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## ON THE MAXIMUM- ENTROPY/AUTOREGRESSIVE MODELING OF TIME SERIES

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MODELING OF TIME SERIES**

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January 1984

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## ABSTRACT

The autoregressive (AR) model of a random process is interpreted in the light of the Prony's relation which relates a complex conjugate pair of poles of the AR process in the  $z$ -plane (or the  $z$  domain) on the one hand, to the complex frequency of one complex harmonic function in the time domain on the other. Thus the AR model of a time series is one that models the time series as a linear combination of complex harmonic functions, which include pure sinusoids and real exponentials as special cases. An AR model is completely determined by its  $z$ -domain pole configuration. The maximum-entropy/autoregressive (ME/AR) spectrum, defined on the unit circle of the  $z$ -plane (or the frequency domain), is nothing but a convenient, but ambiguous visual representation. It is asserted that the position and shape of a spectral peak is determined by the corresponding complex frequency, and the height of the spectral peak contains little information about the complex amplitude of the complex harmonic functions.

## INTRODUCTION

Maximum-entropy (ME) spectral analysis, as the name suggests, seeks to maximize the information entropy of a random process. The method was originally proposed by Burg (1967). Owing to its superior resolution to the conventional Fourier spectral analysis, ME spectrum has since been extensively used in research fields such as speech processing, astronomy, and geophysics. Van den Bos (1971) later showed that the ME spectrum is formally identical to the autoregressive (AR) spectrum, and hence that the ME spectral analysis is equivalent to the AR modeling of a random process (see e.g. Makhoul, 1975). Since the latter is much more succinct in concept, we shall in this study consider the ME spectral analysis from the AR point of view, and speak of the ME/AR spectrum.

Suppose we have an equally spaced, real-valued time series  $\{x(n); n=1,2,\dots,N\}$ , which, as usual, is a realization of a linear random process. The AR model of the time series is the following:

$$x(n) = \sum_{k=1}^M S_k x(n-k) + a(n), \quad n = M+1, \dots, N. \quad (1)$$

where  $M$  is the order of the AR model,  $\{S_k; k=1, 2, \dots, M\}$  are called the AR coefficients, and  $\{a(n); n=1,2,\dots,N\}$  is a realization of a zero-mean, white random process with variance  $\langle a^2 \rangle$ . In this paper we are mainly concerned with the ME/AR modeling of harmonic processes. An important special case is given by a time series generated by an impulsive excitation  $a(0)$  at time zero while all subsequent value of  $a(n)$  are zero. Then  $x(n)$  (normally corrupted by some additive noise) gives the impulse response of the physical system, and equation (1) represents a truncated complex harmonic process (as we shall see later). In any event, the parameters  $M$  and  $\{S_k\}$  completely determine the AR process. In practice, they are to be estimated from  $\{x(n); n=1,2,\dots,N\}$  by minimizing  $\langle a^2 \rangle$ . Equation (1) can then be used (if so wish) in the prediction of future values of the time series by setting  $a(n)=0$ . In fact, (1) represents one important class of models in the linear prediction theory (see e.g. Box & Jenkins, 1970, Makhoul, 1975), and  $\langle a^2 \rangle$  is often called the prediction-error power.

The ME/AR spectrum can be obtained from (1) by taking  $z$  transforms. Thus,

$$X(z) = z^M A(z)/H(z) \quad (2)$$

where  $A(z)$  is the  $z$  transform of  $a(n)$ , and

$$H(z) = z^M - S_1 z^{M-1} - \dots - S_M \quad (3)$$

is a polynomial of degree  $M$  in complex variable  $z$ . The spectrum, therefore, is

$$P(z) = |z^{2M}| \langle a^2 \rangle / |H(z)|^2 \quad (4)$$

evaluated on the unit circle of the complex  $z$ -plane,  $C_0: z = \exp(i\omega)$ ; or, in a more familiar form,

$$P(\omega) = \langle a^2 \rangle / |1 - \sum_{k=1}^M S_k \exp(-i\omega k)|^2 \quad (5)$$

for  $0 < \omega < 2\pi$ , where  $\omega$  is the argument of  $z$ , or the angular frequency.

There has been a number of methods proposed for the estimation of  $\{S_k\}$  and  $\langle a^2 \rangle$  (see e.g. Makhoul, 1975, Ulrych & Bishop, 1975, Ulrych & Clayton, 1976). They are all based on some sort of linear least-squares procedures. The determination of  $M$ , on the other hand, is a problem much more complicated and controversial. However, we are not concerned with any of the estimation methods in this study. In fact, our present concern starts only after the model parameters have been determined or given. Essentially we shall address the following question: What is the meaning of the AR model (1) and the ME/AR spectrum (5)?

#### The AR Model and Prony's Relation

First let us concentrate on the polynomial  $H(z)$ . According to equation (2), the zeros of  $H(z)$  are also the *poles* of the AR process (1). In fact, AR models are often called all-pole models in engineering literatures. Since the polynomial coefficients  $1, -S_1, \dots, -S_M$  are all real (see equation 1), the fundamental theorem of algebra ensures that these poles are either real or in complex conjugate pairs. Suppose, for the time being, that  $M$  is even and all the poles come in complex conjugate pairs. Let  $M=2M_1$ . Then in general we can express  $H(z)$  as

$$H(z) = \prod_{j=1}^{M_1} [z - \exp(i\sigma_j)] [z - \exp(-i\sigma_j^*)] \quad (6)$$

with poles at  $\{\exp(i\sigma_j), \exp(i\sigma_j^*); j=1,2, \dots, M_1\}$ . The meaning of the complex quantities  $\{\sigma_j; j=1,2, \dots, M_1\}$  becomes obvious once we see that the AR time series (1), with  $a(n)=0$ , can be expressed as the following linear combination of  $M_1$  *complex harmonic functions* of time:

$$x(n) = \sum_{j=1}^{M_1} [A_j \exp(in\sigma_j) + A_j^* \exp(-in\sigma_j^*)], \quad n = 1,2, \dots, N. \quad (7)$$

That is, the AR time series (1) (with  $a(n)=0$ ) and the series (7) are equivalent through equation (6). Equation (6), here referred to as the *Prony's relation*, plays the central role in the present study, and is proved in the Appendix. It has been proposed as the basis of a method for harmonic analysis by Chao & Gilbert (1980). If we let  $\sigma_j = \omega_j + i\alpha_j$ , it is obvious from (7) that  $\omega_j$  is simply the angular frequency and  $\alpha_j$  the exponential decay constant of the  $j$ th complex harmonic component in the time series. For this reason,  $\sigma_j$ 's are called the *complex frequencies*. Their geometrical interpretation, according to Prony's relation (6), is given in Figure 1. The complex quantities  $\{A_j; j=1,2, \dots, M_1\}$  in (7) give the amplitude and phase of each complex harmonic component, and are called the complex amplitudes. They contain the same information as that in the  $M$  initial values of the AR time series (1) in the sense that the former can be determined by the latter, and vice versa. We shall see later that the last statement has important implications.

Now let us study the case where some poles lie on the real axis of the  $z$ -plane. For example, if  $M$  is odd, we are bound to have at least one real pole. It is easy to show that real poles also satisfy Prony's relation (in fact, the so-called Prony's method was originally designed for fitting real exponentials, see Appendix); and they can be considered here as special cases of complex poles. Thus, a negative pole corresponds to a component which is an alternating real exponential series (a case of little practical interest), while a positive pole corresponds to a real exponential function of time (it could be a decaying exponential, a constant, or a growing exponential depending on whether the pole is less than, equal to, or greater than 1, respectively).

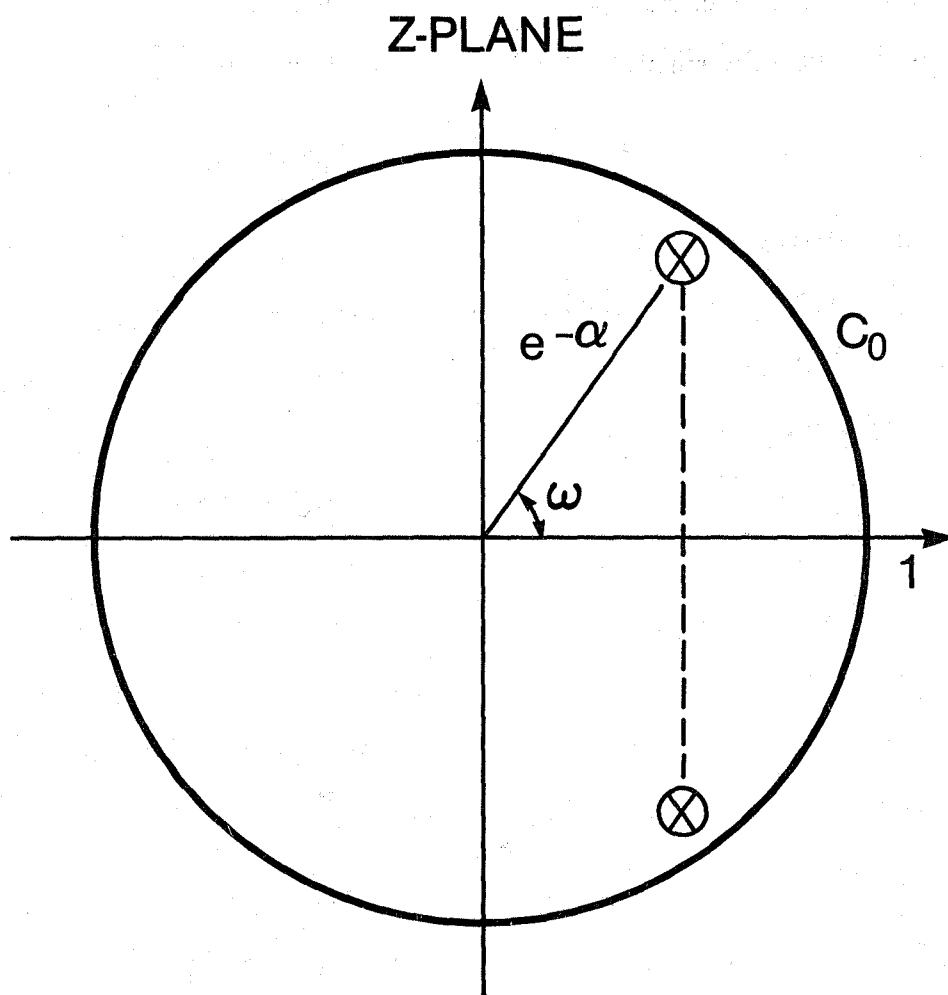


Figure 1. Geometrical interpretation of the position of one complex conjugate pair of poles on the z-plane (denoted by  $\otimes$ ) in relation to the corresponding complex frequency  $\sigma = \omega + i\alpha$ .

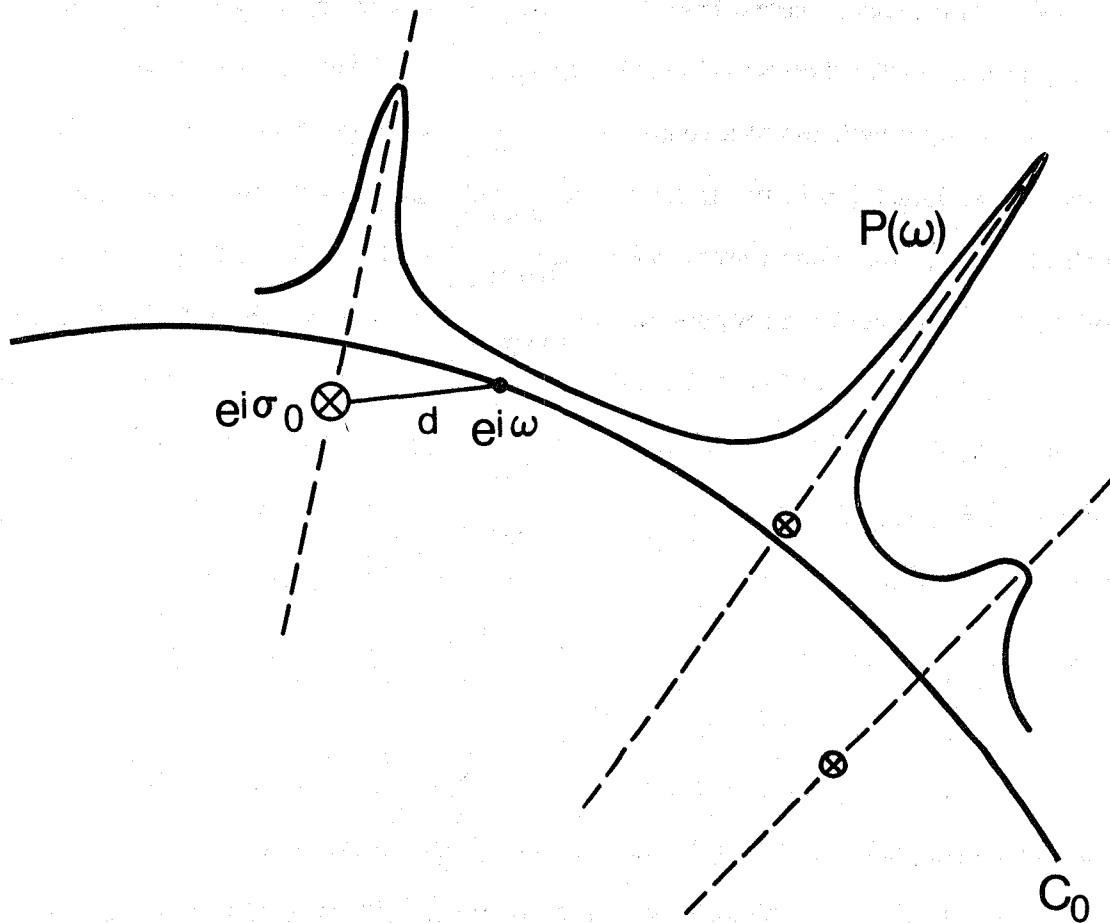


Thus, for ME/AR spectral analysis the Prony's relation in essence provides a connection between the "z-domain" representation and the time-domain representation of a random process. In the z domain a time series is modeled by  $M_1$  complex conjugate pairs of poles, whereas in the time domain it is modeled as a linear combination of  $M_1$  complex harmonic functions. (It follows that the AR linear prediction of a time series is in fact the extrapolation of these complex harmonic functions into the future.) Each complex conjugate pair of poles corresponds to a complex harmonic function, in a way summarized in Figure 1. In general, the number  $M_1$  is much smaller than the number of data points  $N$ . By contrast, the conventional Fourier spectral analysis decomposes a time series into  $N/2$  real harmonic components (or pure sinusoids) at the fixed frequency interval  $2\pi/N$ . [In this sense, the Fourier analysis "models" a time series (signal as well as noise) exactly, in the form of the inverse Fourier transform.] This "brute-force" quantization of frequencies is, of course, what is responsible for the low resolving power of the Fourier analysis. The "data-adaptive" (Lacoss, 1971) ME/AR scheme does not suffer from this limitation and essentially gives a high-resolution "line" spectrum, as we shall see in the next section.

#### The Meaning of the ME/AR Spectrum

The function  $P(z)$  (equation 4) is a positive-valued function defined in the complex z-plane (or the z domain); the value of  $P(z)$  at the poles grows to infinity as an inverse-square function. The ME/AR spectrum  $P(\omega)$  (equation 5) is the "cross section" of  $P(z)$  along the unit circle  $C_0$  (or the frequency domain, which is a one-dimensional subset of the z domain). It can be easily shown from equation (4) or (5) that  $P(\omega)$  is inversely proportional to the square of the distance between the point  $\exp(i\omega)$  (on  $C_0$ ) and any of the poles. Therefore any pole that is close to  $C_0$  will give rise to a conspicuous peak in the spectrum—the closer the pole to  $C_0$  the higher the peak, as depicted schematically in Figure 2.

Now let us study the spectrum in the vicinity of a pole which lies close to  $C_0$  and is isolated on the z-plane in the sense that the position and shape of the associated peak are not affected appreciably



Z-PLANE

Figure 2. Schematic diagram of the ME/AR spectrum in relation to the pole configuration.  $C_0$  is the unit circle.

by the presence of other poles. Let the pole be  $\exp(i\sigma_0)$  where  $\sigma_0 = \omega_0 + i\alpha_0$  (see Figure 2); and we know from above that this pole (along with its complex conjugate) corresponds to a complex harmonic component with frequency  $\omega_0$  and decay constant  $\alpha_0$ . The fact that  $\exp(i\sigma_0)$  lies close to  $C_0$  implies that  $|\alpha_0| \ll 1$ , or that the complex harmonic function has a high quality factor,  $Q \equiv \omega_0 / 2\alpha_0 \gg 1$ . Thus, we have  $P(\omega) = c \langle a^2 \rangle / d^2$ , where  $d$  is the distance between  $\exp(i\omega)$  and  $\exp(i\sigma_0)$ , and  $c$  is essentially a constant of  $\omega$  depending on the distances between  $\exp(i\omega)$  and all other poles, which vary little for  $\omega$  in the vicinity of  $\omega_0$ . Invoking the cosine law and Taylor expansion, we get the following expression for the ME/AR spectrum of a high- $Q$ , complex harmonic function in the vicinity of its frequency:

$$P(\omega) = \frac{c \langle a^2 \rangle}{\alpha_0^2 + (\omega - \omega_0)^2} \quad (8)$$

Equation (8) is a bell-shaped function of  $\omega$  with maximum at  $\omega = \omega_0$ , as shown in Figure 3. This is essentially the basis for the usage of ME/AR spectrum in harmonic analysis of time series.

Simple as it is, equation (8) has some important implications:

- (i) In general, the numerator  $\langle a^2 \rangle$  does contain information about the amplitude of the complex harmonic function. Despite this, however, the spectrum  $P(\omega)$  is largely determined by the denominator, or, the pole configuration—a conspicuous peak merely indicates the presence of a pole lying close to  $C_0$ . Therefore in practice there is little information about the signal amplitude in the ME/AR spectrum. This is indeed a logical derivative of an earlier statement that the information on the complex amplitude is contained in the initial values of the AR model, not in the AR parameters. Therefore it is not surprising that the ME/AR spectral analysis under some circumstances may introduce spurious spectral peaks that does not correspond to any “signal”—they are simply the consequence of spurious poles (due to noises even though the noise level may be low) which happen to be close to  $C_0$ .
- (ii) The shape of the spectral peak is characterized by  $\Delta$ , its full spectral width at half power (see Figure 3). From equation (8) it is immediately seen that  $\Delta = 2|\alpha_0|$ . Hence it is possible

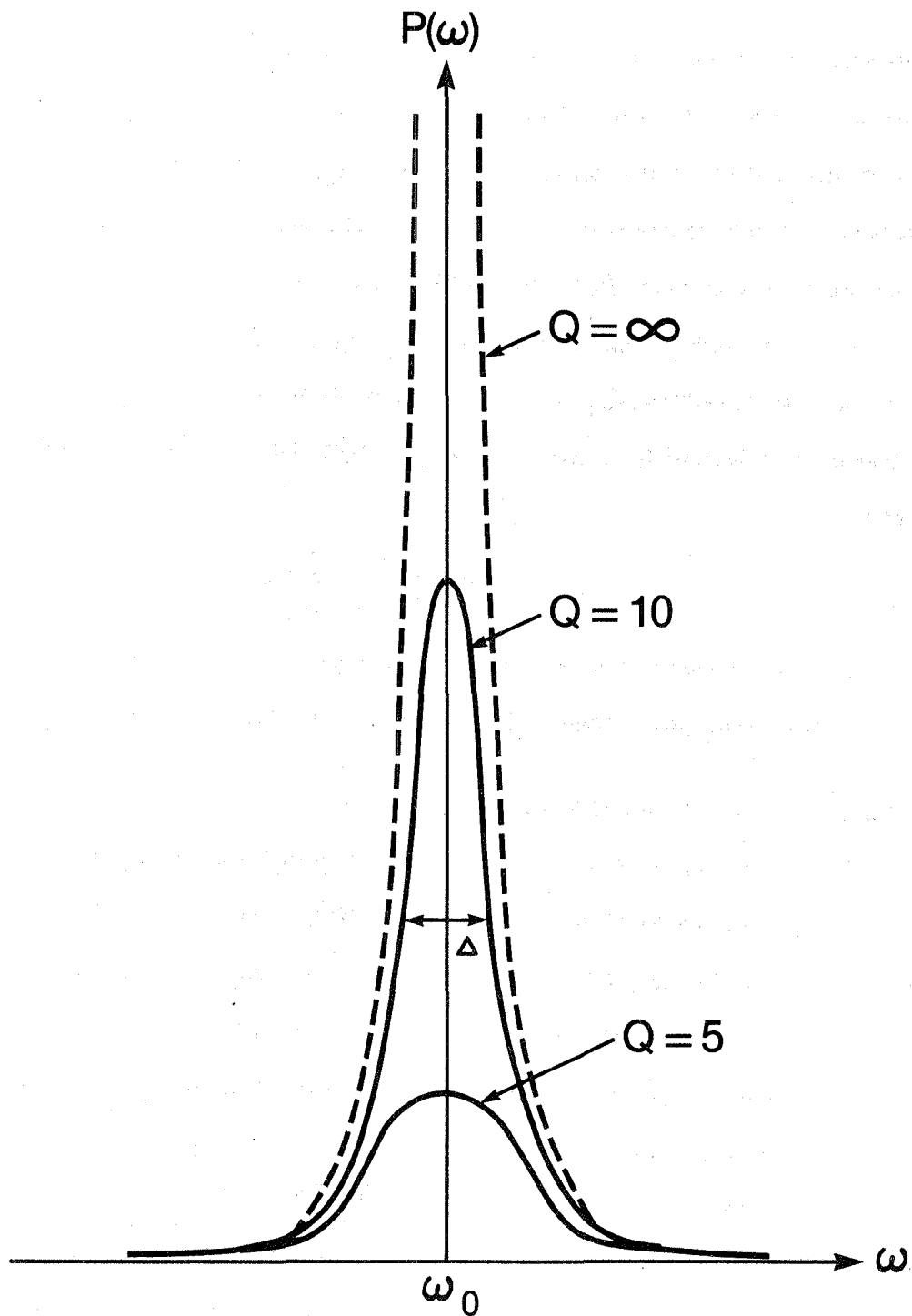


Figure 3. The ME/AR spectrum, as a function of the frequency  $\omega$ , of complex harmonic functions with complex frequency  $\sigma_0 = \omega_0 + i\alpha_0$ .  $Q \equiv \omega_0 / 2\alpha_0$ , and  $\Delta$  is the full spectral width at half power.

to estimate from  $\Delta$  the decay constant  $\alpha_0$ , and hence  $Q$ , of the complex harmonic component, in precisely the same way as would be done in the Fourier spectral analysis. This procedure has been employed by various investigators in studying the Earth's polar motion (Currie, 1974, Vincente & Currie, 1976, Graber, 1976) although, as far as I know, no theoretical justification were previously given. However, since equation (8) depends on the square of  $\alpha_0$ , it cannot distinguish between positive and negative  $\alpha_0$ . In other words, a decaying harmonic function (with a pole inside  $C_0$ ) and a growing harmonic function (with a pole outside  $C_0$ ) will have identical spectrum as long as they have the same  $|\alpha_0|$ , as first pointed out by Chao (1983).

Lacoss, in a pioneering work (Lacoss, 1971), claimed that the ME/AR spectral height is proportional to the square of the signal-to-noise ratio (S/N). This is inconsistent with equation (8) and (i) above. Lacoss (1971) also claimed that the spectral width is inversely proportional to S/N, so that the integrated area under a spectral peak is proportional to S/N. This, again, is inconsistent with our argument (ii) above. In fact, according to our equation (8) the spectral area is  $2\langle a^2 \rangle / |\alpha_0|$ , which, incidentally, grows unbounded as the decay constant  $\alpha_0$  approaches zero (as for a pure sinusoid). At any rate, the term "power density spectrum" seems inappropriate for ME/AR spectrum because it does not reflect the nature of the latter.

One (perhaps provocative) point to be made here is the following. Since the spectrum  $P(\omega)$  is only the "cross section" of the function  $P(z)$  in the frequency domain, it alone does not tell the whole story. Indeed,  $P(\omega)$  is nothing but one convenient, but ambiguous way to display visually the content of a time series. What is important is the pole configuration of equation (2). In practice all the poles, or the zeros of the polynomial  $H(z)$  (equation 3), can be found numerically by means of, say, Bairstow's method (see e.g. Wilkinson, 1965). Bairstow's method looks for real quadratic factors of  $H(z)$  one by one, which can then be further factorized into a complex conjugate pair of poles as the right side of equation (7). The latter, according to Prony's relation, corresponds to a complex harmonic

function whose complex frequency can thus be determined. For example, a pole lying close to  $C_0$  (with absolute value  $\approx 1$ ) can be associated with a quasi-sinusoidal component without looking at the ME/AR spectrum  $P(\omega)$ . Moreover, it becomes pointless to discuss the interference of two spectral peaks in the frequency domain as long as their poles can be determined.

## Conclusions

- (1) The poles of a real autoregressive (AR) process are either real or in complex conjugate pairs.
- (2) Each complex conjugate pair of poles in the  $z$  domain corresponds to one complex harmonic function in the time domain. Special cases include pure sinusoids and real exponentials; the latter correspond to real poles. The relation between the position of the poles and the corresponding complex frequencies is provided by the Prony's relation. Therefore, an AR model is one that models in a least-squares sense the time series as a linear combination of complex harmonic function of time.
- (3) The complete information of the AR parameters of a random process is contained in the pole configuration in the (two-dimensional)  $z$  domain. The maximum-entropy/autoregressive (ME/AR) spectrum, being defined on the (one-dimensional) frequency domain, is only a convenient, but ambiguous visual representation.
- (4) The position and shape of an ME/AR spectral peak is determined by the corresponding complex frequency (which, in turn, is determined by the pole position relative to the unit circle), and the height or area of the spectral peak contains little information about the amplitude of the harmonic components. The latter is contained, instead, in the initial values of the time series.

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APPENDIX  
PRONY'S RELATION

The so-called Prony's method, due to Prony (1795), was originally designed for analyzing a linear combination of (real) exponential functions (see e.g. Froberg, 1969). It can be easily extended to encompass complex harmonic functions, as we shall do presently. The notations used are the same as in the text.

Suppose we have the following equally spaced, real-valued time series:

$$x(n) = \sum_{j=1}^{M_1} [A_j \exp(in\sigma_j) + A_j^* \exp(-in\sigma_j^*)], \quad n = 1, 2, \dots, N. \quad (A1)$$

Let  $\exp(i\sigma_j) \equiv c_j$ . Then

$$\begin{aligned} x(1) &= \sum_j [A_j c_j + A_j^* c_j^*] \\ x(2) &= \sum_j [A_j c_j^2 + A_j^* c_j^{*2}] \\ &\text{-----} \\ x(M) &= \sum_j [A_j c_j^M + A_j^* c_j^{*M}] \end{aligned} \quad (A2)$$

where  $M=2M_1$ . Now form the polynomial of degree M

$$\begin{aligned} H(z) &= \prod_{j=1}^{M_1} (z - c_j) (z - c_j^*) \\ &= z^M - S_1 z^{M-1} - \dots - S_M. \end{aligned} \quad (A3)$$

Note that the coefficients  $1, -S_1, \dots, -S_M$  are real. Then for any  $n > M$ ,

$$\begin{aligned} &x(n) - S_1 x(n-1) - S_2 x(n-2) - \dots - S_M x(n-M) \\ &= \sum_{j=1}^{M_1} A_j c_j^{n-M} [c_j^M - S_1 c_j^{M-1} - \dots - S_M] + [\text{compl. conj.}] \\ &= \sum_{j=1}^{M_1} A_j c_j^{n-M} H(c_j) + [\text{compl. conj.}] \\ &= 0 \quad [\text{because } H(c_j)=0] \end{aligned} \quad (A4)$$

or,

$$x(n) = \sum_{k=1}^M S_k x(n-k), \quad n = M+1, \dots, N. \quad (\text{A5})$$

Thus, equation (A3) relates the linear combination (A1) to the AR time series (A5). It is called the Prony's relation.

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