X_i] [X_i]^T]^{-1} [\emptyset_i] \tag{2-18}$$

2.2 Forcing Functions

As indicated by Eqs. (2-4) and (2-5), the forces may be applied at the dependent DOF as well as the independent DOF. This feature of the equations allows for forces to be applied where either accelerations could not be measured or where an accelerometer produced bad readings and was subsequently discarded. It also allows the user more freedom in

determining which locations are to be identified as independent DOF. If, after data acquisition is completed and a general analysis done, it is determined that a particular DOF is not important, its responses can be reduced out of further calculations, even though it was forced.

In addition to the freedom of location of forces, there is also a freedom in the type of forces which can be applied, subject to one constraint. From Eq. (2-13) it is seen that, if two forces are fully coherent the inverse of the matrix will be singular. However, the restriction is not too severe. If two forcing functions are identical, only one forcing function needs to be included in the force array. The force distribution matrix will handle the duplication in the applied force spectrums. Since, in reality, a structure will have different input impedances at each location, similar signals into the shakers will produce different forces. The best way to avoid any potential problems, however, is to have a low coherence between forces. This can be achieved by using either of two forcing methods, independent random signals at each shaker or simultaneous sine sweeps with each shaker having a different phase shift.

Another type of excitation is a free-decay response. A minor reformulation of the equations is involved, with Eq. (2-11) being modified to the form

$$
- [[\ddot{X}_{i_{Re}}] \ [\ddot{X}_{i_{Im}}]] \ = \ [[\tilde{C}_i] \ [\tilde{K}_i]] \begin{bmatrix} [\dot{X}_{i_{Re}}] & [\dot{X}_{i_{Im}}] \\ [X_{i_{Re}}] & [X_{i_{Im}}] \end{bmatrix} \tag{2-18}
$$

This formulation eliminates the need for measurement of the forces, thus allowing a more general type of excitation. This can also be used in

conjunction with the Randomdec method where a free-decay response is generated from a structure that is being randomly excited.

2.3   Averaging

There are two possible methods to eliminate noise contamination in the eigenvalue results. Both methods involve a type of signal averaging. The first method involves averaging FRFs from several runs to form a single acceleration and force FRF spectrum at each location before insertion into Eq. (2-13).

There is one serious drawback to this method. With random excitations, excessive averaging can produce an FRF spectrum with a uniform value, effectively giving fewer independent columns in Eq. (2-13) with which to do a least-squares fit. For a swept-sine test, this averaging technique should suffice, since the FRF will not tend to become uniform, especially if the signal generator has a phase modulator.

The second method is to string the multiple runs together in the same manner as the individual frequencies were in Eq. (2-8). That is

$$[\ddot{X}(w)] = [[\ddot{X}_1(w)] \mid [\ddot{X}_2(w)] \mid \ldots \mid [\ddot{X}_p(w)]]$$

This increases the number of columns, therefore the accuracy of the least-squares fit, but the required storage space for the matrices is also increased.

2.4   Pseudo-Degrees-of-Freedom

If the structure being tested has more modes in the frequency range of interest than there are measurement locations, there is a technique to

increase the number of DOF based on existing measurement DOF. This method involves sampling extra sets of data and placing each new set into the equations as if they were new DOF. With this approach is an alternate method in which the responses of the originally measured data are partitioned into subsets and correspondingly labeled as different sets of data. In this way the number of DOF is increased without the need for more measurements. This, however, gives fewer columns of data for the least-squares process, since this procedure takes some of the columns in Eq. (2-8) and makes new rows out of them.

2.5   Mode Selection

Proper selection of the true modes of vibration from the eigen-solution of Eq. (2-16) is just as important as any other part of the modal analysis. When the number of modes is overspecified, either through oversight or to compensate for noise contamination, a systematic method is needed to identify the spurious modes. Usually the spurious modes can be eliminated immediately by a quick examination of the eigenvalues. The damping values are often excessively high, low, or even negative. The natural frequencies might have values that are out of the measured frequency range. At other times, however, the spurious mode eigenvalues are within the range of tolerance. Then, an inspection of the eigenvectors is needed. One procedure involves the use of pseudo-DOF. A comparison of the portion of the eigenvector representing the pseudo-DOF with the portion of the eigenvector which corresponds to the independent DOF from which the pseudo DOF were produced can readily determine the validity of the mode. A favorable relationship would be one where there is a high correlation of

eigenvector values at corresponding DOF. Another method would be to analyze results of analyses from several data sets; modes that repeatedly show up would indicate valid modes.

2.6   Time-Domain Analysis

The MSMA procedure just described may also be performed directly in the time domain instead of first transforming the signal into the frequency-domain, although this raises some new problems. The relationship between acceleration, velocity, and displacement is not as simple. To acquire the three quantities there are three possible methods. The first is to measure all three quantities directly. The second is to measure the acceleration and then integrate to obtain the velocity and displacement. In this case, the initial velocity and displacement are needed. The third method is to measure the displacement and then differentiate the signal to obtain the velocity and acceleration. Since none of these procedures is easily accomplished, the frequency-domain approach remains the preferred approach.

Chapter 3

3.1 Simulation Studies

To verify the validity and accuracy of the MSMA technique described in Chapter 2, several computer simulations were developed and examined. The primary concerns evaluated in the tests were: (1) resolution of the modes in high noise, (2) identification of modes less than $\xi_r w_r$ apart, (3) accuracy of dependent DOF, and (4) application of forces at dependent DOF.

A computer program was designed such that the system natural frequencies, damping, and mode shapes, and a noise-to-signal ratio are input. The noise level is calibrated by using the root-mean-square of the response signal over the entire frequency range. Also input are the frequency range, number of independent DOF and their corresponding locations, number of dependent DOF and their locations, number of forced DOF and their locations, and the method of excitation. The program creates the frequency responses for the accelerations and forces and then proceeds to manipulate them in accordance with the equations of Chapter 2.

The first series of tests was on a system with nine natural frequencies, seven of which were below 10 Hz. Two of the frequencies were selected less than $\xi_r w_r$ apart, so that the effects of closely-coupled modes could be observed. The noise-to-signal ratio was selected at 20% in conjunction with an applied random force signal. Changed during the test series were the number of forced DOF: 1, 4, and 8 forces were used. As indicated in Table 1, the mode shapes had magnitudes at each station of

1.0, except for one DOF in each mode whose magnitude was set at 0.01 to simulate a point close to a node line.

A sampling run was first made using a single shaker to obtain an estimate of the number of modes in the 0-10 Hz. range. A check of the FRFs showed that six modes were fairly apparent, so an estimate of eight modes (six for the observed ones and two more for any missed modes or any pseudo or spurious modes created by the noise) did not seem unreasonable.

Three runs were made for each of the forced-DOF conditions plus one additional run utilizing the averaging technique of placing several runs into one large array. The averaged analysis used three individual runs which differed from the previous three runs.

To check the accuracy of the results an error analysis was used that looked at the percentage of error in the natural frequency, damping, and real part of the modal vector compared to the input system characteristics. The natural frequency and damping errors were computed directly. The modal vector, though, was a bit more complicated. The mode shapes all had magnitudes of 1 at each DOF, except for one with .01, so the estimated mode shapes were each scaled such that the largest component had an amplitude of 1 and a phase angle of 0 degrees. The real part of each component was then squared, summed (except for the one component of .01 amplitude, which was omitted from the error analysis), divided by 7 (the corresponding magnitude of the original vector); and then the square root was taken. The final value, which ranged from 0 to 1, was then used directly for the error analysis.

To highlight the results, the errors for three modes were investigated. The first was the 5.9 Hz. mode, which was relatively uncoupled.

The other two modes were the coupled modes, 5.5 Hz. and 5.55 Hz. Referring to the diagrams in Fig. 1, it can be seen that with 20% noise the natural frequency estimates are highly accurate, even for the closely-coupled modes. Also evident in the diagrams is the overall higher accuracy of the parameters corresponding to the 5.9 Hz. mode compared to the 5.5 and 5.55 Hz. modes. Although the errors for the damping and modal vectors are large for the 20% noise single-shaker case, the diagrams show a steady trend towards lower error levels as the number of shakers is increased. In fact, the error levels for the closely-coupled modes are comparable to the error level for the 5.9 Hz. mode, when eight shakers were employed.

Also tested with this simulation model was a set of runs in which the noise-to-signal ratio was changed from 0% to 10% and finally to 20%. To illustrate the effect of noise on the results and the effect of additional shakers, the modal parameters corresponding to the 5.0, 5.5, and 5.55 Hz. modes are tabulated (see Tables 2 through 10). An analysis of the tables shows, once again, a clear trend toward higher levels of accuracy at a given noise level for an increase in the number of shakers. A dramatic result from this set of runs was the absence of the 5.0 Hz mode from the single-shaker analysis when noise was present. As the number of shakers increased though, the 5.0 Hz mode appeared and then increased in accuracy. Overall, by a comparison of the tables, it is seen that the algorithm does not become unstable with increasing noise levels. On the contrary, good results are obtained even at relatively high noise levels when several shakers are used.

The second series of tests analyzed goals (3) and (4). To do this, an 8 DOF system was used, with four modes in the 0-10 Hz. range. In this

case the noise-to-signal ratio was also 20%, but the number of forces was set at 1, 5, and 8 (an initial sampling run indicated three, possibly four, modes present in the 0-10 Hz. range, so five independent DOF were selected).

The previous series of tests was again performed, but since the only substantial errors occurred in the coupled modes, their parameters were the only ones analyzed. In this analysis, only the normalized independent modal vectors and a ratio of the normalized dependent to independent modal vectors were calculated (see Fig. (2)).

Again a convergence toward higher accuracy is seen in these results as the number of applied forces is increased. An important feature of the results is that the accuracy of the dependent DOF, calculated from Eq. (2-17), follows closely the same trends as the independent DOF. Also to be noted in the diagrams is the error levels for five shakers and for eight shakers. Although three shakers are exciting dependent DOF in the eight shaker case, the accuracy is either maintained or improved in almost every instance. Even if there happens to be a higher error indication for an increase in shakers, there is a definite lessening in the error spread.

Two more series of tests were run. The first was a set of runs identical to the first test case with the exception that the forcing functions were swept-sines instead of random excitation. The results of this set were basically identical to the random excitation set. The final series also used the same set-up as the first test, except that no noise was introduced and the test range was from 0-1024 Hz. with 256 data points (a delta frequency of over 4 Hz.). The ninth mode corresponding to 25 Hz. was also deleted. All modes were identified to extremely high accuracy

(the error was less than one part per billion) in spite of the fact that there were seven natural frequencies between two of the data points.

Some preliminary testing of the algorithm utilizing the concept of pseudo-DOF was performed with mixed results. When a large percentage of the needed independent DOF were actually pseudo-DOF, there was a significant degradation in the accuracy of the results. However, if the pseudo-DOF were used sparingly to obtain the number of needed DOF, the algorithm performed on the same level as in the previously cited tests. In either case, the use of pseudo-DOF in spurious mode detection was satisfactory. An increase in the number of pseudo-DOF beyond the number of needed DOF had no ill effect on the algorithm's performance. In fact it helped facilitate the spurious mode identification.

Mode No.	Frequency (Hz)	Damping
1	5.000	0.010
2	5.250	0.010
3	5.500	0.010
4	5.550	0.010
5	5.900	0.010
6	6.200	0.010
7	6.600	0.010
8	15.000	0.010
9	25.000	0.010

DOF MODE	1	2	3	4	5	6	7	8
1	.010	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.000	.010	1.000	1.000	-1.000	-1.000	-1.000	-1.000
3	1.000	1.000	-.010	-1.000	-1.000	-1.000	1.000	1.000
4	1.000	1.000	-1.000	-.010	1.000	1.000	-1.000	-1.000
5	1.000	-1.000	-1.000	1.000	.010	-1.000	-1.000	1.000
6	1.000	-1.000	-1.000	1.000	-1.000	.010	1.000	-1.000
7	1.000	-1.000	1.000	-1.000	-1.000	1.000	-.010	1.000
8	1.000	-1.000	1.000	-1.000	1.000	-1.000	1.000	-.010
9	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

No. of Forces	Force Locations
1	1
4	1-4
8	1-8

Table 1. System Frequencies, Damping Factors, and Mode Shapes (defined at 8 DOF only); Forces and Locations

Undamped Natural Frequency = .500630E+01
    Damping = .143159E-01

Coordinate	Coefficient	
1	.308015E-02	-.277992E-01
2	.994932E+00	-.215889E-01
3	.998240E+00	-.101613E-01
4	.996091E+00	-.180890E-01
5	.998748E+00	-.492194E-02
6	.998139E+00	-.695993E-02
7	.100000E+01	0
8	.999306E+00	-.251417E-02

Undamped Natural Frequency = .550004E+01
    Damping = .100156E-01

Coordinate	Coefficient	
1	.996047E+00	-.769268E-02
2	.996975E+00	-.589179E-02
3	-.735022E-02	.515352E-02
4	-.996900E+00	.602848E-02
5	-.998680E+00	.256406E-02
6	-.998711E+00	.250478E-02
7	.100000E+01	0
8	.999892E+00	-.209626E-03

Undamped Natural Frequency = .554982E+01
    Damping = .999716E-02

Coordinate	Coefficient	
1	-.994959E+00	-.443466E-03
2	-.992961E+00	-.618205E-03
3	.998700E+00	.114035E-03
4	.954467E-02	.389808E-04
5	-.997289E+00	-.238917E-03
6	-.997345E+00	-.234214E-03
7	.100000E+01	0
8	.999725E+00	.238590E-04

No. of Forces = 1   Noise Ratio = 0.0

Table 2

Undamped Natural Frequency = .549815E+01
Damping = .922684E-02

Coordinate	Coefficient	
1	.100000E+01	0
2	.961693E+00	.154095E-01
3	-.111589E+00	-.243182E+00
4	-.953094E+00	-.189219E-01
5	-.866331E+00	.201760E+00
6	-.874222E+00	.225322E+00
7	.852086E+00	-.247594E+00
8	.852671E+00	-.202553E+00

Undamped Natural Frequency = .556574E+01
Damping = .125684E-01

Coordinate	Coefficient	
1	.969798E+00	.718633E-02
2	.100000E+01	0
3	-.601954E+00	.299185E+00
4	-.306331E-01	.115159E+00
5	.663191E+00	-.152995E+00
6	.655221E+00	-.180120E+00
7	-.574999E+00	.258952E+00
8	-.641525E+00	.178636E+00

No. of Forces = 1   Noise Ratio = 0.1

No mode corresponding to 5 Hz was identified

Table 3

Undamped Natural Frequency = .550027E+01
Damping = .979364E-02

Coordinate	Coefficient	
1	.897034E+00	-.161891E-01
2	.100000E+01	0
3	-.263962E+00	-.827695E-01
4	-.717776E+00	.560193E-01
5	-.923468E-01	.215160E+00
6	-.173389E+00	.196507E+00
7	.551836E+00	-.126410E+00
8	.510949E+00	-.138541E+00

Undamped Natural Frequency = .558794E+01
Damping = .110378E-01

Coordinate	Coefficient	
1	-.721310E+00	.649048E-01
2	-.560135E+00	.146199E-01
3	.833390E+00	-.662602E-02
4	-.769025E-01	-.250925E-01
5	-.414973E+00	-.122354E-01
6	-.508665E+00	-.241696E-01
7	.100000E+01	0
8	.947173E+00	.470355E-02

No. of Forces = 1   Noise Ratio = 0.2

No mode corresponding to 5 Hz was identified

Table 4

Undamped Natural Frequency = .499677E+01
Damping = .995318E-02

Coordinate	Coefficient	
1	.140079E-01	-.289103E-03
2	.100000E+01	0
3	.998304E+00	.122927E-03
4	.999471E+00	.385208E-04
5	.997609E+00	.171764E-03
6	.997915E+00	.149431E-03
7	.996893E+00	.223324E-03
8	.997265E+00	.196417E-03

Undamped Natural Frequency = .549998E+01
Damping = .100134E-01

Coordinate	Coefficient	
1	.100000E+01	0
2	.999560E+00	.154268E-02
3	-.112414E-01	.434173E-02
4	-.999586E+00	-.142288E-02
5	-.998737E+00	-.438455E-02
6	-.998722E+00	-.443570E-02
7	.998114E+00	.657796E-02
8	.998166E+00	.639820E-02

Undamped Natural Frequency = .554972E+01
Damping = .999856E-02

Coordinate	Coefficient	
1	-.992035E+00	-.217490E-03
2	-.988883E+00	-.304124E-03
3	.997943E+00	.577118E-04
4	.927480E-02	.239258E-04
5	-.995724E+00	-.113431E-03
6	-.995813E+00	-.110535E-03
7	.100000E+01	0
8	.999567E+00	.122911E-04

No. of Forces = 4   Noise Ratio = 0.0

Table 5

Undamped Natural Frequency = .499588E+01
    Damping = .101836E-01

Coordinate	Coefficient	
1	.180826E-01	.648566E-02
2	.989374E+00	.422064E-03
3	.984854E+00	.601076E-02
4	.100000E+01	0
5	.975333E+00	.364940E-02
6	.987170E+00	-.791028E-03
7	.977551E+00	.291086E-02
8	.975623E+00	-.161876E-03

Undamped Natural Frequency = .549835E+01
    Damping = .979900E-02

Coordinate	Coefficient	
1	-.995591E+00	.353937E-01
2	-.963411E+00	.140834E-01
3	-.193424E-01	-.112736E-01
4	.987221E+00	.125764E-01
5	.952534E+00	=.241061E=01
6	.100000E+01	0
7	-.995252E+00	-.490844E-01
8	-.960618E+00	-.491432E-01

Undamped Natural Frequency = .554825E+01
    Damping = .943557E-02

Coordinate	Coefficient	
1	.993486E+00	-.268232E-01
2	.100000E+01	0
3	-.973952E+00	.261893E-02
4	-.554699E-01	.508969E-02
5	.851456E+00	-.601461E-02
6	.903672E+00	-.198642E-02
7	-.925752E+00	.291869E-01
8	-.905322E+00	.464260E-01

No. of Forces = 4   Noise Ratio = 0.1

Table 6

Undamped Natural Frequency = .500344E+01
Damping = .949072E-02

Coordinate	Coefficient	
1	-.431481E-02	-.383221E-02
2	.981993E+00	-.220043E-02
3	.924128E+00	.172318E-02
4	.944983E+00	.305296E-01
5	.971504E+00	-.134243E-01
6	.100000E+01	0
7	.953668E+00	-.808437E-02
8	.975622E+00	-.367880E-02

Undamped Natural Frequency = .548780E+01
Damping = .959778E-02

Coordinate	Coefficient	
1	.709776E+00	.119457E+00
2	.676447E+00	.170732E-01
3	.129852E+00	-.441760E-01
4	-.915645E+00	-.537032E-01
5	=.857429E+00	=.136665E+00
6	-.941759E+00	-.145444E+00
7	.100000E+01	0
8	.955578E+00	.678429E-01

Undamped Natural Frequency = .554758E+01
Damping = .104780E-01

Coordinate	Coefficient	
1	.992678E+00	.668940E-01
2	.100000E+01	0
3	-.878481E+00	-.152797E+00
4	-.856727E-01	.457745E-01
5	.638683E+00	.176693E+00
6	.692493E+00	.154775E+00
7	-.752810E+00	-.214112E+00
8	-.713608E+00	-.181675E+00

No. of Forces = 4   Noise Ratio = 0.2

Table 7

Undamped Natural Frequency = .499655E+01
            Damping = .985967E-02

Coordinate	Coefficient	
1	.142906E-01	.864394E-03
2	.100000E+01	0
3	.998184E+00	-.367753E-03
4	.999433E+00	-.115383E-03
5	.997441E+00	-.513542E-03
6	.997768E+00	-.446875E-03
7	.996674E+00	-.667799E-03
8	.997072E+00	-.587373E-03

Undamped Natural Frequency = .549997E+01
            Damping = .999536E-02

Coordinate	Coefficient	
1	.100000E+01	0
2	.999341E+00	.535210E-03
3	-.118561E-01	.150597E-02
4	-.999390E+00	-.492155E-03
5	-.998124E+00	-.151908E-02
6	-.998102E+00	-.153676E-02
7	.997187E+00	.228076E-02
8	.997264E+00	.221846E-02

Undamped Natural Frequency = .554991E+01
            Damping = .999709E-02

Coordinate	Coefficient	
1	-.997589E+00	-.458045E-03
2	-.996632E+00	-.638380E-03
3	.999379E+00	.116595E-03
4	.978273E-02	.372708E-04
5	-.998702E+00	-.249320E-03
6	-.998729E+00	-.244781E-03
7	.100000E+01	0
8	.999868E+00	.243785E-04

No. of Forces = 8   Noise Ratio = 0.0

Table 8

Undamped Natural Frequency = .499798E+01
            Damping = .965077E-02

Coordinate	Coefficient	
1	.251566E-01	-.728226E-02
2	.986205E+00	-.423328E-02
3	.100000E+01	0
4	.984109E+00	-.513716E-02
5	.973499E+00	.805894E-03
6	.971472E+00	.548334E-02
7	.973073E+00	.201598E-01
8	.969475E+00	.141926E-01

Undamped Natural Frequency = .550583E+01
            Damping = .102792E-01

Coordinate	Coefficient	
1	-.985845E+00	-.163209E-01
2	-.994625E+00	.324345E-01
3	.276250E-01	-.977544E-02
4	.100000E+01	0
5	.946089E+00	.302677E-01
6	.952970E+00	.644163E-01
7	-.986397E+00	.187762E-02
8	-.959759E+00	-.215829E-01

Undamped Natural Frequency = .554904E+01
            Damping = .999550E-02

Coordinate	Coefficient	
1	.983620E+00	.708132E-02
2	.100000E+01	0
3	-.951970E+00	.111605E+00
4	-.645294E-01	-.112379E+00
5	.875027E+00	-.191167E+00
6	.874207E+00	-.193172E+00
7	-.886737E+00	.218127E+00
8	-.890127E+00	.182806E+00

No. of Forces = 8   Noise Ratio = 0.1

Table 9

Undamped Natural Frequency = .500069E+01
       Damping = .929434E-02

Coordinate	Coefficient	
1	.204826E-03	.548111E-01
2	.995489E+00	.149953E-01
3	.988078E+00	.501376E-01
4	.995600E+00	.579839E-01
5	.995954E+00	.971265E-02
6	.980042E+00	.484835E-02
7	.985900E+00	.174225E-02
8	.100000E+01	0

Undamped Natural Frequency = .549181E+01
       Damping = .786745E-02

Coordinate	Coefficient	
1	.100000E+01	0
2	.978124E+00	.865004E-01
3	-.562677E-02	.274473E-01
4	-.949020E+00	-.470651E-01
5	-.928190E+00	-.187158E+00
6	-.959287E+00	-.153123E+00
7	.969323E+00	.904357E-01
8	.982434E+00	.148072E+00

Undamped Natural Frequency = .555115E+01
       Damping = .100090E-01

Coordinate	Coefficient	
1	-.835000E+00	-.188699E+00
2	-.823659E+00	-.144876E+00
3	.921431E+00	.105953E+00
4	-.618963E-01	.955734E-01
5	-.966994E+00	-.642726E-01
6	-.963982E+00	-.450693E-01
7	.100000E+01	0
8	.969904E+00	.349342E-01

No. of Forces = 8   Noise Ratio = 0.2

Table 10

Mode No.	Frequency (Hz)	Damping
1	5.000	0.010
2	5.250	0.010
3	5.500	0.010
4	5.550	0.010
5	15.000	0.010
6	25.000	0.010

DOF MODE	1	2	3	4	5	6	7	8
1	.010	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.000	.010	1.000	1.000	-1.000	-1.000	-1.000	-1.000
3	1.000	1.000	-.010	-1.000	-1.000	-1.000	1.000	1.000
4	1.000	1.000	-1.000	-.010	1.000	1.000	-1.000	-1.000
5	1.000	-1.000	-1.000	1.000	.010	-1.000	-1.000	1.000
6	1.000	-1.000	-1.000	1.000	-1.000	.010	1.000	-1.000

No. of Forces	Force Locations
1	1
5	1-5
8	1-8

Table 11. System Frequencies, Damping Factors, and Mode Shapes
(defined at 8 DOF only); Forces and Locations
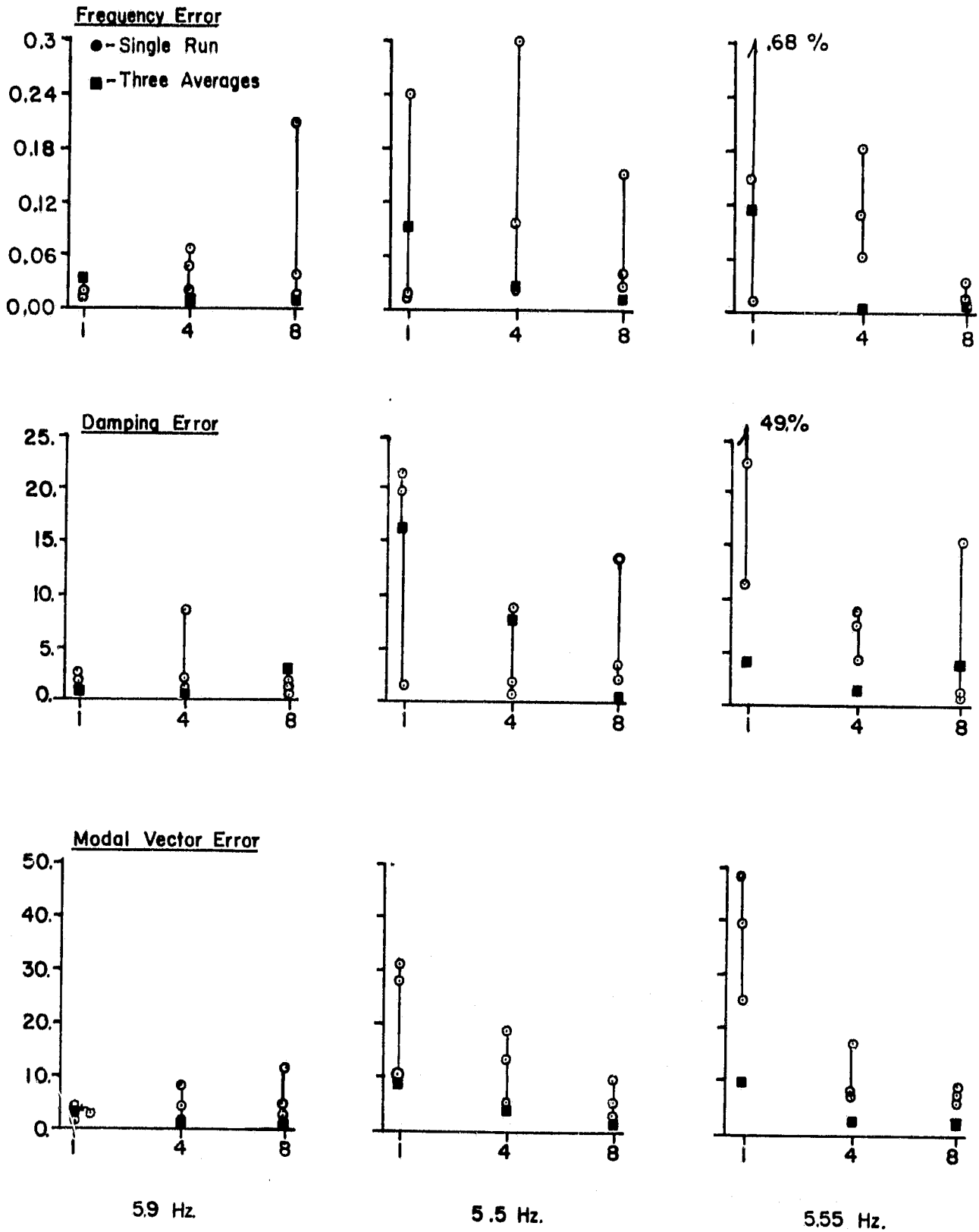
Figure 1. Error Comparisons for Test Series 1
Noise/Signal = 0.2, Random Noise Excitation

Normalized Dependent to Independent Modal Vector Error
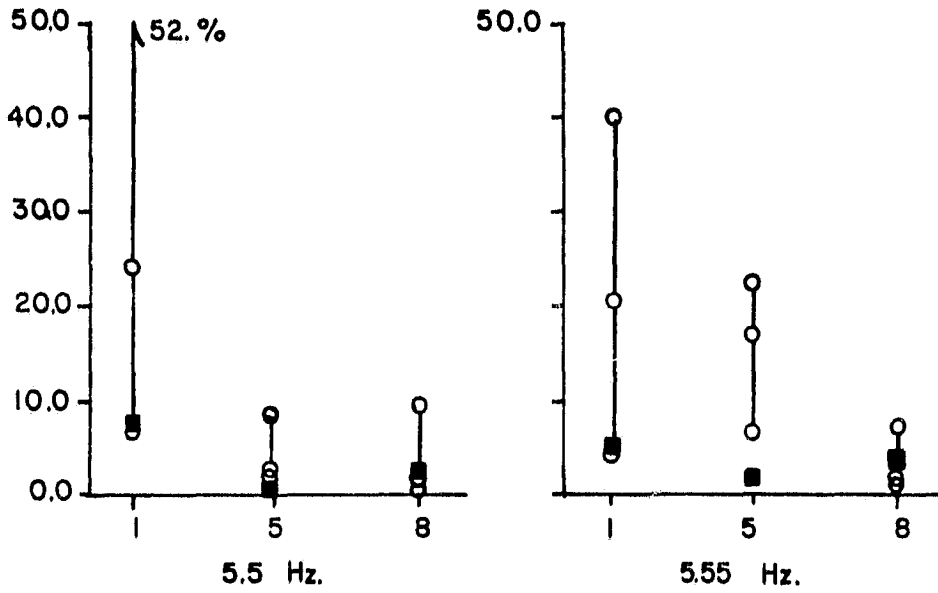
Independent Modal Vector Error
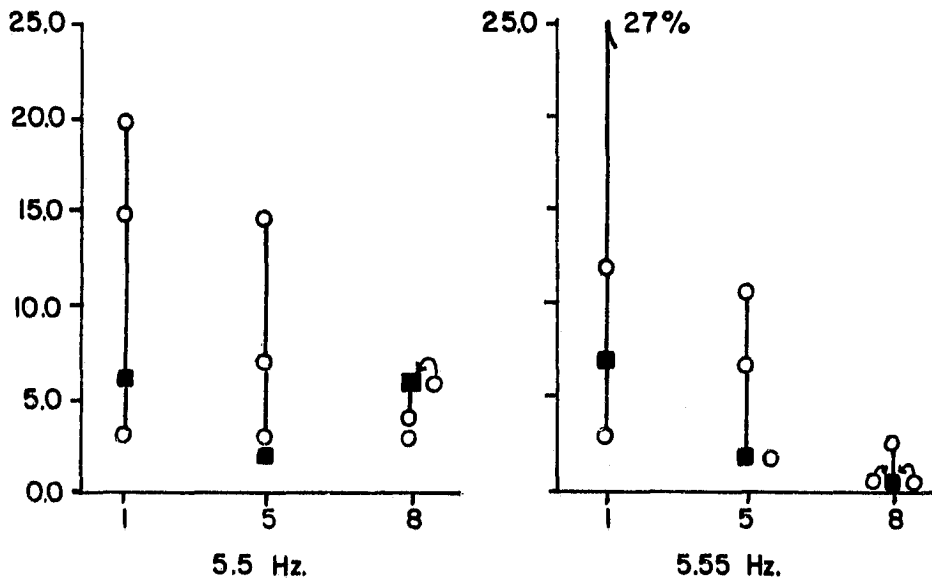
Figure 2.   Error Comparisons for Test Series 2
Noise/Signal = 0.2, Random Noise

Chapter 4

4.1　Conclusions

In this thesis, the theory of a method used to determine an estimation of the modal parameters for a viscously-damped, linear structure has been developed. The MSMA method involves the application and measurement of several non-coherent forces. Verification of the method was demonstrated analytically through an extensive computer simulation study.

The computer simulation showed the MSMA method to be stable. With increasing amounts of noise the results did not diverge dramatically. Also, as the number of applied forces was increased, the results converged to the system values.

Several advantages of the MSMA method were demonstrated by the computer analysis. The first is the ability to locate a mode which has a node line in the vicinity of a shaker by applying several forces. The second is the small amount of user judgement and expertise involved. Basically the only user judgement needed is a knowledge of the approximate number of modes in the analysis range, and that is usually supplied by a previous finite element simulation or can be estimated by examination of a preliminary FRF. The third advantage is the relatively fast analysis time. With the small amount of user interaction the analysis time is basically dependent upon computation speed and total amount of data processed.

4.2   Recommendations for Further Work

There are still many features of the MSMA method that need to be examined.

a.   A modal confidence factor (MCF) should be developed to systematically evaluate the validity of acquired modes. The MCF should be based on a comparison of the eigenvectors corresponding to the pseudo-DOF and the eigenvectors of their companion true-DOF.

b.   A preliminary mode count procedure should be established. Instead of an initial count based on a finite element model, there should be a method of determining an estimate from the structure itself. One method may be to administer SPR tests at a couple of points to obtain a rough estimate, and then increase that by some margin.

c.   To make this method more efficient and more capable of being put into small computers, numerical methods to reduce the size of the data block and required working space should be developed.

d.   Since the $[\tilde{C}_i]$ and $[\tilde{K}_i]$ matrices are system constants, it might prove advantageous to use cross-spectrums of the forces and accelerations to reduce the statistical variance. One form of this technique would modify Eq. (2-9) to obtain

$$-[G(\ddot{X}\dot{X})\ G(\ddot{X}X)\ G(\ddot{X}F)] = [[\tilde{C}_i][\tilde{K}_i][\tilde{D}_{id}]] \begin{bmatrix} G(\dot{\ddot{X}}\dot{X}) & G(\dot{X}X) & G(\dot{X}F) \\ G(X\dot{X}) & G(XX) & G(XF) \\ G(F\dot{X}) & G(FX) & G(FF) \end{bmatrix}$$

where $G(AB)$ is the cross-spectrum between A and B. This equation can now be used to solve for the modal matrices in one of two

fashions. The first would be to average the cross-spectrum matrices for each frequency point into one matrix before carrying out the mathematical operations and subsequent solution for the modal matrices. The second would be to obtain an estimate for the modal matrices for each frequency point and then average those estimates together to obtain the overall estimate.

e. Though the equations from Chapter 2 are applicable to any M, C, and K matrices, the derivation of the equations was for a system with proportional damping. A formal derivation for non-proportional damping should be developed.

f. Alternative methods for the development of pseudo-DOF should be explored and developed. A method which uses data previously acquired and manipulates the data numerically, through a phasing scheme or some other approach, would be highly attractive in that it would eliminate additional testing of the structure to produce the pseudo-DOF.

APPENDIX

Computer Program Listing

```
      PROGRAM RCMB2(TTY,INPUT,OUTPUT,TAPE2=TTY,TAPE5=INPUT,TAPE6=OUTPUT,
     1TAPE1)
*
*
*
*
*     IN THE PROGRAM ALL INTEGERS ARE INPUT IN I5 FORMAT, AND ALL
*     REAL NUMBERS IN F10 FORMAT.
*
*     THE USER IS FIRST ASKED HOW MANY D.O.F. THE SYSTEM HAS( I.E.
*     HOW MANY RESPONSE LOCATIONS).
*     NEXT THE USER IS ASKED HOW MANY MODES ARE TO BE IN THE SYSTEM
*     (THE NUMBER OF MODES DO NOT HAVE TO EQUAL THE NUMBER OF D.O.F.).
*
*     THE FREQUENCIES AND DAMPING RATIOS ARE THEN INPUT.
*     THE MODE SHAPES ARE INPUT AFTER THAT.
*     THE NEXT INPUT IS THE INITIAL FREQUENCY, AND THE FINAL FREQUENCY
*     GIVING THE FREQUENCY RANGE TO BE TESTED IN 256 INCREMENTS.
*     THE PROGRAM WILL OUTPUT THE FREQUENCIES CORRESPONDING TO THE
*     MAXIMUM AND MINIMUM AMPLITUDES IN THE IMAGINARY RESPONSE.
*     THE USER HAS THE CHOICE OF CHOOSING A DIFFERENT RESPONSE POINT
*     (THE REFERENCE POINT IS FIXED AT 1) OR JUST USING THE GIVEN
*     VALUES FOR ESTIMATES.
*
*     THE USER IS THEN ASKED HOW MANY MODES ARE TO BE EXAMINED( THIS
*     CANNOT EXCEED THE NUMBER OF D.O.F.), AND IS THEN REQUESTED TO
*     INPUT THE CORRESPONDING NUMBER OF RESPONSE LOCATIONS, AND THE
*     EXCITATION LOCATION.
*     THE PROGRAM RESPONDS WITH THE EIGENVALUES AND EIGEN VECTORS.
*
*     THIS PROGRAM USES IMSLIBF ROUTINE 'EIGRF'
*     AND 'GGUBFS'
*
*
*
*
      COMMON/FREQ/AM(20),W(20),D(20),SM(20,20),IFOR(10),NF
      COMMON/DATA/AI(256),AR(256),WW(256),F(10,512),UU(256)
      COMMON/PA/S(30,10),SMBKI(30,30),SMBK(10,30),ASIN(10,10)
      COMMON/AR/XX(30,30),SINT(10,10),SMINV(100),SMINVT(100)
      DIMENSION Y(10,512),X(30,512),YX(30,30),XIN(30,30)
      DIMENSION T(10,30),IRESRE(10)
      DIMENSION IDL(10),YD(10,512)
      DIMENSION ITYPE(20),IPHI(20),IRESLOC(20)
```

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```
       DIMENSION ZZZ(100),LB(29),MB(29),SY(400)
       DIMENSION IJK(20),PDPI(20,10)
       DIMENSION PMAG(20,20),THETA(20,20),WK(1000),XMEN(900)
       COMPLEX AW(20),PHI(20,20),ZZ(20,20),Z(400)
       DOUBLE PRECISION DSEED
*      SETTING CONSTANTS
       DSEED=1.D0
       TWOPI=8.*ATAN(1.)
*      INPUTTING DATA
       WRITE(2,80)
       PRINT 80
  80   FORMAT(1X,'ARE YOUR INPUT FREQUENCIES AND SHAPES ON A DATA',
      1/,1X,'FILE? [Y=1:N=0]')
       PRINT 27
       WRITE(2,27)
       READ(2,400)I
       PRINT 301,I
       IF(I.NQ.1) GOTO 600
       PRINT 1
       READ(5,400)NDOF
       PRINT 301,NDOF
       PRINT 2
       READ(5,400)NFR
       PRINT 301,NFR
       PRINT 3
       DO 200 I=1,NFR
       PRINT 4,I
       READ(5,401),W(I),D(I)
       PRINT 300,W(I),D(I)
       W(I)=W(I)*TWOPI
  200  CONTINUE
       PRINT 5
       INT=NDOF/7+1
       IEXTRA=NDOF-(INT-1)*7
       DO 201 I=1,NFR
       PRINT 6,I
       DO 201 JJ=1,INT
       IEND=7
       IF(JJ.EQ.INT) IEND=IEXTRA
       IF(IEND.EQ.0) GOTO 201
       READ(5,401)(SM((JJ-1)*7+J,I),J=1,IEND)
  201  CONTINUE
       DO 202 I=1,NFR
  202  PRINT 300,(SM(J,I),J=1,NDOF)
       GOTO 601
  600  CONTINUE
       WRITE(2,14)
       PRINT 14
  14   FORMAT(1X,'YOU MAY ENTER UP TO 20 MODES AND D.O.F. AND',
      1/,1X,'ANALIZE UP TO 10.',//)
```

```
      PRINT 1
      WRITE(2,1)
      PRINT 27
      WRITE(2,27)
  27  FORMAT(1X,'[ I5 FORMAT ]')
  28  FORMAT(1X,'[ F10 FORMAT ]')
   1  FORMAT(1X,'ENTER NUMBER OF DEGREES OF FREEDOM')
      READ(2,400) NDOF
 400  FORMAT(I5)
      PRINT 301,NDOF
 300  FORMAT(1X,'*',19(1X,F6.3))
 301  FORMAT(1X,'*',2X,4(2X,I5))
      PRINT 2
      WRITE(2,2)
      PRINT 27
      WRITE(2,27)
   2  FORMAT(1X,'ENTER NUMBER OF MODES')
      READ(2,400) NFR
      PRINT 301,NFR
      PRINT 3
      WRITE(2,3)
      PRINT 28
      WRITE(2,28)
   3  FORMAT(1X,'ENTER FREQUENCIES [IN HZ.], DAMPING RATIOS')
      DO 100 I=1,NFR
      PRINT 4,I
      WRITE(2,4)I
   4  FORMAT(1X,'FREQUENCY NO.',I3)
      READ(2,401) W(I),D(I)
 401  FORMAT(7(E10.4))
      PRINT 300,W(I),D(I)
      W(I)=TWOPI*W(I)
 100  CONTINUE
      PRINT 5
      WRITE(2,5)
      PRINT 28
      WRITE(2,28)
   5  FORMAT(1X,'ENTER MODE SHAPES')
      INT=NDOF/7+1
      IEXTRA=NDOF-(INT-1)*7
      DO 101 I=1,NFR
      PRINT 6,I
      WRITE(2,6)I
   6  FORMAT(1X,'ENTER MODE NO.',I3,' SEVEN D.O.F. PER LINE')
      DO 101 JJ=1,INT
      IEND=7
      IF(JJ.EQ.INT) IEND=IEXTRA
      IF(IEND.EQ.0) GOTO 101
      READ(2,401)(SM((JJ-1)*7+J,I),J=1,IEND)
 101  CONTINUE
```

```
         DO 102 I=1,NFR
  102    PRINT 300,(SM(J,I),J=1,NDOF)
  601    CONTINUE
         DO 569 I=1,NFR
         AM(I)=0.0
         DO 569 J=1,NDOF
  569    AM(I)=AM(I)+SM(J,I)*SM(J,I)
  888    CONTINUE
*
*        SETTING PARAMETERS FOR RUN
*
         WRITE(2,7)
         PRINT 7
         PRINT 28
         WRITE(2,28)
    7    FORMAT(1X,'ENTER INITIAL FREQUENCY RANGE IN HZ. ; INIT. FREQ. / ',
        1'FINAL FREQ.')
         READ(2,401) STFR,FIFR
         PRINT 330,STFR,FIFR
  330    FORMAT(1X,'*',2(1X,F12.5))
         STFR=TWOPI*STFR
         FIFR=TWOPI*FIFR
         FIST=FIFR-STFR
         WRITE(2,29)
         PRINT 29
         WRITE(2,27)
         PRINT 27
         READ(2,400) ITWO
         PRINT 301,ITWO
   29    FORMAT(1X,'INPUT THE NUMBER OF FREQUENCY POINTS TO BE ANALIZED',
        1/,1X,'[UP TO 00256]')
         IDTWO=2*ITWO
         ITT=ITWO-1
         TWOM1=FLOAT(ITT)
         WRITE(2,32)
         PRINT 32
         WRITE(2,28)
         PRINT 28
         READ(2,401) ANOISE
         PRINT 300,ANOISE
   32    FORMAT(1X,'INPUT THE PERCENT OF NOISE IN THE FUNCTIONS',
        1/,1X,'[ 100% = 1. ]')
*
*        GENERATING A CLEAN TRANSFER FUNCTION REFERENCED TO DOF 1.
*        PRINTS OUT ALL LOCAL MAXIMUMS AND MINIMUMS IN THE IMAGINARY
*        PLANE.
*
  519    WRITE(2,16)
         PRINT 16
         PRINT 27
```

```
      WRITE(2,27)
  16  FORMAT(1X,'ENTER THE RESPONSE POINT FOR THE INITIAL RUN')
      READ(2,400)JRESP
      PRINT 301,JRESP
*
*
*     INITIALIZE PARAMETERS TO KEEP FROM GETTING A WARNING
*
*
      ANSOLD=0.
      DIFOLD=0.
      WWOLD=0.
      DO 106 J=1,ITWO
      TT=FLOAT(J)
      WW(J)=STFR+(TT-1.)*FIST/TWOM1
      CALL TRAN(WW(J),1,ANS,NFR,JRESP)
      IF(J.EQ.1) GOTO 123
      DIF=ANS-ANSOLD
      IF(J.EQ.2) GOTO 124
      POS=DIF*DIFOLD
      IF(POS.GE.0) GOTO 124
      WRITE(2,12)WWOLD,ANSOLD
      PRINT 12,WWOLD,ANSOLD
 124  DIFOLD=DIF
 123  ANSOLD=ANS
      WWOLD=WW(J)/TWOPI
  12  FORMAT(1X,'W=',E12.6,' AMP=',E12.6)
 106  CONTINUE
      WRITE(2,17)
      PRINT 17
      PRINT 27
      WRITE(2,27)
  17  FORMAT(1X,'IS ANOTHER RESPONSE POINT REQUIRED? [Y=1/N=0]')
      READ(2,400)IANS
      PRINT 301,IANS
      IF(IANS.EQ.1) GOTO 519
      WRITE(2,8)
   8  FORMAT(1X,'ENTER NUMBER OF MODES TO BE EXAMINED')
      PRINT 8
      PRINT 27
      WRITE(2,27)
      READ(2,400) NME
      PRINT 301,NME
      WRITE(2,10)NME
      PRINT 10,NME
      PRINT 27
      WRITE(2,27)
  10  FORMAT(1X,'ENTER THE',I3,
     1' DIFFERENT INDEPENDENT RESPONSE LOCATIONS;')
     1/,1X,'ONE PER LINE')
```

```
          DO 103 I=1,NME
          READ(2,400) IRESLOC(I)
          PRINT 301,IRESLOC(I)
103   CONTINUE
          WRITE(2,26)
          PRINT 26
          WRITE(2,27)
          PRINT 27
 26   FORMAT(1X,'ENTER THE NUMBER OF FORCED INDEPENDENT RESPONSE',
     1/,1X,'LOCATIONS')
          READ(2,400)NF
          PRINT 301,NF
          DO 509 I=1,NF
          WRITE(2,40)
          PRINT 40
          READ(2,400)IFOR(I)
          PRINT 301,IFOR(I)
509   CONTINUE
 40   FORMAT(1X,'INPUT FORCE LOCATION')
          WRITE(2,21)
          PRINT 21
          PRINT 27
          WRITE(2,27)
 21   FORMAT(1X,'ENTER THE NUMBER OF DEPENDENT RESPONSE LOCATIONS')
          READ(2,400)NDL
          PRINT 301,NDL
          IF(NDL.EQ.0) GOTO 505
          WRITE(2,22)NDL
          PRINT 22,NDL
          PRINT 27
          WRITE(2,27)
 22   FORMAT(1X,'ENTER THE',I3,' DEPENDENT RESPONSE LOCATIONS;',
     1/,1X,'ONE PER LINE')
          DO 522 I=1,NDL
          READ(2,400)IDL(I)
522   PRINT 301,IDL(I)
505   CONTINUE
603   CONTINUE
          WRITE(2,602)
          PRINT 602
          WRITE(2,27)
          PRINT 27
          READ(2,400) IQUEST
          PRINT 301,IQUEST
602   FORMAT(1X,'DO YOU WANT SWEPT SINE [1]',
     1/,'OR RANDOM [2] FORCING?')
          IF(IQUEST.NE.1.AND.IQUEST.NE.2) GOTO 603
          IF(IQUEST.EQ.2) GOTO 605
          WRITE(2,606),NF
          PRINT 606,NF
```

```
          DO 604 I=1,NF
          WRITE(2,28)
          READ(2,401)LB(I)
          PRINT 300,LB(I)
 604      CONTINUE
 606      FORMAT(1X,'ENTER THE',I2,' PHASE SHIFTS')
          WRITE(2,608)
          PRINT 608
          WRITE(2,27)
          PRINT 27
          READ(2,400) IAVE
          PRINT 301,IAVE
 608      FORMAT(1X,'HOW MANY AVERAGES DO YOU WANT?')
          DO 576 KK=1,IAVE
          DO 576 I=1,NDOF
          DO 577 J=1,NF
          K=IFOR(J)
 577      IF(K.EQ.I) GOTO 578
          DO 579 J=1,IDTWO
 579      F(I,J)=0.0
          GOTO 580
 578      CONTINUE
          PHII=LB(J)
          CALL ASINE(FIST,IDTWO,STFR,FIFR,PHII,TWOM1,XMEN)
          DO 582 II=1,IDTWO
 582      Z(II)=CMPLX(XMEN(II),0.0)
          CALL FFT(IDTWO,Z,TWOPI)
          DO 581 J=1,ITWO
          JJJ=J+(KK-1)*ITWO
          JJ=JJJ+IAVE*ITWO
          F(K,JJJ)=REAL(Z(J))
          F(K,JJ)=AIMAG(Z(J))
 581      CONTINUE
 580      CONTINUE
 576      CONTINUE
          GOTO 607
 605      CONTINUE
          WRITE(2,608)
          PRINT 608
          WRITE(2,27)
          PRINT 27
          READ(2,400) IAVE
          PRINT 301,IAVE
          DO 568 KK=1,IAVE
          DO 568 I=1,NDOF
          DO 570 J=1,NF
          K=IFOR(J)
 570      IF(K.EQ.I) GOTO 571
          DO 572 J=1,IDTWO
 572      F(I,J)=0.0
```

```
        GOTO 573
  571   CONTINUE
        DO 574 J=1,ITWO
        JJJ=J+(KK-1)*ITWO
        JJ=JJJ+IAVE*ITWO
        ARAN=GGUBFS(DSEED)
        ARAN=ARAN*TWOPI
        F(K,JJJ)=COS(ARAN)
        F(K,JJ)=SIN(ARAN)
  574   CONTINUE
  573   CONTINUE
  568   CONTINUE
  607   CONTINUE
*
*       Y IS AN (NME)*(512) MATRIX CONTAINING THE ACCELERATION RESPONSES
*       THE REAL PARTS ARE IN TNE FIRST 256 COLUMNS AND THE IMAGINARY ARE
*     ' IN THE SECOND 256 COLUMNS.
*
*       X IS AN (2*NME+1)*(512) MATRIX CONTAINING THE VELOCITY RESPONSES
*       IN THE FIRST NME ROWS ( IN THE SAME FASHION AS THE ACCELERATION
*       RESPONSES), THE DISPLACEMENT RESPONSES IN THE SECOND NME ROWS.
*       THE FINAL ROW CONTAINS 1. IN THE FIRST 256 COLUMNS AND 0. IN
*       THE FINAL 256.
*
*
        DO 500 K=1,IAVE
        DO 500 I=1,NME
        IN=IRESLOC(I)
        CALL FRF(ITWO,TWOM1,STFR,FIFR,NFR,IN,ANOISE,DSEED,K,IAVE)
        DO 500 J=1,ITWO
        JJJ=J+(K-1)*ITWO
        JJ=JJJ+IAVE*ITWO
        II=I+NME
        III=II+NME
        X(II,JJ)=AI(J)
        X(II,JJJ)=AR(J)
        X(I,JJJ)=-WW(J)*X(II,JJ)
        X(I,JJ)=WW(J)*X(II,JJJ)
        Y(I,JJJ)=-WW(J)*WW(J)*X(II,JJJ)
        Y(I,JJ)=-WW(J)*WW(J)*X(II,JJ)
  500   CONTINUE
        IDTAVE=IDTWO*IAVE
        DO 575 I=1,NF
        II=I+NME+NME
        K=IFOR(I)
        DO 575 J=1,IDTAVE
        X(II,J)=F(K,J)
  575   CONTINUE
        IF(NDL.EQ.0) GOTO 530
        DO 523 K=1,IAVE
```

```
      DO 523 I=1,NDL
      IN=IDL(I)
      CALL FRF(ITWO,TWOM1,STFR,FIFR,NFR,IN,ANOISE,DSEED,K,IAVE)
      DO 523 J=1,ITWO
      JJJ=J+(K-1)*ITWO
      JJ=JJJ+IAVE*ITWO
      YD(I,JJJ)=-WW(J)*WW(J)*AR(J)
      YD(I,JJ)=-WW(J)*WW(J)*AI(J)
  523 CONTINUE
  530 CONTINUE
      N3=2*NME+NF
      N2=2*NME
      ND=NME-NF
      ND3=3*ND
      NF3=3*NF
*
*
*     XIN=(X)*(TRANSPOSE(X))
*     XIN IS PUT INTO AN ONE DIMENSIONAL ARRAY XMEN SO THAT IT CAN BE
*     INVERTED EASILY, THE INVERTED XMEN IS THEN PUT BACK INTO XIN
*
*
      DO 501 I=1,N3
      DO 501 J=1,N3
      XIN(I,J)=0.
      DO 501 K=1,IDTAVE
      XIN(I,J)=XIN(I,J)+X(I,K)*X(J,K)
  501 CONTINUE
   13 FORMAT(1X,17(E8.2))
      ID=0
      DO 510 J=1,N3
      DO 510 I=1,N3
      ID=ID+1
      XMEN(ID)=XIN(I,J)
  510 CONTINUE
      CALL MINV(XMEN,N3,DETER,LB,MB)
      ID=0
      DO 511 J=1,N3
      DO 511 I=1,N3
      ID=ID+1
      XIN(I,J)=XMEN(ID)
  511 CONTINUE
*
*
*     YX=(Y)*(TRANSPOSE(X))
*     T=(YX)*(XIN) ;  XIN BEING THE INVERTED FORM
*
*
      DO 502 I=1,NME
      DO 502 J=1,N3
```

```
         YX(I,J)=0.
         DO 502 K=1,IDTAVE
  502    YX(I,J)=YX(I,J)+Y(I,K)*X(J,K)
         DO 503 I=1,NME
         DO 503 J=1,N3
         T(I,J)=0.
         DO 503 K=1,N3
  503    T(I,J)=T(I,J)-YX(I,K)*XIN(K,J)
*
*        THE T MATRIX IS COMPOSED OF THREE SUBMATRICES;
*        THE FIRST IS B(NME,NME), THE NEXT IS K(NME,NME), THE
*        LAST IS GAMMA(NME,1).
*
*        SY IS A MATRIX COMPOSED OF B,K,AN IDENTITY MATRIX, AND A
*        NULL MATRIX, AS FOLLOWS:
*
*                        SY = [ B   K  ]
*                             [-I   0  ]
*
*        THE NEGATIVE OF SY IS PUT INTO AN EIGEN SOLVER AND THE VALUES
*        RETURNED ARE OUTPUT.
*
*        THE EIGENVALUES AND VECTORS ARE ALSO GENTERATED USING ONLY THE
*        K MATRIX.
*
*
         IT=-NME
         DO 504 J=1,2*NME
         IT=IT+NME
         DO 504 I=1,NME
         IT=IT+1
  504    SY(IT)=T(I,J)
         IT=0
         N2=2*NME
         DO 506 I=1,N2
         IT=IT+NME
         DO 506 J=1,NME
         IT=IT+1
         SY(IT)=0.
         IF(I.EQ.J) SY(IT)=-1.
  506    CONTINUE
  514    CONTINUE
         DO 518 I=1,N2*N2
         SY(I)=-SY(I)
  518    CONTINUE
         CALL EIGRF(SY,N2,N2,2,AW,Z,N2,WK,IER)
*
*        SCALING EIGENVECTORS
*
         IR=0
```

```
      DO 531 J=1,N2
      DO 531 I=1,N2
      IR=IR+1
      ZZ(I,J)=Z(IR)
  531 CONTINUE
      DO 545 J=1,N2
      ZMAGO=0.0
      DO 545 II=1,NME
      I=II+NME
      ZMAG=CABS(ZZ(I,J))
      PMAG(I,J)=ZMAG
      IF(II.GT.NME) GOTO 42
      IF(ABS(PMAG(I,J)).GT.ZMAGO) IJK(J)=I
      IF(ABS(PMAG(I,J)).GT.ZMAGO) ZMAGO=ABS(PMAG(I,J))
   42 CONTINUE
      RE=REAL(ZZ(I,J))
      AII=AIMAG(ZZ(I,J))
      THETA(I,J)=ATAN2(AII,RE)
  545 CONTINUE
      DO 546 J=1,N2
      DO 546 II=1,NME
      I=II+NME
      K=IJK(J)
      THET=THETA(I,J)-THETA(K,J)
      RE=(PMAG(I,J)*COS(THET))/ABS(PMAG(K,J))
      AII=(PMAG(I,J)*SIN(THET))/ABS(PMAG(K,J))
      ZZ(I,J)=CMPLX(RE,AII)
  546 CONTINUE
*
*
*     TO FIND THE MODE SHAPE COEFFICIENTS FOR THE DEPENDENT D.O.F.
*     USING THE EQUATION:
*
*               PHI(DEP)*INV[PHI(IND)]=[Y*TRANS(X)]*INV[X*TRANS(X)]
*     WHERE [Y]= THE ACCELERATION RESPONSES FOR THE DEPENDENT D.O.F.
*            [X]= THE ACCELERATION RESPONSES FOR THE INDEP. D.O.F.
*
*     PDPI= YX* YY     WHERE YY HAS BEEN INVERTED
*
*
      IF(NDL.EQ.0) GOTO 529
      DO 524 I=1,NME
      DO 524 J=1,NME
      XIN(I,J)=0.0
      DO 524 K=1,IDTAVE
  524 XIN(I,J)=XIN(I,J)+Y(I,K)*Y(J,K)
      DO 525 I=1,NDL
      DO 525 J=1,NME
      YX(I,J)=0.0
      DO 525 K=1,IDTAVE
```

```
 525  YX(I,J)=YX(I,J)+YD(I,K)*Y(J,K)
       IT=0
       DO 526 J=1,NME
       DO 526 I=1,NME
       IT=IT+1
 526  XMEN(IT)=XIN(I,J)
       CALL MINV(XMEN,NME,DETER,LB,MB)
       IT=0
       DO 527 J=1,NME
       DO 527 I=1,NME
       IT=IT+1
 527  XIN(I,J)=XMEN(IT)
       DO 528 I=1,NDL
       DO 528 J=1,NME
       PDPI(I,J)=0.0
       DO 528 K=1,NME
 528  PDPI(I,J)=PDPI(I,J)+YX(I,K)*XIN(K,J)
       ND2=2*NDL
       DO 532 I=1,NDL
       DO 532 J=1,N2
       PHI(I,J)=0.0
       DO 532 K=1,NME
       KK=K+NME
 532  PHI(I,J)=PHI(I,J)+PDPI(I,K)*ZZ(KK,J)
 529  CONTINUE
*
*
*      PRINT OUT THE CONVERGENCE CRITERIA FOR THE EIGEN SOLUTION
*      ( LESS THAN ONE INDICATES GOOD CONVERGENCE)
       WRITE(2,11)WK(1)
       PRINT 11,WK(1)
*      SORTING THE DEPENDENT AND INDEPENDENT D.O.F. INTO
*      CONSECTIVE ORDER
*
*
       ITOT=NDL+NME
       DO 533 I=1,ITOT
       DO 534 J=1,NME
       IF(IRESLOC(J).NQ.I) GOTO 534
       IPHI(I)=J
       ITYPE(I)=1
       GOTO 533
 534  CONTINUE
       IF(NDL.EQ.0) GOTO 533
       DO 541 J=1,NDL
       IF(IDL(J).NQ.I) GOTO 541
       IPHI(I)=J
       ITYPE(I)=2
 541  CONTINUE
```

```
 533  CONTINUE
*
*
*     PRINTING OUT THE EIGEN SOLUTIONS
*
*
      DO 547 K=1,N2
      AW(K)=AW(K)/TWOPI
      SIG=-REAL(AW(K))
      DEL=AIMAG(AW(K))
      SQ=SQRT(SIG*SIG+DEL*DEL)
      DMP=SIG/SQ
      PRINT 24,K,SQ,DMP
  24  FORMAT(///,1X,'MODE NO.',I3,2X,'NATURAL UNDAMPED FREQUENCY= ',
     1E12.6,/,33X,'DAMPING = ',E12.6,
     1//,1X,'COORDINATE',9X,'COEFFICIENT',/)
      DO 535 I=1,ITOT
      J=IPHI(I)
      JJ=J+NME
      IF(ITYPE(I).EQ.2) GOTO 536
      PRINT 23,I,ZZ(JJ,K)
      GOTO 535
 536  PRINT 23,I,PHI(J,K)
 535  CONTINUE
      IF(NMEID.EQ.0) GOTO 43
      DO 44 I=1,NMEID
      II=I+NME+NME
  44  PRINT 23,IRESRE(I),ZZ(II,K)
  43  CONTINUE
 547  CONTINUE
  23  FORMAT(4X,I2,12X,E12.6,2X,E12.6)
*
*
*     SETTING THE EIGEN PROBLEM FOR THE STIFFNESS MATRIX
*        XMEN(NME,NME) = T(NME,[NME TO 2*NME])
      IT=0
*
*
      DO 515 J=1,NME
      DO 515 I=1,NME
      JJ=J+NME
      IT=IT+1
      XMEN(IT)=T(I,JJ)
 515  CONTINUE
      PRINT 31
  31  FORMAT(//,1X,'K MATRIX')
      DO 516 I=1,NME
      PRINT 18,(XMEN(I+(J-1)*NME),J=1,NME)
  18  FORMAT(3X,10E12.6)
 516  CONTINUE
```

```
      CALL EIGRF(XMEN,NME,NME,2,AW,Z,NME,WK,IER)
      IR=0
      DO 537 J=1,NME
      DO 537 I=1,NME
      IR=IR+1
      ZZ(I,J)=Z(IR)
  537 CONTINUE
      DO 555 J=1,NME
      ZMAGO=0.0
      DO 555 II=1,NME
      PMAG(II,J)=REAL(ZZ(II,J))
      IF(II.GT.NME) GOTO 45
      IF(ABS(PMAG(II,J)).GT.ZMAGO) IJK(J)=II
      IF(ABS(PMAG(II,J)).GT.ZMAGO) ZMAGO=ABS(PMAG(II,J))
   45 CONTINUE
  555 CONTINUE
      DO 556 J=1,NME
      DO 556 II=1,NME
      K=IJK(J)
      ZZ(II,J)=ZZ(II,J)/ABS(PMAG(K,J))
  556 CONTINUE
*
*
*     FINDING THE MODE SHAPE COEFFICIENTS FOR THE DEP. D.O.F.
*
*
      IF(NDL.EQ.0) GOTO 544
      DO 538 I=1,NDL
      DO 538 J=1,NME
      PHI(I,J)=0.0
      DO 538 K=1,NME
      PHI(I,J)=PHI(I,J)+PDPI(I,K)*ZZ(K,J)
  538 CONTINUE
  544 CONTINUE
*
*
*     COMPUTING THE DAMPING FACTORS
*     DIAG[INV[PHI(IND)]*B(MATRIX)*PHI(IND)] =[2*DAMP(I)*AW(I)]
*
*
      IT=0
      DO 508 J=1,NME
      DO 508 I=1,NME
      IT=IT+1
      ZZZ(IT)=Z(IT)
  508 PDPI(I,J)=Z(IT)
      DO 542 I=1,NME
      DO 542 J=1,NME
      XIN(I,J)=0.0
      DO 542 K=1,NME
```

```
 542  XIN(I,J)=XIN(I,J)+T(I,K)*PDPI(K,J)
       CALL MINV(ZZZ,NME,DETER,LB,MB)
       IT=0
       DO 543 J=1,NME
       DO 543 I=1,NME
       IT=IT+1
 543  PDPI(I,J)=ZZZ(IT)
       DO 520 I=1,NME
       DO 520 J=1,NME
       THETA(I,J)=0.0
       DO 520 K=1,NME
 520  THETA(I,J)=THETA(I,J)+PDPI(I,K)*XIN(K,J)
*
*
*     CHANGING EIGENVALUES TO HERTZ AND OUTPUTTING SOLUTIONS
*
*
       DO 548 K=1,NME
       AW(K)=AW(K)/(TWOPI*TWOPI)
       RAW=AW(K)
       ICOUNT=0
       IF(RAW.LT.0) ICOUNT=1
       RAW=ABS(RAW)
       RAW=SQRT(RAW)
       DAMP=THETA(K,K)/(2.*TWOPI*RAW)
       PRINT 20,K,RAW,DAMP
       IF(ICOUNT.EQ.1) PRINT 25
  25  FORMAT(1X,'OMEGA SQUARED WAS NEGATIVE',//)
  20  FORMAT(1X,'MODE NO.',I2,'  NATURAL FREQ.(HZ)= ',E12.6,
     12X,'DAMPING= ',E12.6,/)
       DO 539 I=1,ITOT
       J=IPHI(I)
       IF(ITYPE(I).EQ.2) GOTO 540
       PRINT 23,I,ZZ(J,K)
       GOTO 539
 540  PRINT 23,I,PHI(J,K)
 539  CONTINUE
       IF(NMEID.EQ.0) GOTO 548
       DO 47 I=1,NMEID
       II=I+NME
  47  PRINT 23, IRESRE(I),ZZ(II,K)
 548  CONTINUE
       WRITE(2,19)
       PRINT 19
       PRINT 27
       WRITE(2,27)
  19  FORMAT(1X,'DO YOU WANT TO REPEAT THIS WITH THE SAME FUNCTIONS?',
     1'    Y=1 N=0')
       READ(2,400)IT
       PRINT 301,IT
```

```
      IF(IT.EQ.1) GOTO 888
11    FORMAT(2X,E12.6,2X,E12.6)
      END




      SUBROUTINE TRAN(F,IX,Y,NFR,J)
      COMMON/FREQ/AM(20),W(20),D(20),SM(20,20),IFOR(10),NF
      Y=0.
      DO 10 I=1,NFR
      Y=Y+(SM(J,I)*SM(IX,I)*2.*D(I)*(F/W(I))/
     1((1-(F/W(I))**2)**2+(2.*D(I)*F/W(I))**2))
10    CONTINUE
      RETURN
      END


      SUBROUTINE FRF(ITWO,TWOM1,STFR,FIFR,NFR,I,ANOISE,DSEED,KAY,IAVE)
      COMMON/FREQ/AM(20),W(20),D(20),SM(20,20),IFOR(10),NF
      COMMON/DATA/AI(256),AR(256),WW(256),F(10,512),UU(256)
      DOUBLE PRECISION DSEED
*
*     DEVELOPES THE DISPLACEMENT FRFS
*
      RANGE=FIFR-STFR
      DELT=RANGE/TWOM1
      NTIM=ITWO
      X=0.0
      DO 101 K=1,ITWO
      KFOUR=K+(KAY-1)*ITWO
      KKK=KFOUR+IAVE*ITWO
      TK=FLOAT(K)
      WW(K)=STFR+(TK-1.)*DELT
      AI(K)=0.
      AR(K)=0.
      DO 101 KK=1,NF
      ISH=IFOR(KK)
      DO 101 J=1,NFR
      AK=AM(J)*W(J)*W(J)
      HI=(SM(ISH,J)*SM(I,J)/AK)/
     1((1.-(WW(K)/W(J))**2)**2+(2.*D(J)*WW(K)/W(J))**2)
      ANUM=(1.-(WW(K)/W(J))**2)*HI
      BNUM=-2.*D(J)*(WW(K)/W(J))*HI
      AI(K)=AI(K)+ANUM*F(ISH,KKK)+BNUM*F(ISH,KFOUR)
      AR(K)=AR(K)+ANUM*F(ISH,KFOUR)-BNUM*F(ISH,KKK)
      X=X+(AI(K)*AI(K)+AR(K)*AR(K))
101   CONTINUE
```

```
      X=SQRT(X)
      C=ANOISE*X/ITWO
      DO 102 K=1,ITWO
      DSE=GGUBFS(DSEED)
      ANOI=2.*(2.*DSE-1.)*C
      AMAG=AI(K)*AI(K)+AR(K)*AR(K)
      AMAG=SQRT(AMAG)
      AI(K)=AI(K)+ANOI*AI(K)/AMAG
      AR(K)=AR(K)+ANOI*AR(K)/AMAG
  102 CONTINUE
      RETURN
      DO 110 J=1,NTIM
      T=FLOAT(J)
      T=8.*ATAN(1.)*T/RANGE
      UU(J)=0.0
      DO 110 K=1,ITWO
      AN=FLOAT(K)-1.
      IF(K.EQ.1) AR(K)=AR(K)/2.
      UU(J)=UU(J)+AR(K)*COS(T*DELT*AN)*2.-2.*AI(K)*SIN(T*DELT*AN)
      IF(K.EQ.1) AR(K)=2.*AR(K)
  110 CONTINUE
      WRITE(1,111) (UU(J),J=1,NTIM)
  111 FORMAT(4E14.6)
      RETURN
      END



      SUBROUTINE MINV(AMAT,N,D,L,M)
      DIMENSION AMAT(900),L(29),M(29)
*
*     MATRIX INVERSION SUBROUTINE
*
   42 CONTINUE
      D=1.
      NK=-N
      DO 80 K=1,N
      NK=NK+N
      L(K)=K
      M(K)=K
      KK=NK+K
      BIGA=AMAT(KK)
      DO 20 J=K,N
      IZ=N*(J-1)
      DO 20 I=K,N
      IJ=IZ+I
   10 IF(ABS(BIGA)-ABS(AMAT(IJ))) 15,20,20
   15 BIGA=AMAT(IJ)
      L(K)=I
      M(K)=J
   20 CONTINUE
```

```
        J=L(K)
        IF(J-K) 35,35,25
25      KI=K-N
        DO 30 I=1,N
        KI=KI+N
        HOLD=-AMAT(KI)
        JI=KI+J-K
        AMAT(KI)=AMAT(JI)
30      AMAT(JI)=HOLD
35      I=M(K)
        IF(I-K) 45,45,38
38      JP=N*(I-1)
        DO 40 J=1,N
        JK=NK+J
        JI=JP+J
        HOLD=-AMAT(JK)
        AMAT(JK)=AMAT(JI)
40      AMAT(JI)=HOLD
45      IF(BIGA) 48,46,48
46      D=0.0
        RETURN
48      DO 55 I=1,N
        IF(I-K) 50,55,50
50      IK=NK+I
        AMAT(IK)=AMAT(IK)/(-BIGA)
55      CONTINUE
        DO 65 I=1,N
        IK=NK+I
        HOLD=AMAT(IK)
        IJ=I-N
        DO 65 J=1,N
        IJ=IJ+N
        IF(I-K) 60,65,60
60      IF(J-K) 62,65,62
62      KJ=IJ-I+K
        AMAT(IJ)=HOLD*AMAT(KJ)+AMAT(IJ)
65      CONTINUE
        KJ=K-N
        DO 75 J=1,N
        KJ=KJ+N
        IF(J-K) 70,75,70
70      AMAT(KJ)=AMAT(KJ)/BIGA
75      CONTINUE
        D=D*BIGA
        AMAT(KK)=1./BIGA
80      CONTINUE
        K=N
100     K=(K-1)
        IF(K) 150,150,105
105     I=L(K)
```

```
      IF(I-K) 120,120,108
108   JQ=N*(K-1)
      JR=N*(I-1)
      DO 110 J=1,N
      JK=JQ+J
      HOLD=AMAT(JK)
      JI=JR+J
      AMAT(JK)=-AMAT(JI)
110   AMAT(JI)=HOLD
120   J=M(K)
      IF(J-K) 100,100,125
125   KI=K-N
      DO 130 I=1,N
      KI=KI+N
      HOLD=AMAT(KI)
      JI=KI-K+J
      AMAT(KI)=-AMAT(JI)
130   AMAT(JI)=HOLD
      GOTO 100
150   RETURN
      END



      SUBROUTINE ASINE(FIST,IDTWO,STFR,FIFR,PHI,TWOM1,XMEN)
      DIMENSION XMEN(512)
*
*     DEVELOPES TIME HISTORY OF SWEPT-SINE EXCITATION
*
      DEL=FIST/TWOM1
      TO=1./((TWOM1+1.)*DEL)
      DO 10 I=1,IDTWO
      TI=FLOAT(I)
      F=STFR+DEL*(TI-1.)
      T=TO*(TI-1.)/TWOM1
 10   XMEN(I)=(3.-T/TO)*SIN(F*T+PHI)
      RETURN
      END



      SUBROUTINE FFT(IDTWO,Z,TWOPI)
      COMPLEX Z(512),U,T,W
*
*     DEVELOPES FRF OF THE SWEPT-SINE EXCITATION
*
      II=IDTWO
      M=0
      PI=TWOPI/2.
  1   M=M+1
      II=II/2
```

```
       IF(II.GE.2) GOTO 1
       MV2=IDTWO/2
       NM1=IDTWO-1
       J=1
       DO 7 I=1,NM1
       IF(I.GE.J) GOTO 5
       T=Z(J)
       Z(J)=Z(I)
       Z(I)=T
  5    K=MV2
  6    IF(K.GE.J) GOTO 7
       J=J-K
       K=K/2
       GOTO 6
  7    J=J+K
       DO 20 L=1,M
       LE=2**L
       LE1=LE/2
       U=(1.,0.)
       W=CMPLX(COS(PI/LE1),-SIN(PI/LE1))
       DO 20 J=1,LE1
       DO 10 I=J,IDTWO,LE
       IP=I+LE1
       T=Z(IP)*U
       Z(IP)=Z(I)-T
  10   Z(I)=Z(I)+T
  20   U=U*W
       RETURN
       END
```

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