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Estimation of parameters in linear structural relationships: Sensitivity to the choice of the ratio of error variances

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Summary

Maximum likelihood estimation of parameters in linear structural relationships under normality assumptions requires knowledge of one or more of the model parameters if no replication is available. The most common assumption added to the model definition is that the ratio of the error variances of the response and predictor variates is known. This article investigates the use of asymptotic formulae for variances and mean squared errors as a function of sample size and the assumed value for the error variance ratio.

Some key words: Errors in variables; Identifiability; Regression

1. Introduction

Linear structural relationships are linear models between two stochastic variates (Y,X) in which both variates are measured with error. Let $Y_i = \alpha + \beta X_i$ and define observable variates

$$x_i = X_i + u_i \qquad y_i = Y_i + v_i, \quad i = 1,2,\ldots,n. \tag{1.1}$$

Assume further that independently $X \sim N(\mu_X, \sigma_X^2)$, $u \sim N(0, \sigma_u^2)$, and $v \sim N(0, \sigma_v^2)$.

Under these model assumptions it is well known that in the absence of replication no consistent estimators of α and β exist because the model lacks identifiability (e.g., Madansky 1959). Geary (1942) showed that when (u,v) are jointly normally distributed, if X possesses a finite cumulant of order greater than two then β is identifiable in the joint distribution of (x,y); thus, nonnormal distributions for X generally allow consistent estimation of β. Reiersol (1950) strengthened this result by proving that if (i) u and v are independently distributed or (ii) (u,v) is bivariate normal, nonnormality of X is a necessary and sufficient condition for identifiability of β. Reiersol's results are summarized in Table 1. Note especially that α is identifiable (and estimable) when β is identifiable; consequently, the focus of this article is on the estimation of the slope parameter β.

[Insert Table 1]

The identifiability conditions displayed in Table 1 pertain to linear structural models in which none of the model parameters are known. Kendall and Stuart (1977, Chapter 29) detail various solutions to the likelihood equations when one or more of the variances in model (1.1) are known. Of important theoretical interest is the assumption that $\lambda = \sigma_v^2/\sigma_u^2$, the ratio of error variances, is known. Under this assumption the joint distribution of (y,x) is identifiable and the likelihood equations have a unique solution for β:

$$\hat{\beta} = s(\lambda) + \delta(s_{xy})[s^2(\lambda) + \lambda]^{1/2}, \quad s(\lambda)=(s_y^2-\lambda s_x^2)/(2s_{xy}) \qquad (1.2)$$

where s_y^2, s_x^2, and s_{xy} are the sample variances and covariance, respectively, and $\delta(s_{xy})$=sign(s_{xy}). This solution is consistent and asymptotically normal and all estimators of the model variances are assured to be nonnegative. While this estimator is commonly used to estimate the slope parameter, few theoretical or simulation studies have been conducted to evaluate the adequacy of asymptotic variance formulae for finite sample sizes or the sensitivity of estimator (1.2) to erroneous selection of the variance ratio λ.

In this article both of the above topics are investigated. Section 2 contains asymptotic variance and mean squared error formulae for estimator (1.2) for both correct and incorrect choices of λ. Section 3 presents the results of a simulation study in which the sample size and the assumed value of λ are varied for several model configurations. Replication of observations is discussed in Section 4 and concluding remarks are made in Section 5.

2. Asymptotic Properties

Asymptotically (i.e., replacing sample moments by their parameter values),

$$\partial\hat{\beta}/\partial\lambda = -\beta t/(\beta^2+\lambda) \ , \qquad (2.1)$$

where $t = \sigma_u^2/\sigma_X^2$ is a "noise-to-signal ratio" for the observable predictor variable x. The rate of change of $\hat{\beta}$ with respect to λ is thus seen to depend on the true values of β, λ, and t. Figure 1 illustrates the general features of equation (2.1): $\hat{\beta}$ is relatively insensitive to the true value of λ for large values of λ and small values of t. Together these two conditions imply that σ_u^2, the error variance for the observable variate x, is small. In other words, under the conditions for which the linear structural model (1.1) is usually proposed (i.e., t moderate to large or λ small to moderate--each implying that σ_u^2 is nonnegligible), the estimator (1.2) can be very sensitive to the true value of λ.

[Insert Figure 1]

A similar perspective on the sensitivity of (1.2) to the value of λ is obtained by assuming λ is stochastic rather than constant. Lindley and El-Sayyad (1968) suggest assuming a uniform (k^{-1},k) prior for λ if the two measurement errors are believed to be of the same magnitude. Alternative proposals might include $N(k,\sigma_\lambda^2)$ or Chisquare(k) priors. Using statistical differentials (e.g., Serfling 1980) one can approximate the expectation of (2.1) using a three-term Taylor series expansion of $\partial\hat{\beta}/\partial\lambda$. The approximate expectations are, respectively,

$$-2\beta t[(2\beta^2 + k + k^{-1})^{-1} + (k-k^{-1})^2\{3(2\beta^2 + k + k^{-1})^3\}^{-1}] \quad (2.2)$$

$$-\beta t[(\beta^2 + k)^{-1} + \sigma_\lambda^2(\beta^2 + k)^{-3}] \quad (2.3)$$

$$-\beta t[(\beta^2 + k)^{-1} + 2k(\beta^2 + k)^{-3}] \ . \quad (2.4)$$

Graphs of equations (2.2) to (2.4) as a function of k are variants of Figure 1, all resulting in the same general conclusion: the slope estimator (1.2) is relatively insensitive to the true value of λ only when t is close to zero and k is large.

The slope estimator (1.2) is asymptotically unbiased when the error variance ratio is known. Again applying the method of statistical differentials, the asymptotic variance of (1.2) (ignoring terms of $0(n^{-2})$) is

$$n^{-1}[(\beta^2 + \lambda)t + \lambda t^2] , \tag{2.5}$$

which reduces to equation (9) of Robertson (1974) when $\lambda = 1$. For comparative purposes, the asymptotic mean squared error for the least squares estimator (s_{xy}/s_x^2) of the slope parameter under the assumptions accompanying model (1.1) is (c.f., Richardson and Wu 1970, equations (2.24) and (2.25))

$$\beta^2 t(n^{-1} + t)(1 + t)^{-2} + n^{-1}\lambda t(1 + t)^{-1} . \tag{2.6}$$

Note that the least squares estimator is the maximum likelihood estimator when $\sigma_u^2 = 0$, in which case (2.6) reduces to $\sigma_v^2/(n\sigma_X^2)$.

The foregoing expressions enable one to assess the sensitivity of the linear structural estimator (1.2) to the true value of the ratio of error variances. In application it is also of interest to examine the sensitivity of (1.2) to an erroneous choice of λ. When λ is incorrectly specified, (1.2) is no longer asymptotically unbiased. Ignoring terms of $0(n^{-2})$, the asymptotic expectation and variance of (1.2) using an assumed value λ^* for the ratio of error variances are, respectively,

$$E(\hat{\beta}) = g_\lambda(\lambda^*) + \delta[g_\lambda(\lambda^*)][g_\lambda^2(\lambda^*) + \lambda^*]^{1/2} \tag{2.7}$$

$$\mathrm{var}(\hat{\beta}) = n^{-1}(\beta^2+\lambda^*)^{-2}[3\beta^2 t^2(\lambda-\lambda^*)^2+(\beta^2+\lambda^*)^2\{(\beta^2+\lambda)t+\lambda t^2\}] \tag{2.8}$$

where $g_\lambda(\lambda^*) = [(\beta^2-\lambda^*)\sigma_X^2+(\lambda-\lambda^*)\sigma_u^2]/[2\beta\sigma_X^2]$ and $\delta[g_\lambda(\lambda^*)]=\mathrm{sign}[g_\lambda(\lambda^*)]$. When $\lambda^* = \lambda$, $\mathrm{bias}(\hat{\beta}) = 0$ and equation (2.8) reduces to (2.5).

Figures 2 and 3 compare the asymptotic mean squared errors of the structural model estimator (1.2) with the least squares estimator, the latter mean squared error calculated from equation (2.6). In Figure 2 the true variance ratio λ is assumed known and β, σ_X^2, and σ_v^2 are fixed at 3, 5, and 10, respectively. Unless λ is extremely small, corresponding to relatively small error in the response variable, the structural model estimator has a smaller asymptotic mean squared error than least squares, with the improvement offered by the structural model estimator increasing with the sample size and decreasing with λ.

[Insert Figure 2]

If λ^* is chosen incorrectly, Figure 3 demonstrates that the benefits of using structural model estimators over least squares diminishes as λ^* differs from λ. For this figure the model parameters are set at $(\beta, \sigma_X^2, \lambda) = (3, 5, 6)$. Both Figures 2 and 3 are illustrative of a general conclusion which can be drawn from a comparison of the asymptotic mean squared errors: λ^* must be in relative proximity to the true value λ for the structural model estimator to be a substantial improvement over least squares.

[Insert Figure 3]

3. Simulation Results

In each of Tables 2 to 5, 1000 replications of samples of size n were generated from model (1.1) with normal variates generated by I.M.S.L. subroutine GGNML on a C.D.C. 6600 computer. Table 2 compares the average $\hat{\beta}$ calculated using equation (1.2) with the true value of β. In this table β, σ_X^2, and σ_u^2 are fixed at 3, 5, and 5, respectively, so that varying σ_v^2 the results are only a function of λ and n. For samples of size 50 and 100 the maximum relative error in estimating β using the correct value of λ^* is 4%. Incorrectly choosing λ^* larger than the true variance ratio results in underestimation of β whereas too small a selection of λ^* results in overestimation of β.

[Insert Table 2]

Estimated and asymptotic mean squared errors are compared in Table 3. Estimated mean squared errors are computed from the usual formula,

$$\text{mse} = \Sigma(\hat{\beta}-\beta)^2/1000 \quad ,$$

and asymptotic mean squared errors are obtained from equation (2.5) (recall that $\hat{\beta}$ is asymptotically unbiased when λ is known) using the true values of β, λ, and t. The ratios in Table 3 corresponding to correct assumed values of λ^* indicate that use of asymptotic formulae for moments of structural model estimators cannot be recommended for samples of size 100 or less. Even when λ^* is chosen correctly and the true model parameters are inserted in the asymptotic formulae, samples of size 100 result in errors of 15-30% between sample mean squared errors and those calculated from equation (2.5).

[Insert Table 3]

Tables 4 and 5 display ratios of sample and asymptotic mean squared errors for samples of size 200 for a variety of values of

β, λ, and t. When λ^* is chosen correctly the ratios are much closer to 1.0 in these tables than in Table 3. If a relative error of approximately 10% or less is acceptable, samples of size 200 could be considered minimally acceptable for a wide range of model parameters.

[Insert Tables 4 and 5]

Tables 3 to 5 also demonstrate that λ^* must be selected near its true value for the asymptotic variance formula (2.5) to provide a reasonable assessment of the variability of $\hat{\beta}$. When β is small it is especially undesirable to choose values of λ^* which are less than the true ones. The deleterious effects of erroneous selection of λ^* decrease with larger values of β and smaller values of t.

4. Replication

Replication of observations for one or more specific values of X allows consistent estimation of β when λ is unknown. Dorff and Gurland (1961) investigate four analysis of variance estimators of β in functional equation models (the X_i are assumed to be unknown constants) when one or more of the X_i are replicated. On the basis of asymptotic mean squared error comparisons when an equal number of replications is available for each X_i, they prefer an estimator similar to equation (1.2) in which the following estimator of λ is inserted in place of the true value:

$$\hat{\lambda} = w_{yy}/w_{xx}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{r} (y_{ij}-\bar{y}_{i.})^2 / \sum_{i=1}^{n} \sum_{j=1}^{r} (x_{ij}-\bar{x}_{i.})^2, \qquad (4.1)$$

where r replicates for both x and y are available for each of the n X_i. Under the structural model assumptions (1.1) this estimator of β is consistent and has asymptotic variance equal to

$$m(\beta^2+\lambda)+\lambda nm^2(\beta^2-\lambda)^2(\beta^2+\lambda)^{-2} + 4\lambda^2m^2\beta^2(\beta^2+\lambda)^{-2}[rn/(r-1)] \qquad (4.2)$$

where m = t/rn. Equation (4.2) corresponds to $var_A(b_4)$ of Dorff and Gurland (1961).

Barnett (1970) derives the maximum likelihood estimator of β for functional models. Dolby (1976) derives maximum likelihood estimators of β for an "ultrastructural" model which includes the structural and functional models as special cases. When an equal number of replicates are available for each X_i, he shows that the maximum likelihood estimator of β is again of the form (1.2) with the following estimator of λ:

$$\hat{\lambda} = (s_{yy}w_{xy}-s_{xy}w_{yy})/(s_{xx}w_{xy}-s_{xy}w_{xx}) \;, \qquad (4.3)$$

where $s_{yy} = r\Sigma(\bar{y}_{i.}-\bar{y}_{..})^2$, etc. The two estimators of λ, equations (4.1) and (4.3), are asymptotically equivalent since $plim(w_{xy}) = 0$; therefore, the asymptotic variance of $\hat{\beta}$ using (4.3) to estimate λ is given by equation (4.2). Although the maximum likelihood estimator and the analysis of variance estimator are asymptotically equivalent, the latter estimator might be preferable with small sample sizes since λ estimated by (4.3) can be negative (see Dolby (1976)).

Tables 6 and 7 compare empirical and asymptotic properties of the analysis of variance structural model estimator based on equation (4.1). The summary statistics displayed in these tables are computed from 1000 simulated experiments with $(\beta,\sigma_x^2,\sigma_u^2)=(3,5,5)$ and r = 2 and 5 replicates. The analysis of variance estimator produces satisfactory agreement (relative error less than 10%) between the average estimate of β and its true value for sample sizes as small as n = 20 with r = 2 replicates. Agreement between the empirical and asymptotic mean squared errors again requires a sample size of at least n = 200 (total sample size N = nr) for an empirical relative error of approximately 10% or less.

[Insert Tables 6 and 7]

Erroneous use of least squares when the predictor variable is measured with error is especially unwarranted when λ can be estimated with replicated observations. Figure 4 illustrates that estimator (1.2) using equation (4.1) for $\hat{\lambda}$ substantially improves estimator accuracy over least squares, even for small sample sizes. The model parameters used in the construction of this figure are the same as those used in Figure 2. Maximum likelihood estimation of β using equation (4.3) results in simulation results comparable to Tables 6 and 7 and mean squared error improvement over least squares equal to that displayed in Figure 4.

[Insert Figure 4]

5. Discussion

The results presented in Sections 3 and 4 are only a portion of a larger study in which simulations and asymptotic comparisons were conducted for a wide range of model parameters. The tables and figures are illustrative of the overall results. In general the effective use of asymptotic properties of the structural estimator (1.2) requires a large sample size and accurate selection of the variance ratio λ when the variance ratio cannot be estimated from replicated observations. Correct selection of λ and a large sample size also assures a smaller mean squared error than least squares unless the variance ratio is very small. Incorrect selection of λ, especially the selection of too small a value, compromises the effectiveness of the structural model estimator relative to least squares. Replication of observations for one or more specific values of X is an effective alternative to least squares when all model parameters are unknown provided that the total sample size is sufficiently large.

The investigations reported in this paper assume that the true predictor variable is normally distributed. If X is nonnormal all model parameters, including λ, are usually estimable. Unfortunately the derivation of maximum likelihood estimators is theoretically intractable for many important distributions; e.g., $X \sim \text{Beta}(a,b)$ and (u,v) normally distributed. Although moment estimators of β are available, they are not unique and are often inefficient. Alternatives to moment estimators are currently under investigation and will be reported in the near future.

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Table 1. Identifiability conditions for linear structural models

(a) Identifiability of β

(i) X is nonnormally distributed and either u and v are independent or (u,v) is bivariate normal

(ii) X is normally distributed and the distribution of neither u nor v is divisible by a normal distribution

(b) β is identifiable

(i) α is identifiable

(ii) if u and v are independent and the characteristic functions of u, v, and X are continuous, all other model parameters are identifiable

(iii) if (u,v) is normally distributed, all other model parameters are identifiable iff

  (1) the distribution of X (and Y) is not divisible by a normal distribution, and

  (2) either u=0 or v=0

Table 2. Ratio of simulated and asymptotic expectations of structural model slope estimator

True λ	Assumed λ 0.2	0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.0
					(a) n = 20				
0.2	1.16	1.04	0.99	0.95	0.89	0.77	0.70	0.65	0.62
1.0	1.21	1.13	0.92	1.02	0.99	0.81	0.73	0.68	0.66
2.0	1.40	1.29	1.15	1.09	1.09	0.90	0.79	0.73	0.69
10.0	2.49	2.08	1.93	2.20	2.30	2.26	1.43	1.25	1.10
					(b) n = 50				
0.2	1.02	0.99	0.94	0.90	0.86	0.74	0.68	0.64	0.62
1.0	1.11	1.09	1.02	0.98	0.93	0.81	0.72	0.68	0.64
2.0	1.23	1.19	1.14	1.08	1.02	0.87	0.78	0.71	0.67
10.0	2.27	2.19	2.13	2.04	1.98	1.62	1.37	1.19	1.04
					(c) n = 100				
0.2	1.01	0.98	0.93	0.89	0.85	0.74	0.67	0.64	0.61
1.0	1.10	1.07	1.01	0.96	0.92	0.79	0.72	0.67	0.64
2.0	1.22	1.17	1.12	1.07	1.02	0.87	0.77	0.71	0.67
10.0	2.16	2.12	2.03	1.93	1.85	1.56	1.33	1.14	1.03

Table 3. Ratio of simulated and asymptotic (using true λ) mean squared errors of the structural model slope estimator

True λ	Assumed λ 0.2	0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.0
					(a) n = 20				
0.2	108.11	2.69	1.94	2.02	1.31	2.48	2.68	2.83	3.23
1.0	9.91	3.32	550.84	2.72	1.99	1.66	1.81	2.29	2.58
2.0	35.78	10.48	80.04	90.50	3.42	2.38	2.24	1.80	2.24
10.0	4553.13	996.18	3087.32	5074.26	357.29	1563.05	43.71	92.26	25.27
					(b) n = 50				
0.2	1.27	1.15	1.20	1.35	1.86	3.81	5.29	6.54	7.40
1.0	2.21	1.92	1.53	1.13	1.25	2.25	3.69	4.72	5.70
2.0	3.79	3.08	2.35	1.81	1.29	1.44	2.37	3.45	4.18
10.0	36.15	30.49	30.84	42.32	26.79	18.65	4.80	4.64	1.69
					(c) n = 100				
0.2	1.19	0.99	1.37	1.94	3.07	6.89	10.42	13.12	14.69
1.0	2.27	1.63	1.15	1.16	1.44	4.28	7.04	9.35	11.00
2.0	5.07	3.69	2.29	1.61	1.15	2.06	4.33	6.37	7.98
10.0	48.73	46.22	38.95	31.49	27.20	12.54	5.58	2.02	1.29

Table 4. Ratio of simulated and asymptotic (using true λ) mean squared errors of the structural model slope estimator: $n = 200$, $t = .1$

	Assumed λ								
True λ	0.2	0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.0
					(a) $\beta = 0.1$				
0.2	1.10	1.23	1.44	1.48	1.38	1.57	1.49	1.54	1.62
1.0	9.31	1.40	1.10	1.08	0.97	1.08	0.98	1.01	1.06
2.0	118.04	3.44	1.38	1.14	0.98	0.97	1.00	1.03	0.94
10.0	3046167.58	2823896.24	6151.68	9.22	3.85	1.55	1.08	1.04	0.98
					(b) $\beta = 1.0$				
0.2	1.08	1.65	3.49	5.09	6.35	9.13	10.93	11.47	12.11
1.0	5.45	2.30	1.06	1.46	1.88	4.13	5.28	6.15	6.45
2.0	16.43	7.97	2.90	1.35	1.07	1.98	2.85	3.58	3.87
10.0	137.29	95.94	53.21	31.12	18.68	4.15	1.95	1.20	1.06
					(c) $\beta = 10.0$				
0.2	1.03	1.06	1.03	1.06	1.04	1.07	1.10	1.09	1.03
1.0	1.06	1.03	1.03	1.04	1.04	1.06	1.01	1.13	1.15
2.0	1.07	1.07	1.09	0.95	1.02	1.07	1.02	1.07	1.15
10.0	1.26	1.22	1.23	1.08	1.22	1.16	1.13	1.13	1.10

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Table 5. Ratio of simulated and asymptotic (using true λ) mean squared errors of the structural model slope estimator: $n = 200$, $t = 1$

True λ	Assumed λ 0.2	0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.0
					(a) $\beta = 0.1$				
0.2	1.08	0.99	1.18	1.31	1.33	1.40	1.48	1.45	1.42
1.0	1.57	31592.99	1.08	0.67	0.56	0.51	0.51	0.53	0.50
2.0	11789245.73	238241.92	57182.60	3.27	1.40	0.51	0.44	0.41	0.39
10.0	38414594.50	4864779.56	2293212.56	795824.42	703991.66	89744.73	694.63	2.28	1.12
					(b) $\beta = 1.0$				
0.2	1.03	5.62	15.38	21.15	24.16	29.89	31.91	32.85	33.34
1.0	39.90	11.69	1.17	2.76	5.10	10.67	12.91	13.71	14.54
2.0	128.86	72.34	19.83	4.11	1.19	3.97	6.00	7.16	7.46
10.0	12090.97	11811.55	31704.10	136241.72	837.30	227.65	17.88	2.83	1.12
					(c) $\beta = 10.0$				
0.2	1.01	1.09	1.05	1.13	1.02	1.20	1.49	1.82	2.28
1.0	1.13	1.13	1.10	1.12	1.06	1.17	1.30	1.65	2.21
2.0	1.25	1.21	1.07	1.11	1.11	1.15	1.20	1.52	1.91
10.0	2.67	2.83	2.76	2.61	2.48	1.91	1.47	1.25	1.11

Table 6. Ratio of simulated and asymptotic expectations of the analysis of variance estimator

Number of Replicates	Lambda 0.2	0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.0
				(a) n = 20					
r = 2	1.0343	1.0360	1.0307	1.0319	1.0282	1.0405	1.0702	1.0689	1.0505
r = 5	1.0177	1.0128	1.0100	1.0141	1.0202	1.0120	1.0056	1.0102	1.0247
				(b) n = 50					
r = 2	1.0142	1.0076	1.0131	1.0055	1.0230	1.0190	1.0102	1.0180	1.0175
r = 5	1.0053	1.0048	1.0018	1.0066	1.0063	1.0061	1.0051	1.0035	0.999
				(c) n = 100					
r = 2	1.0006	1.0072	1.0056	1.0034	1.0049	1.0026	1.0083	1.0059	1.0146
r = 5	1.0004	1.0024	1.0017	1.0007	1.0038	1.0012	1.0045	1.0032	1.0030
				(d) n = 200					
r = 2	1.0005	1.0022	1.0002	1.0023	1.0014	1.0022	1.0049	1.0018	1.0029
r = 5	1.0010	1.0008	1.0011	0.0997	1.0017	1.0026	1.0005	0.9964	1.0027

Table 7. Ratio of simulated and asymptotic mean squared errors of the analysis of variance estimator

Number of Replicates	Lambda								
	0.2	0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.0
				(a) n = 20					
r = 2	1.6781	1.7403	1.6348	1.3932	1.7450	1.7214	2.1859	2.4212	1.6236
r = 5	1.3629	1.2243	1.2485	1.3055	1.3357	1.2587	1.2387	1.2177	1.2627
				(b) n = 50					
r = 2	1.1411	1.0843	1.1497	1.1170	1.2940	1.2411	1.1274	1.1141	1.1059
r = 5	1.1298	1.0739	1.2029	1.0366	1.1035	1.1369	1.1322	1.0393	1.0660
				(c) n = 100					
r = 2	1.0447	1.0593	1.0784	1.1257	1.0233	1.0819	1.0075	0.9960	1.1219
r = 5	1.0641	0.9633	0.9321	1.0525	1.0880	1.0852	1.0763	1.0121	1.0370
				(d) n = 200					
r = 2	1.0193	1.0608	1.0197	1.0691	0.9976	0.9992	1.0150	1.0163	0.9888
r = 5	1.0368	1.0040	1.0477	0.9999	1.0315	0.9850	1.0320	1.0216	1.0032

Fig. 1. Asymptotic rate of change of the maximum likelihood slope estimator as a function of $b = \beta^2$, the variance ratio λ, and the noise to signal ratio t.

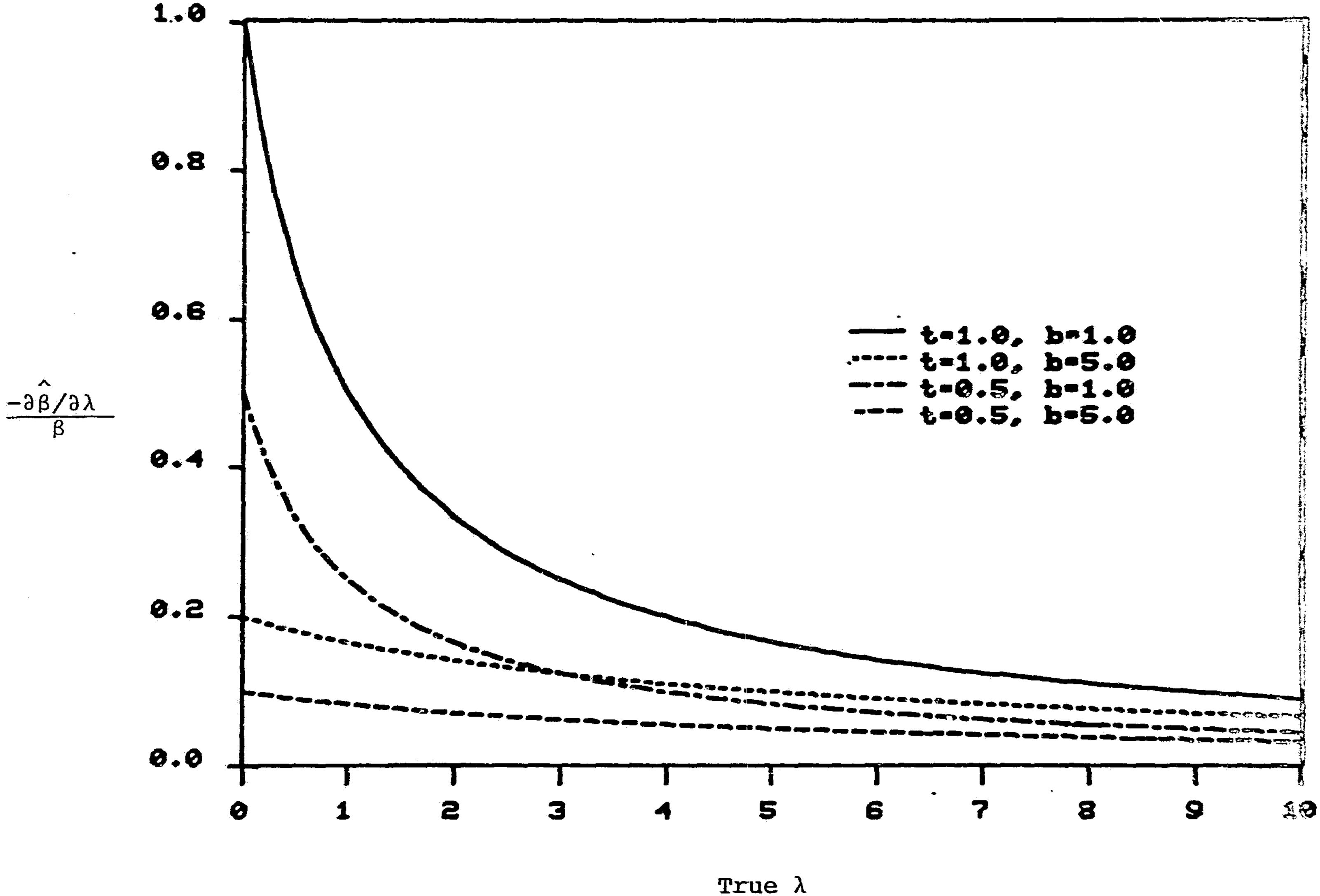

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Fig. 2. Ratio of the asymptotic mean squared error of the structural model slope estimator and the least squares estimator

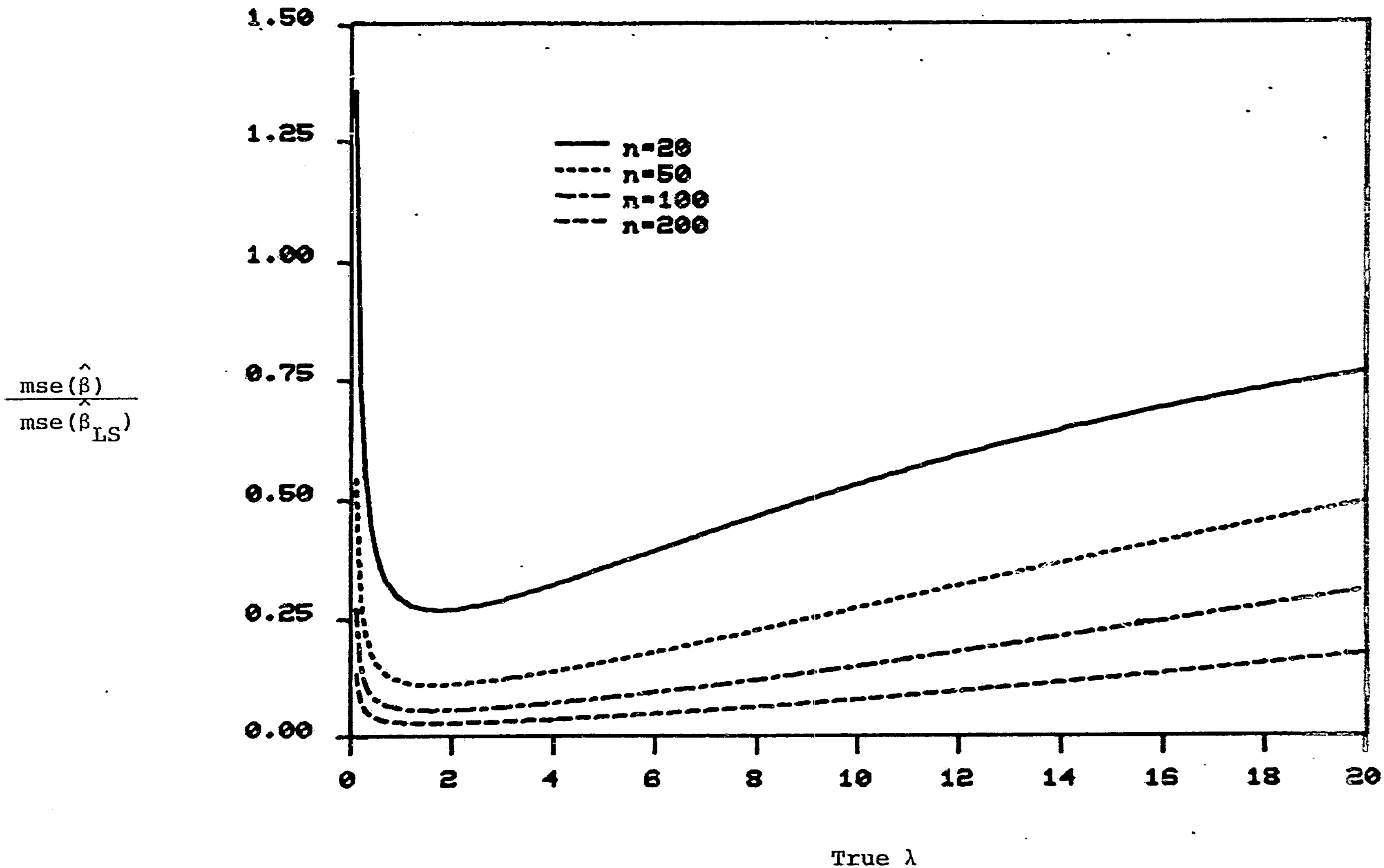


Fig. 3. Ratio of the asymptotic mean squared errors of the structural model slope estimator and the least squares estimator when λ is selected incorrectly (true λ = 6)

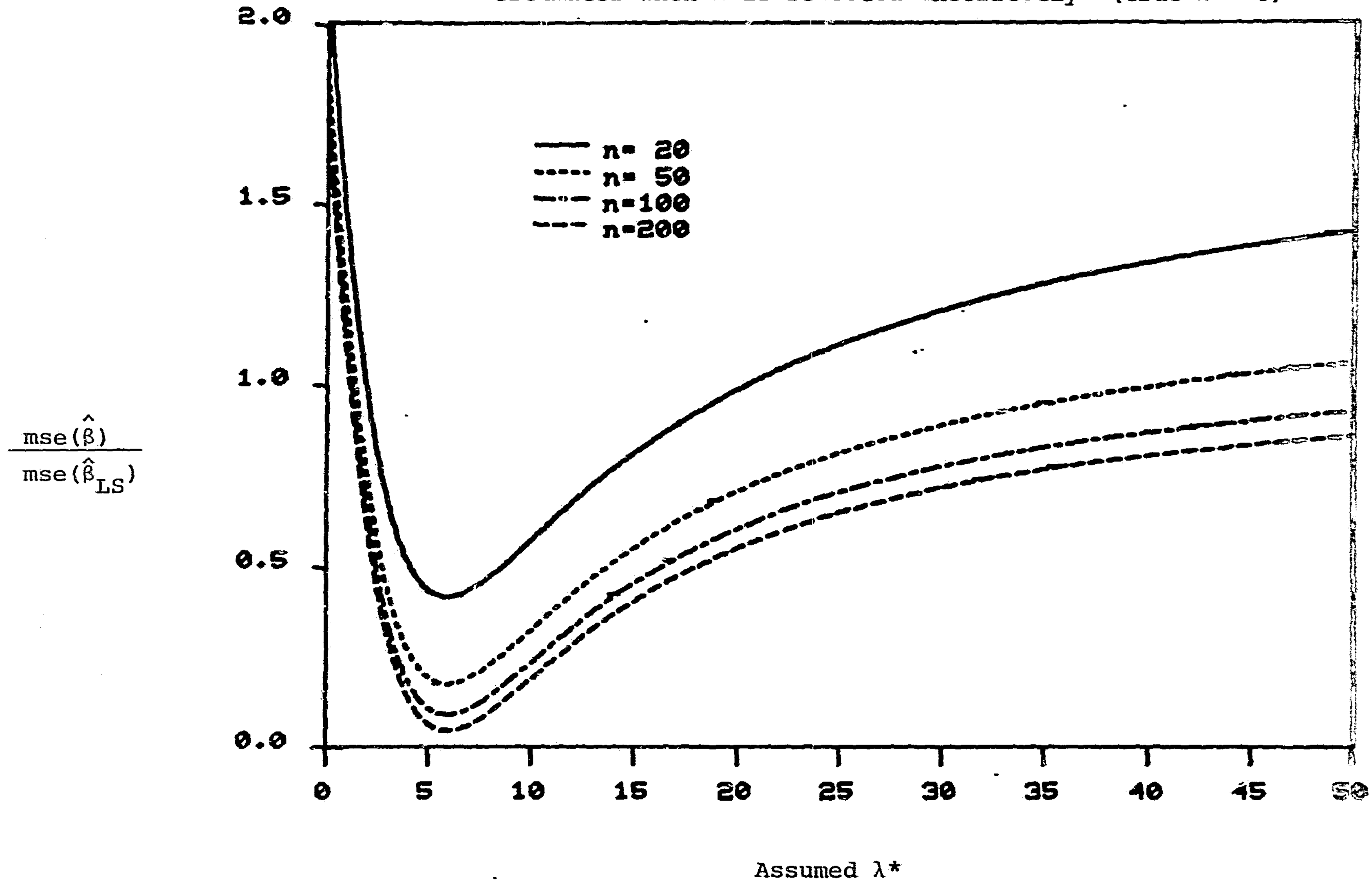
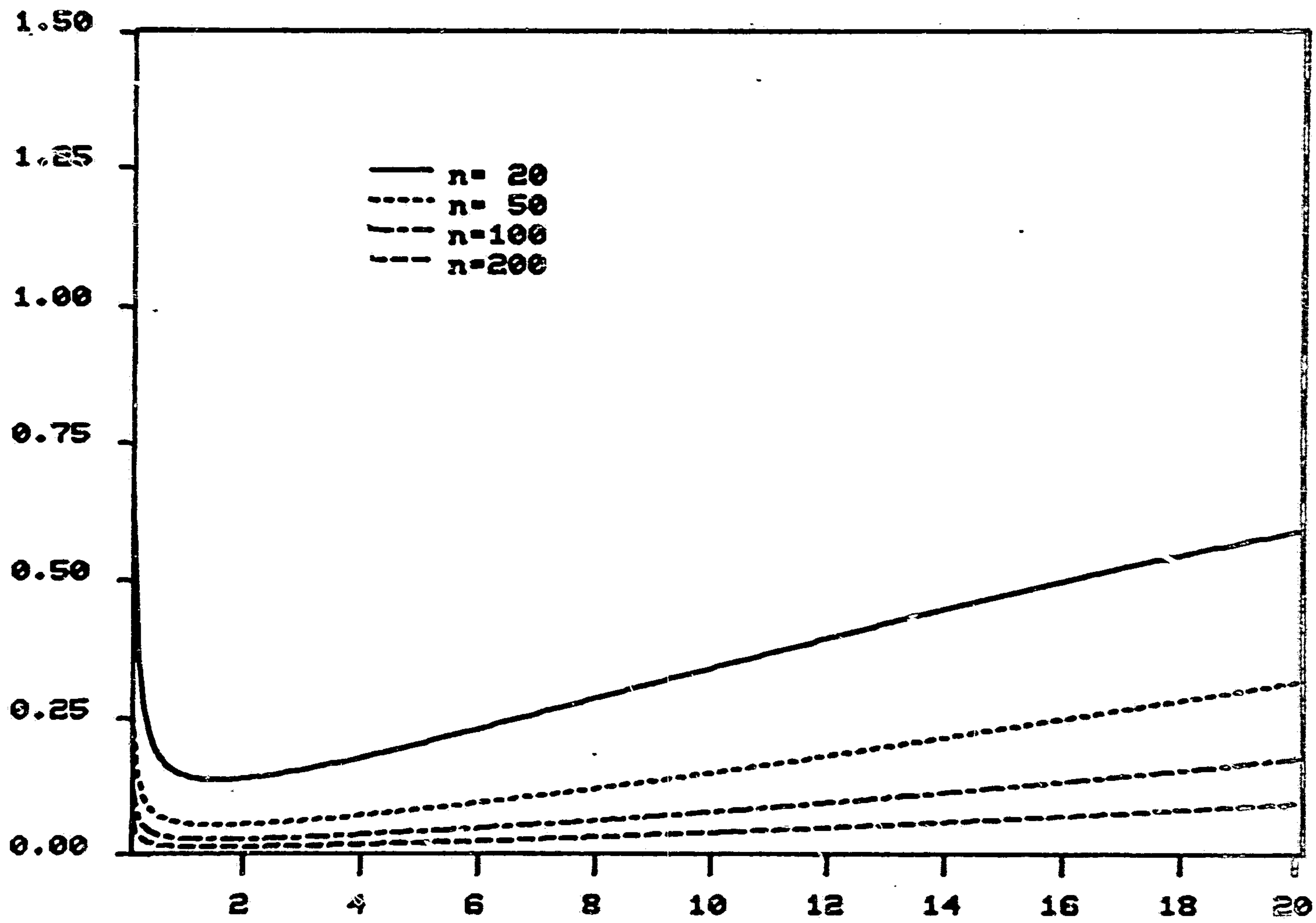


Fig. 4. Ratio of the asymptotic mean squared errors of the analysis of variance estimator and the least squares estimator, $r = 2$



$\frac{mse(\hat{\beta})}{mse(\hat{\beta}_{LS})}$

True λ