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STATISTICAL MODELING OF SPACE SHUTTLE ENVIRONMENTAL DATA

Final Report

by



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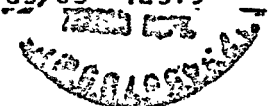
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GEORGE C. MARSHALL
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MARSHALL SPACE FLIGHT CENTER,
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J. D. Tubbs and D. W. Brewer
Principal Investigators

PREFACE

The report contains the results of several papers related to modeling using a class of the bivariate gamma distribution. The separate papers contain loosely related subjects pertaining to this problem. Since the separate papers were prepared at different times during the contract period and have been submitted for publication in the open literature and each paper is intended to be self-contained, there is some redundancy in tables and illustrations.

Each of the papers in this report were extensions and/or generalizations of the results given in NASA TM- 82483, entitled "A Bivariate Gamma Probability Distribution with Application to Gust Modeling," by O. E. Smith, S. I. Adelfang, and J. D. Tubbs. A modification of this paper is currently under review by Communications in Statistics.

The first paper in this report, entitled "A Note on the Ratio of Positively Correlated Gamma Variates," has been accepted for publication in Communications in Statistics and it presents some new analytical results using a class of the bivariate gamma distribution. Comparable results were available in the open literature using a different class of the bivariate gamma.

The second paper is entitled "A Method for Determining if Unequal Shape Parameters are Necessary in a Bivariate Gamma Distribution" and is an application of the results given in the first paper and addresses questions concerning

hypothesis tests for equality of shape parameters from correlated gamma distributed variates. This paper is currently under review by Technometrics.

The third paper, entitled "A Differential Equations Approach to the Modal Location for a Family of Bivariate Gamma Distribution," contains extensive analytical results for the location of the mode as a function of the free parameters. To the authors' knowledge this is the only such representation for a non-gaussian bivariate distribution. This paper has been submitted to SIAM J. on Scientific and Statistical Computing.

The fourth paper is a report summarizing the analysis of some wind gust data using the analytical results developed in relationship to the modeling application.

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A NOTE ON THE RATIO OF POSITIVELY CORRELATED GAMMA VARIATES

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ABSTRACT

Mielke and Flueck (1976) derived the density function and corresponding moments for the ratio of correlated gamma distributed variates. They considered a class of bivariate gamma distributions suggested by Cherian (1941) and David and Fix (1961). Recently, Lee, Holland, and Flueck (1979) derived some additional distributional results using this class of functions. This paper derives similar results using a different class of bivariate gamma distributions.

1. INTRODUCTION

Mielke and Flueck (1976) derived the distributional results for the ratio, R , of correlated gamma distributed variables. There are several classes of the bivariate gamma distribution [three are summarized in Mardia (1970) and an additional two in Johnson and

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Kotz (1972)]. Mielke and Flueck (1976) derived the distributional results for the ratio, R , of correlated gamma distributed variables using the Cherian-David-Fix class of bivariate gamma random variables [Cherian (1941) and David and Fix (1961)]. That is, let X , Y , and P denote independent gamma random variables with common scale parameter λ and respective shape parameters $\alpha - \xi$, $\beta - \xi$, and ξ , for $0 < \xi < \min(\alpha, \beta)$.

Then it can be shown that the bivariate probability density function for $U = X + P$ and $V = Y + P$ is given by

$$f_{U,V}(u,v) = \frac{\exp[-(u+v)]}{K} \int_0^{\min(u,v)} p^{\xi-1} (u-p)^{\alpha-\xi-1} (v-p)^{\beta-\xi-1} e^{-p} dp \quad (1.1)$$

for $K = \Gamma(\alpha-\xi)\Gamma(\beta-\xi)\Gamma(\xi)$

when the scale parameter λ is assumed to be unity. Mielke and Flueck (1976) showed that (1.1) can be written as

$$f_{U,V}(u,v) = \begin{cases} \frac{u^{\alpha-1} v^{\beta-\xi-1} e^{-(u+v)}}{\Gamma(\alpha)\Gamma(\beta-\xi)} F_1^*(\xi, 1+\xi-\alpha; \beta, u/v, -u) & \text{if } 0 < u < v \\ \frac{u^{\alpha-\xi-1} v^{\beta-1} e^{-(u+v)}}{\Gamma(\alpha-\xi)\Gamma(\beta)} F_1^*(\xi, 1+\xi-\alpha; \beta, v/u, -v) & \text{if } 0 < v < u \end{cases} \quad (1.2)$$

where $F_1^*(a,b,c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{n+m} (b)_n}{(c)_{m+n} m! n!} x^m y^n$, is a "degenerate"

two variable hypergeometric function [Gradshteyn and Ryzhik (1967), p. 1067] and $(a)_n = \Gamma(a+n)/\Gamma(a)$. Thus, U and V are gamma random variables with shape parameters α and β and positive dependence parameter ξ . In particular, $E(U) = \text{Var}(U) = \alpha$, $E(V) = \text{Var}(V) = \beta$, and $\text{Cov}(U,V) = \xi$.

Mielke and Flueck (1976) derived the density function for $R = U/V$ using a change of variables. That is,

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$$f_R(r) = \begin{cases} \frac{r^{\alpha-1}(1+r)^{\xi-\alpha-\beta}}{B(\alpha-\xi, \beta)} F_1(\xi, \alpha+\beta-\xi, 1+\xi-\beta, \alpha; r/1+r, r) & \text{if } 0 < r < 1 \\ \frac{r^{\alpha-\xi-1}(1+r)^{\xi-\alpha-\beta}}{B(\alpha-\xi, \beta)} F_1(\xi, \alpha+\beta-\xi, 1+\xi-\alpha, \beta; 1/1+r, 1/r) & \text{if } 1 < r \end{cases} \quad (1.3)$$

$$\text{where } F_1(a, b, c, d; x, y) = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n} (b)_m (c)_n}{(d)_{m+n} m! n!} x^m y^n, \quad |x| < 1 \quad |y| < 1$$

is a two variable hypergeometric function [Gradshteyn and Ryzhik, (1967), p. 1053], and $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. In addition, they show that the integral moments of R are given by

$$E(R^s) = \sum_{j=0}^s \binom{s}{j} \frac{(\alpha-\xi)_j (\xi)_{s-j}}{(\beta-j)_s} \quad \text{for } s \geq 0. \quad (1.4)$$

In particular,

$$E(R) = \alpha\beta - \xi / \beta(\beta-1), \quad \beta > 1 \quad (1.5)$$

$$E(R^2) = \frac{\xi(\xi+1)}{\beta(\beta+1)} + \frac{2(\alpha-\xi)\xi}{\beta(\beta-1)} + \frac{(\alpha-\xi+1)(\alpha-\xi)}{(\beta-1)(\beta-2)}, \quad \beta > 2$$

Recently, Lee, Holland, and Flueck (1979) were able to obtain comparable results for density of R using the Cherian-David-Fix class of densities by expressing f_R as a weighted difference of hypergeometric functions. The purpose of this paper is to derive comparable results for R using a different class of the bivariate gamma distribution. This class is a special case of the one suggested by Jensen (1970) as modified by Gunst and Webster (1973). The next section contains a brief discussion of this class of distributions. In section 3 the derivation of f_R is given using this class of functions. Section 4 outlines a possible application for the probability function in the area of hypothesis testing for the equality of shape parameters in the presence of correlation.

2. GUNST AND WEBER CLASS OF BIVARIATE GAMMAS

Gunst and Weber (1973) proposed a computationally feasible method for deriving the joint density function for the bivariate chi-square distribution. Since the chi-square is a special case of the gamma, this method was used for the bivariate gamma case. That is, a bivariate gamma density function for U and V with common scale parameter $\lambda = 1$ and shape parameters α, β , ($\alpha < \beta$) is given by

$$f(u,v) = \frac{u^{\alpha-1} v^{\beta-1} e^{-[(u+v)/(1-\eta)]}}{(1-\eta)^\alpha \Gamma(\alpha) \Gamma(\beta-\alpha)} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\eta^{j+k} \Gamma(\beta-\alpha+k) (uv)^j v^k}{(1-\eta)^{2j+k} \Gamma(\beta+j+k) j! k!} \quad (2.1)$$

where $\eta = \rho \sqrt{\beta/\alpha}$, ρ is the correlation coefficient between the variables U and V. Gunst and Webster (1973) suggested this class of densities in that they are computationally tractable and do not involve mathematical functions, such as Laguerre polynomials or convoluted sums [Jensen (1970) and Kibble (1941)]. Smith, Adelfang, and Tubbs (1982) discuss this class of densities in greater detail.

In the next section the distributional properties for the ratio, R, are derived using the Gunst-Webster class of bivariate gammas.

3. RATIO OF CORRELATED GAMMA VARIATES

By letting $R = U/V$ and $S = U+V$, the joint pdf for R and S can easily be shown to be

$$f_{R,S}(r,s) = c_1 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_2 \frac{s}{(1+r)^2} \left[\frac{sr}{1+r} \right]^{\alpha+j-1} \left[\frac{s}{1+r} \right]^{\beta+j+k-1} e^{-s/(1-\eta)} \quad (3.1)$$

where $c_1 = [(1-\eta)^\alpha \Gamma(\alpha) \Gamma(\beta-\alpha)]^{-1}$, $c_2 = \frac{\eta^{j+k} \Gamma(\beta-\alpha+k)}{(1-\eta)^{2j+k} \Gamma(\beta+j+k) j! k!}$. Hence,

by integrating over S the pdf for R becomes

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$$f_R(r) = (1-\eta)^\beta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_j c_k B(\alpha+j, \beta+j+k) r^{\alpha+j-1} / (1+r)^{\alpha+\beta+2j+k} \quad (3.2)$$

where $c_j = (\alpha)_j \eta^j / j!$, $c_k = (\beta-\alpha)_k \eta^k / k!$, $(a)_n = \Gamma(a+n) / \Gamma(a)$, and

$$B(a,b) = \Gamma(a)\Gamma(b) / \Gamma(a+b).$$

Whenever the shape parameters are equal then the density function for R is given by

$$f_R(r) = (1-\eta)^\alpha \sum_{j=0}^{\infty} c_j B(\alpha+j, \alpha+j) r^{\alpha+j-1} / (1+r)^{2\alpha+2j} \quad (3.3)$$

From (3.2) it can be shown that the m^{th} raw moment for R is given by

$$E(R^m) = (1-\eta)^\beta \sum_{j=0}^{\infty} c_j \sum_{k=0}^{\infty} c_k B(\alpha+j+m, \beta+j+k-m) / B(\alpha+j, \beta+j+k) \quad (3.4)$$

if $m < \beta$. In which case, it follows that

$$E(R) = (1-\eta)^\beta \sum_j c_j \sum_k c_k (\alpha+j) / (\beta+j+k-1) \quad (3.5)$$

and

$$E(R^2) = (1-\eta)^\beta \sum_j c_j \sum_k c_k (\alpha+j)(\alpha+j+1) / (\beta+j+k-1)(\beta+j+k-2) \quad (3.6)$$

Whenever $\eta = 0$, then

$$E(R) = \alpha / (\beta-1), \quad E(R^2) = \alpha(\alpha+1) / (\beta-1)(\beta-2) \quad (3.7)$$

which agrees with the values given by Mielke and Flueck (1976) whenever $\xi = 0$ and with Lee, Holland, and Flueck (1979) whenever $a = 0$.

Lee, Holland, and Flueck (1979) discuss some of the mathematical properties for the density of R for various values of

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α, β , and η . They demonstrated that the density can be ∞ at $r=1$ whenever either of the shape parameters is less than one. However, in the Gunst-Webster construction by assuming that $\alpha > 1$ and $\alpha < \beta$ the density function given in equation (3.2) is stable. Figures 1-4 illustrate the various shapes that $f_R(r)$ has as a function of the three parameters.

A definite computational advantage of equation (3.2) versus equation (1.3) stems from the ability to compute the tail probabilities for R . By letting $a=\alpha+j$ and $b=\beta+j+k$, we have

$$F_R(r_0) = (1-\eta)^\beta \sum_j c_j \sum_k c_k P[F_{2a,2b} \leq br_0/a] \quad (3.8)$$

where $F_{r,s}$ denotes a random variable from an F-distribution with r and s degrees of freedom. Note if $\eta = 0$, then (3.7) becomes

$$F_R(r_0) = P[F_{2\alpha,2\beta} \leq \beta r_0/\alpha] \quad (3.9)$$

which agrees with the well known results concerning the ratio of independent chi-squares. Furthermore, if $\eta \neq 0$ and $\alpha = \beta$ then (3.7) becomes

$$F_R(r_0) = (1-\eta)^\alpha \sum_j c_j P[F_{2(\alpha+j),2(\alpha+j)} \leq r_0] \quad (3.10)$$

which is similar to an expression given by Johnson and Kotz (1972), Chapter 40, Section 3.

4. APPLICATION

In this section an application is given for computing the cdf of R , given by equation (3.7). Diagram 1 defines the area given in equation (4.1).

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FIGURE 1. DENSITY FUNCTION FOR $R = U/V$ $a = 1.5$, $b = 1.5$

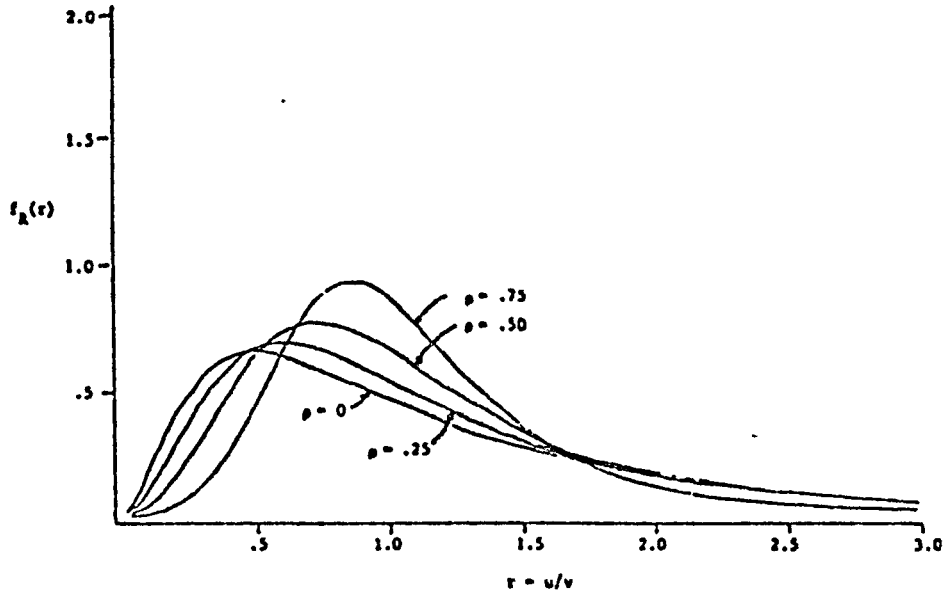
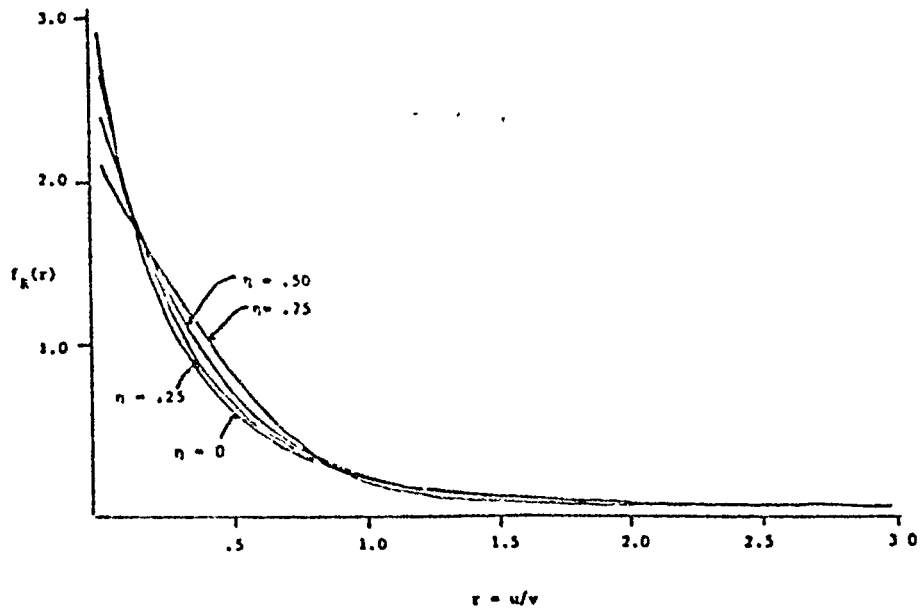


FIGURE 2. DENSITY FUNCTION FOR $R = U/V$ $a = 1.0$, $b = 3.0$



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FIGURE 3. DENSITY FUNCTION FOR $R = U/V$ $\alpha = 2.0$, $\beta = 3.0$

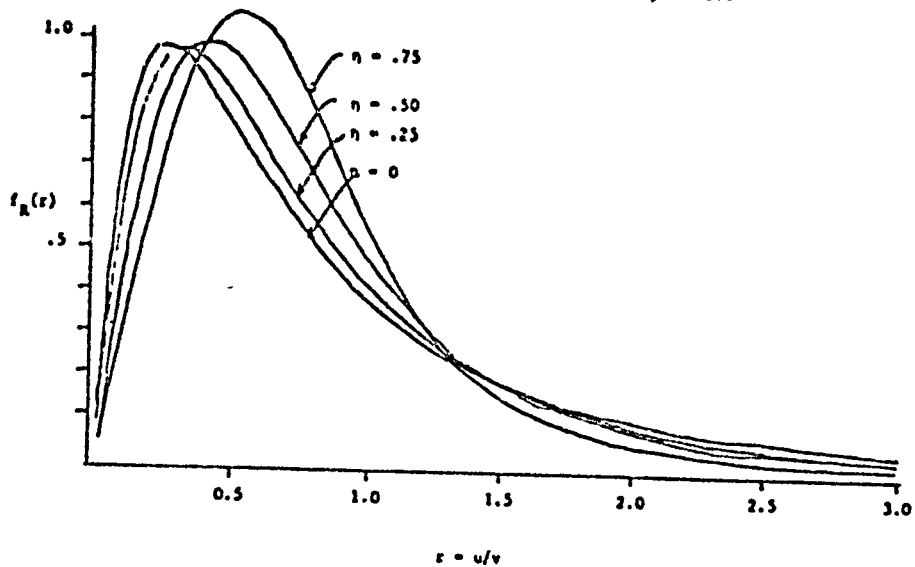
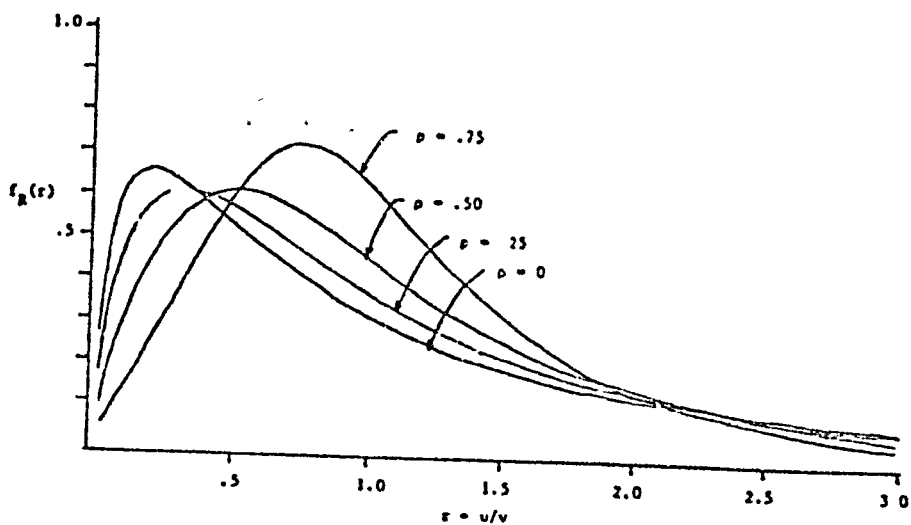


FIGURE 4. DENSITY FUNCTION FOR $R = U/V$ $\alpha = 3.0$, $\beta = 3.0$



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By letting $\cot \theta_0 = U/V = r_0$ and $G(\theta) = 1 - F_R(r_0)$, one has

$$G(\theta) = (1-\eta)^\beta \sum_j c_j \sum_k c_k P[F_{2b,2a} \leq (a/b)\tan \theta] \quad (4.1)$$

Figure 5 contains the graph of the function $G(\theta)$ versus θ for $\alpha = 1$ and $\beta = 1, 2, \text{ or } 3$ and $\eta = 0, .25, .50, \text{ and } .75$. From this figure and other cases which are not included one observes that whenever $\alpha = \beta$ then $G(45^\circ) = .5$ and $G(45^\circ) < .5$ whenever $\alpha < \beta$. This observation and additional properties were used in developing a test for the hypothesis

$$H_0: \alpha = \beta \quad \text{vs.} \quad H_A: \alpha < \beta \quad (4.2)$$

The procedure is presented in Tubbs (1983) and uses the Cramer-Von Mises criteria for testing (4.2). That is, define

$$W_n = n \int (F_R(x) - F_n(x))^2 dF_R(x) \quad (4.3)$$

where $F_R(x)$ is the cdf for the null distribution given in (3.10). $F_n(x)$ is the empirical distribution for $r_i = u_i/v_i$ and the r_i 's are arranged in increasing order. Whenever H_0 is true, then W_n is distribution free and has a convenient computational form given by

$$W_n = \frac{1}{12n} + \sum_{i=1}^n \left\{ Z_i - \frac{(2i-1)}{2n} \right\}^2 \quad (4.4)$$

where $Z_i = F_R(r_i)$. H_0 is rejected if W_n exceeds a specified critical point. Tubbs (1983) considers the properties of this test procedure in greater detail.

5. CONCLUSIONS AND SUMMARY

This paper derives both the density and the distribution functions for the ratio of positively correlated gamma variates using a modification of Jensen's bivariate gamma distribution. The expression for the moments differ from those given by either Mielke and Flueck (1976) or Lee, Holland, and Flueck (1979).

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However, all the expressions are identical whenever the variates are uncorrelated. A principal advantage found in this representation stems from the ability to compute the CDF of the ratio. The value of the CDF for the ratio was shown to have potential application to the problem of testing for equality of shape parameters in a particular family of the bivariate gamma distribution.

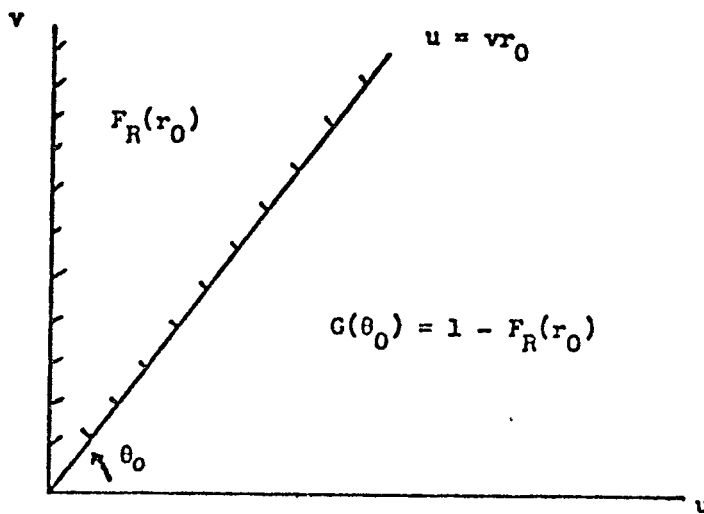
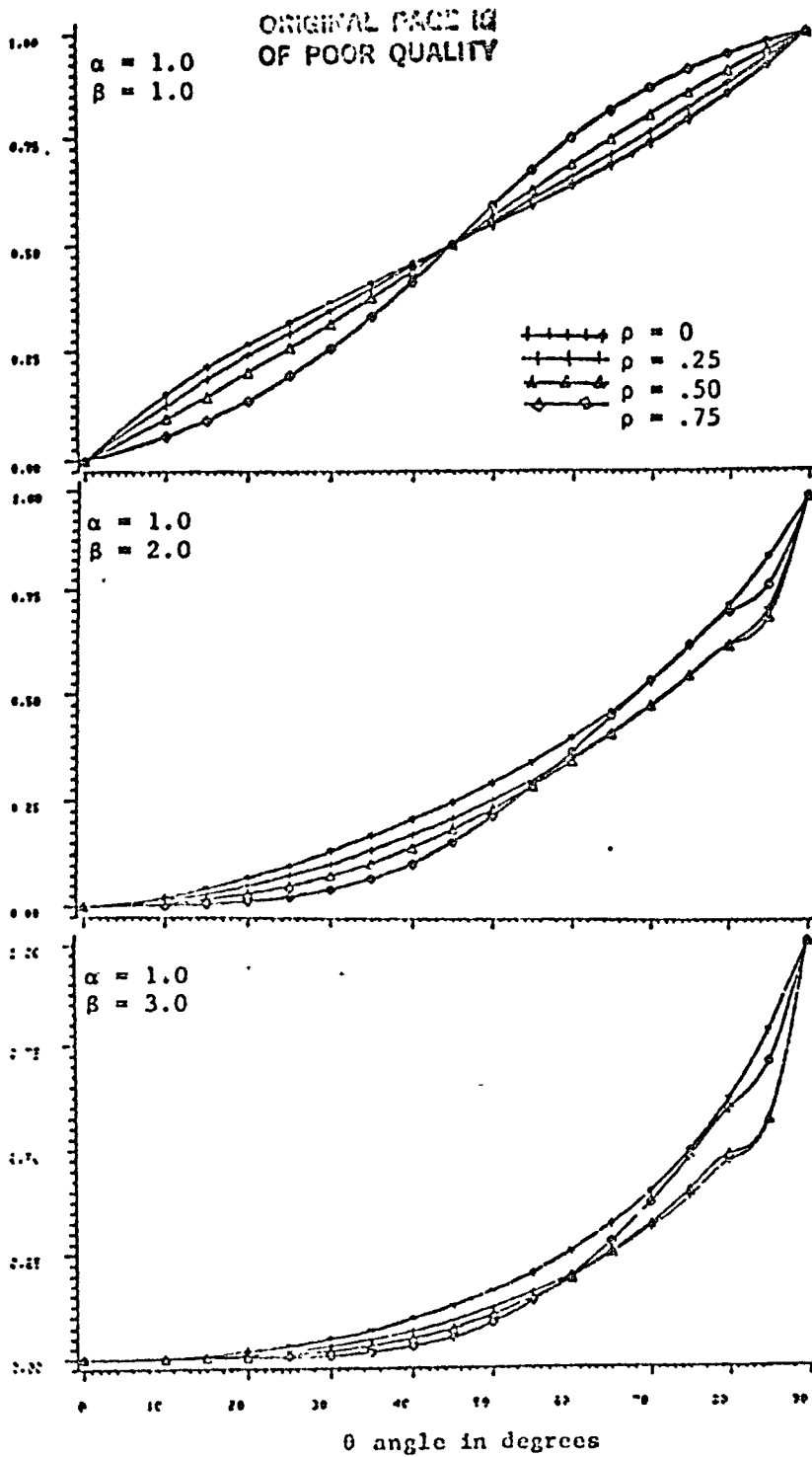


Diagram 1



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A METHOD FOR DETERMINING IF UNEQUAL SHAPE
PARAMETERS ARE NECESSARY IN A BIVARIATE
GAMMA DISTRIBUTION

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ABSTRACT

A procedure for aiding an experimentalist in deciding between four and five parameters in a Jensen's type bivariate gamma distribution is presented. The procedure is based upon the properties of the CDF for the ratio of correlated gamma distributed variates. The criteria of interest is posed in a test of hypothesis setting and results are presented using the Cramér-Von Mises test of fit.

1. INTRODUCTION

Smith and Adelfang (1981) discuss the applicability of a bivariate gamma distribution as a parametric model for wind gust amplitude and length. In modeling this bivariate data with a gamma distribution, it was necessary to find a distribution that would allow for correlation between the random variables X and Y when the marginal distributions are univariate gammas with possibly unequal shape and scale parameters. That is, $X \sim G(\gamma_X, \beta_X)$ and

$Y - G(\gamma_y, \beta_y)$ where the probability density function for
 $Z - G(\gamma, \beta)$ is given by

$$f_Z(z) = \beta^\gamma z^{\gamma-1} e^{-\beta z} / \Gamma(\gamma). \quad (1.1)$$

A brief survey of the open literature reveals that there are several classes of the bivariate gamma distribution. One need only consult Mardia (1970) and Johnson and Kotz (1972) to find five classes of the bivariate gamma distribution [Kibble (1941), Cherion (1941), McKay (1934), Jensen (1970), and Moran (1969)]. Of these classes only Jensen (1970) and Moran (1969) allow for unequal shape parameters and both of these have computational limitations which affect their utility to the experimentalist. Recently, McAllister, Lee, and Holland (1981) and McAllister (1983) have addressed the limitations with Jensen's model and provided results which overcome many of the computational difficulties. However, at the time of Smith et al. (1983) development these results were not available. Hence, they modified a bivariate chi-square model given by Gunst and Webster (1973). This allows for possibly unequal shape parameters and is computationally tractable. The model is not as general as that given by Jensen (1970), however, one can derive the bivariate model given by Kibble (1941) as a special case whenever the shape parameters are equal. In this paper, the unequal shape parameter model will be referred to as the five-parameter model and the equal shape

case as the four-parameter model. Smith, Adelfang, and Tubbs (1983) discuss the properties of these distributions and it is apparent that the four-parameter has numerous computational advantages over the five-parameter model. So if one assumes that the data is correctly modeled by this class of the bivariate gamma distribution, a question of practical interest becomes, How does one decide if the five-parameters are really necessary? The purpose of this paper is to present a procedure which would aid the experimentalist in answering the above question. The problem is posed in a hypothesis testing setting. That is, test the hypothesis

$$H_0: \gamma_x = \gamma_y \quad (1.2)$$

versus

$$H_1: \gamma_x < \gamma_y. \quad (1.3)$$

It should be noted that the proposed method is not an omnibus test of fit for the bivariate gamma against all other possible models. Instead the procedure is intended for deciding between the four or five parameter models as given in Smith, Adelfang, and Tubbs (1983).

The next section contains the distributional results needed for the test of hypothesis (1.2). The test procedure is given in section 3 and evaluated in section 4. Section 5 contains a summary and remarks concerning some of the limitations of the procedure.

2. DISTRIBUTIONAL RESULTS

Smith, Adelfang, and Tubbs (1983) modified a bivariate-Chi square distribution given by Gunst and Webster (1973) and obtained the density function given by

$$f(x,y) = K_1/K_2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_{jk} x^j (ny)^{j+k} \quad (2.1)$$

where

$$K_1 = x^{\gamma_x-1} y^{\gamma_y-1} \exp\{-(x+y)/(1-\eta)\},$$

$$K_2 = (1-\eta)^{\gamma_x} \Gamma(\gamma_x) \Gamma(\gamma_y - \gamma_x),$$

$$c_{jk} = \eta^{j+k} \Gamma(\gamma_y - \gamma_x + k) / \{(1-\eta)^{2j+k} \Gamma(\gamma_y + j + k) j! k!\},$$

and $x = X\beta_x$, $y = Y\beta_y$, β_x , β_y are known scale parameters,

$\eta = \rho \sqrt{\gamma_y/\gamma_x}$, ρ is the correlation coefficient between the variables X and Y. The joint probability distribution function is given by

$$\begin{aligned} F(x_0, y_0) &= \Pr[X \leq x_0, Y \leq y_0] \\ &= J \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_{jk} H(\gamma_x + j, x_0/(1-\eta)) \\ &\quad \cdot H(\gamma_y + j + k, y_0/(1-\eta)) \end{aligned} \quad (2.2)$$

where

$$J = (1-\eta)^{\gamma_y} / \Gamma(\gamma_x) \Gamma(\gamma_y - \gamma_x),$$

$$d_{jk} = \eta^{j+k} \Gamma(\gamma_y - \gamma_x + k) / \Gamma(\gamma_y + j + k) j! k!$$

$$H(a, x) = \int_0^{\infty} t^{a-1} e^{-t} dt.$$

Equations (2.1) and (2.2) are for the unequal shape parameters and will be referred to as the five-parameter model. It should be re-emphasized that this model is not completely general in that one assumes that $\gamma_y > \gamma_x$ and the correlation between variables X and Y are restricted to the interval $[0, n\sqrt{\gamma_x/\gamma_y}]$ for $n \in [0, 1]$.

If $\gamma_x = \gamma_y = \gamma$ then it can be shown that (2.1) and (2.2) reduce to the well known functions given by Kibble (1941). That is, the density function is given by

$$f(x,y) = (xy)^{\gamma-1} \exp\{-((x+y)/(1-n))\} / \Gamma(\gamma) \cdot \sum_{j=0}^{\infty} (nxy)/(1-n)^2)^j / \Gamma(\gamma+j) j! \quad (2.3)$$

and the distribution function becomes

$$F(x,y) = (1-n)^{\gamma} / \Gamma(\gamma) \sum_{j=0}^{\infty} n^j / \Gamma(\gamma+j) j! \cdot H(\gamma+j, x/(1-n)) H(\gamma+j, y/(1-n)). \quad (2.4)$$

Equations (2.3) and (2.4) will be referred to as the four parameter model. A comparison of the distribution function given in (2.2) and (2.4) reveals that there are distinct differences in terms of the computational complexity. Thus for computational reasons the experimentalist would like to know how much greater does $\hat{\gamma}_y$ have to exceed $\hat{\gamma}_x$ before equation (2.2) is really necessary. Ideally he would like to answer this question before using both (2.2) and (2.4) then selecting the results which are more gratifying. In order to address this issue, this

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paper considers the problem of testing hypothesis (1.2) versus (1.3) using an univariate random variate given by the ratio of X to Y, $R = X/Y$. Tubbs and Smith (1983) derive the density and distribution functions for R whenever the bivariate density is either (2.1) or (2.3). That is, if equation (2.1) holds then the density function for R is given by

$$f_R(r) = (1-n)^Y \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_j c_k B(a,b)^{-1} r^{a-1} / (1+r)^{a+b} \quad (2.5)$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, $c_j = (a)_j n^j / j!$,

$c_k = (b-a)_k n^k / k!$, $a = \gamma_x + j$, $b = \gamma_y + j + k$, and $(a)_n = \Gamma(a+n)/\Gamma(a)$.

The distribution function for R is given by

$$\begin{aligned} F_R(r_0) &= \Pr[X/Y \leq r_0] \\ &= (1-n)^Y \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_j c_k \Pr[F_{2a,2b} \leq br_0/a] \quad (2.6) \end{aligned}$$

where $F_{r,s}$ denotes a random variable from an F-distribution with r and s degrees of freedom. The corresponding functions whenever $\gamma_x = \gamma_y = \gamma$ are given by

$$f_R(r) = (1-n)^Y \sum_{j=0}^{\infty} c_j B(a,a)^{-1} r^{a-1} / (1+r)^{2a} \quad (2.7)$$

and

$$F_R(r_0) = (1-n)^Y \sum_{j=0}^{\infty} c_j \Pr[F_{2a,2a} \leq r_0] \quad (2.8)$$

where $a = \gamma + j$.

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3. HYPOTHESIS TESTING

Since $R = X/Y$ is a univariate random variable it is informative to graph $F_R(r)$ versus r . However, since $r > 0$ a more meaningful graph can be produced by letting $\theta = \cot^{-1}r$ and $G(\theta_0) = 1 - F_R(r_0)$ where $\theta_0 = \cot^{-1}r_0$. The area corresponding to $F_R(r_0)$ is shown in diagram 1. Furthermore, it follows that

$$G(\theta_0) = (1-\eta)^{\gamma_y} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_j c_k \Pr[F_{2a, 2b} \leq (a/b) \tan \theta_0] \quad (3.1)$$

in the five-parameter model and

$$G(\theta_0) = (1-\eta)^{\gamma} \sum_{j=0}^{\infty} c_j \Pr[F_{2\epsilon, 2a} \leq \tan \theta_0] \quad (3.2)$$

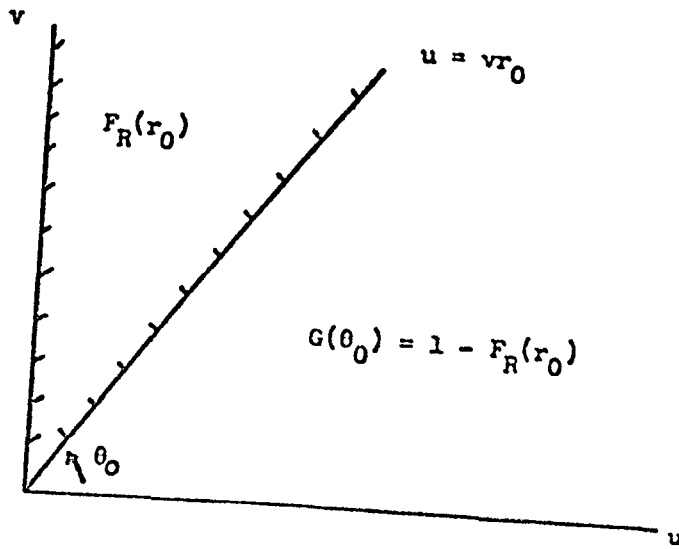
in the four-parameter case.

Since θ is restricted to the finite interval $(0, \pi/2)$, it is somewhat instructive to plot $G(\theta)$ versus θ as functions of the free parameters, γ_x , γ_y and η . As in Tubbs and Smith (1983) the scale parameters are assumed to be known and hence equal to one. This restriction will be addressed later in the paper. Figures 1-3 contain some of the illustrative cases.

From these plots one observes that $G(45^\circ) = .5$ whenever the four-parameter model holds and $G(45^\circ) < .5$ in the five-parameter models. Rather than just using this observation a function was selected to measure the distance between these distribution functions. The Cramér-Von Mises

DIAGRAM 1

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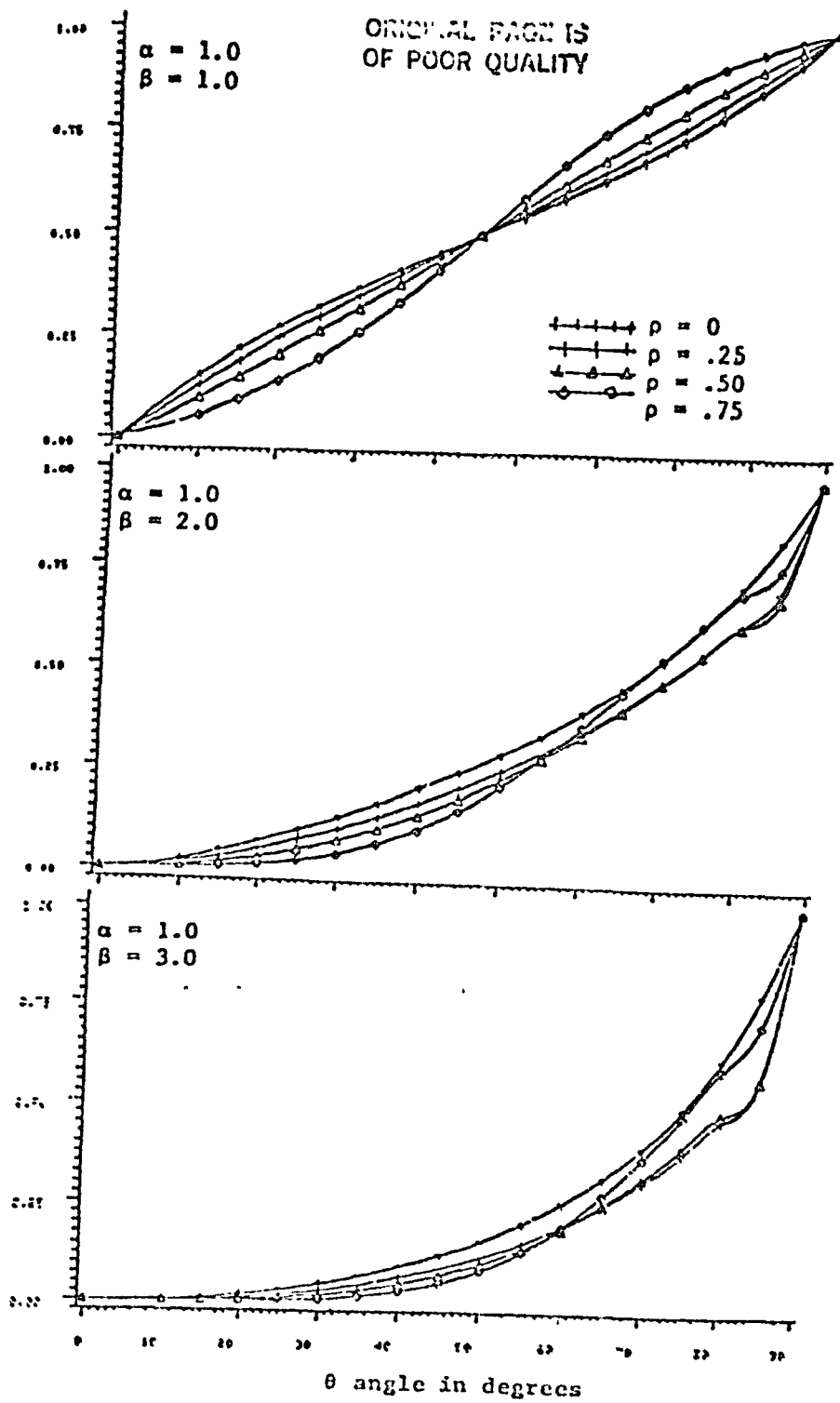


FIGURE 1. $F_R(\theta)$ vs. θ

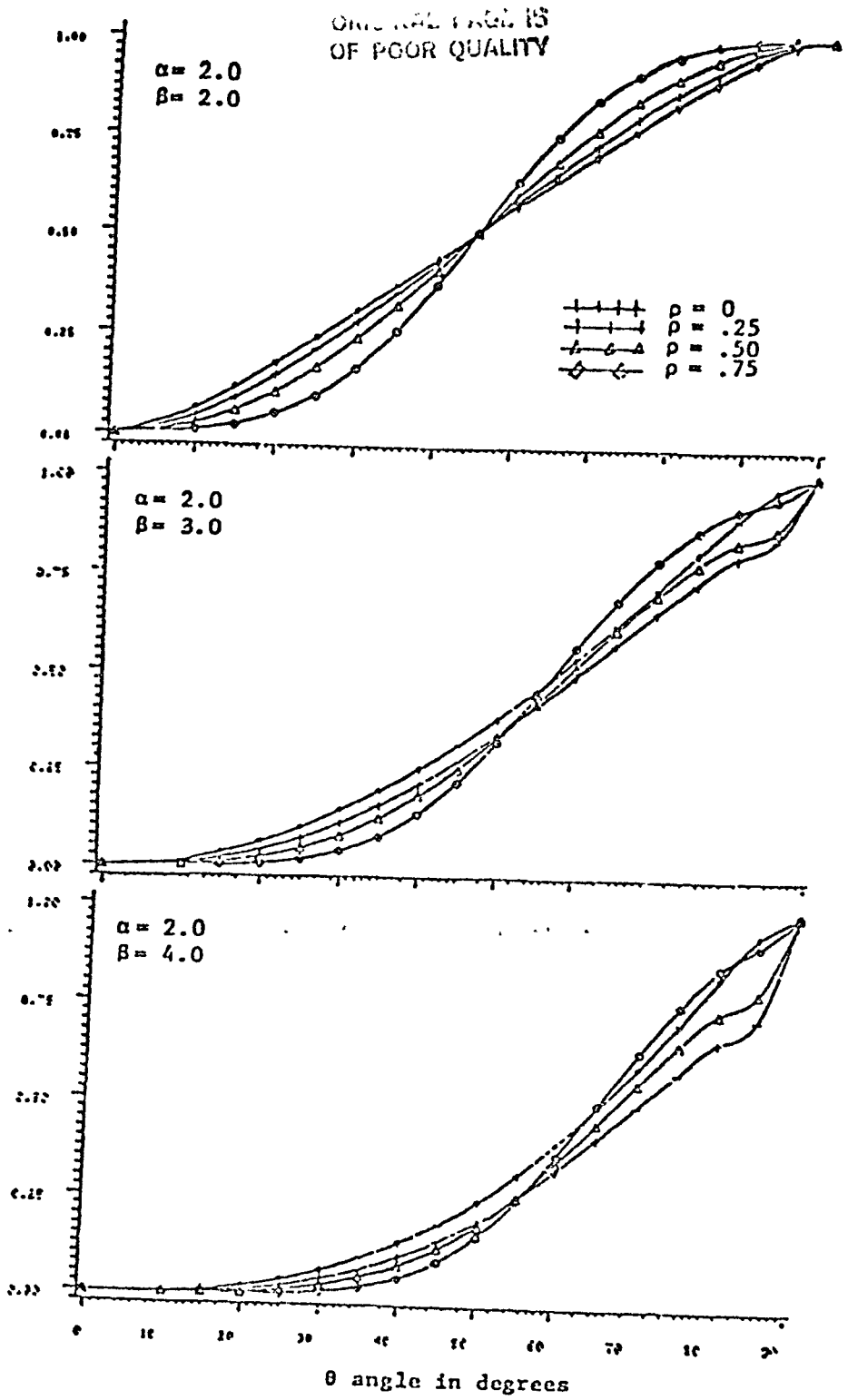


FIGURE 2. $F_R(\theta)$ vs. θ

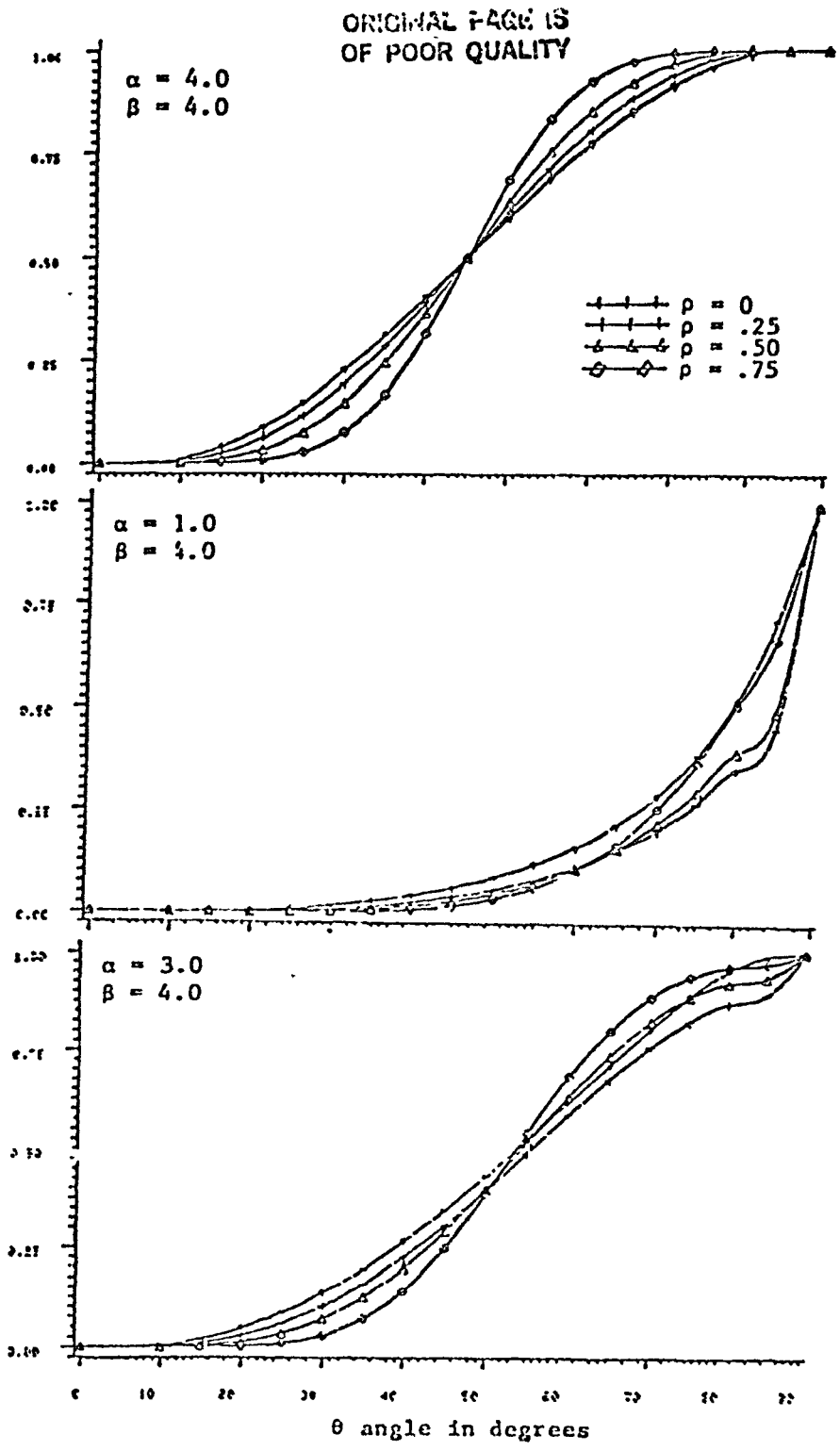


FIGURE 3. $F_R(\theta)$ vs. θ

type goodness-of-fit procedure was selected since the test is distribution free whenever the parameters are specified. Furthermore the test statistic is easy to compute.

Let

$$W_n = n \int_0^{\pi/2} (G(\theta) - G_n(\theta))^2 dG(\theta) \quad (3.3)$$

where $G(\theta)$ is given in (3.2), $G_n(\theta)$ is the empirical distribution function of $\theta_i = \tan^{-1}(r_i)$, $r_i = X_i/Y_i$ are arranged in increasing order. Whenever hypothesis (1.2) is true, then W_n has the convenient computational form given by

$$W_n = 1/12n + \sum_{i=1}^n \left\{ z_i - \frac{(2i-1)}{2n} \right\}^2 \quad (3.4)$$

where $z_i = G(\theta_i)$. Furthermore, from Anderson and Darling (1951) one can reject (1.2) whenever W_n exceeds a specified critical point. These critical points are given from Anderson and Darling's asymptotic distribution. Stephen (1976) defines a procedure for modifying the critical points for small samples, however, the underlying problem of modeling bivariate data will probably dictate large sample sizes.

4. EVALUATION OF THE TEST PROCEDURE

In this section the procedure defined in the previous section is evaluated. The evaluation is performed in two parts. The intent of the first part was to determine whether or not the procedure even works. That is, are the

apparent visual differences between the function $G(\theta)$ as seen in Figures 1-3 significant in the "Cramér-Von Mises" metric. The second part of the evaluation concerns the robustness of the procedure to the nuisance parameters.

In the first part, let

$$D_n(\delta) = n \int_0^{\pi/2} \{G(\theta) - A(\theta)\}^2 dG(\theta) \quad (4.1)$$

where $G(\theta)$ is given in (3.2) and $A(\theta)$ is given by (3.1) when $\gamma_y = \gamma_x + \delta$, for $\delta > 0$. For positive integers n , compute.

$$\alpha_n(\delta) = P_r[W_n > D_n(\delta)]. \quad (4.2)$$

If the alternative hypothesis given by

$$H_1: \gamma_x < \gamma_y = \gamma_x + \delta \quad (4.3)$$

holds, then the expected value of W_n in (3.3) is given by $D_n(\delta)$. Hence, $\alpha_n(\delta)$ is the expected type I error of testing hypothesis (1.2) as a function of δ . Table 1 contains the value of $\alpha_n(\delta)$ for various values of the parameters. The $\alpha_n(\delta)$'s were computed using Tiku's approximation to the asymptotic distribution of W_n [Tiku (1965)].

For example, from Table 1 one would expect the test to reject integer ($\delta=1$) differences between the shapes for X and Y at the 95% significant level whenever $n > 50$.

The procedure used to generate the values in Table 1 is somewhat unconventional, however, they do indicate that

the test procedure would be sensitive to differences in the shape parameters that exceed unity. A Monte Carlo simulation was also performed. The results are not reported in the interest of space and since the simulation was quite limited. A detail simulation is very expensive due to the computational costs in computing the null distribution $G(\theta)$ needed in evaluating type I errors. It is especially costly to simulate any type II errors. In spite of these restrictions upon the simulation's merit, the results were supportive of the expected results given in Table 1.

The second part of the evaluation is concerned with the question of robustness of the test to the unspecified parameters, namely, ρ and β_x, β_y . In order to determine the sensitivity of the test to the misspecified correlation coefficient ρ , the following distance was evaluated for different values of $\gamma_x = \gamma_y$.

$$D_n(\rho) = n \int_0^{\pi/2} \{G(\theta) - B(\theta)\}^2 dG(\theta) \quad (4.4)$$

where $G(\theta)$ is given in equation (3.2) when $\rho = 0$, and $B(\theta)$ is given by equation (3.2) whenever $\rho > 0$, for $\rho = .25(.25).75$. Table 4 contains the type I errors $\alpha_n(\rho)$ given by

$$\Pr[W_n > D_n(\rho)] = \alpha_n(\rho) \quad (4.5)$$

for different values of n and $\gamma_x = \gamma_y = \gamma$. $\alpha_n(\rho)$ in

Table 1. Tail Probabilities for $\alpha_n(\delta)^*$

γ_x	n	n	$\delta = .25$.50	.75	1.00	1.25	1.50	1.75	2.00
1	0	20	1.00	.45	.22	.12	.06	.04	.02	.02
		50	.53	.12	.02	.01				
		100	.25	.02	.01					
	.25	20	1.00	.41	.18	.09	.05	.03	.02	.01
		50	.48	.09	.02	.01				
		100	.21	.01						
	.50	20	.87	.33	.13	.96	.03	.02	.01	.01
		50	.41	.06	.01					
		100	.15	.01						
	.75	20	.69	.21	.07	.02	.01	.01		
		50	.27	.02						
		100	.06							
2	0	20	1.00	.82	.48	.27	.17	.13	.07	.04
		50	1.00	.36	.13	.04	.02			
		100	.60	.13	.02	.01	.01			
	.25	20	1.00	.72	.40	.22	.13	.07	.04	.03
		50	.91	.29	.09	.01				
		100	.52	.09	.01					
	.50	20	1.00	.59	.30	.15	.08	.04	.03	.01
		50	.73	.21	.05	.02				
		100	.41	.05	.01					
	.75	20	1.00	.40	.16	.06	.03	.01		
		50	.52	.09	.01					
		100	.24	.02						
3	0	20	1.00	1.00	.66	.43	.28	.18	.12	.08
		50	1.00	.55	.25	.11	.04	.02	.01	
		100	.87	.26	.07	.01				
	.25	20	1.00	1.00	.56	.35	.21	.13	.08	.05
		50	1.00	.46	.18	.07	.02	.01		
		100	.59	.11	.01					
	.75	20	1.00	.56	.26	.12	.05	.02	.01	
		50	.72	.18	.04	.01				
		100	.39	.04	.01					

*if $\alpha_n(\delta) < .01$, then the entry is left blank.

(4.5) is the expected type I error as a function of the nuisance parameter ρ . It should be mentioned that the distribution for W_n is not the same as that given by Anderson and Darling asymptotic approximation since the nuisance parameter ρ is unspecified [cp., Stephen (1976)], however, it does not appear feasible to follow Darling's procedure for computing the exact distribution whenever ρ and β_x, β_y are replaced by their consistent estimators. In spite of this shortcoming, equation (4.5) is used. However, Stephens (1976) showed that the asymptotic approximation given by Anderson and Darling is conservative as compared to his fitted distribution in the family of normal distributions [Stephens (1976) Table 4, p. 367] and the extreme value distributions [Stephens (1977) Table 1, p. 687]. Thus, it seems reasonable that equation (4.5) is also conservative, that is, if $\alpha_n(\rho)$ is the true value for the l.h.s. of equation (4.5), then $\alpha(\rho) < \alpha_n(\rho)$.

Table 2. Type 1 Errors for Unspecified

γ	n	$\rho=.25$.50	.75
1	20	1.00	1.00	.65
	50	1.00	.87	.25
	100	1.00	.50	.06
2	20	1.00	1.00	.55
	50	1.00	.74	.18
	100	1.00	.41	.04
3	20	1.00	1.00	.53
	50	1.00	.70	.16
	100	1.00	.37	.03

From Table 2, it follows that the procedure is only sensitive to ρ whenever $\rho = .75$ and $n > 50$. This observation was also supported in the simulation study.

In order to determine the sensitivity of the test to the scale parameters, the distance given by

$$D_n(s) = n \int_0^{\pi/2} \{G(\theta) - C(\theta)\}^2 dG(\theta) \quad (4.6)$$

where $G(\theta)$ is given in (3.2) and $C(\theta)$ is given by (3.?) whenever $\tan\theta = sr$, $s = \beta_x/\beta_y = .90(.02)1.10$. Errors in either of the scale parameters can be considered by varying s in (3.2). Table 3 contains the expected type I errors given by

$$\Pr[W_n > D_n(s)] = \alpha_n(s) \quad (4.7)$$

for different values of n , γ , and ρ .

From Table 3 one observes that the procedure appears to be resilient to errors in the scale parameter and that one might have a type I error when $\gamma = 3$, $\rho = .75$, and $n = 100$ at the 95% significance level. In addition it also appears that the results are symmetric about $s = 1$.

5. CONCLUSIONS AND SUMMARY

A procedure is outlined for determining whether a four or five parameter bivariate gamma model is appropriate. The procedure was evaluated and three

Table 3. Type 1 Errors for Misspecified Scales

γ	ρ	n	.90	.92	.94	1.06	1.08	1.10	
1	0	50	1.0	1.0	1.0	1.0	1.0	1.0	
		100	1.0	1.0	1.0	1.0	1.0	1.0	
	.25	50	1.0	1.0	1.0	1.0	1.0	1.0	
		100	.89	1.0	1.0	1.0	1.0	.87	
	.50	50	1.0	1.0	1.0	1.0	1.0	1.0	
		100	.67	.96	1.0	1.0	1.0	.68	
	.75	50	.69	1.0	1.0	1.0	1.0	.73	
		100	.37	.57	.89	.83	.60	.40	
	2	0	50	.98	1.0	1.0	1.0	1.0	1.0
			100	.56	.79	1.0	1.0	.80	.57
		.25	50	.78	1.0	1.0	1.0	1.0	.80
			100	.44	.64	1.0	1.0	.66	.46
.50		50	.58	.81	1.0	1.0	.84	.61	
		100	.38	.46	.74	.76	.49	.31	
.75		50	.31	.49	.78	.80	.52	.34	
		100	.09	.22	.44	.46	.24	.11	
3		0	50	.68	.98	1.0	1.0	1.0	1.0
			100	.36	.55	.86	.88	.57	.38
		.25	50	.55	.77	1.0	1.0	.80	.57
			100	.26	.43	.70	.72	.45	.28
	.50	50	.37	.57	.89	.92	.60	.41	
		100	.14	.28	.52	.54	.30	.16	
	.75	50	.17	.32	.57	.59	.34	.19	
		100	.03	.10	.27	.28	.12	.04	

The values of $s = (.96, 1.04)$ were omitted since the Type 1 error was 1.0 for all parameters. Likewise whenever $n=20$.

different functions were evaluated in order to determine the procedure's feasibility and sensibility to the nuisance parameters. Admittedly, the evaluation is very limited and there are several limitations which would prohibit this type of procedure as an omnibus test of fit. However, the results appear to be promising to the experimentalist interested in obtaining insight into the stated problem.

There are several nonparametric procedures for testing (1.2) versus (1.3) and perhaps these are not as sensitive to the nuisance parameters. However, the proposed procedure is based upon "measuring" significant departures of the parametric distributions function which are vital to the modelers' primary objective.

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A DIFFERENTIAL EQUATIONS APPROACH TO THE
MODAL LOCATION FOR A FAMILY OF
BIVARIATE GAMMA DISTRIBUTIONS

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ABSTRACT

Analytical and numerical computational methods are given for determining the location of the mode as a function of the parameters of a class of the bivariate gamma distribution.

I. INTRODUCTION

Smith, Adelfang, and Tubbs (1983) derived some computational results for a family of bivariate distributions. In their paper they consider the location of the mode as a function of the shape parameters, γ_1 and γ_2 , and the dependence coefficient η . The purpose of this paper is to consider this problem in greater detail. That is, the paper will consider analytical and numerical computational

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methods for locating the modal values for the class of density functions given in Smith, Adelfang, and Tubbs (1983). The general density function is given by:

$$f(t_1, t_2; \gamma_1, \gamma_2, \eta) = \frac{t_1^{\gamma_1-1} t_2^{\gamma_2-1} e^{-(t_1+t_2)}}{(1-\eta)^{\gamma_1} \Gamma(\gamma_1) \Gamma(\gamma_2-\gamma_1)} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} \quad (1.1)$$

$$\text{where } a_{jk} = \frac{\eta^{j+k} \Gamma(\gamma_2-\gamma_1+k) (t_1 t_2)^j t_2^k}{(1-\eta)^{2j+k} \Gamma(\gamma_2+j+k) j! k!}$$

$t_1 = \beta_1 x$, $t_2 = \beta_2 y$, β_1, β_2 are scale parameters, $\gamma_2 > \gamma_1 > 1$ are shape parameters, and $0 < \eta < 1$ is associated with the correlation coefficient ρ by the equation $\eta = \rho \sqrt{\gamma_2/\gamma_1}$. We will assume without loss of generality that $\beta_1 = \beta_2 = 1$.

We will concentrate on the special case $\gamma_1 = \gamma_2 = \gamma$ of (1.1) for which the distribution function reduces to

$$f(t_1, t_2; \gamma, \eta) = \frac{(t_1 t_2)^{\gamma-1} e^{-(t_1+t_2)}}{(1-\eta)^\gamma \Gamma(\gamma)} \sum_{j=0}^{\infty} \frac{\eta^j (t_1 t_2)^j}{(1-\eta)^{2j} \Gamma(\gamma+j) j!} \quad (1.2)$$

This is the form given by Kibble (1941).

Smith and Adelfang (1981) used the above class of density functions in modeling wind gust data for the ascent flight of the Space Shuttle. A parametric model was selected in that the parameters are used to establish engineering constraints for the shuttle payload system. Thus, the modal location and value

were of interest to this particular application. The authors are not aware of any other results, either analytical or numerical, for the modal location for non-Gaussian multivariate distributions. The closest related work is in the area of density and mode estimation [3.g. Sager (1978, 1979), de Beauville (1978), and Eddy (1980)].

In Section 2 we will derive some qualitative results concerning the behavior of the modal location of (1.2) as a function of (γ, n) . Section 3 presents analogous results for another borderline case $\gamma_1 = 1, \gamma_2 \geq 2$ of (1.1). In Section 4 we present some numerical procedures based on the theoretical investigations of the previous sections. The general case $\gamma_2 > \gamma_1 > 1$ is considered in Section 5. We present some numerical tabulations for the modal location of (1.1) as a function of (γ_1, γ_2, n) and consider some numerical interpolations from the borderline cases considered in Sections 2 and 3.

2. EQUAL SHAPE PARAMETERS - ANALYTICAL METHODS

Lemma 1. The function $f(t_1, t_2; \gamma, n)$ defined by (1.2) attains its maximum in the region $R_+^2 = \{(t_1, t_2) : t_1 \geq 0, t_2 \geq 0\}$ on the line $t_1 = t_2$.

Proof: Since f is integrable and continuous over R_+^2 , it is clear that f attains its maximum on R_+^2 . Choose any constant $c > 0$. Let $h(t) = f(t, c-t; \gamma, n)$, $0 < t < c$. Then from (1.2) we have

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$$h(t) = \sum_{j=0}^{\infty} K_j(\gamma, \eta) t^{\gamma+j-1} (c-t)^{\gamma+j-1},$$

where $K_j(\gamma, \eta) > 0$ is independent of t . Therefore,

$$h'(t) = \sum_{j=0}^{\infty} K_j(\gamma, \eta) (\gamma+j-1) t^{\gamma+j-2} (c-t)^{\gamma+j-2} (c-2t).$$

Since $h'(t) > 0$ for $0 < t < c/2$ and $h'(t) < 0$ for $c/2 < t < c$, h attains its maximum at $t = c/2$. Therefore $f(t_1, t_2)$ attains its maximum along any line $t_1 + t_2 = c$ at the point $(c/2, c/2)$.

This completes the proof.

Define $g(t; \gamma, \eta) = f(t, t; \gamma, \eta)$. Then by Lemma 1 it is sufficient to find the point on $t \geq 0$ at which g attains its maximum value. Using (1.2) one can show that

$$g(t; \gamma, \eta) = c(\gamma, \eta) e^{-2t/(1-\eta)} h(t) \quad (2.1)$$

where $c(\gamma, \eta) = [(1-\eta)(\sqrt{\eta})^{\gamma-1} \Gamma(\gamma)]^{-1}$, and $h(t) = t^{\gamma-1} I_{\gamma-1}(p(\eta)t)$, where $I_{\mu}(z)$ denotes the modified Bessel function with index μ , and $p(\eta) = 2\sqrt{\eta}/(1-\eta)$.

Using [Abramowitz and Stegun (1964), Eqn. 9-6-28] it is not difficult to show that $h'(t) = p(\eta)t^{\gamma-1} I_{\gamma-2}(p(\eta)t)$, therefore $f(\tau, \tau; \gamma, \eta)$ is the mode at the bivariate gamma distribution given by (1.2) if and only if $g'(\tau) = 2g(\tau)/(1-\eta)$ or

$$\sqrt{\eta} I_{\gamma-2}(p(\eta)\tau) = I_{\gamma-1}(p(\eta)\tau), \quad (2.2)$$

where $p(\eta) = 2\sqrt{\eta}/(1-\eta)$.

With the aid of (2.1), we may prove the following theorem.

Theorem 1. For fixed $\gamma > 1$, let $\tau(\eta)$ denote the value at which $f(\tau(\eta), \tau(\eta); \gamma, \eta)$ is a maximum. Then τ is continuously differentiable for $0 \leq \eta < 1$ and satisfies the initial value problem

$$\begin{aligned}\tau'(\eta) &= (\tau/2\eta)((2\tau - 2\gamma + 3)^{-1} - (1+\eta)(1-\eta)^{-1}) \\ \tau(0) &= \gamma - 1.\end{aligned}\tag{2.3}$$

Proof. It is easy to show directly from (2.1) that g attains its maximum at $t = \gamma - 1$ when $\eta = 0$, so that $\tau(0) = \gamma - 1$. Furthermore $\frac{\partial g}{\partial t}$ is continuously differentiable for $0 < \eta < 1$ and computation shows that $\frac{\partial^2 g}{\partial t^2} \neq 0$ at $t = \gamma - 1$ and $\eta = 0$. Therefore, $\tau(\eta)$ is continuously differentiable in a neighborhood of $\eta = 0$ by the implicit function theorem. The proof will be completed by differentiating both sides of (2.1) with respect to η . After some simplification and solving for $\tau'(\eta)$ this yields

$$\tau'(\eta) = (1-\eta)I_{\gamma-2}(p(\eta)\tau)/(4\eta q(\eta)) - (1+\eta)\tau/2\eta(1-\eta)\tag{2.4}$$

where $q(\eta) = I'_{\gamma-1}(p(\eta)\tau) - \sqrt{\eta} I_{\gamma-2}(p(\eta)\tau)$.

By [Abramowitz and Stegun (1964), Eqn. 9-2-26],

$$I'_{\gamma-1}(p(\eta)\tau) = I_{\gamma-2}(p(\eta)\tau) - (\gamma-1)I_{\gamma-1}(p(\eta)\tau)/p(\eta)\tau$$

$$I'_{\gamma-1}(p(\eta)\tau) = I_{\gamma-1}(p(\eta)\tau) + (\gamma-2)I_{\gamma-2}(p(\eta)\tau)/p(\eta)\tau$$

substituting these expressions into $q(\eta)$ and using (2.2) yields after some simplification

$$q(\eta) = (1-\eta)(2\tau - 2\gamma + 3)I_{\gamma-2}(p(\eta)\tau)/2\tau.$$

Substituting this expression into (2.4) completes the proof of Theorem 1.

The nonlinear differential equation (2.3) cannot be solved in general in closed form. Some numerical solutions are given

in Section 4. However (2.3) does give information regarding the qualitative and limiting behavior of $\tau(\eta)$ for $\gamma > 1$. In the special case $\gamma = 3/2$, (2.3) reduces to a linear differential equation which can be solved directly by standard methods.

Corollary 1. If $\gamma = 3/2$, then

$$\tau(\eta) = \frac{(1-\eta)}{4\sqrt{\eta}} \ln \left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}} \right).$$

This result can also be obtained directly from (2.2) using the fact that $I_\mu(z)$ can be expressed in terms of hyperbolic functions when $\mu = \pm 1/2$.

Since the differential equation (2.3) is singular at $\eta = 0$, its numerical solution requires some additional knowledge of the behavior of $\tau(\eta)$ near $\eta = 0$. This is provided by the following corollary.

Corollary 2. The function $\tau(\eta)$ is continuously differentiable at $\eta = 0$ and satisfies

$$\tau'(0) = -(\gamma-1)/\gamma, \quad \gamma > 1. \quad (2.5)$$

Proof. The continuous differentiability of τ at $\eta = 0$ was considered in the proof of Theorem 1. Choose $\eta > 0$, then by the mean value theorem there is a number $\xi \in (0, \eta)$ such that

$$\tau(\eta) = \tau(0) + \eta\tau'(\xi) = \gamma-1 + \eta\tau'(\xi).$$

Substituting this expression into (2.3) and simplifying yields

$$\tau'(\eta) = \frac{(\gamma-1+\eta\tau'(\xi))(1+(1+\eta)\tau'(\xi))}{(1-\eta)(1+2\eta\tau'(\xi))}.$$

Letting $\eta \rightarrow 0$ and using the continuity of $\tau'(\eta)$ we have

$$\tau'(0) = -(\gamma-1)(1+\tau'(0)).$$

Solving this equation for $\tau'(0)$ yields (2.5)

We will write $\tau(\eta, \gamma)$ when we wish to emphasize the dependence of the modal location on γ . Theorem 1 and Corollary 2 may be used to obtain several of the qualitative and asymptotic properties of the function $\tau(\eta, \gamma)$ in the region $0 < \eta < 1$, $\gamma > 1$. These are summarized in the following theorem.

Theorem 2. The modal location function $\tau(\eta, \gamma)$ has the following properties:

- (i) $\tau(\eta, \gamma)$ is a decreasing function of η for fixed $\gamma > 1$;
- (ii) $\lim_{\eta \rightarrow 1} \tau(\eta, \gamma) = \max\{\gamma - \frac{3}{2}, 0\}$ for $\gamma > 1$;
- (iii) $\tau(\eta, \gamma) - (\gamma - \frac{3}{2})$ is a decreasing function of γ for fixed $\eta \in (0, 1)$ and $\gamma > 1$;
- (iv) $\lim_{\gamma \rightarrow \infty} \tau(\eta, \gamma) - (\gamma - \frac{3}{2}) = \frac{1-\eta}{2(1+\eta)}$, for $0 \leq \eta \leq 1$.

Proof: We will show that $\tau'(\eta) < 0$ for $0 \leq \eta < 1$. Suppose not, then since $\tau'(0) < 0$ by Corollary 2, there is a point $\xi > 0$ such that $\tau'(\xi) = 0$ and $\tau'(\eta) < 0$ for $0 < \eta < \xi$. Let $w(\eta) = \tau(\eta) - (\gamma - \frac{3}{2})$ and $z(\eta) = \frac{1-\eta}{2(1+\eta)}$, then from (2.3) it is easy to see that

$$\tau'(\eta) = \frac{\tau(\eta)}{4\eta} \left[\frac{1}{w(\eta)} - \frac{1}{z(\eta)} \right],$$

so $\tau'(\xi) = 0$ if and only if $w(\xi) = z(\xi)$.

Let $h = w - z$. Note that $z'(\eta) = -(1+\eta)^{-2}$ so that $h'(0) = w'(0) - z'(0) = \tau'(0) + 1 > 0$ and $h(0) = w(0) - z(0) = 0$. Therefore since $h(\xi) = 0$ and $h(\eta) > 0$ for $0 < \eta < \xi$, we must have $h'(\xi) \leq 0$. However, $h'(\xi) = w'(\xi) - z'(\xi) = \tau'(\xi) - z'(\xi) = -z'(\xi) = (1+\xi)^{-2} > 0$. This contradiction proves (i). Furthermore, we have that $w(\eta) > z(\eta)$ for $0 < \eta < 1$.

We will now consider the proof of (iii). Fix $\gamma_1 > \gamma_2 > 1$ and let $f(\eta) = w(\eta, \gamma_1) - w(\eta, \gamma_2)$ where as before $w(\eta, \gamma) = \tau(\eta, \gamma) - (\gamma - \frac{3}{2})$. We wish to show that $f(\eta) < 0$ for $0 < \eta < 1$. Clearly $f(0) = 0$ and by (2.5) $f'(0) = \frac{1}{\gamma_1} - \frac{1}{\gamma_2} < 0$. Assume to obtain a contradiction that there is a point $\xi \in (0, 1)$ such that $f(\xi) = 0$. If, in addition, we assume ξ is the first such point, then $f(\eta) < 0$ for $0 < \eta < \xi$ so $f'(\xi) \geq 0$. However, using (2.6) at both γ_1 and γ_2 and the fact that $w(\xi, \gamma_1) = w(\xi, \gamma_2)$ it is not difficult to show that

$$f'(\xi) = \frac{(\gamma_1 - \gamma_2)}{4\xi} \left[\frac{1}{w(\xi)} - \frac{1}{z(\xi)} \right].$$

Since $\gamma_1 > \gamma_2$ and $w(\xi) > z(\xi)$, it follows that $f'(\xi) < 0$. This contradiction completes the proof of (iii).

Now we turn to the proof of (ii). First consider the case $1 < \gamma \leq 3/2$. Since τ is decreasing in η and positive for $0 \leq \eta < 1$ we know that $\tau^* = \lim_{\tau \rightarrow 1} \tau(\eta)$ exists, where the limits at 1 are always from the left. Assume to obtain a contradiction that $\tau^* > 0$. Then it is not difficult to show using (2.3) that

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$$\tau'(n) \leq \frac{1}{4n} - \frac{\tau^*(1+n)}{2n(1-n)}$$

Therefore, for $\frac{1}{2} \leq n < 1$ we have

$$\tau'(n) \leq \frac{1}{2} - \frac{\tau^*}{2(1-n)}$$

Integrating both sides of this inequality from $\frac{1}{2}$ to n yields

$$\tau(n) \leq \tau\left(\frac{1}{2}\right) + \frac{1}{2} + \frac{\tau^*}{2} \ln(1-n)$$

for $\frac{1}{2} \leq n < 1$. However, this implies that $\tau(n) \rightarrow -\infty$ as $n \rightarrow 1$, a contradiction.

The case $\gamma > \frac{3}{2}$ follows easily from (iii) and the proof of (i) because for $\gamma \geq \frac{3}{2}$

$$z(n) \leq \tau(n) - (\gamma - \frac{3}{2}) \leq \tau(n, \frac{3}{2})$$

and both $z(n)$ and $\tau(n, \frac{3}{2})$ approach zero as $n \rightarrow 1$.

Finally, we consider the proof of (iv). Let $u(n, \gamma) = w(n, \gamma) - z(n)$ for $0 \leq n \leq 1$ and $\gamma \geq \frac{3}{2}$. From the proof of (i) we know that $u(n, \gamma) \geq 0$. From (2.0) we obtain

$$w'(n, \gamma) = \frac{1}{4n} \left[1 - \frac{w(n, \gamma)}{z(n)} \right] + \frac{(\gamma - \frac{3}{2})}{4n} \left[\frac{z(n) - w(n, \gamma)}{w(n, \gamma)z(n)} \right]$$

so that

$$w'(n, \gamma) \leq -(\gamma - \frac{3}{2})u(n, \gamma).$$

Therefore,

$$u'(n, \gamma) = w'(n, \gamma) - z'(n) \leq -(\gamma - \frac{3}{2})u(n, \gamma) + 1.$$

From this inequality we obtain

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$$\frac{d}{dn} \left[u(n, \gamma) e^{(\gamma - \frac{3}{2})n} \right] \leq e^{(\gamma - \frac{3}{2})n}$$

$$u(n, \gamma) e^{(\gamma - \frac{3}{2})n} - u(0, \gamma) \leq \frac{1}{(\gamma - \frac{3}{2})} \left[e^{(\gamma - \frac{3}{2})n} - 1 \right].$$

Therefore

$$0 \leq u(n, \gamma) \leq \frac{1}{(\gamma - \frac{3}{2})}.$$

This implies that $u(n, \gamma) \rightarrow 0$ as $\gamma \rightarrow \infty$ and completes the proof of Theorem 2.

3. UNEQUAL SHAPE PARAMETERS--THE CASE $\gamma_1 = 1$

In this section we consider another "borderline" case of the general bivariate gamma distribution, the case $\gamma_1 = 1$. For technical reasons we will limit our discussion to the range $\gamma_2 \geq 2$ and for brevity let $\gamma_2 = \gamma$. Then the function given by (1.1) reduces to

$$f(t_1, t_2; 1, \gamma, n) = \frac{t_2^{\gamma-1} e^{-s_2}}{(1-n)\Gamma(\gamma-1)} \sum_{j=0}^{\infty} e^{-s_1} c_j \frac{s_1^j}{j!} \quad (3.1)$$

where

$$c_j = \sum_{k=0}^{\infty} s_3^{j+k} \frac{\Gamma(\gamma+k-1)}{k! \Gamma(\gamma+j+k)}, \quad j = 0, 1, 2, \dots, \quad (3.2)$$

and where $s_1 = \frac{t_1}{1-n}$, $s_2 = \frac{t_2}{1-n}$, and $s_3 = n s_2$.

The following lemma allows us to restrict our attention to the line $t_1 = 0$.

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Lemma 2. The function $f(t_1, t_2; l, \gamma, n)$ given by (3.1) for $\gamma \geq 2$, takes on its maximum value in the region $t_1 \geq 0$, $t_2 \geq 0$ on the line $t_1 = 0$.

Proof: Since f is continuous and integrable in the first quadrant, we know it takes on its maximum value at some point (t_1^*, t_2^*) . We will prove that $t_1^* = 0$ by showing that for any fixed $t_2 > 0$, $f(t_1, t_2)$ is a decreasing function of t_1 . This is equivalent to showing that the function

$$g(s) = \sum_{j=0}^{\infty} e^{-s} c_j \frac{s^j}{j!}$$

is a decreasing function on $s \geq 0$ where c_j is given by (3.2). Note that

$$\begin{aligned} g'(s) &= -e^{-s} \sum_{j=0}^{\infty} c_j \frac{s^j}{j!} + e^{-s} \sum_{j=0}^{\infty} c_j \frac{j s^{j-1}}{j!} \\ &= -e^{-s} \sum_{j=0}^{\infty} c_j \frac{s^j}{j!} + e^{-s} \sum_{j=0}^{\infty} c_{j+1} \frac{s^j}{j!} \\ &= -e^{-s} \sum_{j=0}^{\infty} \frac{s^j}{j!} (c_j - c_{j+1}). \end{aligned}$$

Therefore $g'(s) < 0$ for $s \geq 0$ if $c_{j+1} < c_j$ for $j = 0, 1, 2, \dots$

To this end note that

$$\begin{aligned} c_{j+1} &= \sum_{k=0}^{\infty} s_3^{j+1+k} \frac{\Gamma(\gamma+k-1)}{k! \Gamma(\gamma+j+1+k)} \\ &= \sum_{k=1}^{\infty} s_3^{j+k} \frac{\Gamma(\gamma+k-2)}{(k-1)! \Gamma(\gamma+j+k)} \\ &= \sum_{k=1}^{\infty} s_3^{j+k} \frac{\Gamma(\gamma+k-1)}{k! \Gamma(\gamma+j+k)} \cdot \frac{k}{\gamma+k-2} < c_j \end{aligned}$$

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since $\gamma \geq 2$ implies that $\frac{k}{\gamma+k-2} \leq 1$ for $k = 1, 2, 3, \dots$. This completes the proof of Lemma 2.

According to the preceding lemma, the mode of the bivariate gamma distribution in this case is the point $(0, \mu)$ where μ is the point on $t \geq 0$ where the following function is a maximum:

$$g(t) = t^{\gamma-1} e^{-t/(1-\eta)} h(t)$$

where

$$h(t) = \sum_{k=0}^{\infty} \left(\frac{\eta t}{1-\eta}\right)^k \frac{\Gamma(\gamma+k-1)}{k! \Gamma(\gamma+k)} = \sum_{k=0}^{\infty} \left(\frac{\eta t}{1-\eta}\right)^k \frac{1}{k! (\gamma+k-1)}.$$

Note that

$$\begin{aligned} \frac{d}{dt} \left(\left(\frac{\eta t}{1-\eta}\right)^{\gamma-1} h(t) \right) &= \frac{d}{dt} \left(\sum_{k=0}^{\infty} \left(\frac{\eta t}{1-\eta}\right)^{\gamma+k-1} \frac{1}{k! (\gamma+k-1)} \right) \\ &= \frac{\eta}{1-\eta} \sum_{k=0}^{\infty} \left(\frac{\eta t}{1-\eta}\right)^{\gamma+k-2} \frac{1}{k!} \\ &= \frac{\eta}{1-\eta} \left(\frac{\eta t}{1-\eta}\right)^{\gamma-2} e^{\eta t/(1-\eta)}. \end{aligned}$$

Therefore,

$$\begin{aligned} h(t) &= \left(\frac{1-\eta}{\eta t}\right)^{\gamma-1} \frac{\eta}{1-\eta} \int_0^t \left(\frac{\eta s}{1-\eta}\right)^{\gamma-2} e^{\eta s/(1-\eta)} ds \\ &= \frac{1}{t^{\gamma-1}} \int_0^t s^{\gamma-2} e^{\eta s/(1-\eta)} ds \end{aligned}$$

so that the function we wish to maximize is

$$g(t) = e^{-t/(1-\eta)} \int_0^t s^{\gamma-2} e^{\eta s/(1-\eta)} ds. \quad (3.3)$$

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Lemma 4. Let $\mu(\eta)$, or when necessary $\mu(\eta, \gamma)$, denote the value for which $f(0, \mu(\eta, \gamma), 1, \gamma, \eta)$ is a maximum where f is defined by (3.1) and (3.2). Then $\mu(0) = \gamma - 1$, $\mu(\eta) \geq \gamma - 2$ for $0 \leq \eta < 1$, and satisfies the equation

$$g(\mu) = (1-\eta)\mu^{\gamma-2}e^{-\mu}, \quad 0 \leq \eta < 1, \quad (3.4)$$

where g is defined by (3.3).

Proof: It is easy to see from (3.3) that g attains its maximum on $[0, \infty)$ at a point $t^* > 0$ for which $g'(t^*) = 0$ and $g''(t^*) \leq 0$.

Differentiating (3.3) we obtain

$$g'(t) = -\frac{1}{1-\eta} g(t) + t^{\gamma-2} e^{-t}$$

and

$$g''(t) = -\frac{1}{1-\eta} g'(t) + t^{\gamma-3} e^{-t} (\gamma-2-t).$$

Therefore $g'(\mu) = 0$ implies (3.4) and $g''(\mu) \leq 0$ implies that $\mu \geq \gamma - 2$. Since when $\eta = 0$, $g(t) = e^{-t} \frac{t^{\gamma-1}}{\gamma-1}$ it is easy to see that $\mu(0) = \gamma - 1$. This completes the proof of the lemma.

With the aid of these preliminaries we may prove the following theorem in the spirit of Theorem 1.

Theorem 3. For fixed $\gamma \geq 2$, let $\mu(\eta)$ denote the value of which $f(0, \mu(\eta), 1, \gamma, \eta)$ is a maximum. Then μ is continuously differentiable on $0 \leq \eta < 1$, $\mu'(0) = -1 + \frac{1}{\gamma}$, and on $0 < \eta < 1$ μ satisfies the initial value problem

$$\mu'(\eta) = -\frac{\mu(\mu - (\gamma - 1) + \eta)}{\eta(1-\eta)(\mu - (\gamma - 2))}; \quad (3.5)$$

$$\mu(0) = \gamma - 1.$$

Proof: As in the proof of Theorem 1 the continuous differentiability of μ in a neighborhood of $\eta=0$ may be proved by applying the implicit function theorem to (3.4). This differentiability will be extended to all of $[0,1)$ by proving that (3.5) holds. Let $g(t,\eta)$ denote the function defined by (3.3) and let g_t and g_η denote its partial derivatives with respect to t and η , respectively. Then differentiating both sides of (3.4) with respect to η we obtain

$$g_t(\mu, \eta)\mu' + g_\eta(\mu, \eta) = (1-\eta)e^{-\mu}\mu^{\gamma-3}(\gamma-2-\mu)\mu' - e^{-\mu}\mu^{\gamma-2}. \quad (3.6)$$

By definition $g_t(\mu, \eta) = 0$ and direct differentiation of (3.4) and integration by parts yields for $0 < \eta < 1$ that

$$\begin{aligned} g_\eta(t, \eta) &= -\frac{t}{(1-\eta)^2} g(t, \eta) \\ &\quad + \frac{1}{(1-\eta)^2} e^{-t/(1-\eta)} \int_0^t s^{\gamma-1} e^{ns/(1-\eta)} ds \\ &= -\frac{t}{(1-\eta)^2} g(t, \eta) \\ &\quad + \frac{1}{(1-\eta)^2} e^{-t/(1-\eta)} \left[\frac{1-\eta}{\eta} t^{\gamma-1} e^{nt/(1-\eta)} \right. \\ &\quad \left. - \frac{1-\eta}{\eta} (\gamma-1) \int_0^t s^{\gamma-2} e^{ns/(1-\eta)} ds \right] \end{aligned}$$

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$$= -\frac{t}{(1-\eta)^2} g(t, \eta) + \frac{1}{\eta(1-\eta)} t^{\gamma-1} e^{-t} - \frac{\gamma-1}{\eta(1-\eta)} g(t, \eta).$$

Therefore, using (3.4) we obtain

$$\begin{aligned} g_{\eta}(\mu, \eta) &= -\frac{\mu}{1-\eta} \mu^{\gamma-2} e^{-\mu} + \frac{1}{\eta(1-\eta)} \mu^{\gamma-1} e^{-\mu} \\ &\quad - \frac{\gamma-1}{\eta} \mu^{\gamma-2} e^{-\mu} \\ &= \mu^{\gamma-2} e^{-\mu} \left(-\frac{\mu}{1-\eta} + \frac{\mu}{\eta(1-\eta)} - \frac{\gamma-1}{\eta} \right) \\ &= \mu^{\gamma-2} e^{-\mu} \frac{\mu - (\gamma-1)}{\eta}. \end{aligned}$$

Substituting this expression into (3.6) and simplifying yields (3.5). For $\eta = 0$, an easy calculation shows that

$$g_{\eta}(t, 0) = -\frac{1}{\gamma(\gamma-1)} e^{-t} t^{\gamma}$$

from which substitution into (3.6) with $\eta = 0$ and $\mu = \gamma - 1$ shows that $\mu'(0) = -1 + \frac{1}{\gamma}$. This completes the proof of Theorem 3.

The following corollary exploits the fact that (3.5) reduces to a linear differential equation when $\gamma = 2$.

Corollary 3. If $\gamma = 2$, then

$$\mu(\eta) = \frac{1-\eta}{\eta} \ln\left(\frac{1}{1-\eta}\right), \quad 0 < \eta < 1.$$

Proof. For $\gamma = 2$ equation (3.5) reduces to

$$\mu' + \frac{\mu}{\eta(1-\eta)} = \frac{1}{\eta}$$

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which is easily solved in closed form by standard methods to show the desired result. This result is also easily derived directly from (3.4).

It is interesting to note that the translated modal location function $v(n) = \mu(n) - (\gamma - 2)$ satisfies the differential equation

$$v'(n) = \frac{v(n) + \gamma - 2}{n} \left(\frac{1}{v(n)} - \frac{1}{1-n} \right)$$

$$v(0) = 1, v'(0) = -1 + \frac{1}{\gamma},$$

whereas the translated modal location function $w(n) = \tau(n) - (\gamma - \frac{3}{2})$ of Section 2 satisfies the analogous differential equation

$$w'(n) = \frac{w(n) + \gamma - \frac{3}{2}}{4n} \left(\frac{1}{w(n)} - \frac{2(1+n)}{1-n} \right)$$

$$w(0) = \frac{1}{2}, w'(0) = -1 + \frac{1}{\gamma}.$$

For this reason μ behaves in a manner similar to τ . Its properties are stated in the following theorem. Since the proof of this theorem is entirely analogous to the proof of Theorem 2, it is omitted.

Theorem 4. The modal location function $\mu(n, \gamma)$ has the following properties:

- (i) $\mu(n, \gamma)$ is a decreasing function of n for fixed $\gamma \geq 2$;
- (ii) $\lim_{n \rightarrow 1} \mu(n, \gamma) = \gamma - 2$ for $\gamma \geq 2$;



- (iii) $\mu(\eta, \gamma) - (\gamma - 2)$ is a decreasing function of γ for $\gamma \geq 2$ and fixed $\eta \in (0, 1)$;
- (iv) $\lim_{\gamma \rightarrow \infty} (\mu(\eta, \gamma) - (\gamma - 2)) = 1 - \eta$ for $0 \leq \eta \leq 1$.

4. NUMERICAL RESULTS

In this section we present some quantitative results based on the results of the previous sections. Table 1 shows the value of the modal location function for equal shape parameters $\tau(\eta, \gamma)$ for various values of η and γ . Table 2 shows values of the translated modal location function $w(\eta, \gamma) = \tau(\eta, \gamma) - (\gamma - \frac{3}{2})$. This table illustrates the qualitative behavior of this function derived in Theorem 2. The limiting values of $\eta = 1$ and $\gamma = \infty$ are taken from Theorem 2.

The values in Tables 1 and 2 were computed using Theorem 1. Specifically, a fourth-order Runge-Kutta algorithm was used to compute an approximate solution of the differential equation (2.3) on the interval $0 \leq \eta < 1$ for each specified value of γ . Since equation (2.3) is singular at $\eta = 0$, Corollary 2 was used to replace the initial condition $\tau(0) = \gamma - 1$ by the approximate initial condition

$$\tau(h) = \gamma - 1 - \left(\frac{\gamma - 1}{\gamma}\right) h$$

where h is the step size of the numerical method. Figure 1 shows the data of Table 2 in graphical form and illustrates the behavior of the function $w(\eta, \gamma)$ derived in Theorem 2.

Tables 3 and 4 show the corresponding results for the modal location function $\mu(n, \gamma)$ for the case $\gamma_1 = 1$, $\gamma_2 = \gamma$ and its associated translate $v(n, \gamma) = \mu(n, \gamma) - (\gamma - 2)$. These tables were computed by the same methods as Tables 1 and 2 except using the results of Section 3. Figure 2 illustrates the qualitative behavior of the function $v(n, \gamma)$ as indicated by Theorem.4.

Note that the differential equations (2.3) and (3.5) allow the modal location to be computed recursively in n for a fixed value of γ as a dynamic process in a time scale measured by the modified correlation coefficient n . Error in the computation is introduced through the discretization of this continuous evolutionary process. A more conventional computation of the modal location would require an independent calculation for each value of n with error introduced through the truncation of the series representation (1.2) of the distribution function. This error becomes particularly troublesome as $n \rightarrow 1$.

Table 1. Selected values of the modal location function $\tau(n, \gamma)$ for equal shape parameters.

γ/n	0	.1	.3	.5	.7	.9	1
1.1	.1000	.0908	.0720	.0525	.0322	.0110	.0000
1.3	.3000	.2765	.2268	.1724	.1117	.0412	.0000
1.5	.5000	.4660	.3931	.3116	.2169	.0958	.0000
2.0	1.0000	.9491	.8412	.7277	.6127	.5279	.5000
3.0	2.0000	1.9334	1.8034	1.6862	1.5739	1.5268	1.5000
10.0	9.0000	8.9152	8.7754	8.6697	8.5891	8.5624	8.5000

Table 2. Selected values of the translated modal location function $w(n, \gamma)$ for equal shape parameters.

γ/n	0	.1	.3	.5	.7	.9	1
1.1	.5000	.4908	.4720	.4525	.4322	.4110	.4000
1.3	.5000	.4765	.4268	.3724	.3117	.2412	.2000
1.5	.5000	.4660	.3931	.3116	.2169	.0958	.0000
2.0	.5000	.4491	.3412	.2277	.1127	.0279	.0000
3.0	.5000	.4334	.3034	.1862	.0939	.0268	.0000
10.0	.5000	.4152	.2754	.1697	.0891	.0264	.0000
∞	.5000	.4091	.2692	.1667	.0882	.0263	.0000

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Table 3. Selected values of the modal location function $\mu(n, \gamma)$
for $\gamma_1 = 1, \gamma_2 = \gamma \geq 2$.

$\gamma \backslash n$	0	.1	.3	.5	.7	.9	1
2.0	1.0000	.9482	.8322	.6931	.5160	.2558	.0000
3.0	2.0000	1.9310	1.7767	1.5936	1.3702	1.1111	1.0000
10.0	9.0000	8.9085	8.7170	8.5155	8.3081	8.1011	8.0000

Table 4. Selected values of the translated modal location function
 $v(n, \gamma)$ for $\gamma_1 = 1, \gamma_2 = \gamma \geq 2$.

$\gamma \backslash n$	0	.1	.3	.5	.7	.9	1
2.0	1.0000	.9482	.8322	.6931	.5160	.2558	.0000
3.0	1.0000	.9310	.7767	.5936	.3702	.1111	.0000
10.0	1.0000	.9085	.7170	.5155	.3081	.1011	.0000
∞	1.0000	.9000	.7000	.5000	.3000	.1000	.0000

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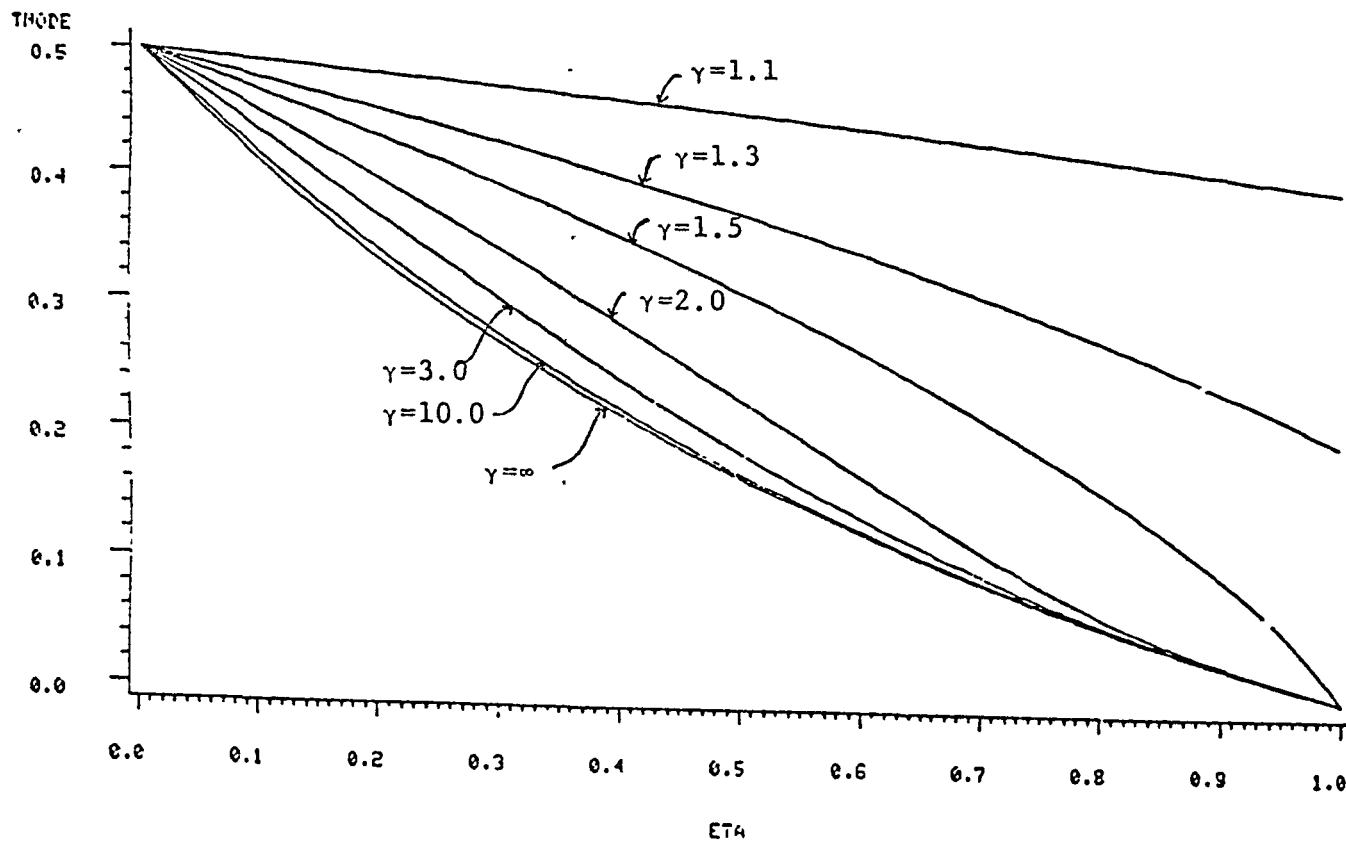


Figure 1. Qualitative behavior of the translated modal location function $w(n, \gamma)$ for equal shape parameters.

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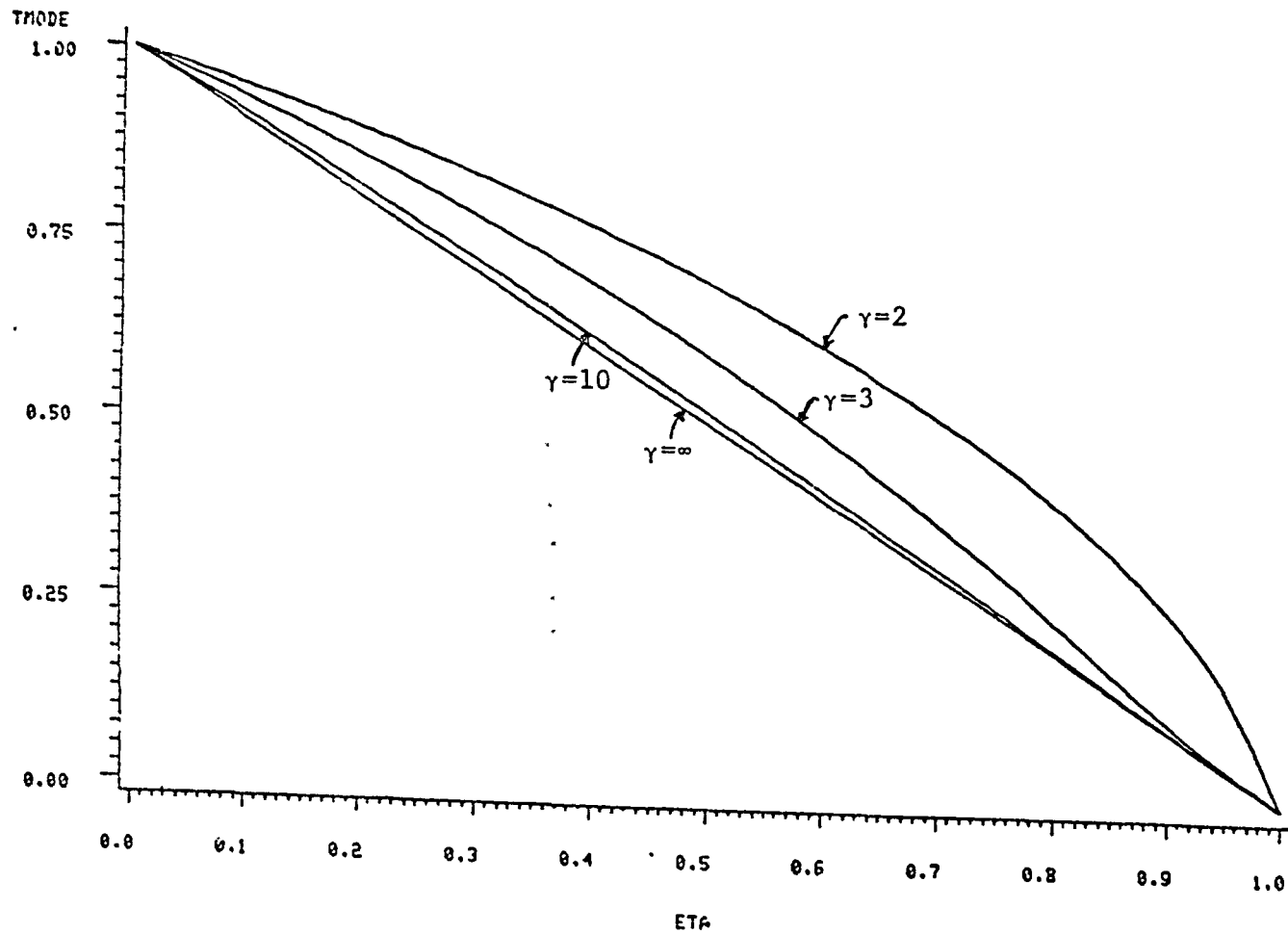


Figure 2. Qualitative behavior of the translated modal location function $v(\eta, \gamma)$ for $\gamma_1=1$, $\gamma_2=\gamma \geq 2$.

5. UNEQUAL SHAPE PARAMETERS

In this section we briefly consider the mode of the general bivariate gamma distribution given by (1.1). By setting the partial derivatives of $f(t_1, t_2; \gamma_1, \gamma_2, \eta)$ with respect to t_1 and t_2 equal to zero, one finds that f attains its maximum at the point (t_1, t_2) whose coordinates satisfy

$$t_1 = \frac{1-\eta}{S} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{jk} (\gamma_1 + j - 1) \quad (5.1)$$

and

$$t_2 = \frac{1-\eta}{S} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{jk} (\gamma_2 + j + k - 1) \quad (5.2)$$

where $S = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{jk}$

and a_{jk} given as in (1.1) depends on t_1 and t_2 .

Table 5 shows selected values of the modal location for the case $\gamma_2 = 3$. They were computed by truncating each of the series in (5.1) and (5.2) to about fifty terms and simultaneously iterating on these equations until an approximate solution is obtained. These computations become unreliable as $\eta \rightarrow 1$ and the truncation error becomes unacceptable.

Figure 3 gives a graphical representation of the change in modal location with η and γ_1 for fixed $\gamma_2 = 3$. It is interesting to note for a fixed η the extent to which the modal location may be approximated by linear interpolation between the borderline cases discussed in Sections 2 and 3.

More specifically, we have the empirical approximations

$$t_1 = \frac{\gamma_1^{-1}}{\gamma_2^{-1}} \tau(n, \gamma_2) \quad (5.3)$$

and

$$t_2 = \mu(n, \gamma_2) + \frac{\gamma_1^{-1}}{\gamma_2^{-1}} (\tau(n, \gamma_2) - \mu(n, \gamma_2)) \quad (5.4)$$

where τ and μ are as defined in Sections 2 and 3, respectively. This empirical relationship is a subject for further investigation.

Table 5. Location of the mode using (5.1) and (5.2) with $\gamma_2 = 3$. Approximate values computed using (5.3) and (5.4) are denoted by *. When these values are equal to two decimal places, only one is given.

$\eta \backslash \gamma_1$	1	1.5	2	2.5	3
0	(0, 2.00)	(.50, 2.00)	(1.00, 2.00)	(1.50, 2.00)	(2.00, 2.00)
.25	(0, 1.82)	(.46, 1.82)	(.92, 1.83)	(1.38, 1.83)	(1.84, 1.84)
.50	(0, 1.59)	(.42, 1.62)	(.84, 1.64)	(1.26, 1.66)	(1.69, 1.69)
.75	(0, 1.31)	(.40, 1.38) (.39, 1.37)*	(.80, 1.44) (.70, 1.44)*	(1.19, 1.51) (1.18, 1.51)*	(1.58, 1.58)
.85	(0, 1.18)	(.41, 1.26) (.39, 1.27)*	(.80, 1.36) (.77, 1.36)*	(1.17, 1.45) (1.16, 1.45)*	(1.54, 1.54)

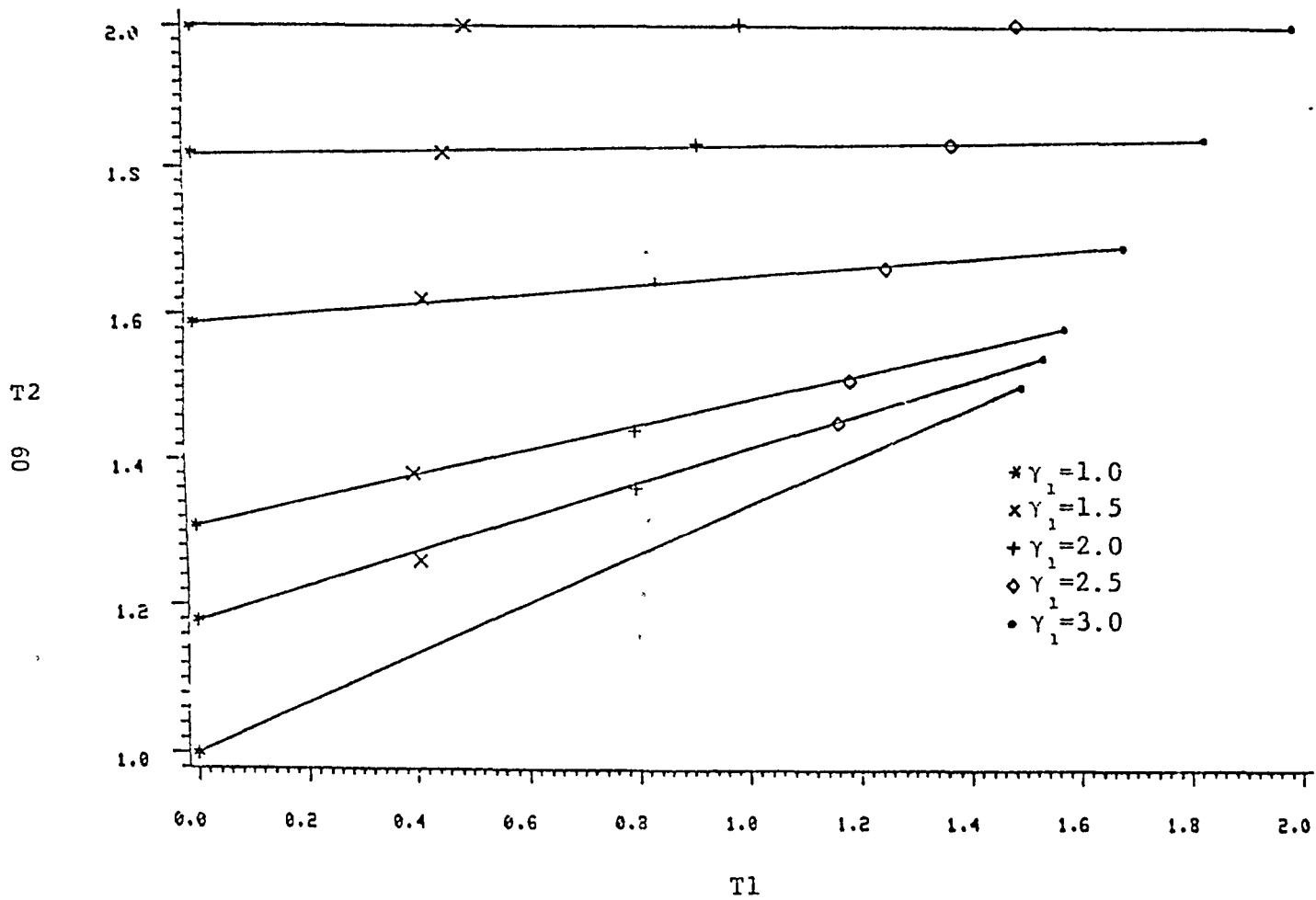


Figure 3. Graphical representation of the values in Table 5.

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ANALYSIS OF WIND GUST DATA

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ABSTRACT

This paper summarizes the analysis of wind gust data using statistical and mathematical procedures which were developed for the bivariate gamma distribution.

1. INTRODUCTION

Adelfang and Smith (1981) discuss the use of the gamma distribution in modeling gust data at Cape Canaveral, Florida. Smith and Adelfang (1981) treated gust amplitude and length scale as the variables of the bivariate gamma distribution. Smith, Adelfang, and Tubbs (1983) presented some useful analytical and computational results for a class of the bivariate gamma and applied some of these results to the wind gust data. The purpose of this paper is to analyze the wind gust data using some additional analytical results obtained for the bivariate gamma distribution.

2. DATA

The data used in this paper consists of absolute gust



magnitude and gust length for both the zonal and meridional components. The 150 wind profiles were filtered using the band pass filter for wavelengths within 420-2470 meter band. Data were available for the reference altitudes: 4Km, 6Km, 8Km, 10Km, 12Km, and 14Km. The data was paired into bivariate components for both the zonal and meridional components, denoted by the pairs (Au,Lu) and (Av,Lv), respectively.

3. ANALYTICAL PROCEDURES

The data were partitioned according to reference altitudes, then the 150 observations were analyzed using both univariate and multivariate techniques. Simple descriptive univariate techniques were generated using PROC UNIVARIATE in SAS. These procedures were used to help in the assessment of the marginal distribution. The multivariate descriptive procedures consisted of bivariate scatter plots and contour plots.

Goodness of fit tests consisted of a univariate test for marginal normality generated by SAS, two tests for bivariate normality as discussed in Meredith and Tubbs (1981), and a bivariate test for the gamma distribution. The latter procedure is a bivariate Chi-square type test which uses the computational results for the distribution function as presented in Smith, Adelfang, and Tubbs (1983).

Parameter estimates for the bivariate gamma distribution were evaluated. These estimates were then used in

generating the three-dimensional bivariate gamma density function plots and the modal locations were estimated using the results given by Brewer, Tubbs, and Smith (1983).

4. RESULTS

The results are summarized in Tables 1-7. Additional results are given in Appendices A and B.

Tables 1-6 summarize the results for both the test of fit and the parameter estimates for the bivariate gamma distribution. There are two main tests for bivariate normality and both of these are discussed in Meredith and Tubbs (1981). The first is a procedure proposed by Rincon-Gallardo et al. (1979). Since this procedure transforms the data to a univariate test for uniformity three different tests for uniformity are used. The second test for bivariate normality is based upon a procedure proposed by Cox and Small (1978).

The bivariate test for the gamma distribution is a Chi-square type test of fit. Thus, this procedure has the usual difficulties of selecting the number of cells and cell location that are associated with this type of test. In the interest of time and space a fixed procedure was applied for all the data sets. Namely, the marginal distributions were partitioned according to the .05, .25, .50, .75, and .90 quantiles based upon the gamma parameter estimates. This particular choice affected the results for some of the data sets, however, it seemed a reasonable global choice.

The univariate gamma parameters were estimated using a maximum likelihood procedure presented by Greenwood and Durand (1960) and discussed in Tubbs and Brewer (1981).

Appendix A contains the results for the univariate descriptive statistics. Appendix B contains plots for each data set. The density functions were generated using the gamma parameter estimates. The contour plots are level slices of the density function and are not equal probability contours. The location of the mode is denoted by the symbol + and this value is computed using the analytical results given in Brewer et al. (1983). Table 7 summarizes the results for the modal location.



Table 1. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 4 Km.

<u>Multivariate Test</u>		<u>(Au, Lu)</u>	<u>(Av, Lv)</u>	
Normality	Cramer-Von Mises	.2062	.2144	
	Watson's U ²	.2023*	.2058*	
	K - S	.0618	.0551	
	Cox	11.45**	26.07***	
Gamma	Chi-Square	33.3	53.00***	
<u>Univariate Test</u>				
Normality	Au	.077*		
	Lu	.068		
	Av	.090**		
	Lv	.077*		
<u>Parameter Estimates</u>		$\hat{\gamma}$	$\hat{\beta}$	$\hat{\rho}$
	Au	3.402	2.430	.2280
	Lu	5.275	.007	
	Av	2.808	1.891	.3415
	Lv	4.429	.006	

* denotes that test is significant at .05 level.
 ** denotes that test is significant at .01 level.
 *** denotes that test is significant at .001 level.



Table 2. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 6 Km.

<u>Multivariate Test</u>		<u>(AuLu)</u>	<u>(Av, Lv)</u>
Normality	Cramer-Von Mises	.3806	.2942
	Watson's U ²	.2411*	.2208*
	K - S	.0897	.0623
	Cox	24.4***	31.8***
Gamma	Chi-Square	72.03***	57.14***

Univariate Test

Normality	Au	.047
	Lu	.081*
	Av	.054
	Lv	.072

Parameter Estimates

	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\rho}$
Au	2.577	1.916	.4217
Lu	4.102	.005	
Av	3.168	2.030	.2506
Lv	4.895	.005	

* denotes that test is significant at .05 level.
 ** denotes that test is significant at .01 level.
 *** denotes that test is significant at .001 level.

Table 3. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 8 Km.

<u>Multivariate Test</u>		<u>(Au, Lu)</u>	<u>(Av, Lv)</u>	
Normality	Cramer-Von Mises	1.090***	.490*	
	Watson's U ²	.721***	.409***	
	K - S	.102*	.083	
	Cox	10.67**	8.67*	
Gamma	Chi-Square	34.74	52.57**	
 <u>Univariate Test</u>				
Normality	Au	.114**		
	Lu	.108**		
	Av	.107**		
	Lv			
<u>Parameter Estimates</u>		$\hat{\gamma}$	$\hat{\beta}$	$\hat{\rho}$
	Au	3.023	2.113	.3396
	Lu	4.149	.005	
	Av	2.922	1.811	.4253
	Lv	4.614	.005	

* denotes that test is significant at .05 level.
 ** denotes that test is significant at .01 level.
 *** denotes that test is significant at .001 level.

Table 4. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 10 Km.

<u>Multivariate Test</u>		<u>(Au, Lu)</u>	<u>(Av, Lv)</u>
Normality	Cramer-Von Mises	.129	.753**
	Watson's U ²	.126	.469**
	K - S	.052	.104*
	Cox	8.23*	8.58*
Gamma	Chi-Square	45.04**	36.29

Univariate Test

Normality	Au	.073*
	Lu	.063
	Av	.073*
	Lv	.109**

Parameter Estimates

	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\rho}$
Au	3.041	1.909	.4345
Lu	5.203	.006	
Av	2.522	1.300	.3890
Lv	4.543	.005	

* denotes that test is significant at .05 level.
 ** denotes that test is significant at .01 level.
 *** denotes that test is significant at .001 level.

Table 5. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 12 Km.

<u>Multivariate Test</u>		<u>(Au, Lu)</u>	<u>(Av, Lv)</u>	
Normality	Cramer-Von Mises	.465*	.398	
	Watson's U ²	.391**	.329**	
	K - S	.077	.076	
	Cox	25.04***	23.70***	
Gamma	Chi-Square	52.54**	41.24*	
 <u>Univariate Test</u>				
Normality	Au	.116**		
	Lu	.112**		
	Av	.042**		
	Lv	.085**		
 <u>Parameter Estimates</u>				
		$\hat{\gamma}$	$\hat{\beta}$	$\hat{\rho}$
	Au	2.155	.969	.2983
	Lu	4.113	.005	
	Av	2.612	1.051	.3066
	Lv	4.023	.005	

* denotes that test is significant at .05 level.
 ** denotes that test is significant at .01 level.
 *** denotes that test is significant at .001 level.

Table 6. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 14 Km

Multivariate Test

Normality	Cramer-Von Mises		1.359***	.616*
	Watson's U ²		.647***	.357**
	K - S		.134**	.084
	Cox		14.6***	17.9***
Gamma	Chi-Square	48	48.29**	42.97*

Univariate Test

Normality	Au	.090**
	Lu	.096**
	Av	.085**
	Lv	.061

Parameter Estimates

	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\rho}$
Au	3.803	1.549	.2420
Lu	4.725	.005	
Av	3.324	1.161	.3171
Lv	5.057	.006	

* denotes that test is significant at .05 level.
 ** denotes that test is significant at .01 level.
 *** denotes that test is significant at .001 level.

Table 7. Modal Location

<u>Altitude</u>	<u>Variables</u>	<u>Method*</u>			
		I		II	
4000	(Au, Lu)	.939	581.9	.939	581.7
	(Av, Lv)	.874	516.7	.873	516.7
6000	(Au, Lu)	.747	539.6	.745	539.4
	(Av, Lv)	1.008	729.4	1.007	729.4
8000	(Au, Lu)	.875	571.0	.874	570.8
	(Av, Lv)	.965	639.8	.960	639.8
10000	(Au, Lu)	.984	624.8	.977	625.2
	(Av, Lv)	1.064	624.8	1.058	624.8
12000	(Au, Lu)	1.083	556.8	1.080	556.8
	(Av, Lv)	1.402	546.2	1.400	546.2
14000	(Au, Lu)	1.713	705.4	1.713	705.4
	(Av, Lv)	1.868	626.0	1.865	626.0

* Method I Truncation of a double series.
 Method II Interpolation.

5. SUMMARY

The data sets are discussed according to reference altitude.

4 Km. The normal or the gamma are not rejected for the zonal (u) components. As discussed in Meredith and Tubbs (1981) the Cox and Small procedure is sensitive to symmetry and is not recommended for this data. The gamma distribution was rejected for the v-component and normality was accepted. However, marginal normality was rejected at the .01 level for the absolute gust magnitude (A_v).

6 Km. The bivariate gamma was rejected in both wind components and normality was not rejected.

8 Km. Normality was rejected for both wind components. The bivariate gamma was accepted in the u-component but not for the v-component.

10 Km. The u-component appears to be normal whereas the gamma is accepted in the v-component.

12 Km. Both distributions appear to be suspect for the u-component and the gamma is accepted for v. Normality is also rejected for v by considering the marginal distributions.

14 Km. Neither distribution is acceptable for u and the gamma is perhaps better for v.

6. CONCLUSIONS

The wind gust data was analyzed using some new procedures for the bivariate gamma distribution. The analysis was meant to be informative, in that it represents examples for some of the analytical procedure. The analysis is not meant to be completely thorough. Hence, there are still some unresolved questions concerning the applicability of the bivariate gamma for modeling wind gust data. One suspects that neither the normal nor the gamma are completely appropriate, however, perhaps both could provide acceptable results for defining engineering constraints.

As mentioned in the paper the test for gamma is a Chi-square type procedure which has inherent problems which does not lend itself to easy data independent analysis. Instead it requires judicious selection of parameters. This analysis did not take advantage of this option, hence, the rejection of the gamma could be attributable to poor cell location choices.

Every data set was analyzed using a test for equality of shape parameters as proposed by Tubbs (1983). This hypothesis of equal shape parameters was rejected in every case.

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Tubbs, J. D. and Brewer, D. W., (1981). Space shuttle environmental analysis statistical models. Final NASA report by Department of Mathematics, University of Arkansas, Fayetteville, Arkansas, under NASA8-33224.



APPENDIX A

Univariate summary statistics generated using
PROC UNIVARIATE is the Statistical Analysis
System.



The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

ALTITUDE = 4000



OPTIONAL SUMMARY
OF POOR QUALITY

VARIABLE=AU

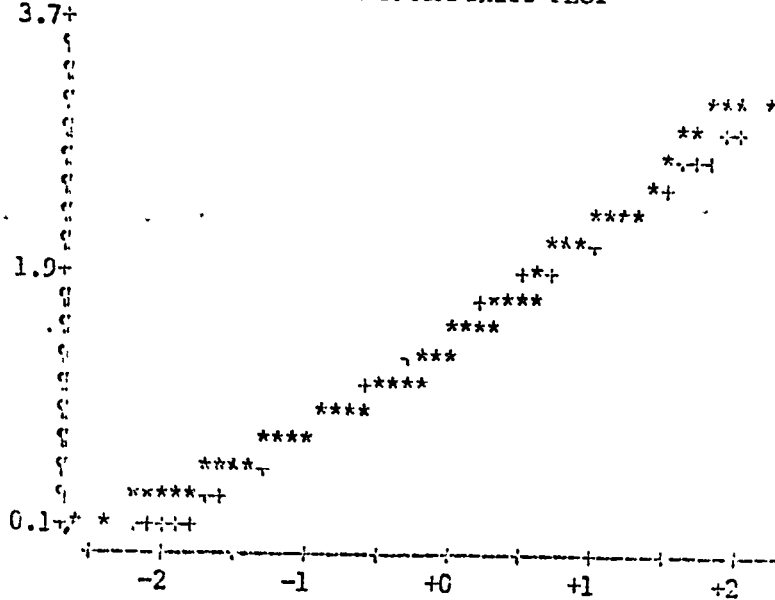
STATISTICS		QUANTILES (DEF=4)	
N	150	SUM	209.976
MEAN	1.39984	VARIANCE	0.499023
STD DEV	0.706416	KURTOSIS	0.239240
MIN	0.05	Q1	0.8775
Q1	0.8775	Q2	1.7825
Q2	1.7825	Q3	3.67
Q3	3.67	MAX	3.40989
MIN	0.05	50% MED	1.295
25% Q1	0.8775	10% Q1	0.59
75% Q3	1.7825	5% Q1	0.3985
90% Q3	2.339	1% Q1	0.06836
95% Q3	2.863	RANGE	3.62
99% Q3	3.40989	Q3-Q1	0.905
PROB'D	0.024	MODE	0.68

EXTREMES

LOWEST	HIGHEST
0.05	3.12
0.086	3.13
0.28	3.13
0.3	3.16
0.33	3.67

STEM	LEAF	NO. OF LEAF
36	7	1
34		0
32		0
30	2336	4
28	047	3
26	3	1
24	52	2
22	00170034556	11
20	210125799	10
18	12311	5
16	14777913345889	14
14	1235667899003566889	20
12	1112340725677	13
10	123334567789055070	18
8	1334567801112350089	19
6	23688891235699	14
4	347367899	9
2	0356	5
0	59	2

OPTIONAL PROBABILITY PLOT



MULTIPLY STEM LEAF BY 10**01

ORIGINAL PRINTING
OF POOR QUALITY

VARIABLE=LU

NO ENTS

QUANTILES(DEF=4)

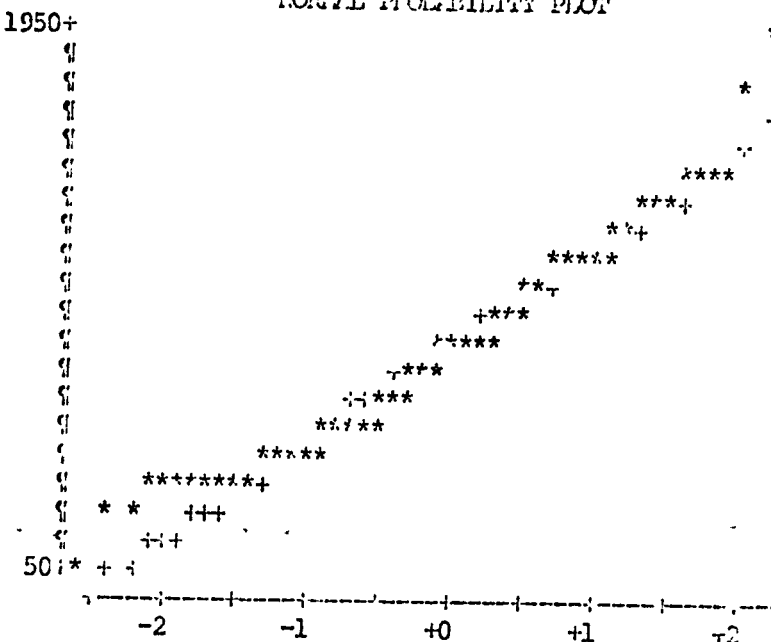
N	150	SUM WGT	150	100% MAX	1974.9	99%	1964.6
MEAN	808.968	SUM	121345	75% Q3	1059	95%	1415.38
STD DEV	344.625	VARIANCE	118808	50% MED	786.45	90%	1277.49
SKEWNESS	0.694665	KURTOSIS	0.642141	25% Q1	537.3	10%	395.91
USS	115666718	CSS	17702335	0% MIN	88.3	5%	347.845
CV	42.606	STD MEAN	28.1434			1%	158.629
T-MEAN=0	28.7445	PROB'ITY	0.0001	RANGE	1886.6		
SGM PAIK	5662.5	PROB'AS%	0.0001	Q3-Q1	521.7		
TU, = 0	150			MODE	702.4		
D:FCRVL	0.0689676	FPCRD	0.000				

EXTREMES

NORMAL PROBABILITY PLOT

LOWEST	HIGHEST
88.3	1439.1
226.2	1486.7
246.8	1792.7
304.7	1554.7
334.4	1974.9

STEM LEAF	#	COMPL
19 57	2	0
18		
17 9	1	
16		
15		
14 03449	5	
13 12349	5	
12 01789	5	
11 01113334556679	14	
10 00123666778	11	
9 00112346699	11	
8 0112233334556777899	19	
7 000335566778899	15	
6 1222456889999	12	
5 00122344456667777899	21	
4 000122245677789	15	
3 03455567799	11	
2 35	2	
1		
0 9	1	



MULTIPLY STEM LEAF BY 10**402



ORIGINAL DATA
OF POOR QUALITY

VARIABLE=AV

MOMENTS

QUANTILES (DEF=4)

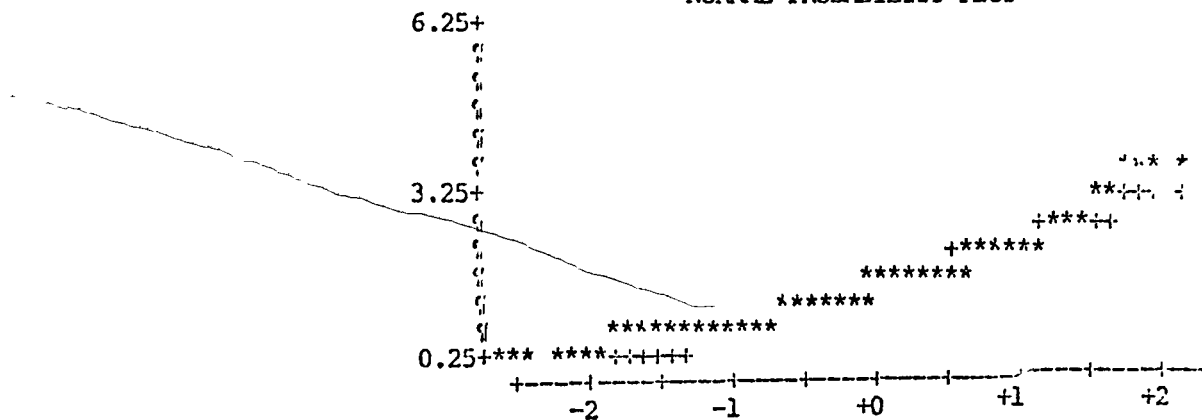
N	150	SUM WGT	150	100% MAX	5.96	99%	4.93488
MEAN	1.4844	SUM	222.66	75% Q3	1.86	95%	3.012
STD DEV	0.822324	VARIANCE	0.676217	50% MED	1.405	90%	2.461
SKEWNESS	1.59111	KURTOSIS	5.65816	25% Q1	0.9	10%	0.673
USS	431.273	CSS	100.756	0% MIN	0.03	5%	0.51
CV	55.3977	STD MEAN	0.0671425			1%	0.03
T-MEAN=0	22.1082	PROB'ITY	0.0001	RANGE	5.93		
SGN RAN	5662.5	PROB'ISS	0.0001	Q3-Q1	0.96		
MIN = 0	150			MODE	1		
D:NORMAL	0.0905899	PROB'D	1-0.01				

DISTRIBS

LOWEST	HIGHEST
0.03	3.54
0.03	3.54
0.07	3.8
0.1	3.95
0.15	5.96

REL. LEAF	↑ FREQ PLOT
0 0	1
5	*
5	
4	
4	
3 5509	4 0
3 11	2 %
2 55556799	6 %
2 00011111222222334	17 %
1 555555566666667777777788888888999999	39 +---+
1 00000000000011111111222222233333444444	39 *---*
0 55566666777777888888888899999999	34 +---+
0 001112	6 %

NORMAL PROBABILITY PLOT



STATISTICS

QUANTILES (DEF=4)

N	150	SUM WGT	150	100% MAX	2017.1	99%	1865.52
MEAN	723.427	SUM	108514	75% Q3	932.275	95%	1315.47
STD DEV	329.434	VARIANCE	108527	50% MED	667.1	90%	1152.19
SEMI-DEV	0.789547	KURTOSIS	1.17774	25% Q1	500.975	10%	350.09
USS	94672440	CSS	16170518	0% MIN	96.1	5%	244.26
CV	45.538	STD LEAF	26.8982			1%	104.566
T-MEAN=0	26.895	PROBABILITY	0.0001	RANGE	1921		
SGN PARAM	5662.5	PROBABILITY	0.0001	Q3-Q1	431.3		
NUM C=0	150			MODE	601.1		
D:NONVAL	0.0776302	PROB D	0.024				

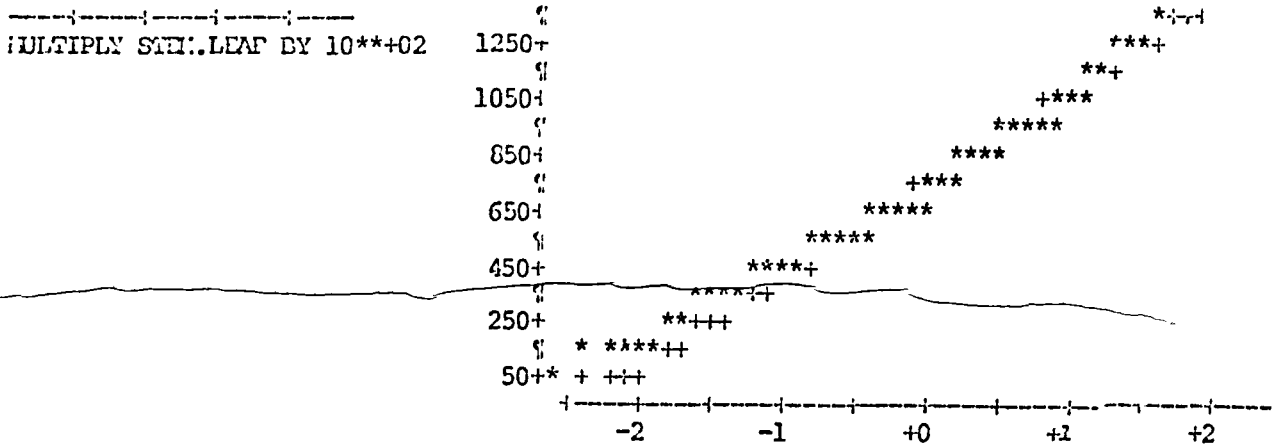
EXTREMES

LOWEST	HIGHEST
96.1	1427.3
112.7	1477.5
122.7	1490.1
134	1719.9
167.1	2017.1

ORIGINAL PAGES
OF POOR QUALITY

STEM LEAF	#	COMPLOT
20 2	1	0
19		
18		
17 2	1	0
16		
15		
14 1369	4	0
13 6	1	0
12 567788	6	0
11 45558	5	0
10 0237779	7	0
9 001223344555667788	18	0
8 011344566769	12	0
7 002222334555789	15	0
6 00000112333344566777999	24	0
5 002223334444567788	20	0
4 112233333344446799	17	0
3 1223555688	10	0
2 278	3	0
1 012377	6	0
0		

NORMAL PROBABILITY PLOT



The next four pages contain the summary statistics for each of the univariate variables A_u , L_u , A_v , and L_v , respectively. The reference altitude is

ALTITUDE = 6000



WRDLEF=AU

MOMENTS

ORIGINAL PAGE IS OF POOR QUALITY

QUANTILES (DEF=4)

N	150	SUM VASTS	150	100% MAX	4.22	99%	3.83239
MEAN	1.34487	SUM	201.73	75% Q3	1.785	95%	2.5415
STD DEV	0.735961	VARIANCE	0.541639	50% MED	1.33	90%	2.209
SKEWNESS	0.630477	KURTOSIS	0.966481	25% Q1	0.8	10%	0.36
USS	352.004	CSS	80.7041	0% MIN	0.03	5%	0.2655
CV	54.7237	STD DEVIATION	0.060091			1%	0.0656999
TRUNC=0	22.3605	PROB. INT.	0.0001	RANGE	4.19		
SGR PAIR	5662.5	PROB. DIST.	0.0001	Q3-Q1	0.985		
MUL. C=0	150			MODE	1.67		
D: NORMAL	0.0470466	PROB. D	0.15				

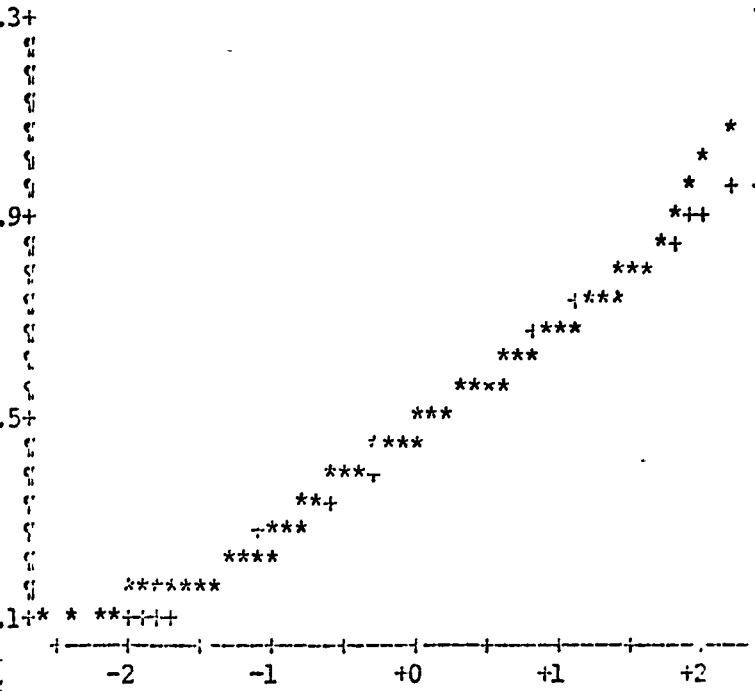
EXTREMES

LOWEST	HIGHEST
0.03	2.83
0.1	3.08
0.15	3.26
0.17	3.46
0.2	4.22

NORMAL PROBABILITY PLOT

STEM: LEAF

42 2	1	0	
40			
38			
36			
34 6	1	0	
32 6	1		1.5+
30 8	1		
28 3	1		
26 8	1		
24 1118	4		
22 012901	6		
20 1244483348	10		
18 02340157778	11		0.1+*
16 0011477777899902238	19		
14 234556677644	12		
12 1223344450222333555599	22		
10 134466667145	12		
8 001224591588	12		
6 33670055	8		
4 012790335668	12		
2 016770124666	12		
0 3057	4		



MULTIPLY STEM:LEAF BY 10**-01

ORIGINAL PAGE IS
OF POOR QUALITY

VAR IASLE=LU

MOMENTS

QUANTILES (DEF=4)

N	150	SUM WGTs	150	100% MAX	1646.5	99%	1635.59
MEAN	748.856	SUM	112328	75% Q3	940.675	95%	1470.54
STD DEV	351.218	VARIANCE	123354	50% MED	711.3	90%	1262.96
SKINNESS	0.552222	KURTOSIS	-0.132913	25% Q1	482.975	10%	323.72
USS	102497549	CSS	18379753	0% MIN	65.9	5%	225.55
CV	46.9006	STD LEAF	28.6768			1%	104.15
T-MEAN=0	26.1136	PROB'ITY	0.0001	RANGE	1580.6		
SGN RANK	5662.5	PROB'ISS	0.0001	Q3-Q1	457.7		
NUN * = 0	150			MODE	634.3		
PROB'AL	0.0812055	PROB'D	0.016				

ENTRIES

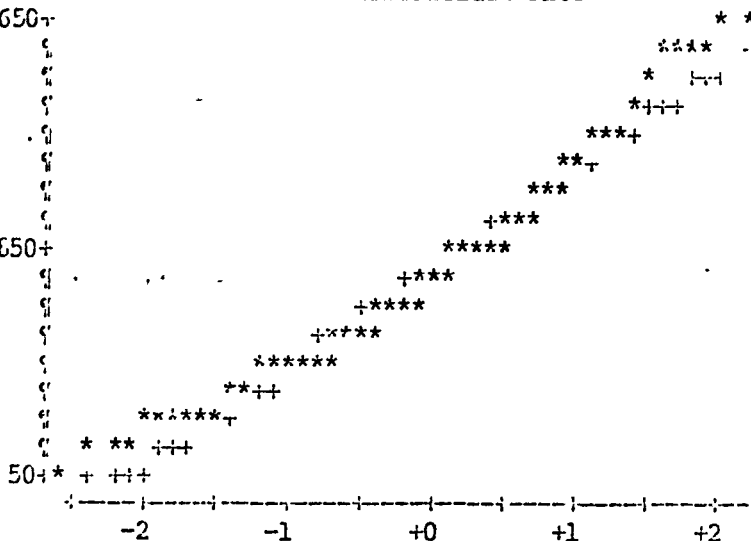
LOWEST	HIGHEST
65.9	1560.4
140.9	1582.9
151.8	1607.6
194.8	1625.1
205.2	1646.5

STEM LEAF

+ FREQ PLOT

16	135	3	0	1650+
15	5568	4		
14	1	1		
13	0177	4		
12	0003799	7		
11	34568	5		
10	112244557	9		
9	0000112345778	13	+-----+	
8	012333455566788899	18		650+
7	0124667788999	13	*-----*	
6	01133344445579	14		
5	0111222234445556789	19		
4	112244555666788899	19	+-----+	
3	0012234799	10		
2	1123688	7		
1	459	3		
0	7	1		50+*

NORMAL PROBABILITY PLOT



MULTIPLY STEM LEAF BY 10**+02

ORIGINAL TABLE
OF POOR QUALITY

VARIABLE-AV

STATISTICS

QUANTILES (DEF=4)

N	150	SUM	150	100% MAX	4.16	99%	3.9254
MEAN	1.55953	SUM	233.93	75% Q3	2.07	95%	3.0305
STD DEV	0.788246	VARIANCE	0.621332	50% MED	1.495	90%	2.632
SKENESS	0.565903	KURTOSIS	0.305502	25% Q1	0.9675	10%	0.612
USS	457.4	CSS	92.5785	0% MIN	0.07	5%	0.334
CV	50.5437	STD MEAN	0.06436	RANGE	4.09	1%	0.1057
T:MEAN=0	24.2314	PROB'ITY	0.0001	Q3-Q1	1.1025		
SGN RANK	5662.5	PROB'IS%	0.0001	MODE	0.71		
INT: = 0	150						
D:NORMAL	0.054269	PROB'D	0.15				

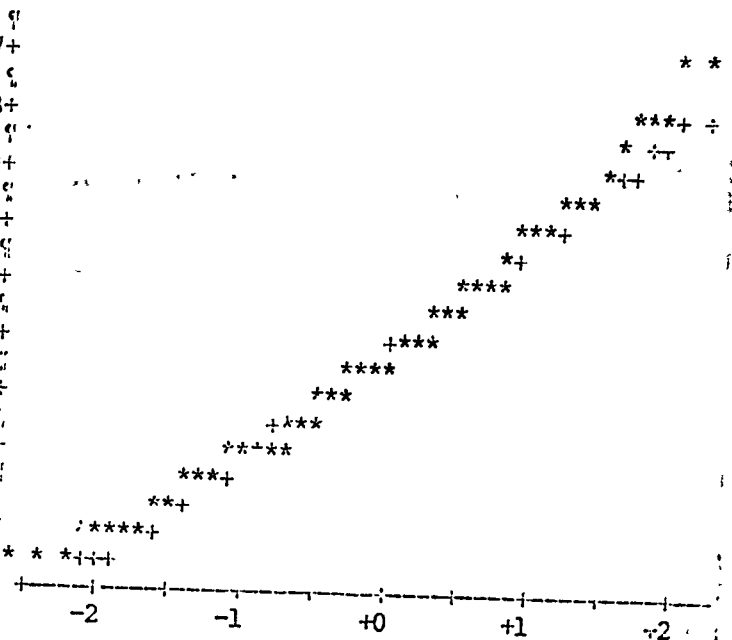
EXTREMES

LOWEST	HIGHEST
0.07	3.32
0.14	3.37
0.19	3.6
0.2	3.7
0.21	4.16

STEM: LEAF	#	COMPLET
40 6	1	0
38		4.1+
36 00	2	{
34		{
32 127	3	{
30 8	1	{
28 69	2	{
26 447116	6	{
24 23446756	8	{
22 47059	5	{
20 013446001234566	15	-----+
18 1467859	7	{
16 02335579991356999	17	{
14 122233348811122479	18	*-----*
12 24568899146789	13	{
10 25669013467	11	{
8 00113589013467788	17	-----+
6 1356111267	10	{
4 87889	5	{
2 017979	6	{
0 749	3	{

MULTIPLY STEM: LEAF BY 10** - 01

NORMAL PROBABILITY PLOT



ORIGINAL PAGE IS
OF POOR QUALITY

VARIABLE=LV

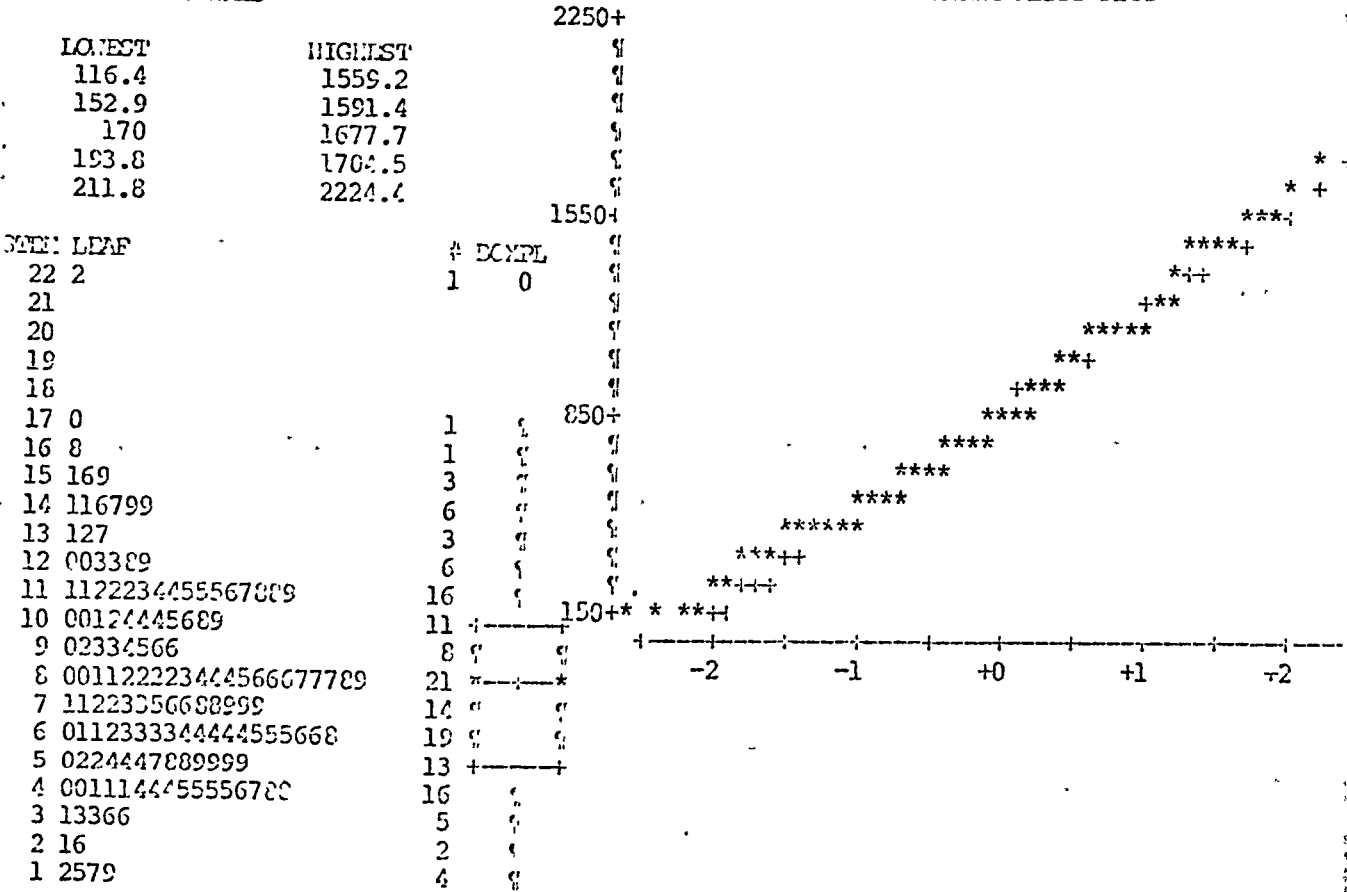
LOCINTS

QUANTILES (DEF=4)

N	150	SUM WGTIS	150	100% MAX	2224.4	99%	1959.25
MEAN	833.661	SUM	125049	75% Q3	1095.12	95%	1491.33
STD DEV	359.396	VARIANCE	129165	50% MED	805.7	90%	1307.57
SKEWNESS	0.614472	KURTOSIS	0.699756	25% Q1	588.275	10%	411.82
USS	123494168	CSS	19245651	0% MIN	116.4	5%	318.755
CV	43.1106	STD MEAN	29.3446			1%	135.015
TR:MEAN=0	28.4094	PROB:PTP	0.0001	RANGE	2108		
SGM:RANK	5662.5	PROB:PSI	0.0001	Q3-Q1	506.85		
NUM:O=0	150			MODE	642.3		
D:HOPE:AL	0.0717385	PROB:PD	0.057				

EXTREMES

NORMAL PROBABILITY PLOT



MULTIPLY STEM:LEAF BY 10**#02

The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

ALTITUDE = 8000

ORIGINAL PAGE IS
OF POOR QUALITY

VARIABLE=AU

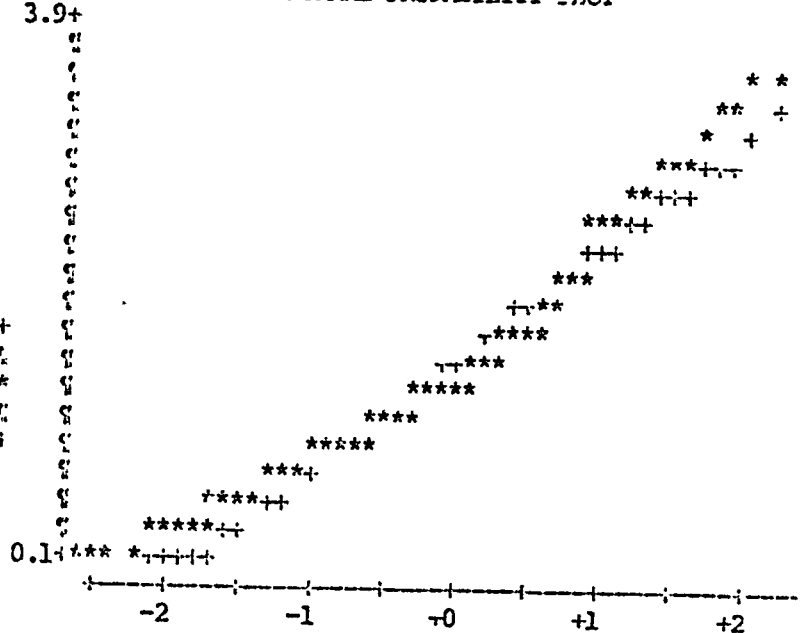
MOMENTS				QUANTILES (DEF=4)			
N	150	SUM WGT	150	100% MAX	3.87	99%	3.6507
MEAN	1.43033	SUM	214.55	75% Q3	1.76	95%	2.9645
STD DEV	0.780668	VARIANCE	0.609442	50% MED	1.295	90%	2.627
SKEWNESS	0.004124	KURTOSIS	0.252035	25% Q1	0.87	10%	0.57
USS	357.685	CSS	90.8069	0% MIN	0.09	5%	0.3655
CV	54.5794	STD MIN	0.0637413			1%	0.09
TRHEW=0	22.4397	PROB1TY	0.0001	RANGE	3.78		
3RD MOM	5662.5	PROB1ST	0.0001	Q3-Q1	0.89		
4TH MOM	150			MODE	0.8		
DISTRIBUT	0.10402	PROB1D	0.01				

ENTRIES

LOWEST	HIGHEST
0.09	3.2
0.09	3.27
0.12	3.41
0.25	3.44
0.32	3.87

ENTR: LEAF	#	DOXPLOT	
38 7	1	0	3.9+
36			
34 14	2	0	
32 07	2	0	
30 7	1	0	
28 32367	5	1	
26 03578	5	1	
24 24489017	8	1	
22 8	1	1	
20 110346	6	1	
18 18012	5	1	
16 112222401125558	15	+-----+	
14 011334679933779	15	+ + +	
12 01274556779001125668	20	*-----*	
10 011222347891273457	18	+ + +	
8 00000227779013488	17	+-----+	
6 14502335688	11	+	
4 0136667779	10	+	
2 52889	5	+	
0 992	3	+	0.1+***

NORMAL PROBABILITY PLOT



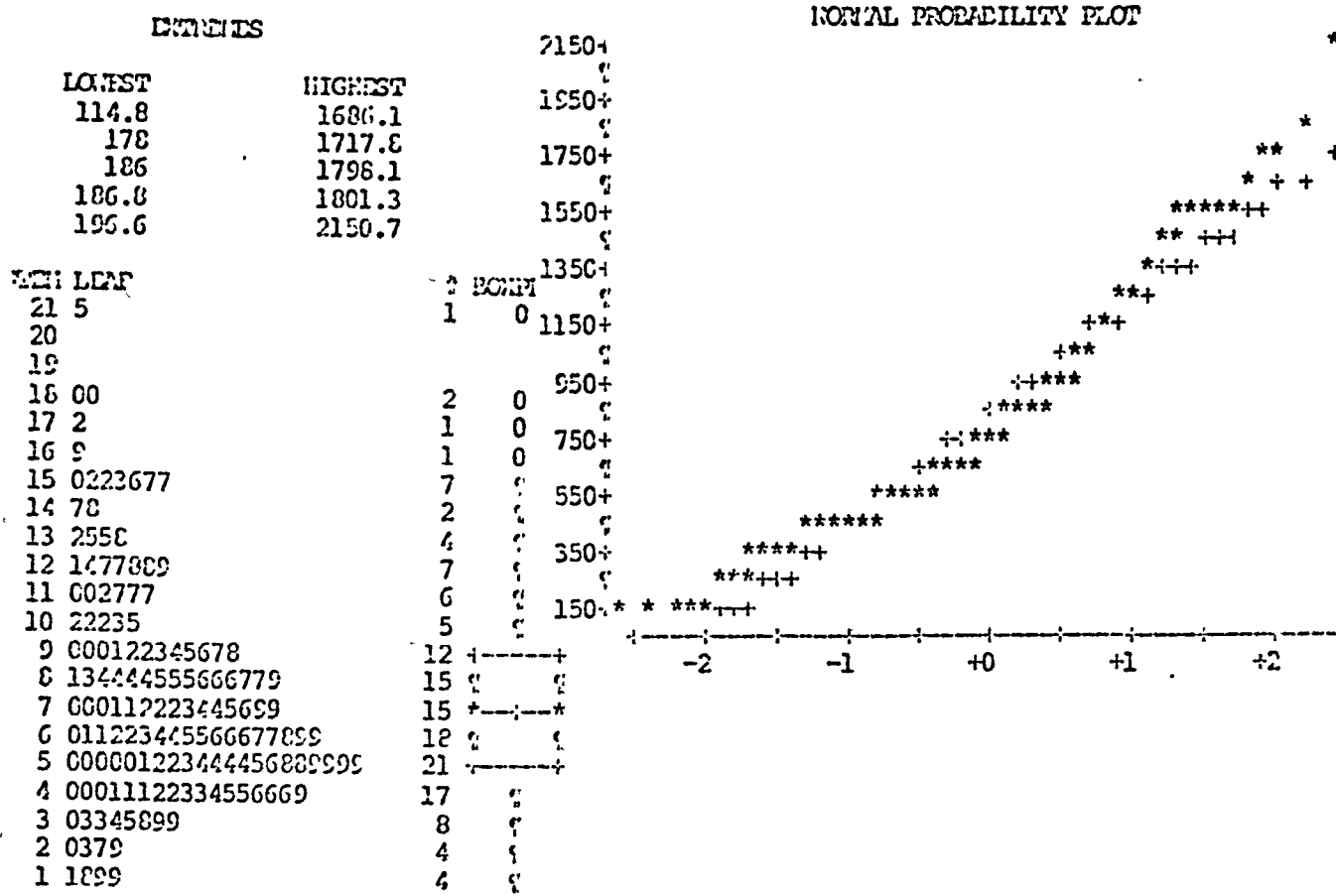
MULTIPLY ENTR:LEAF BY 10**-01

C-2

VARIABLE=LU

ORIGINAL FILE #1
OF POOR QUALITY

STATISTICS				QUANTILES (DEF=4)			
N	150	SUM WGT5	150	100% MAX	2150.7	99%	1972.5
MEAN	792.407	SUM	118873	75% Q3	975.725	95%	1566.55
STD DEV	391.179	VARIANCE	153021	50% MED	703.95	90%	1374.45
SEMI-RANGE	0.893196	RUMOSIS	0.522044	25% Q1	503.075	10%	391.84
USS	117005567	CSS	22800141	0% MIN	114.0	5%	282.655
CV	49.3609	STD MEAN	31.9396			1%	147.032
TRUNC=0	24.812	PROB. QTY	0.0001	RANGE	2035.9		
SCN PAIR	5662.5	PROB. USE	0.0001	Q3-Q1	472.65		
TRUNC=0	150			MODE	114.0		
DEVIATION	0.108007	PROB. D	10.01				



MULTIPLY SIZE..LEAF BY 10**+02

ORIGINAL PAPER
OF POOR QUALITY

VARIABLE=AV

STATISTICS		QUANTILES (DEF=4)					
N	150	SUM	150	100% MAX	3.95	99%	3.9092
MEAN	1.61353	SUM	242.03	75% Q3	2.2225	95%	3.2645
STD DEV	0.872606	VARIANCE	0.761444	50% MED	1.48	90%	2.919
SKEWNESS	0.558956	KURTOSIS	-0.351765	25% Q1	0.96	10%	0.531
USS	503.979	CSS	113.455	0% MIN	0.1	5%	0.3355
CV	54.0806	STD LEAF	0.0712481			1%	0.1153
T:MEAN=0	22.6467	PROB:CTM	0.0001	1% GE	3.85		
SGM:PAK	5662.5	PROB:CS4	0.0001	Q3-Q1	1.2625		
TU: C= 0	150			LCDE	1.04		
D:NONVAL	0.0807145	PROB:D	0.017				

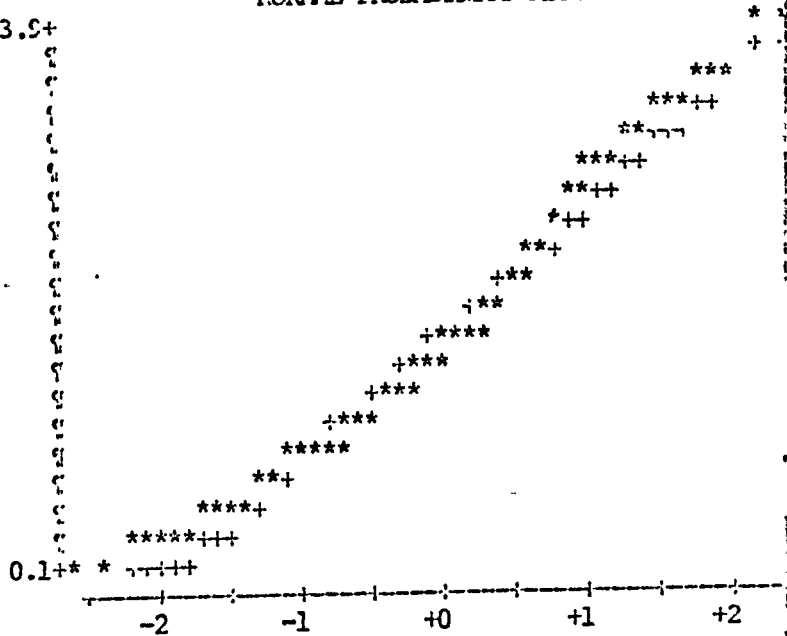
EXTREMES

LOWEST	HIGHEST
0.1	3.43
0.13	3.48
0.23	3.53
0.29	3.87
0.29	3.95

STEM: LEAF	#	PROBLOT
38 75	2	3.9+
36		
34 383	3	
32 3677	4	
30 688	3	
28 124771267	9	
26 244	3	
24 090458	6	
22 23670239	8	
20 000024	6	
18 27789559	8	
16 273300001344599	16	
14 145560125679	12	
12 0112555677013359	16	
10 0244457802569	13	
8 0677891144566779	16	
6 157003479	9	
4 02479134	8	
2 399034	6	
0 03	2	0.1+*

MULTIPLY STEM:LEAF BY 10** -01

NORMAL PROBABILITY PLOT



VARIABLE=LV

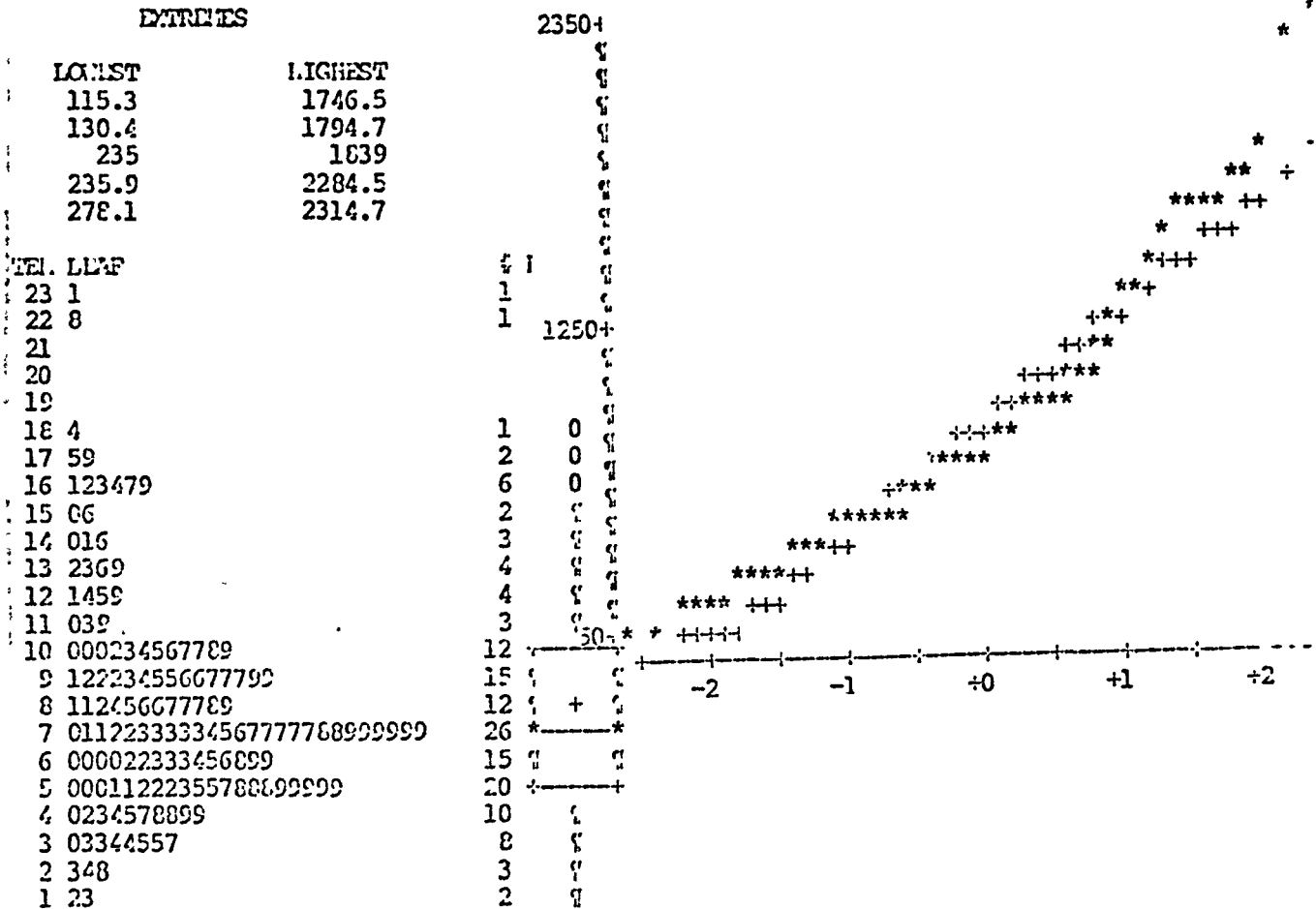
ORIGINAL PAGE IS
OF POOR QUALITY

MOMENTS

QUANTILES (REF=4)

1	150	SUM MGTS	150	100% MAX	2314.7	99%	2299.3
MEAN	842.257	SUM	126339	75% Q3	1001.9	95%	1654.59
STD DEV	396.748	VARIANCE	159000	50% MED	773.8	90%	1412.32
SKEWNESS	1.12027	KURTOSIS	1.65767	25% Q1	580.875	10%	423.16
SS	130100556	CSS	23690944	0% MIN	115.3	5%	320.81
CV	47.3427	STD MEAN	32.5576			1%	123.001
MEAN=0	25.8692	PROB'ITY	0.0001	RANGE	2199.4		
GI: RANK	5662.5	PROB'ESC	0.0001	Q3-Q1	421.025		
UI: = 0	150			MODE	115.3		
NONORMAL	0.107964	PROEID	10.01				

NORMAL PROBABILITY PLOT



MULTIPLY STE:LEAF BY 10**+02

The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

ALTITUDE = 10,000

ORIGINAL PAGE IS
OF POOR QUALITY

VARIABLE=AU

MOMENTS

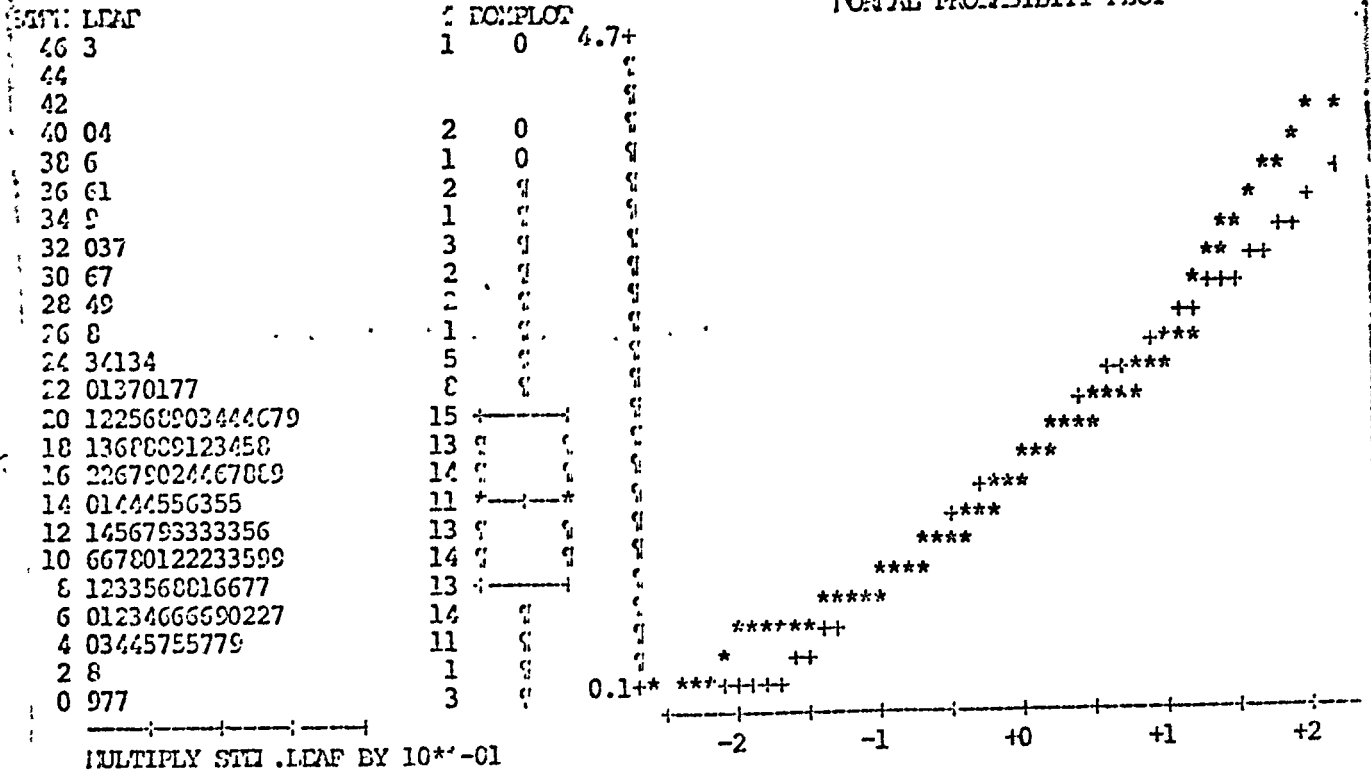
QUANTILES(DEF=4)

N	150	SUM WGT	150	100% MAX	4.73	99%	4.42909
MEAN	1.59273	SUM	238.91	75% Q3	2.0825	95%	3.414
STD DEV	0.879401	VARIANCE	0.773346	50% MED	1.45	90%	2.756
SKEWNESS	0.923389	KURTOSIS	1.03196	25% Q1	0.9025	10%	0.591
USS	495.748	CSS	115.229	0% MIN	0.09	5%	0.44
CV	55.2133	STD MEAN	0.0718028			1%	0.1308
T-MEAN=0	22.1821	PROB'ITY	0.0001	RANGE	4.64		
SGH RANK	5662.5	PROB'ITY	0.0001	Q3-Q1	1.18		
TRUNC=0	150			MODE	1.33		
D-NORMAL	0.0733202	PROB'D	0.046				

EXTREMES

LOWEST	HIGHEST
0.09	3.71
0.17	3.96
0.17	4
0.38	4.14
0.4	4.73

NORMAL PROBABILITY PLOT



ORIGINAL PAGE IS
OF POOR QUALITY

VARIABLE=LU

MOMENTS				QUANTILES(DEF=4)			
N	150	SUM WGT	150	100% MAX	2466	99%	2203.29
MEAN	972.983	SUM	138447	75% Q3	1175.47	95%	1549.76
STD DEV	385.088	VARIANCE	148293	50% MED	899.25	90%	1463.3
SKWNESS	0.591313	KURTOSIS	0.982827	25% Q1	671.925	10%	418.73
CV	149.00139	CS	2205508	0% MIN	173.5	5%	299.57
CV	41.7221	STD MEAN	31.4423			1%	196.858
MEAN=0	29.3548	PROB'ITY	0.0001	RANGE	2292.5		
GEN PAIR	5662.5	PROB'KEY	0.0001	Q3-Q1	503.55		
UN = 0	150			MODE	854.2		
NONNPL	0.0627019	PROB'D	10.15				

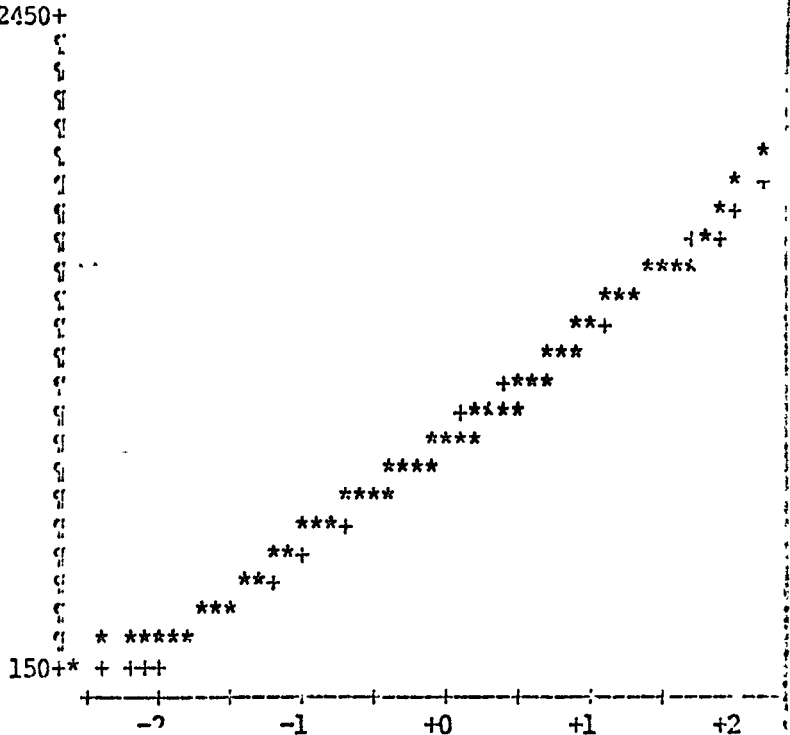
EXTREMES

LOWEST	HIGHEST
173.5	1665.4
219.3	1737.3
224.7	1817.6
250.3	1950.9
276.7	2466

STEM: LEAF	#	LOWPLOT
24 7	1	0 2450+
23		
22		
21		
20		
19 5	1	0
18 2	1	
17 4	1	
16 7	1	
15 094557	6	
14 0266788	7	
13 136788	6	
12 023445679	9	
11 0233678899	10	-----
10 000112224556789	15	
9 000122334455557899	18	+
8 0234556666678899	16	-----*
7 0112234456788999	16	
6 113466677889	13	-----
5 23446799	8	
4 12445559	8	
3 0017899	7	
2 22599	5	
1 7	1	

MULTIPLY STEM:LEAF BY 10**+02

NORMAL PROBABILITY PLOT

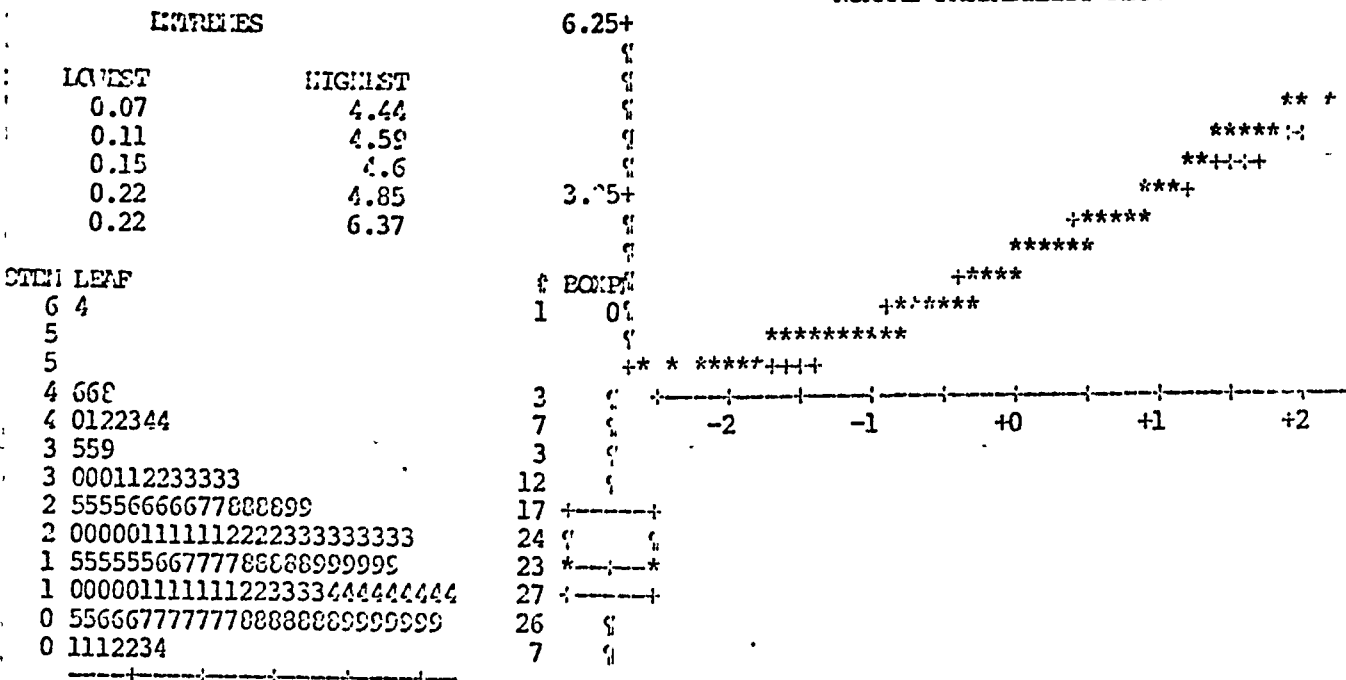


OF POOR QUALITY

VARIABLE-AV

MOMENTS		QUANTILES (DEF=4)					
N	150	SUM WGTs	150	100% MAX	6.37	99%	5.59478
MEAN	1.9398	SUM	290.97	75% Q3	2.5725	95%	4.284
STD DEV	1.13058	VARIANCE	1.29637	50% MED	1.825	90%	3.349
SKEWNESS	0.877054	KURTOSIS	0.909461	25% Q1	1.015	10%	0.691
USS	757.583	CSS	193.159	0% MIN	0.07	5%	0.4665
CV	58.6959	STD MEAN	0.0929649			1%	0.0903999
T:MEAN=0	20.8659	PROB'ITY	0.0001	RANGE	6.3		
SGN RANK	5662.5	PROB'ISS	0.0001	Q3-Q1	1.5575		
NUM = 0	150			MODE	0.89		
D:NORMAL	0.0739582	PROB'D	0.043				

NORMAL PROBABILITY PLOT



ORIGINAL PAGE IS
OF POOR QUALITY

VARIABLE=LV

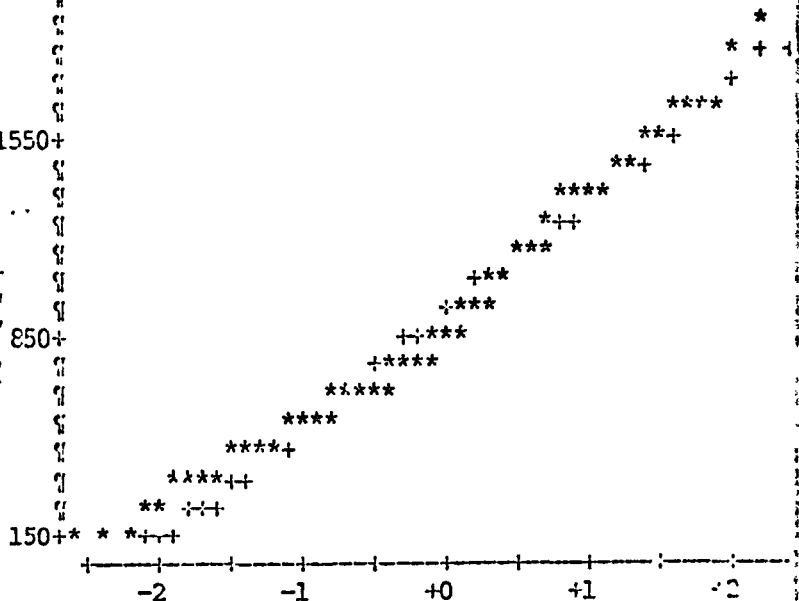
MOMENTS		QUANTILES(DEF=4)					
N	150	SUM	133024	100% MAX	2221.7	99%	2091.19
MEAN	886.825	SUM	133024	75% Q3	1172.22	95%	1596.77
STD DEV	399.875	VARIANCE	159900	50% MED	799.65	90%	1427.67
SKEWNESS	0.58784	KURTOSIS	0.042399	25% Q1	601.825	10%	409.93
USS	141794019	CSS	23825143	0% MIN	106.1	5%	338.805
CV	45.0907	STD MEAN	32.6497			1%	127.775
T-MEAN=0	27.1618	PROB. > T	0.0001	RANGE	2115.6		
SGN RANK	5662.5	PROB. < S	0.0001	Q3-Q1	570.4		
MU1 = 0	150			MODE	959.5		
D:KORVAL	0.109902	PROB. D	0.01				

EXTREMES

LOWEST	HIGHEST
106.1	1663.3
148.6	1695.8
171.9	1821.4
223.1	1965.8
250.7	2221.7

STEM: LEAF	#	COMPL	2250+
22 2	1	0	
21			
20			
19 7	1		
18 2	1		
17 0	1		
16 166	3		1550+
15 2678	4		
14 33678	5		
13 012555777889	12		
12 046799	6		
11 00134477889	11	+	
10 1245669	7		
9 011456666799	12		850+
8 0000111567779	13	+	
7 0000011222445778899	19	*	
6 0000111223366677788	15	+	
5 45555566778	11		
4 00001244555777	14		
3 23579	5		
2 25	2		150+*
1 157	3		

NORMAL PROBABILITY PLOT



MULTIPLY STEM:LEAF BY 10**+02

The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

ALTITUDE = 12,000

CLASSIFICATION
OF POOR QUALITY

FIELD=AU

MOMENTS

QUANTILES (DEF=4)

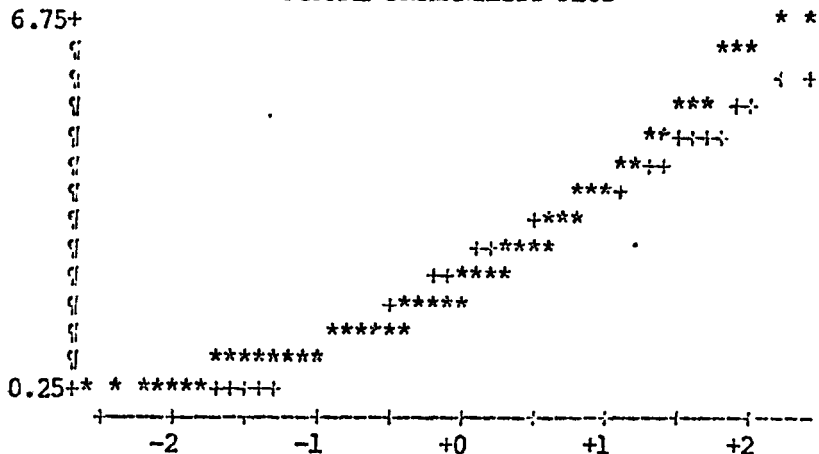
	150	SUM WGT	150	100% MAX	6.7	99%	6.6796
MEAN	2.22487	SUM	333.73	75% Q3	2.8625	95%	5.2585
STD DEV	1.45363	VARIANCE	2.11305	50% MED	1.835	90%	4.29
KURTOSIS	1.08163	KURTOSIS	0.915241	25% Q1	1.1075	10%	0.735
SS	1057.35	CSS	314.845	0% MIN	0.06	5%	0.382
V	65.3358	STD MEAN	0.110689			1%	0.0651
MEAN=0	18.7454	PROB: 0.0001	0.0001	RANGE	6.64		
GR: RMI	5662.5	PROB: 0.0001	0.0001	C3-Q1	1.755		
UI: C=0	150			LCDE	0.97		
NONAL	0.116829	PROB: 0.01	0.01				

EXTREMES

LOWEST	HIGHEST
0.06	6.29
0.07	6.31
0.16	6.34
0.2	6.66
0.23	6.7

ITEM LEAF	%	DOXPLOT
6 77	2	0
6 333	3	0
5		
5 2234	4	0
4 56789	5	0
4 01223	5	0
3 556678899	9	0
3 001122223	9	0
2 5555555666777788	10	+
2 00000122223344444	19	+ +
1 5555566677777778888889999	26	* + + +
1 000000000011111122222233334	29	+ + + + +
0 556666789999999	15	0
0 11222344	8	0

NORMAL PROBABILITY PLOT



VARIABLE=LU

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MOMENTS

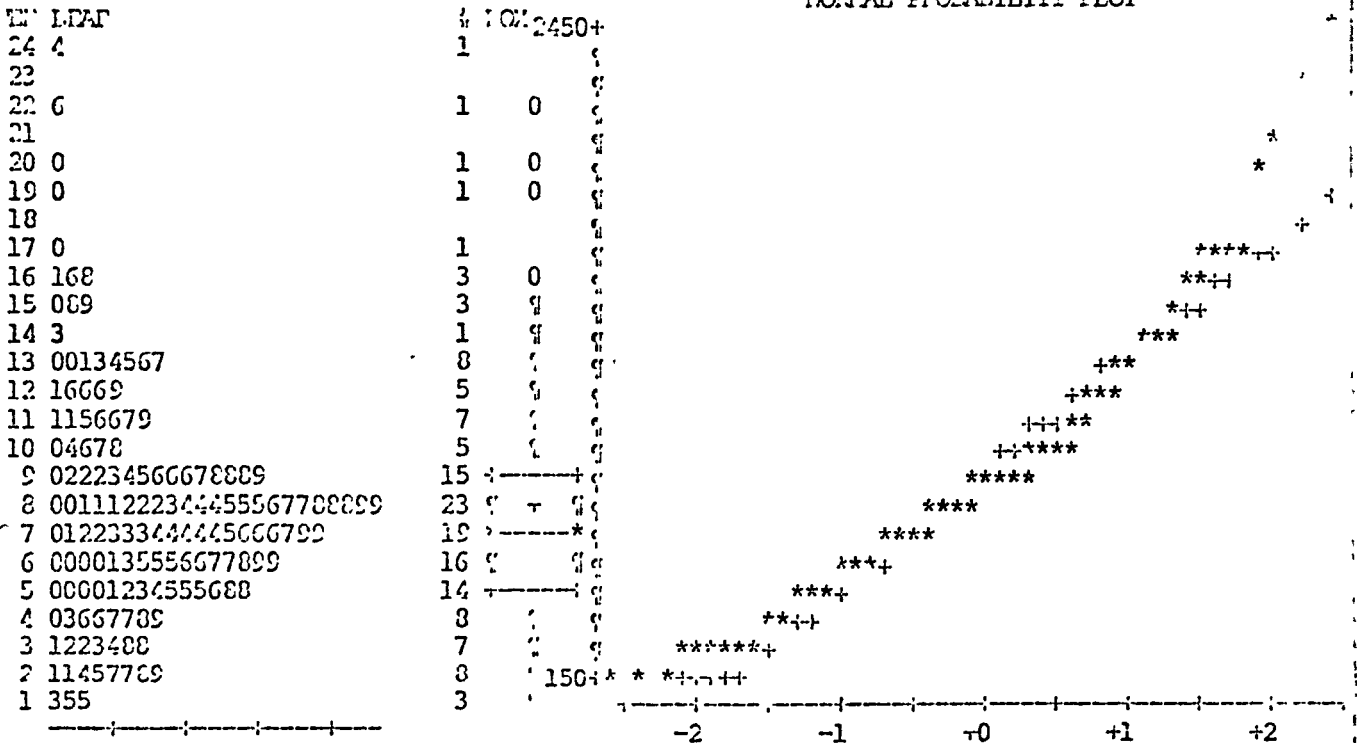
QUANTILES (DEF=4)

	150	SUM	150	100% MAX	2445	99%	2351.52
MEAN	637.755	SUM	125663	75% Q3	989.15	95%	1634.44
STD DEV	408.852	VARIANCE	167160	50% MED	793.95	90%	1352.38
SKEWNESS	1.05082	KURTOSIS	1.90283	25% Q1	556.375	10%	326.89
SS	130181790	CSS	24906658	0% MIN	130.7	5%	258.33
V	48.8033	STD MEAN	33.3826			1%	138.707
STDEV=0	25.0955	PROB<=T	0.0001	RANGE	2314.3		
MIN PANK	5062.5	PROB<=S	0.0001	Q3-Q1	432.775		
MIN C=0	150			MODE	809.1		
MINIMAL	0.112428	PROB<D	1-0.01				

EXTREMES

LOWEST	HIGHEST
130.7	1695.9
146.4	1900.8
153	2005
206.7	2261.7
212.4	2445

NORMAL PROBABILITY PLOT



ORIGINAL PROC 13
OF POOR QUALITY

VARIABLE=AV

MOMENTS

QUANTILES (DEF=4)

N	150	SUM WGTs	150	100% MAX	6.11	99%	6.0794
MEAN	2.4854	SUM	372.81	75% Q3	3.3425	95%	5.259
STD DEV	1.39227	VARIANCE	1.92841	50% MED	2.18	90%	4.639
SKENESS	0.600256	KURTOSIS	-0.185831	25% Q1	1.485	10%	0.734
USS	1215.4	CSS	288.823	0% MIN	0.1	5%	0.4675
CV	56.0178	STD MEAN	0.113678			1%	0.1051
T:MEAN=0	21.8635	PROB:PT9	0.0001	RANGE	6.01		
SKN RATIO	5062.5	PROB:PS9	0.0001	Q3-Q1	1.8575		
MEAN = 0	150			ICDE	2.07		
DI:NONVAL	0.0923982	PROB:D	0.01				

EXTREMES

LOWEST	HIGHEST
0.1	5.57
0.11	5.61
0.34	5.88
0.37	6.05
0.4	6.11

STEM LEAF

```

6 01
5 669
5 2234
4 55666667788
4 114
3 555566677889
3 0000001223344
2 5555566667788899
2 0001111111111111222233333444
1 55556667778888899999
1 00011222333334444
0 555667788999
0 1134444

```

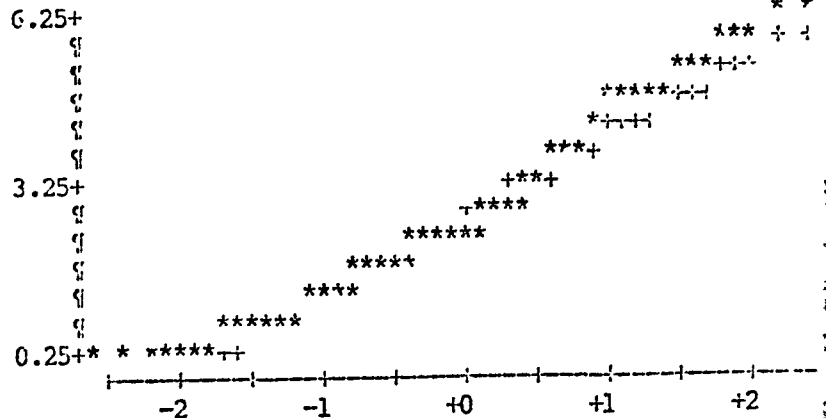
BOXPLOT

```

2 0
3 0
4 0
11 0
3 0
12 0
14 +-----+
17 0 0
27 +-----*
21 0 0
16 +-----+
13 0
7 0

```

NORMAL PROBABILITY PLOT



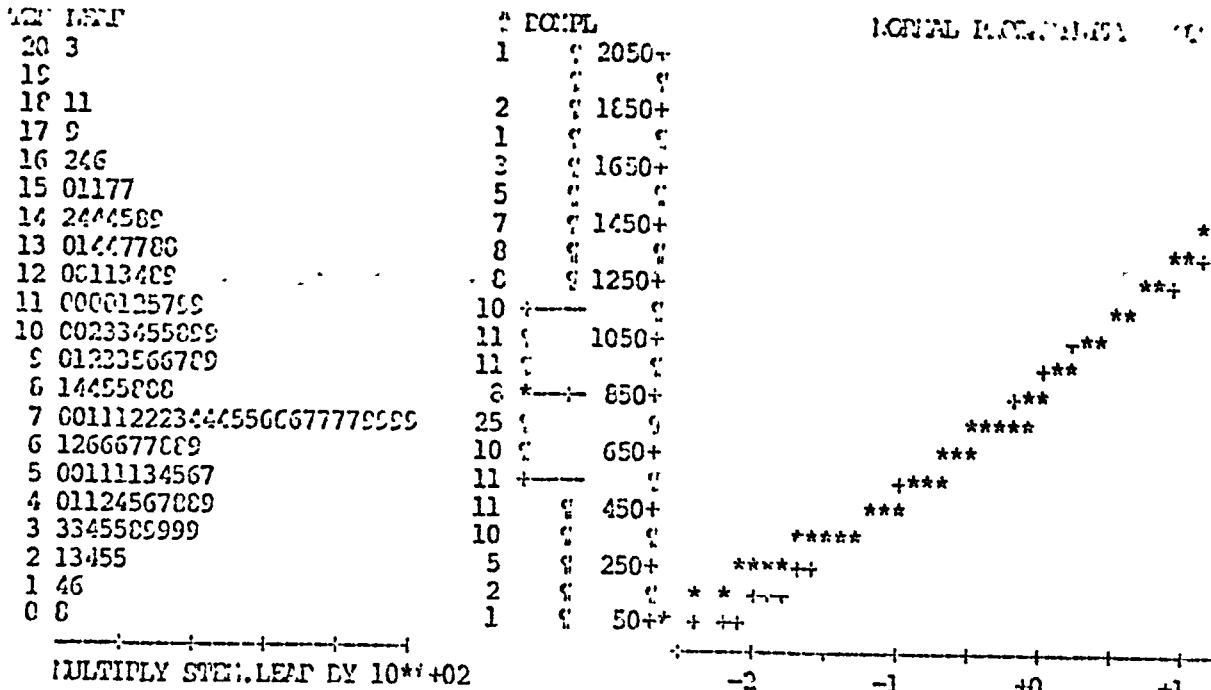
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VARIABLE=LV

MOMENTS		QUANTILES (DEF=4)					
N	150	SUM WGT	150	100% MAX	2029.6	99%	1918.67
MEAN	679.899	SUM	131965	75% Q3	1177.37	95%	1596.11
STD DEV	407.078	VARIANCE	165712	50% MED	602.1	90%	1453.14
KURTOSIS	0.37671	KURTOSIS	-0.428665	25% Q1	544.4	10%	309.15
MIN	140024399	CSS	24691150	0% MIN	83.6	5%	250.125
MAX	46.2642	STD LEAF	33.2378			1%	111.089
MEAN=0	26.4729	PROD: Q1	0.0001	RANGE	1946		
GM: PAK	5662.5	PROD: Q3	0.0001	Q3-Q1	632.975		
U: 0=0	150	PROD: MAX		MODE	83.6		
NONNOR	0.084998	PROD: D	10.01				

EXTREMES

LOWEST	HIGHEST
83.6	1655.4
137.5	1785.3
155.2	1809.5
213.4	1812.1
232.8	2029.6



The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

ALTITUDE - 14,000

VARIABLE=AU

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MOMENTS

QUANTILES(DEF=4)

N	150	SUM WGTs	150	100% MAX	7.76	99%	6.50538
MEAN	2.45373	SUM	368.06	75% Q3	3.265	95%	4.59
STD DEV	1.22695	VARIANCE	1.5054	50% MED	2.28	90%	3.96
SKEWNESS	0.738069	KURTOSIS	1.32473	25% Q1	1.61	10%	1.061
USS	1127.42	CSS	224.304	0% MIN	0	5%	0.6265
CV	50.0032	STD MEAN	0.10018			1%	0.153
TR:LEW=0	24.4933	PROB:VT	0.0001	RANGE	7.76		
SGN PAIR	5587.5	PROB:WS	0.0001	Q3-Q1	1.655		
MU: C= 0	149			MODE	2.28		
EMPIRICAL	0.0902606	PROB:D	10.01				

EXTREMES

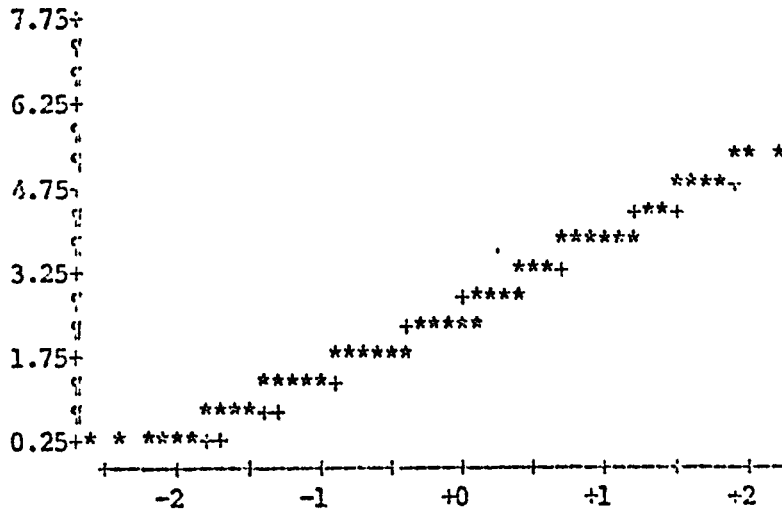
LOWEST	HIGHEST
0	4.79
0.3	5.03
0.41	5.2
0.45	5.3
0.45	7.76

STEM: LEAF

BOXPLOT

7 0	1 0
7	
6	
0	
5	
5 023	3
4 5577E	5
4 0001223	7
3 5557777788889999	17
3 0000111222223334	16
2 55556677999999	15
2 00000001112233333333333333444	20
1 55555556666777778888888999	29
1 0111111222333444	16
0 56677889	8
0 03444	5

NORMAL PROBABILITY PLOT



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VARIABLE=LU

STATISTICS		QUANTILES (DEF=4)					
N	150	SUM	135484	100% MAX	2294	99%	2122.28
MEAN	903.229	SUM	135484	75% Q3	1077.6	95%	1799.22
STD DEV	417.218	VARIANCE	174071	50% MED	857.45	90%	1528.01
STANDARD ERR	0.812246	KURTOSIS	0.412736	25% Q1	589.85	10%	422.14
USS	148309808	CSS	25936505	0% MIN	250.8	5%	303.975
CV	46.1918	STD MEAN	34.0657			1%	251.157
PERCENT	26.5143	PROB'ITY	0.0001	PROB	2643.2		
REL FREQ	5662.5	PROB'ITY	0.0001	Q3-Q1	487.75		
U. % = 0	150			MODE	250.8		
PROB'AL	0.0964196	PROB'D	0.01				

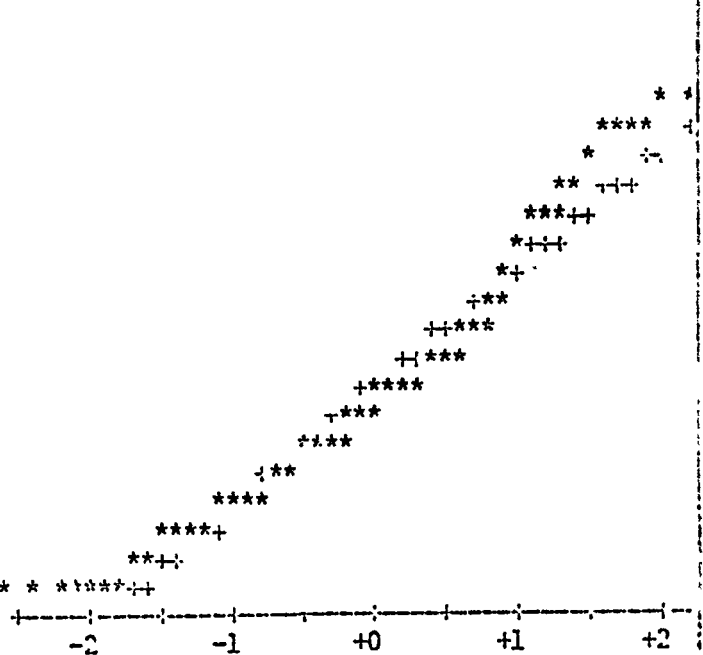
EXTREMS

LOWEST	HIGHEST
250.8	1839.3
251.5	1851.8
267.1	1929
267.2	1957.3
290.1	2294

REL. FREQ	COUNT OF
22 9	1 0 2250+
21	0
20	0
19 36	2 0 2050+
18 1245	4 0
17 8019	4 0 1850+
16 7	1 0
15 02345	5 0 1650+
14 688	3 0
13 24689	5 0 1450+
12 5068	4 0
11 612489	7 0 1250+
10 1223355677888	13 0
9 09122223345556667789	20 0 1050+
8 112233556688889	13 0
7 00011223334446678889	21 0 850+
6 055555788	9 0
5 2223456667799	13 0 650+
4 122233336789	13 0
3 0011688	7 0 450+
2 55779	5 0

MULTIPLY STD. LEV BY 10**-.02

NORMAL PROBABILITY PLOT



VARIABLE=AV

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STATISTICS

QUANTILES (DEF=4)

N	150	SUM MEANS	150	100% MAX	8.75	99%	8.42869
MEAN	2.8624	SUM	429.36	75% Q3	3.58	95%	5.929
STD DEV	1.5036	VARIANCE	2.26141	50% MED	2.77	90%	4.84
SKEWNESS	0.952836	KURTOSIS	1.8421	25% Q1	1.885	10%	1.041
USS	1565.95	CSS	336.951	0% MIN	0	5%	0.7665
CV	52.5363	STD LEM	0.122785			1%	0.0407999
T:MEAN=0	23.3123	PROB LTY	0.0001	RANGE	8.75		
SCN RAN:	5587.5	PROB WGT	0.0001	Q3-Q1	1.695		
.UM, S=0	149			LCDF	1.95		
D:NONPAR	0.0052261	PROB D	16.01				

STATISTICS

LO EST	HIGHEST
0	6.05
0.06	6.26
0.25	7.21
0.46	6.12
0.63	6.75

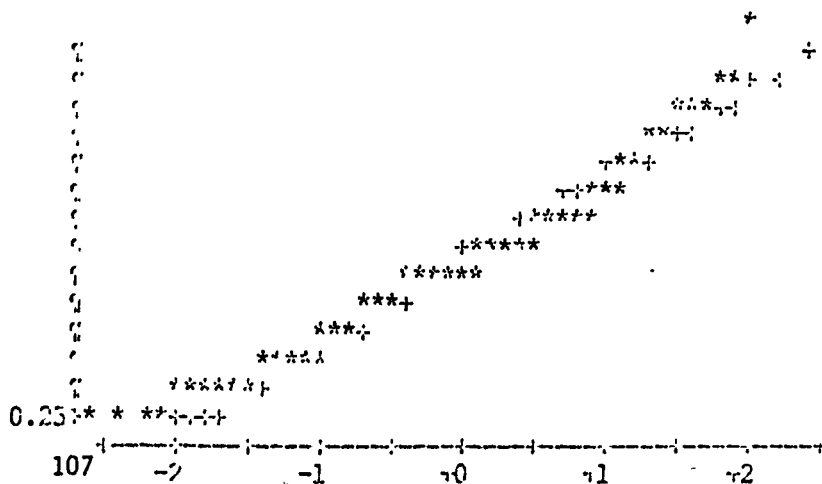
DATA LIST

EXAM PLOT

0 7
 0 1
 7
 7 2
 6
 6 03
 5 6009
 5 0112
 4 66789
 4 01112334
 3 555566667777888899
 3 00000011112233445566
 2 55555566666777778888899999
 2 0000011112222333334
 1 5555666777888999
 1 00001112223344
 0 56778899
 0 012

17
 21
 26
 17
 16
 13
 9
 3

NONPAR MONTECARLO PLOT



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MOIENS

QUANTILES (DEF=4)

N	150	SUM WGT	150	100% MAX	1996.4	99%	1988.75
MEAN	858.725	SUM	128809	75% Q3	1083.62	95%	1566.57
STD DEV	373.779	VARIANCE	139711	50% MED	826.65	90%	1385.02
SQUARES	0.056117	MOSIOSIS	0.273074	25% Q1	584.825	10%	421.8
JSS	131428326	CS	20816946	0% MIN	143.7	5%	343.3
CV	43.5272	STD MEAN	30.519			1%	161.346
COEFF-0	28.1374	PROB-0	0.0001	REL SE	1852.7		
CH RISK	5062.5	EXPRO-0	0.0001	CS-Q1	498.8		
TH = 0	150			PROB	642		
PROB-0	0.0617395	PROB-D	0.15				

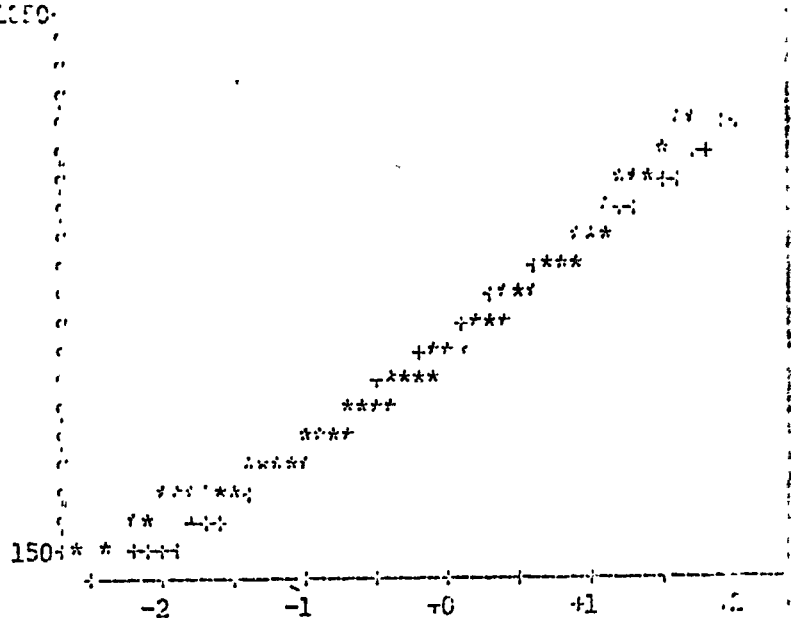
EXPER MS

LO SET	HIGHEST
173.7	1044.2
178.3	1741.5
222.5	1830
277.8	1981.4
323.3	1996.4

REL. MEAN

20 0	1	100%
15 8	2	
11 4	3	
17 4	4	
16 12	5	
15 23	6	
14 24678	7	
13 7782	8	
12 0234075	9	
11 0144457899	10	
10 1119336678899	11	
9 001334666779	12	
8 0002444556667788	13	
7 011732246678888	14	
6 004445667889999	15	
5 00444555668899	16	
4 002233246778889	17	
3 24445579	18	
2 24	19	
1 48	20	150%

NORMAL PROBABILITY PLOT



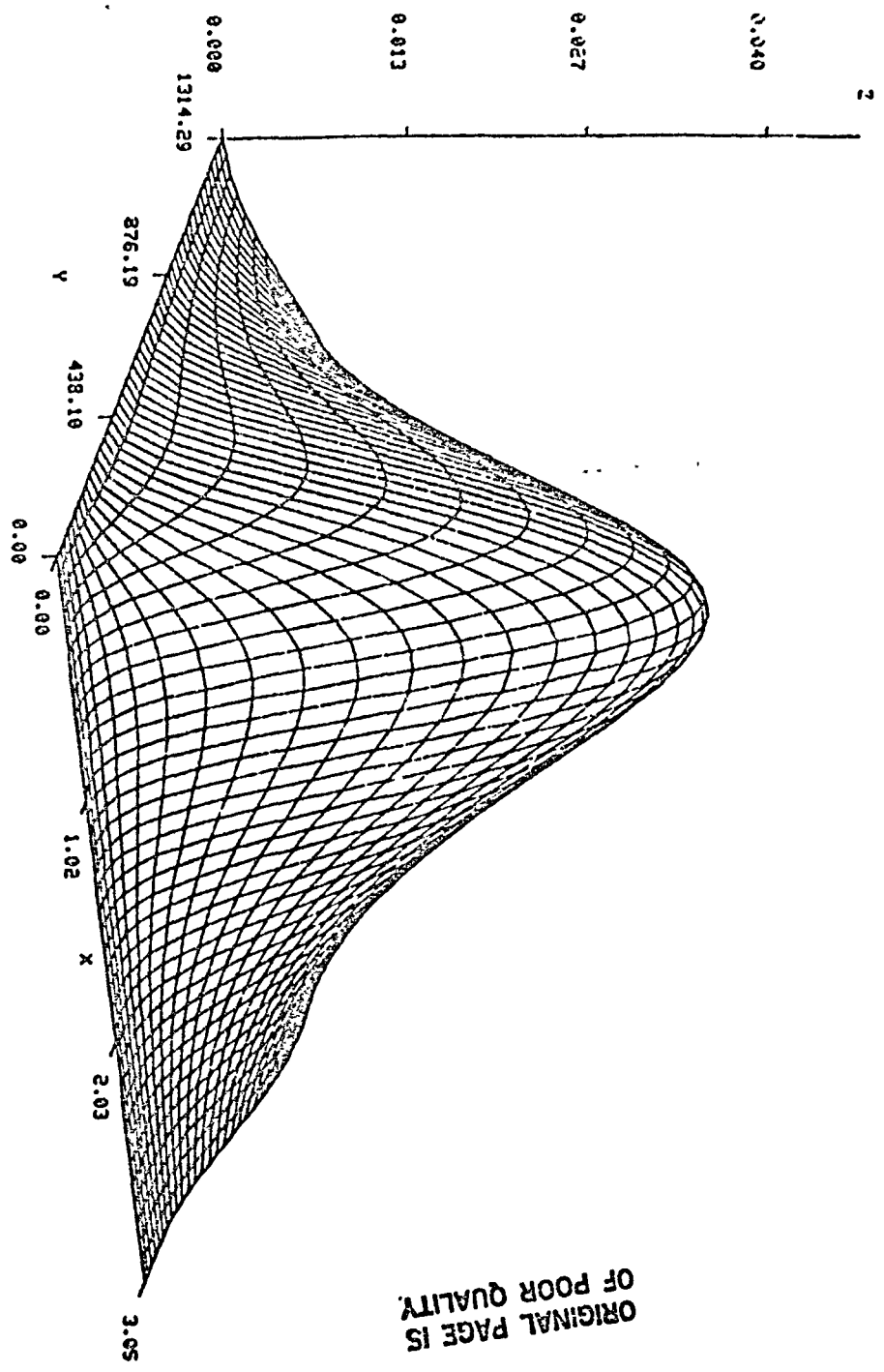
PROB-0 0.0617395 BY 10**+02

APPENDIX B

Plots for the bivariate gamma density function, scatter plots, and contours are given for each data set.

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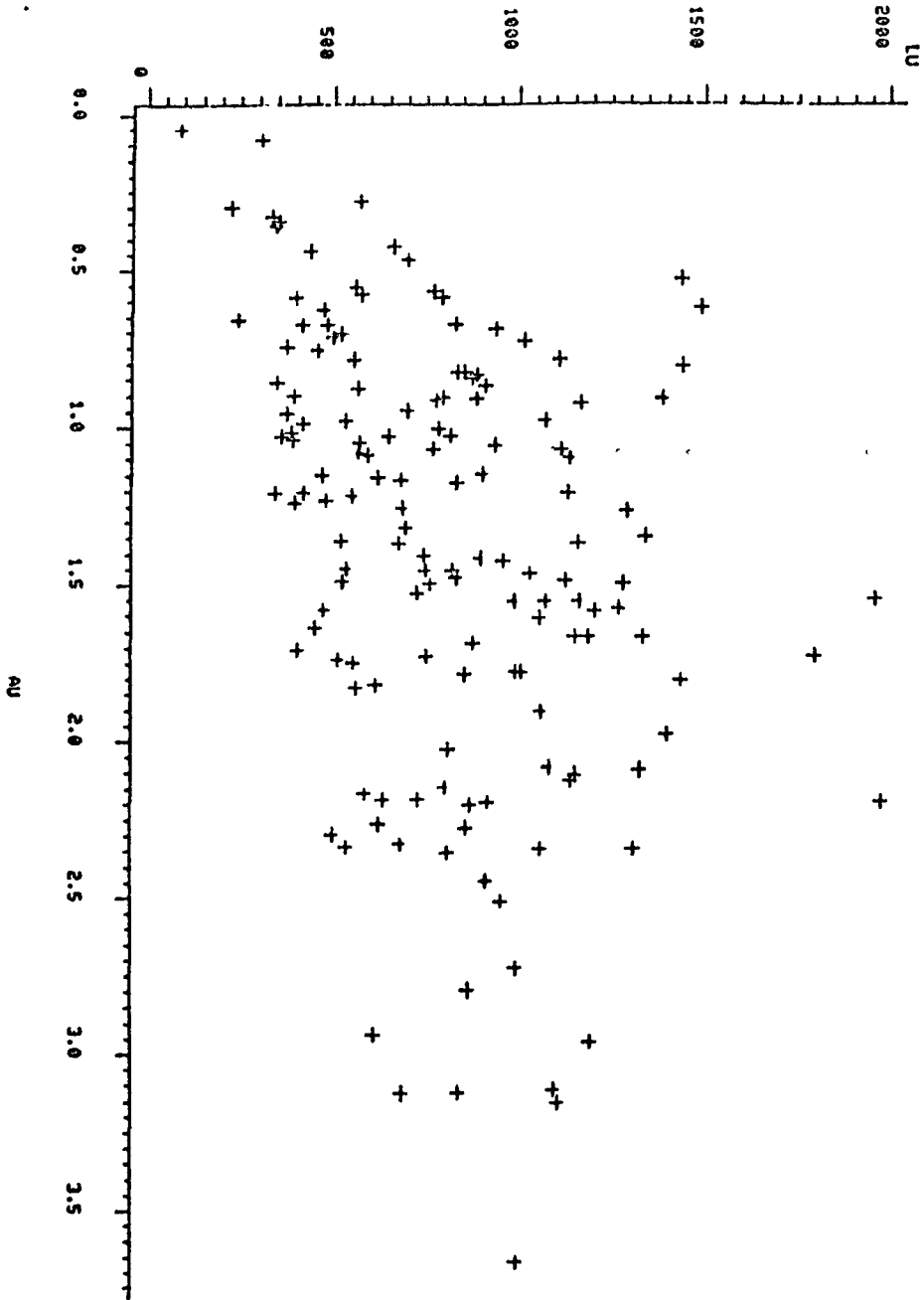
BIVARIATE GAMMA DENSITY
VARIABLES (A,U,L,U)
ALTITUDE = 4000



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SCATTER PLOT OF RAW DATA

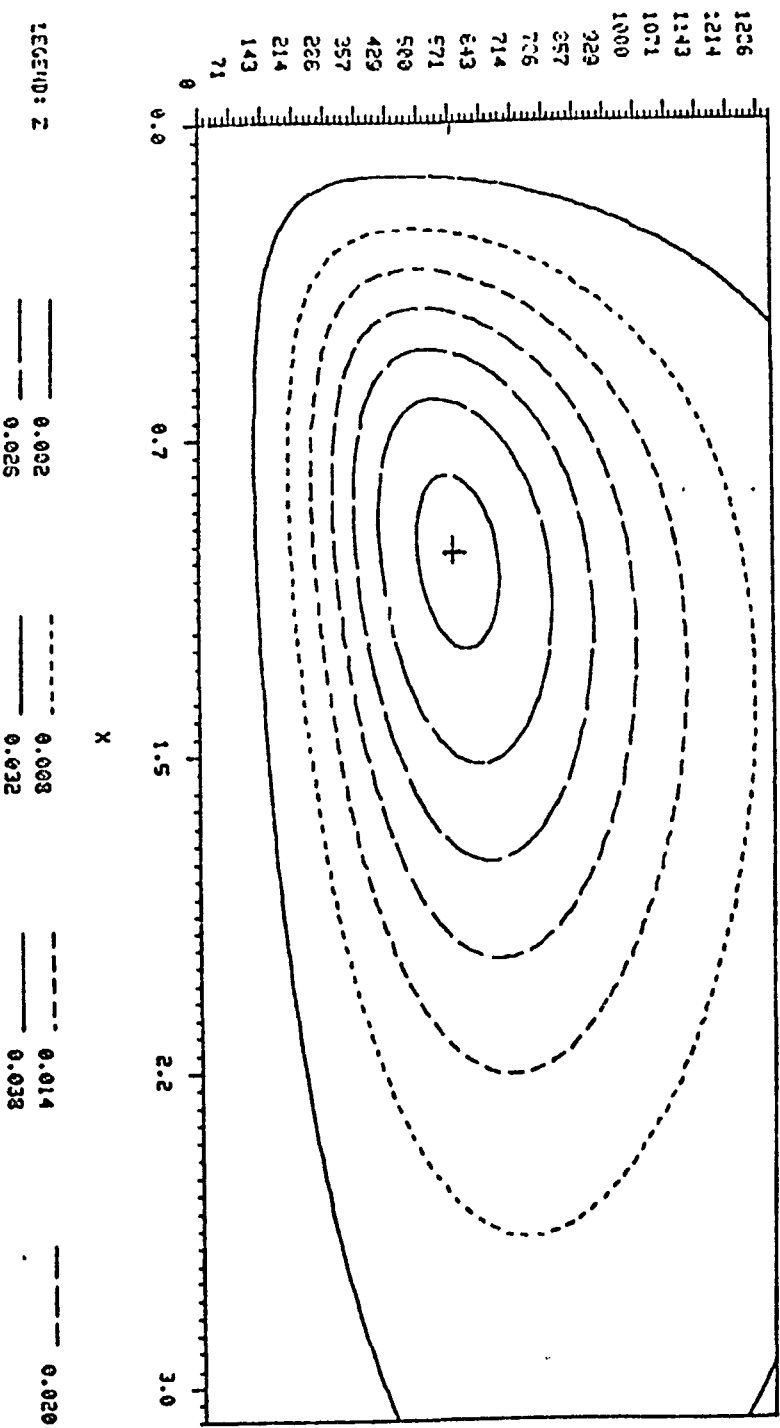
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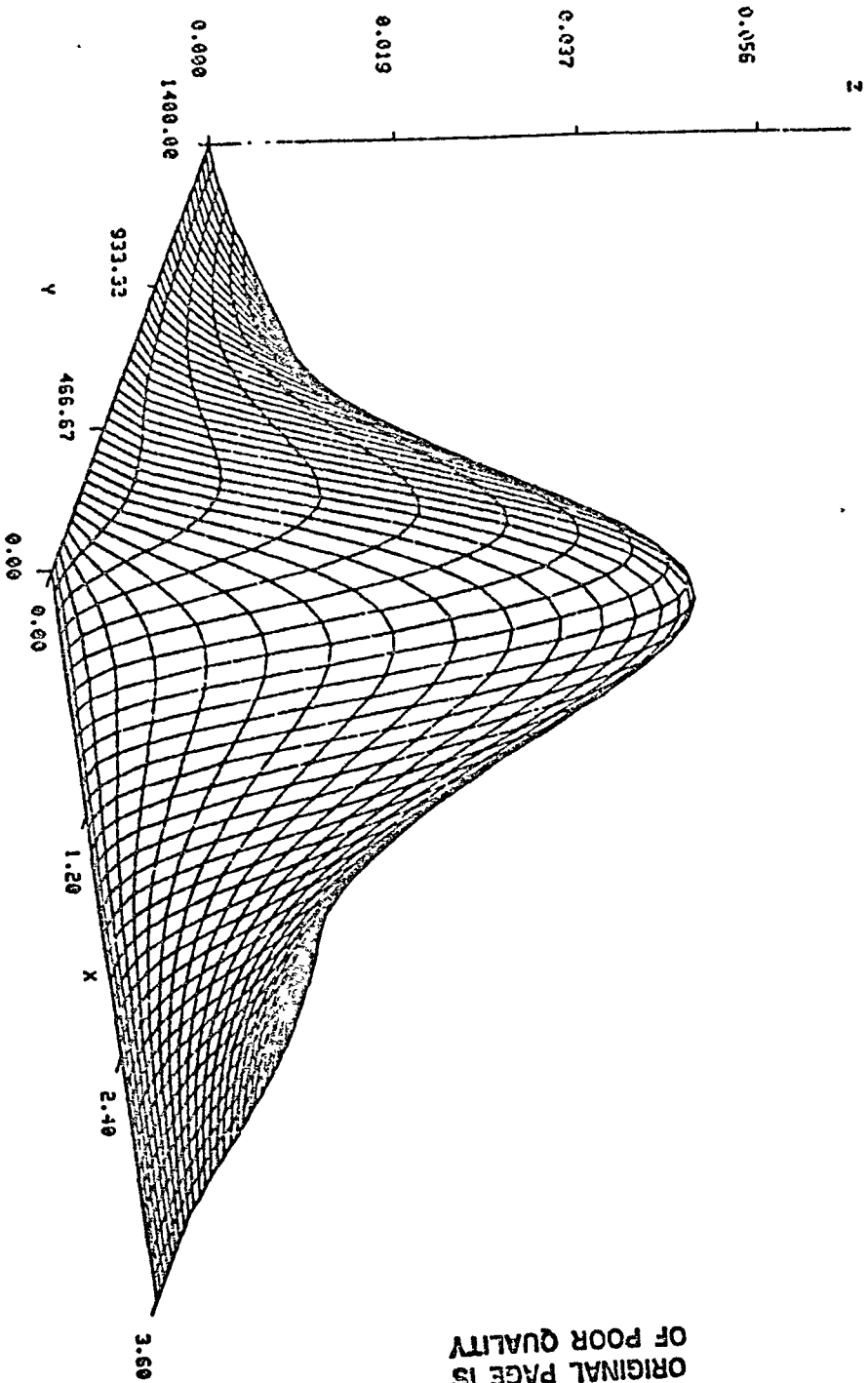
CONTOUR PLOTS

VARIABLES (AULLU)
ALTITUDE = 4000



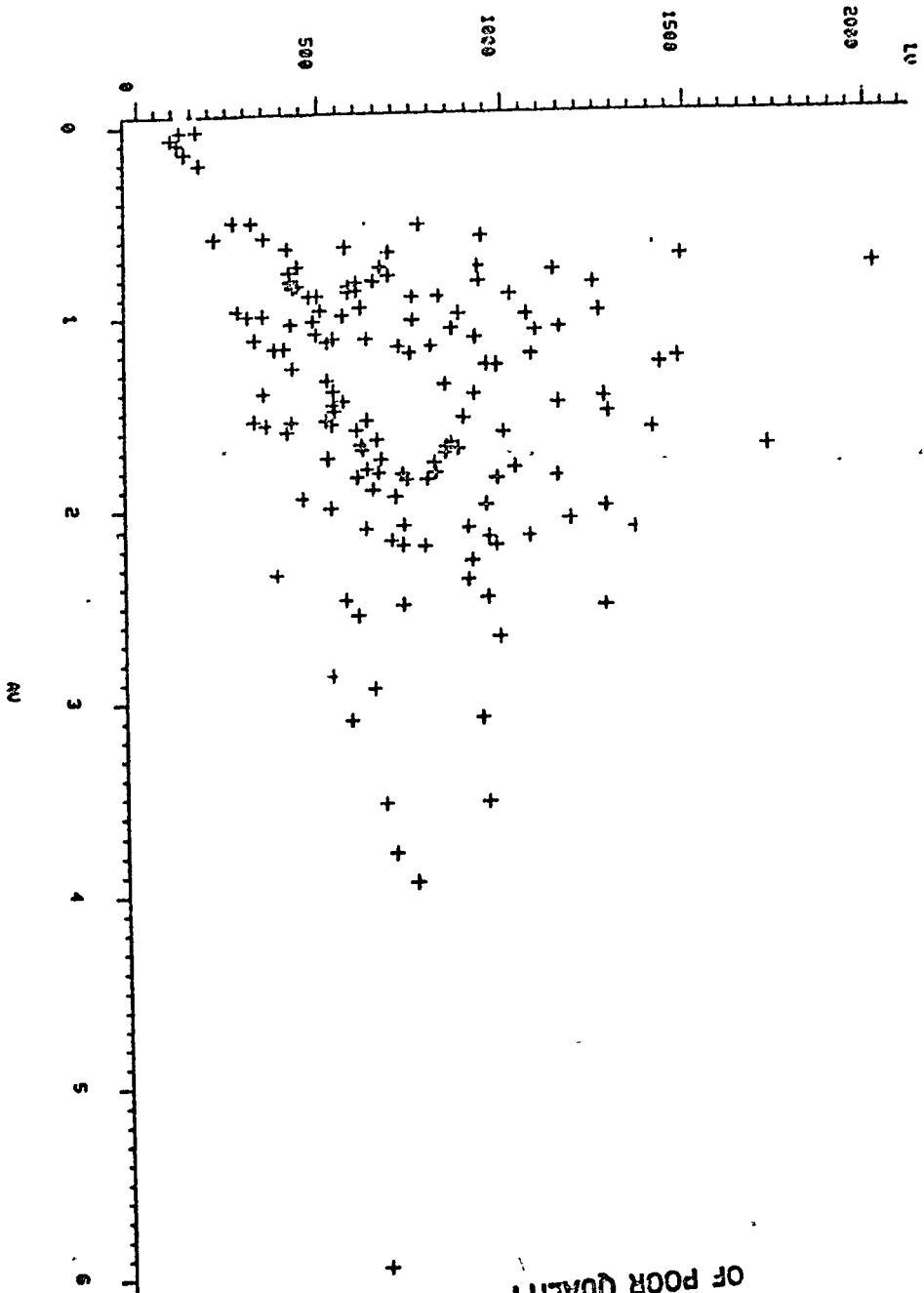
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VARIABLES (AV, LV)
ALTITUDE = 4000



SCATTER PLOT OF RAW DATA

REFERENCE ALTITUDE = 4000

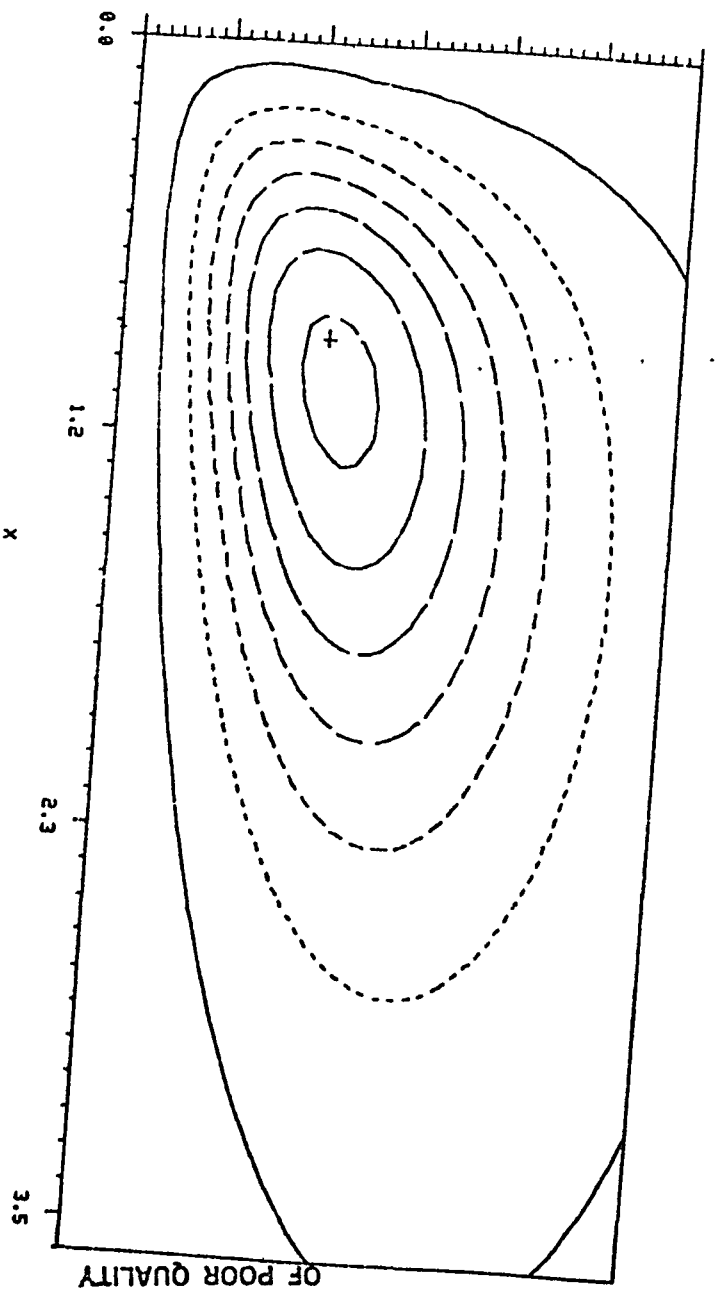


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CONTOUR PLOTS VARIABLES (AV/LV) ALTITUDE = 4000

- 1400.0000
- 1166.6667
- 933.3333
- 700.0000
- 466.6667
- 233.3333
- 0.0000

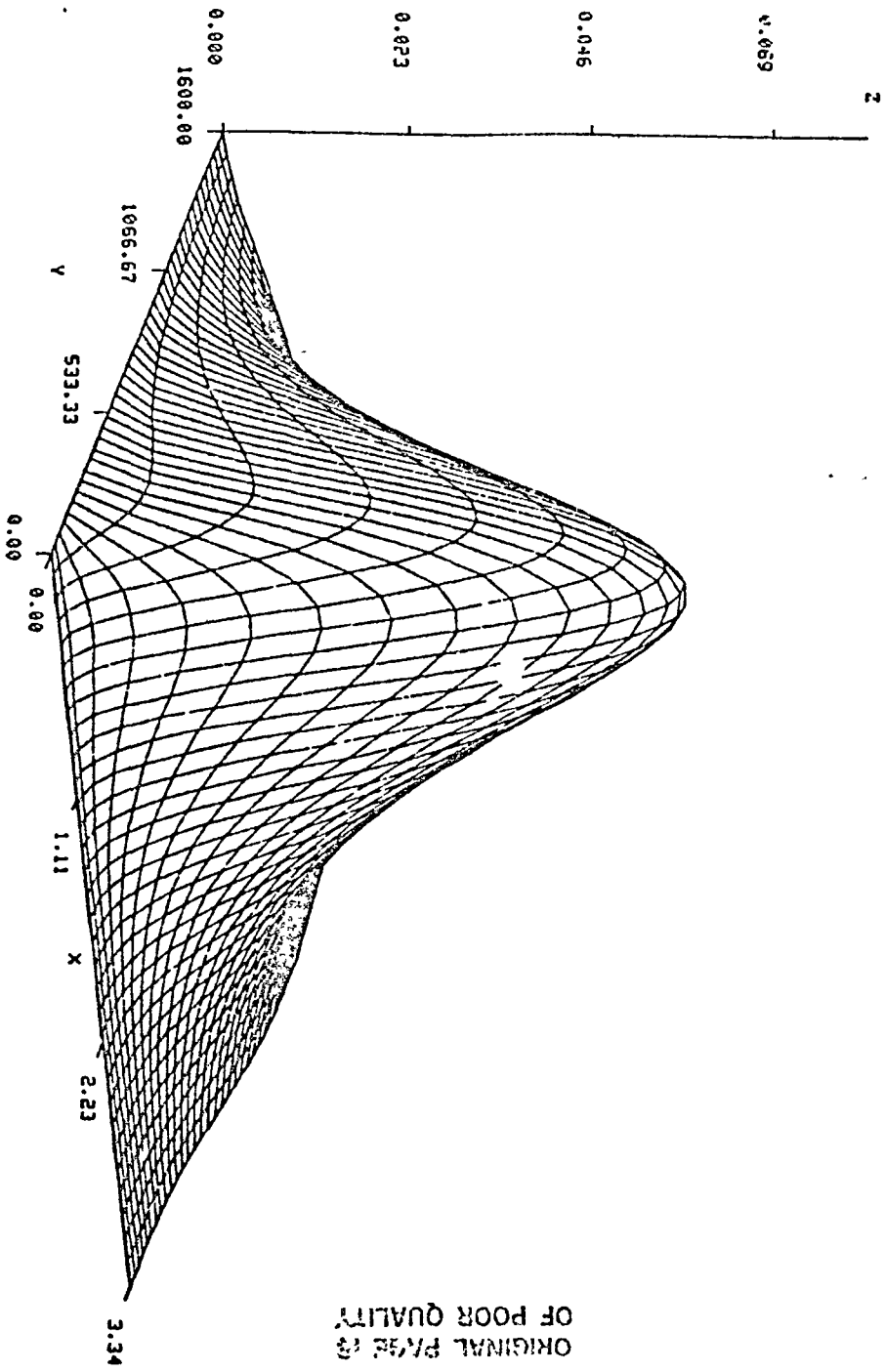
LEGEND: 2



- 0.003
- 0.036
- 0.011
- 0.044
- 0.019
- 0.053
- 0.028

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BIVARIATE GAMMA DENSITY
 VARIABLES (A,U,U)
 ALTITUDE = 6000

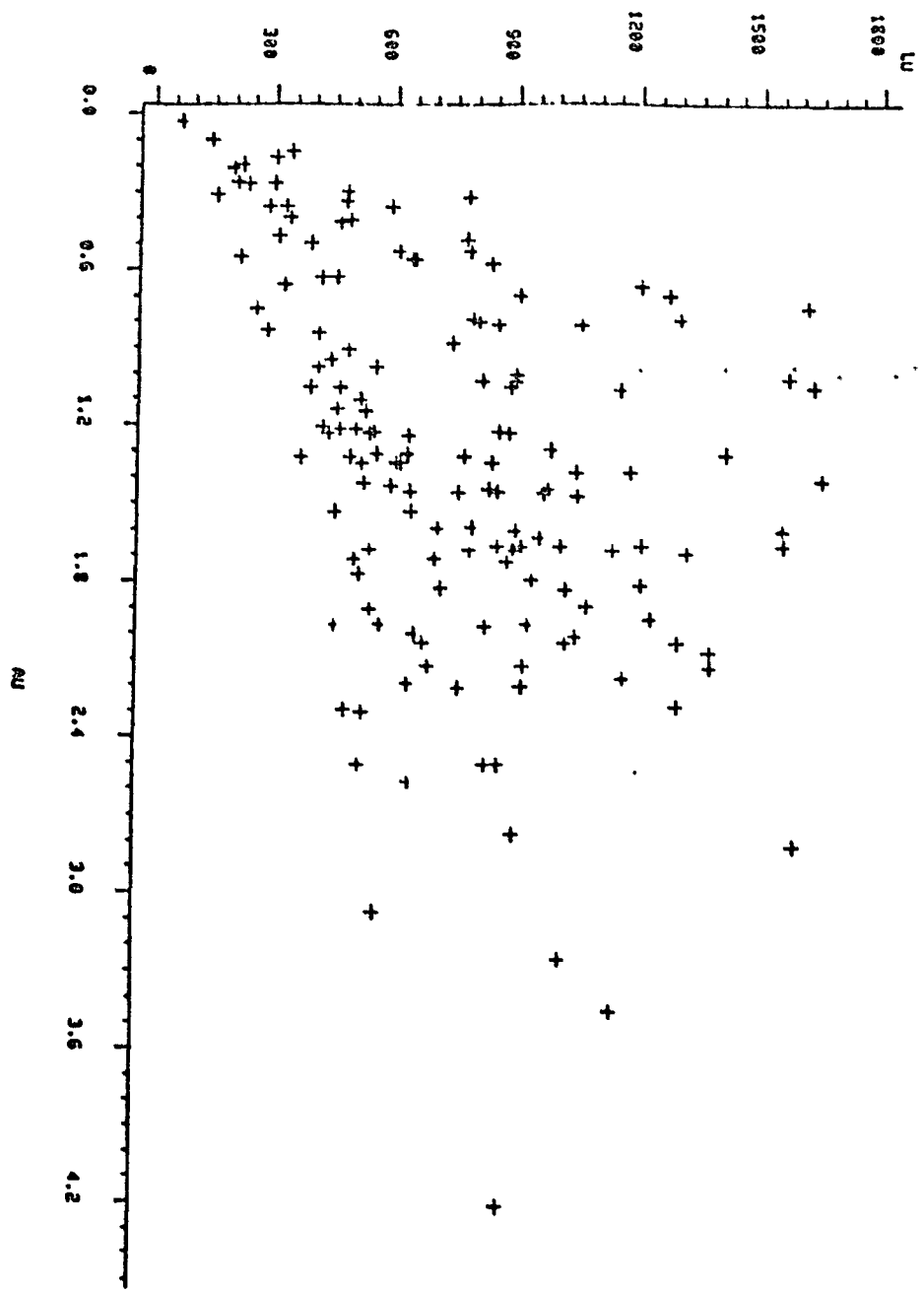


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SCATTER PLOT OF RAW DATA

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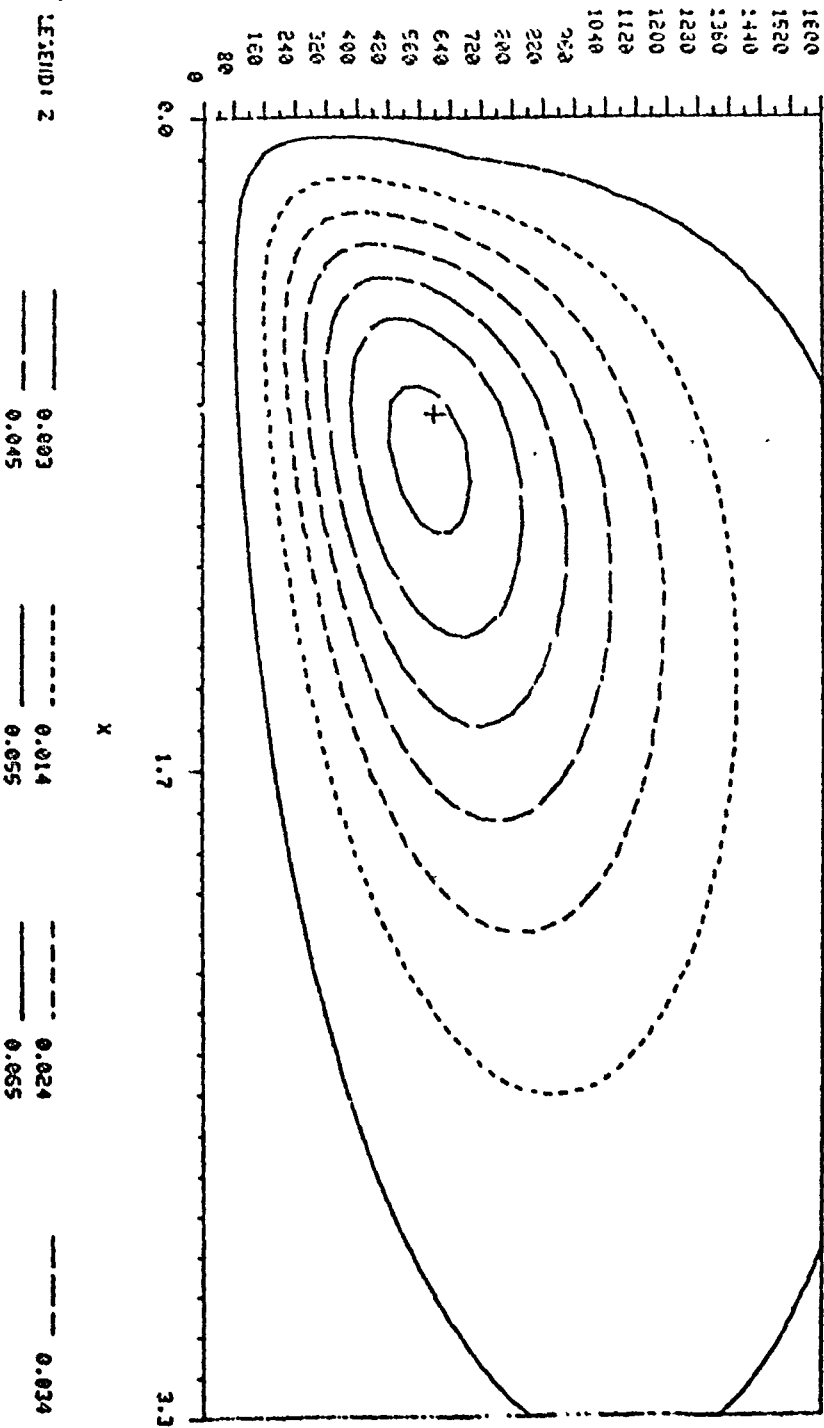


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CONTOUR PLOTS

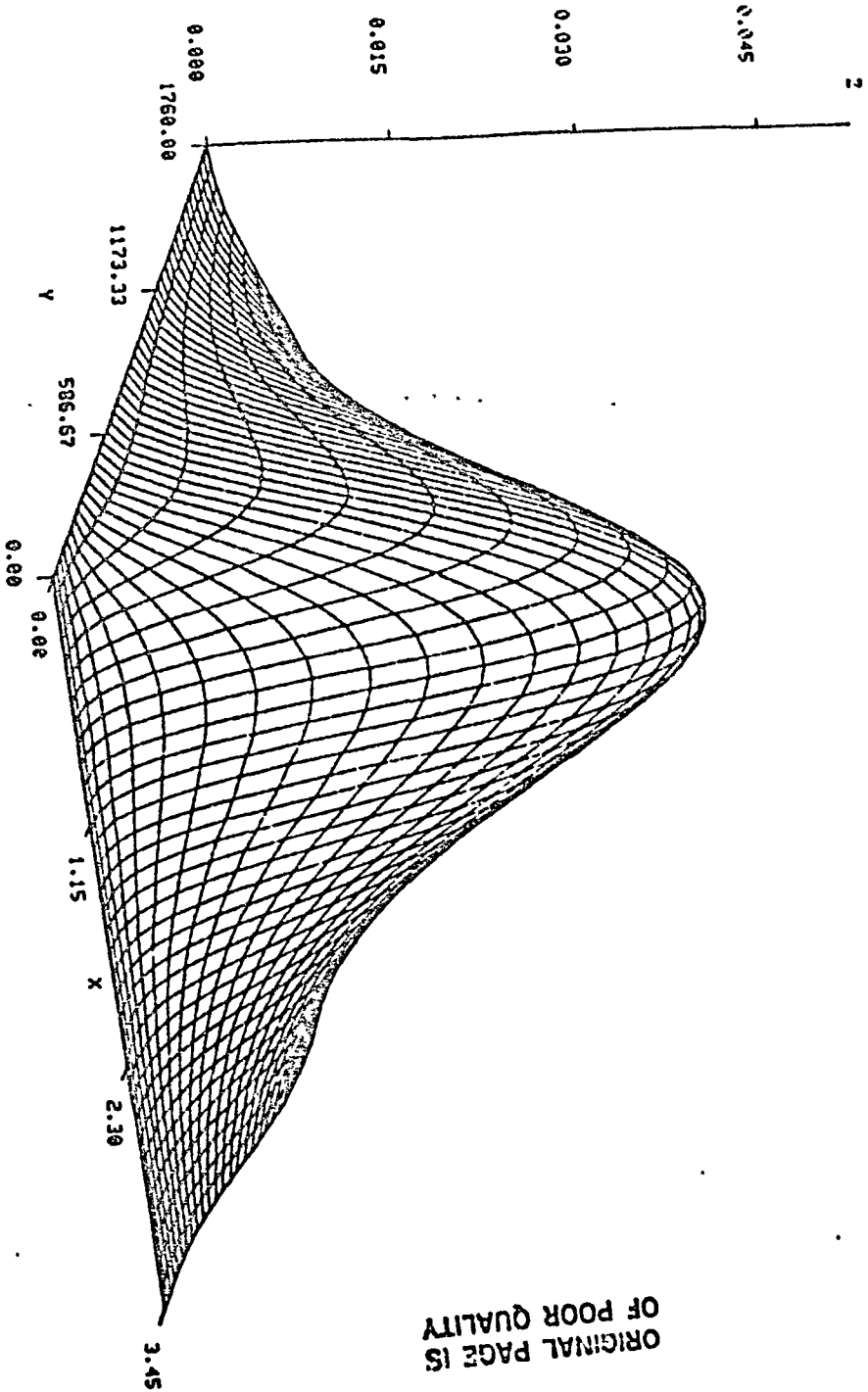
VARIABLES (A₁, U)

ALTITUDE = 6000



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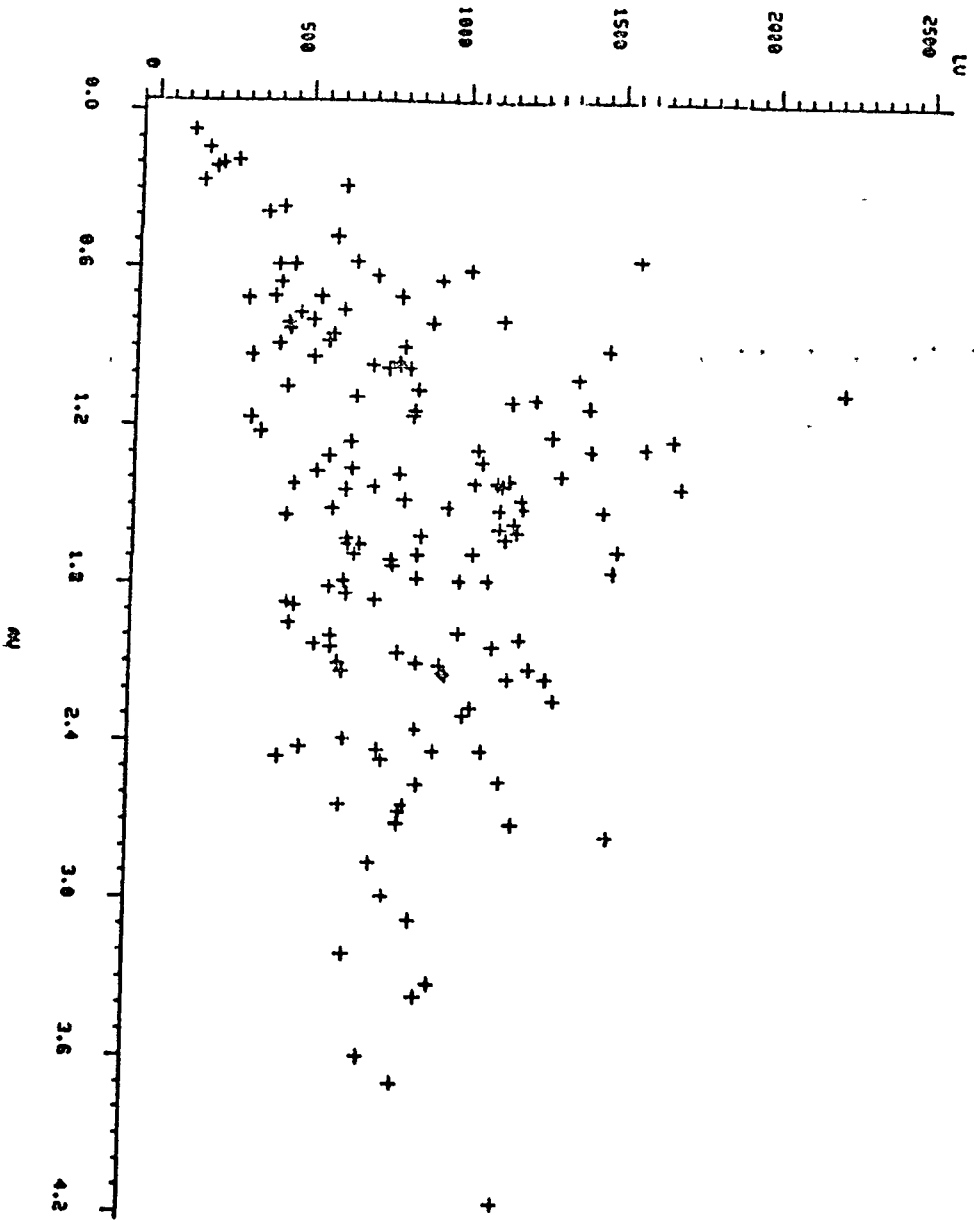
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VARIABLES (AV, LV)
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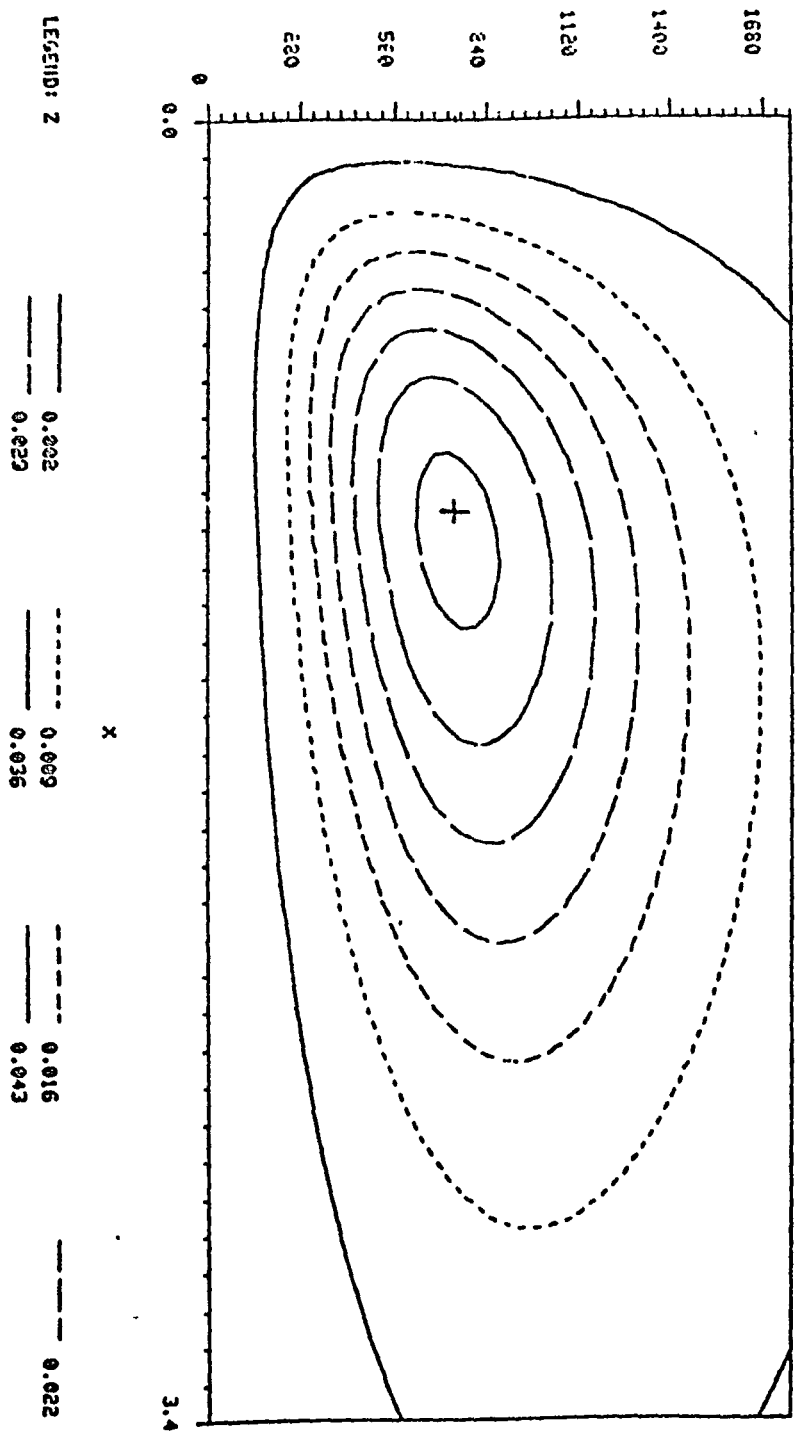
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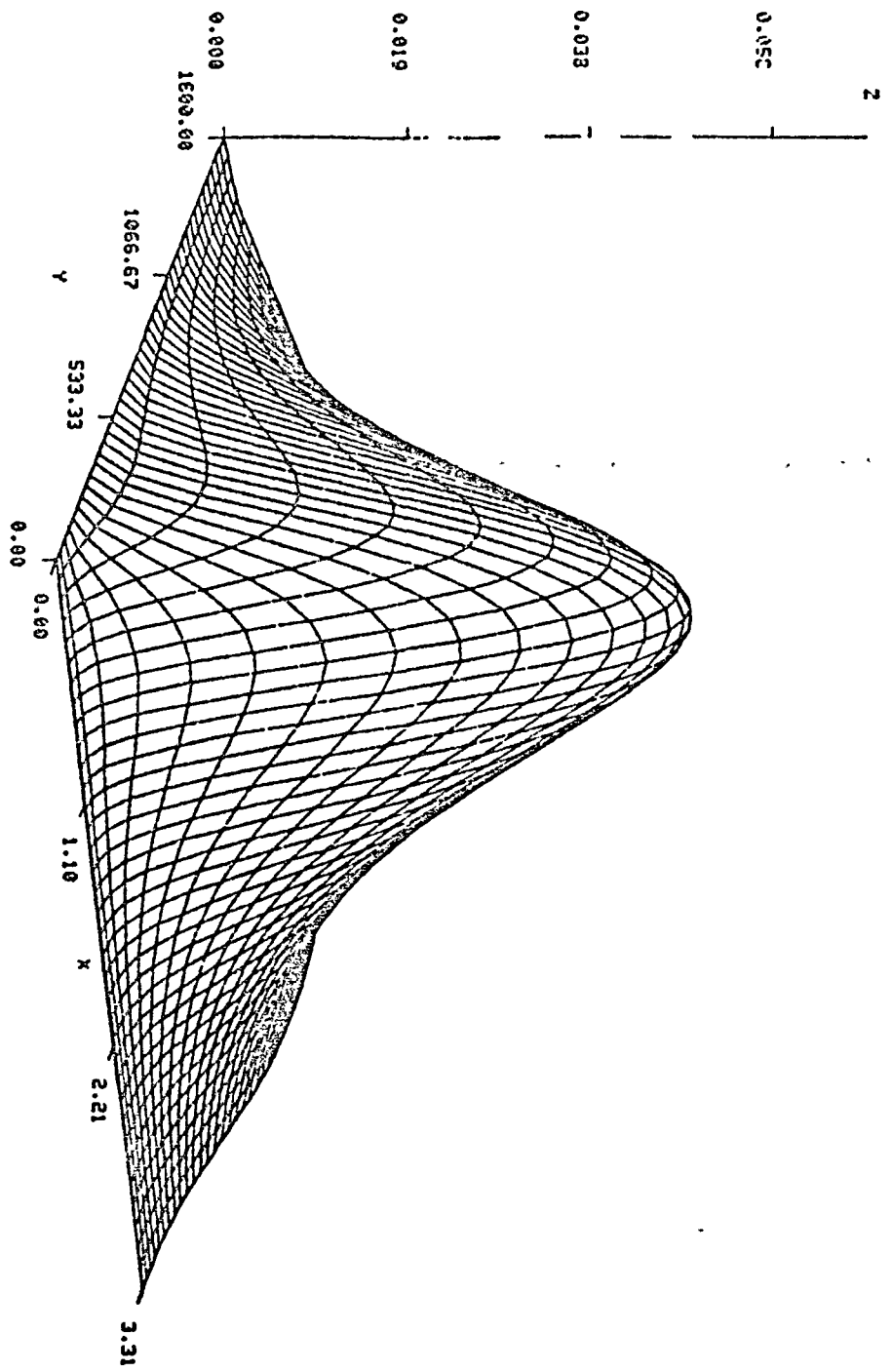
CONTOUR PLOTS

VARIABLES (AV, LV)
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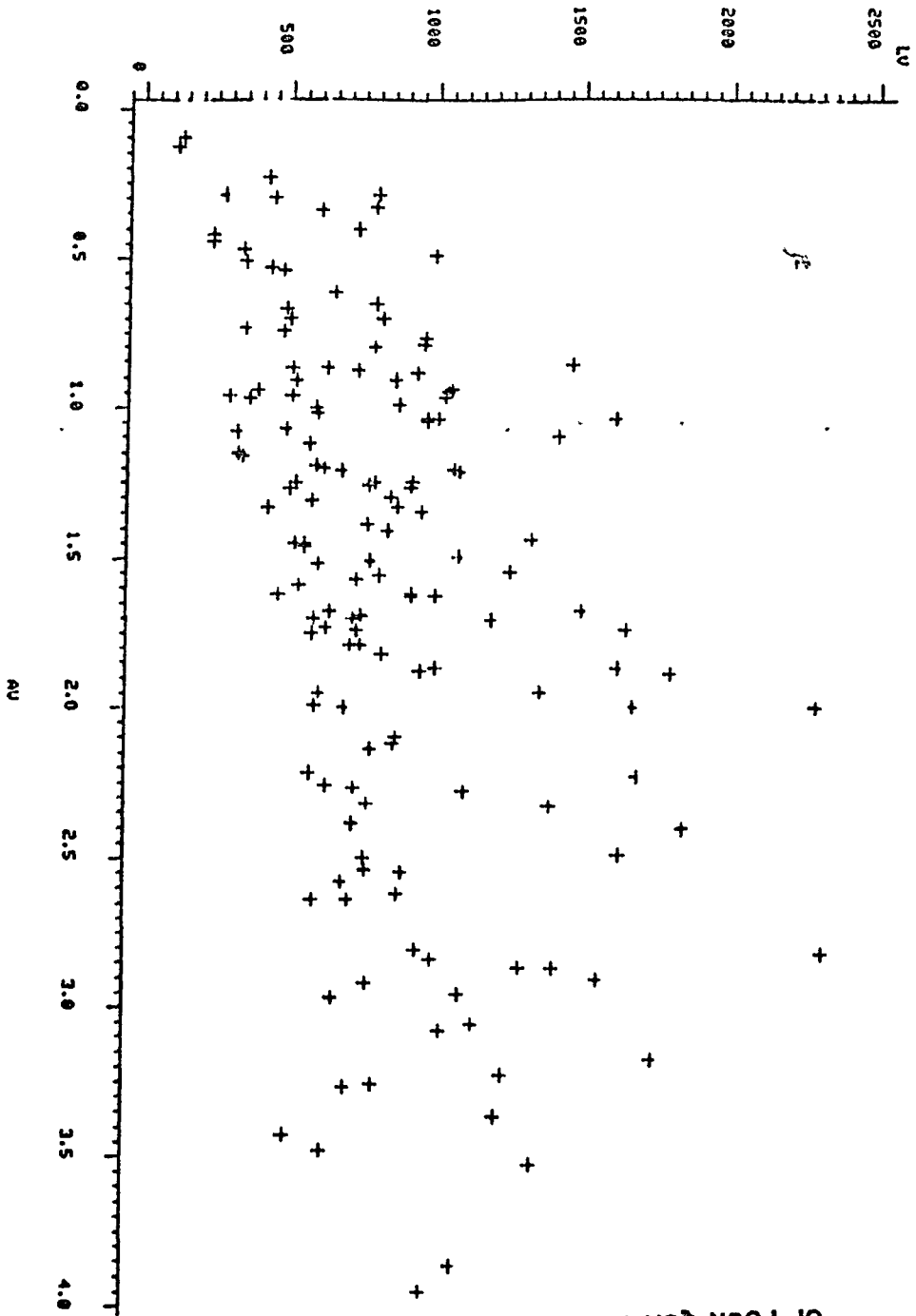
BIVARIATE GAMMA DENSITY
VARIABLES (AU,LU)
ALTITUDE = 8000



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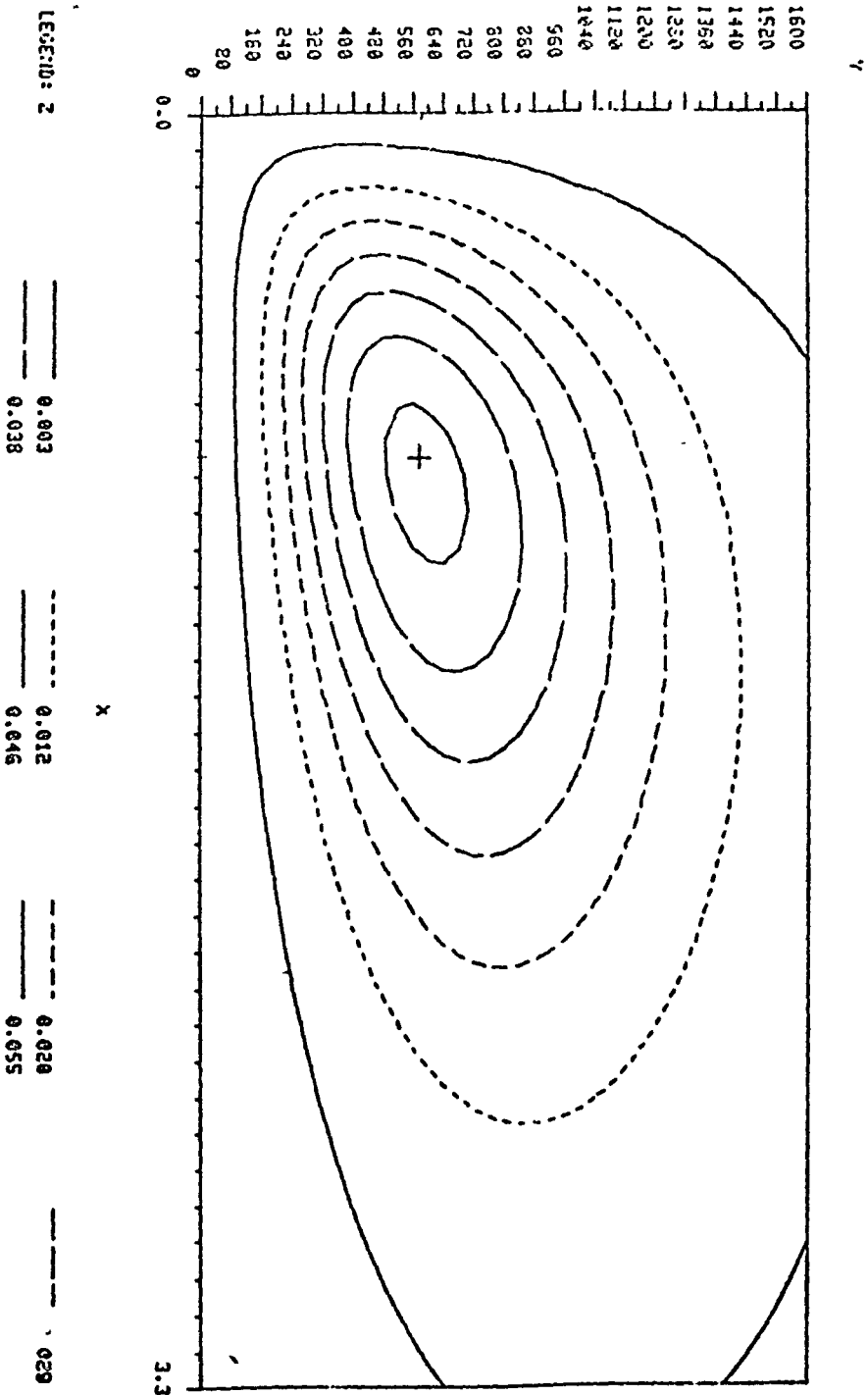
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CONTOUR PLOTS

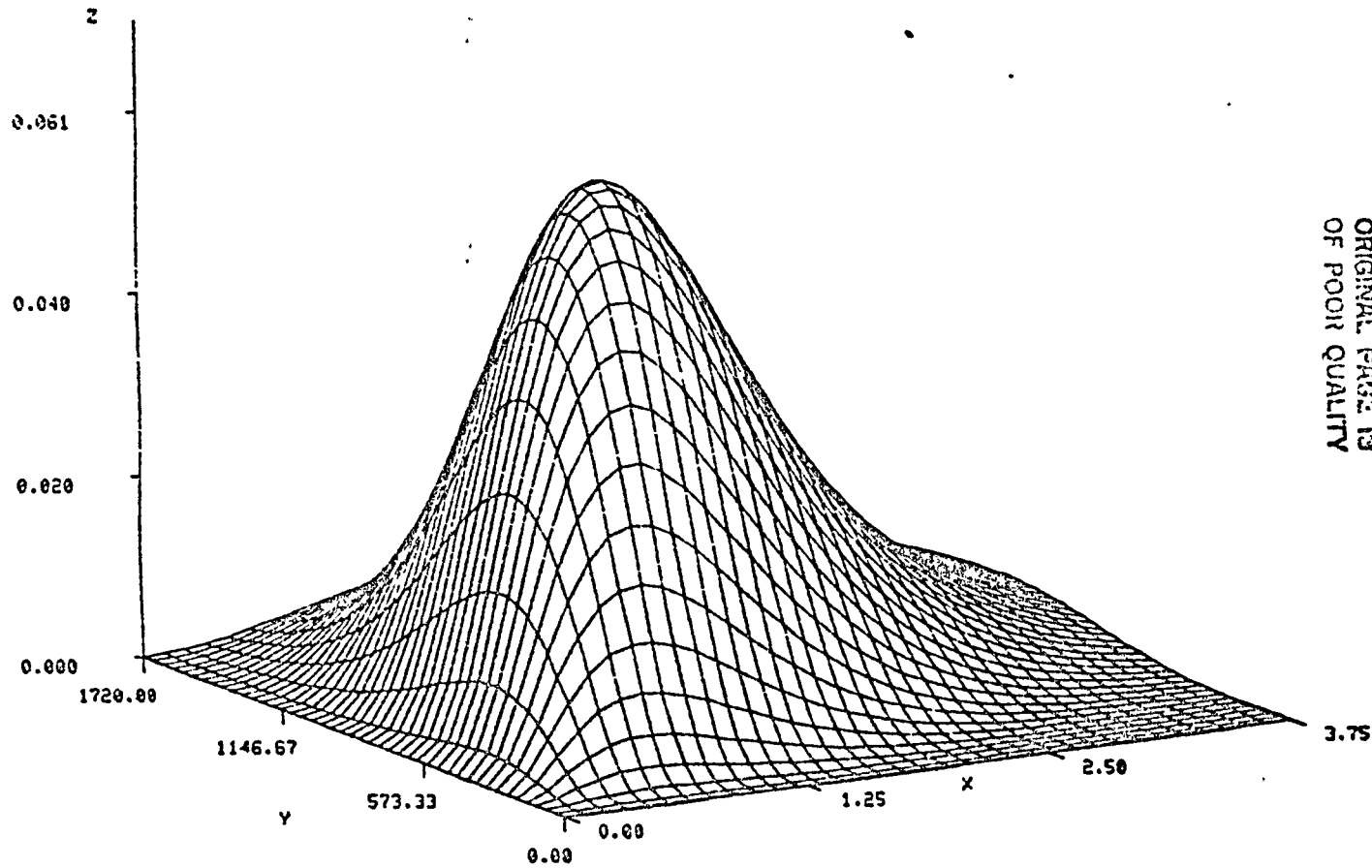
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BIVARIATE GAMMA DENSITY

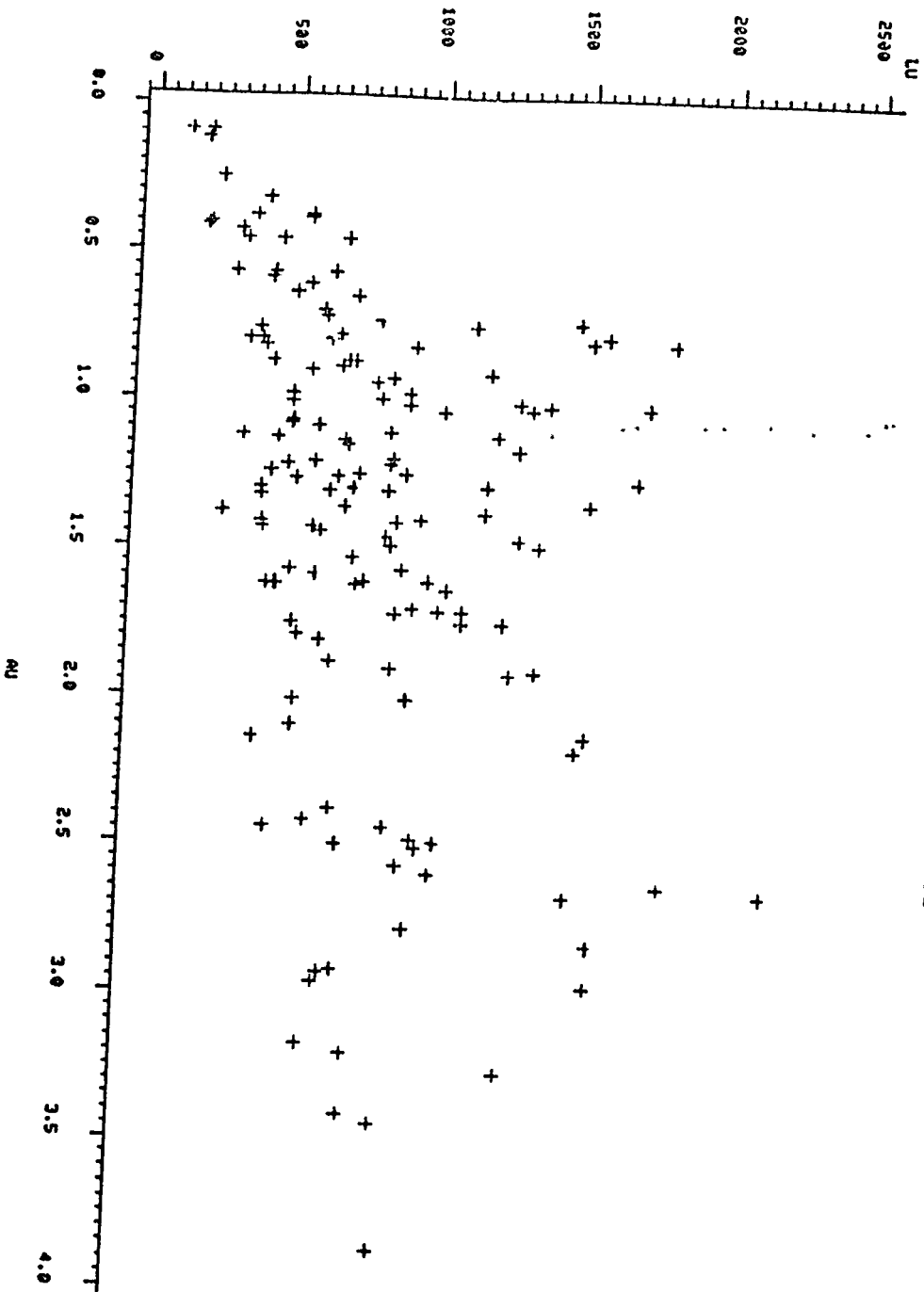
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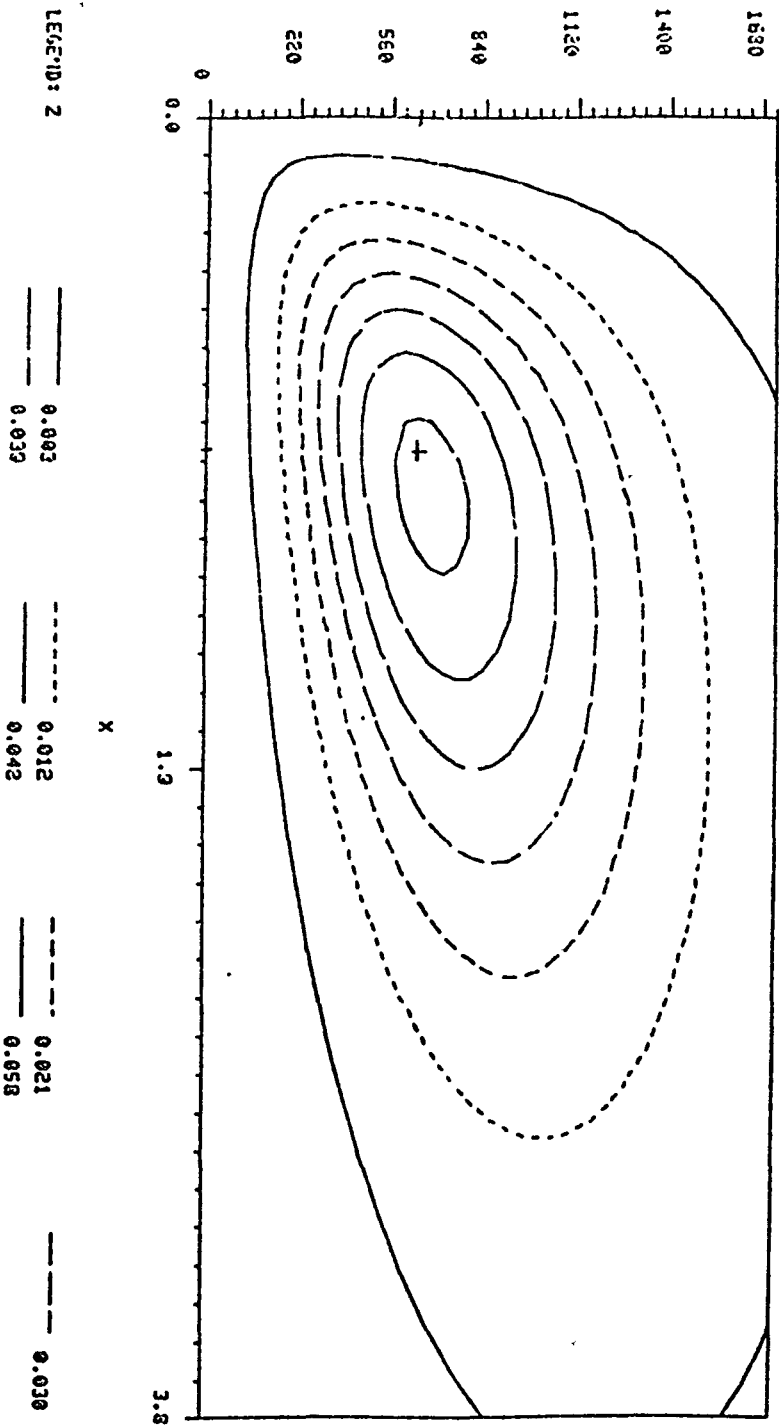
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CONTOUR PLOTS

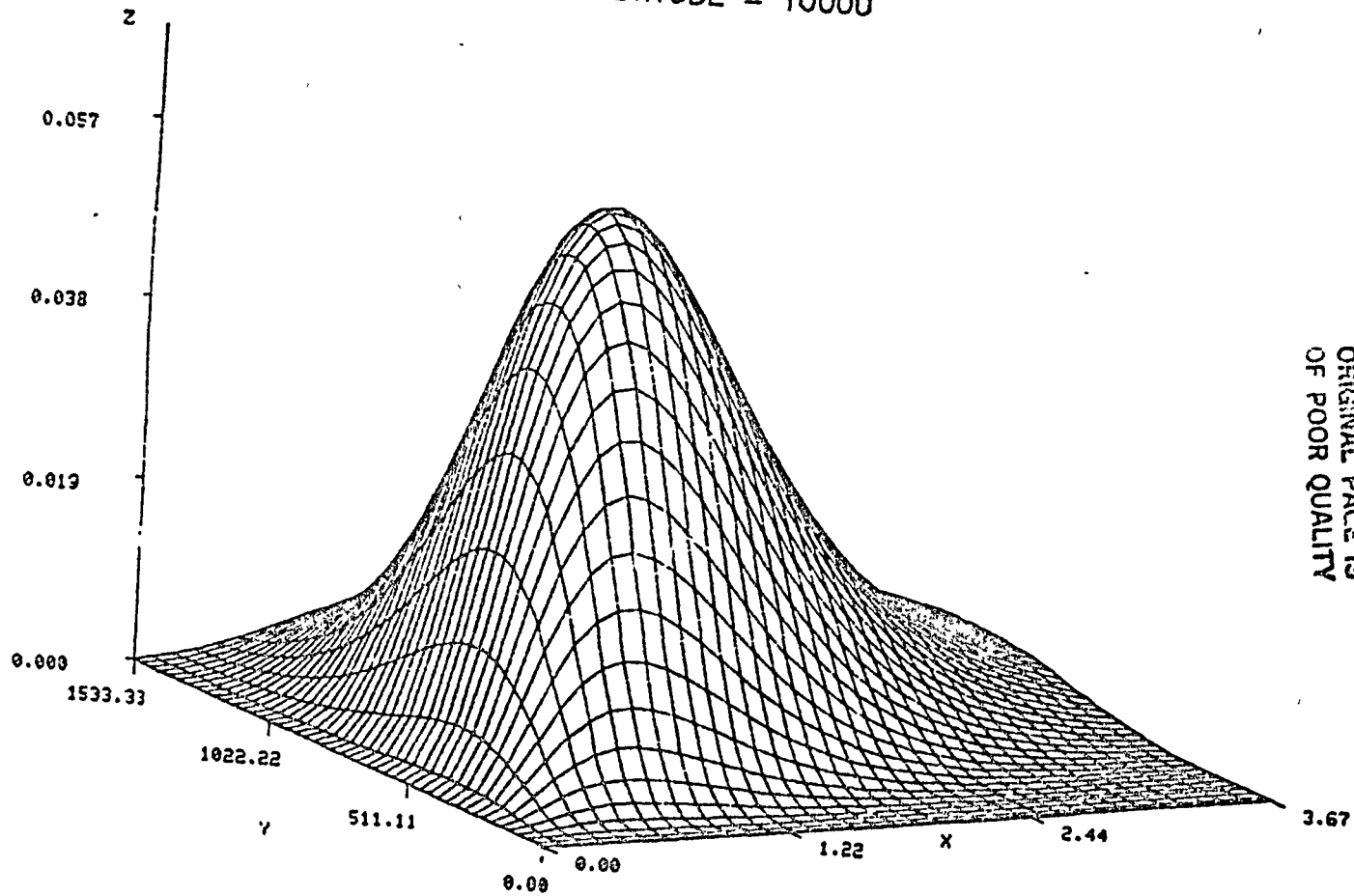
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BIVARIATE GAMMA DENSITY

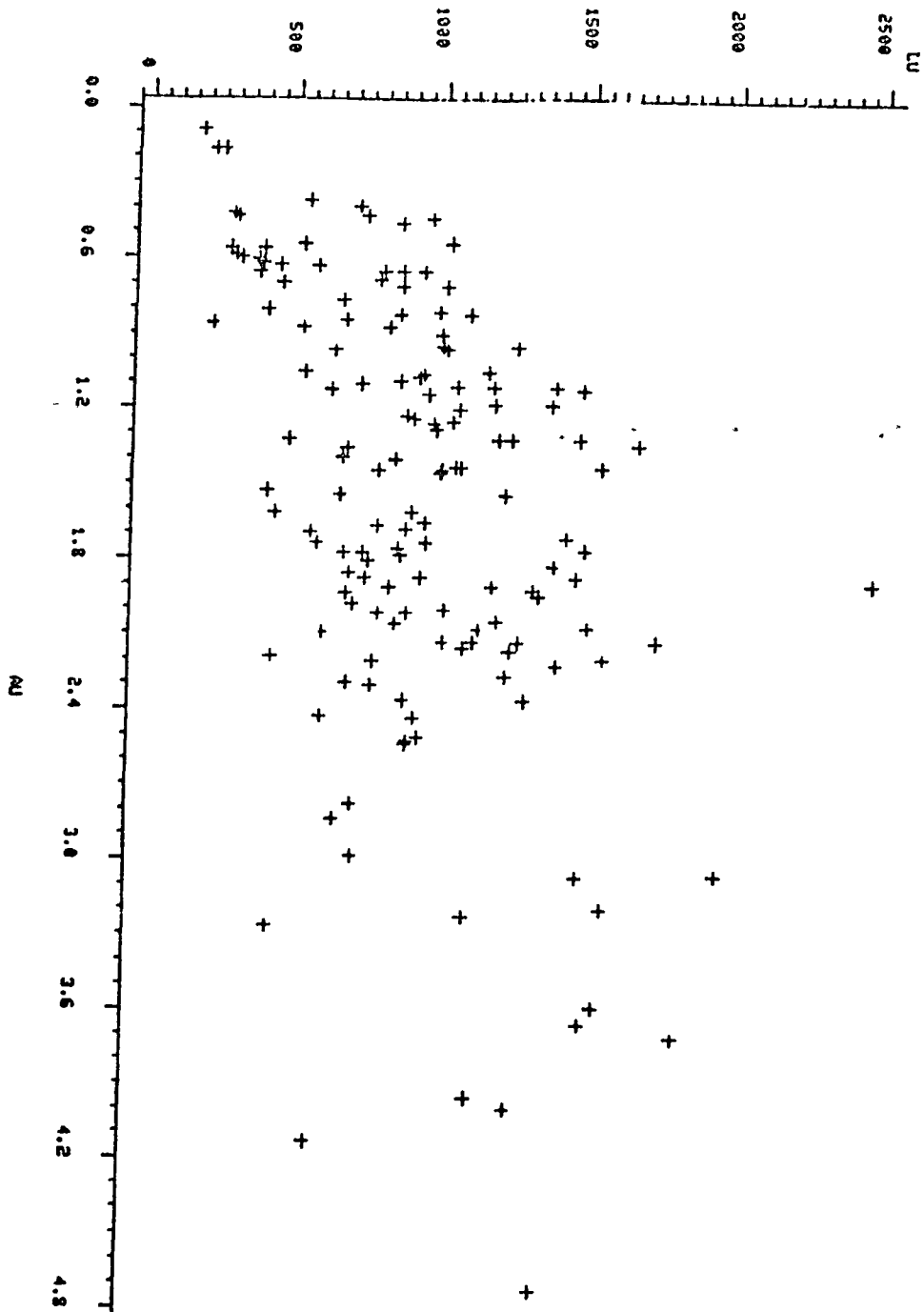
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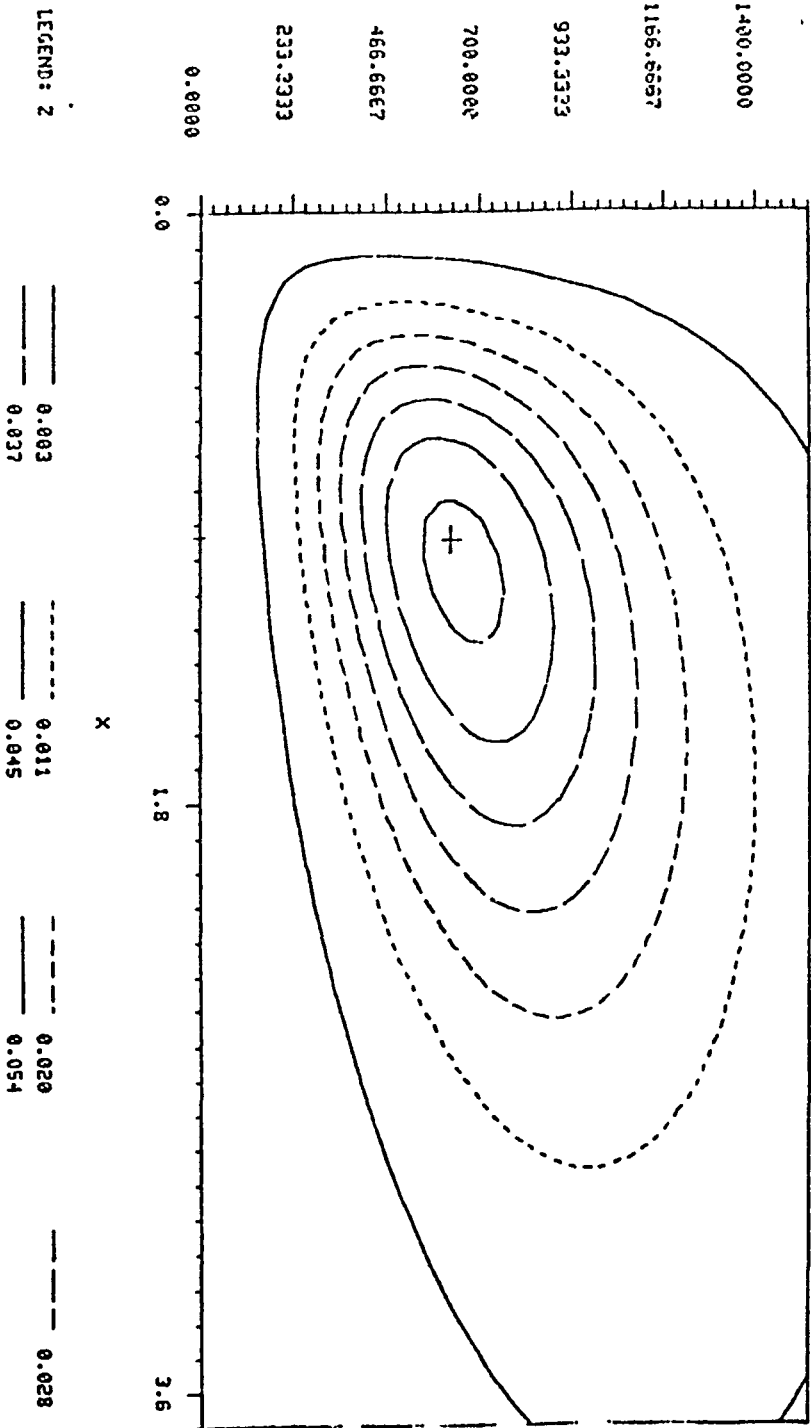
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CONTOUR PLOTS

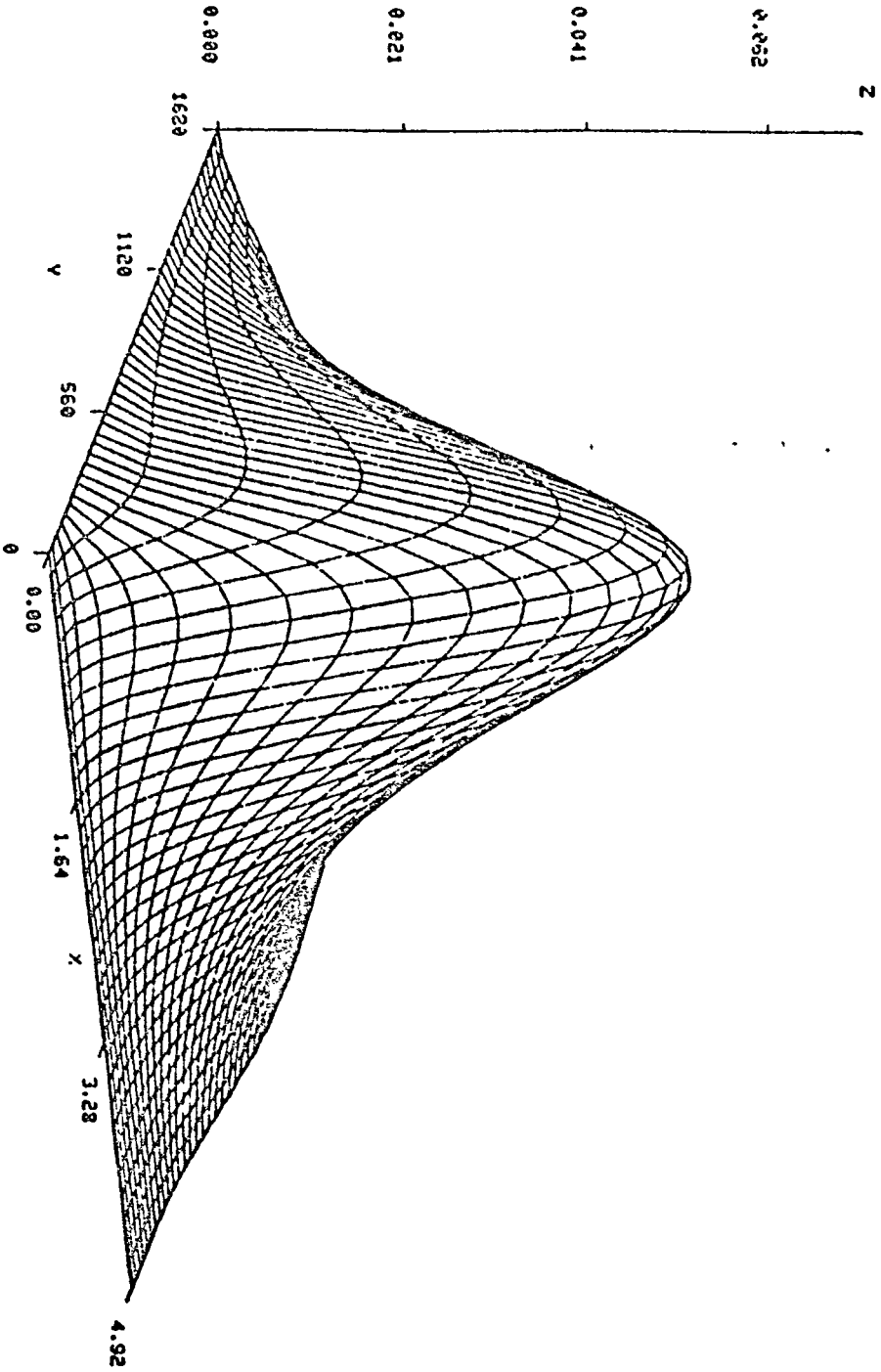
VARIABLES (AU,LU)
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OF POOR QUALITY

BIVARIATE GAMMA DENSITY

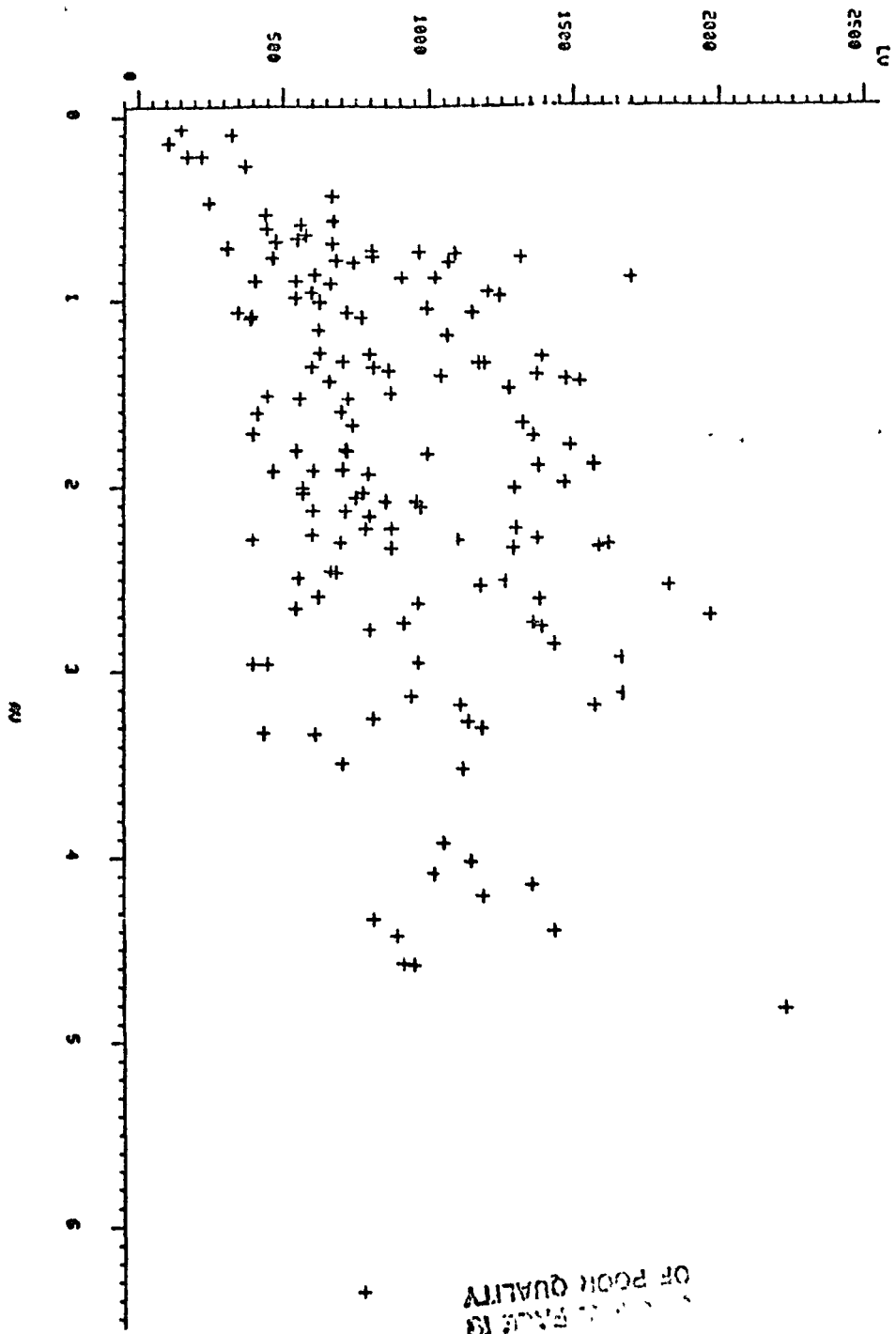
VARIABLES (AV, LV)
ALTITUDE = 10000



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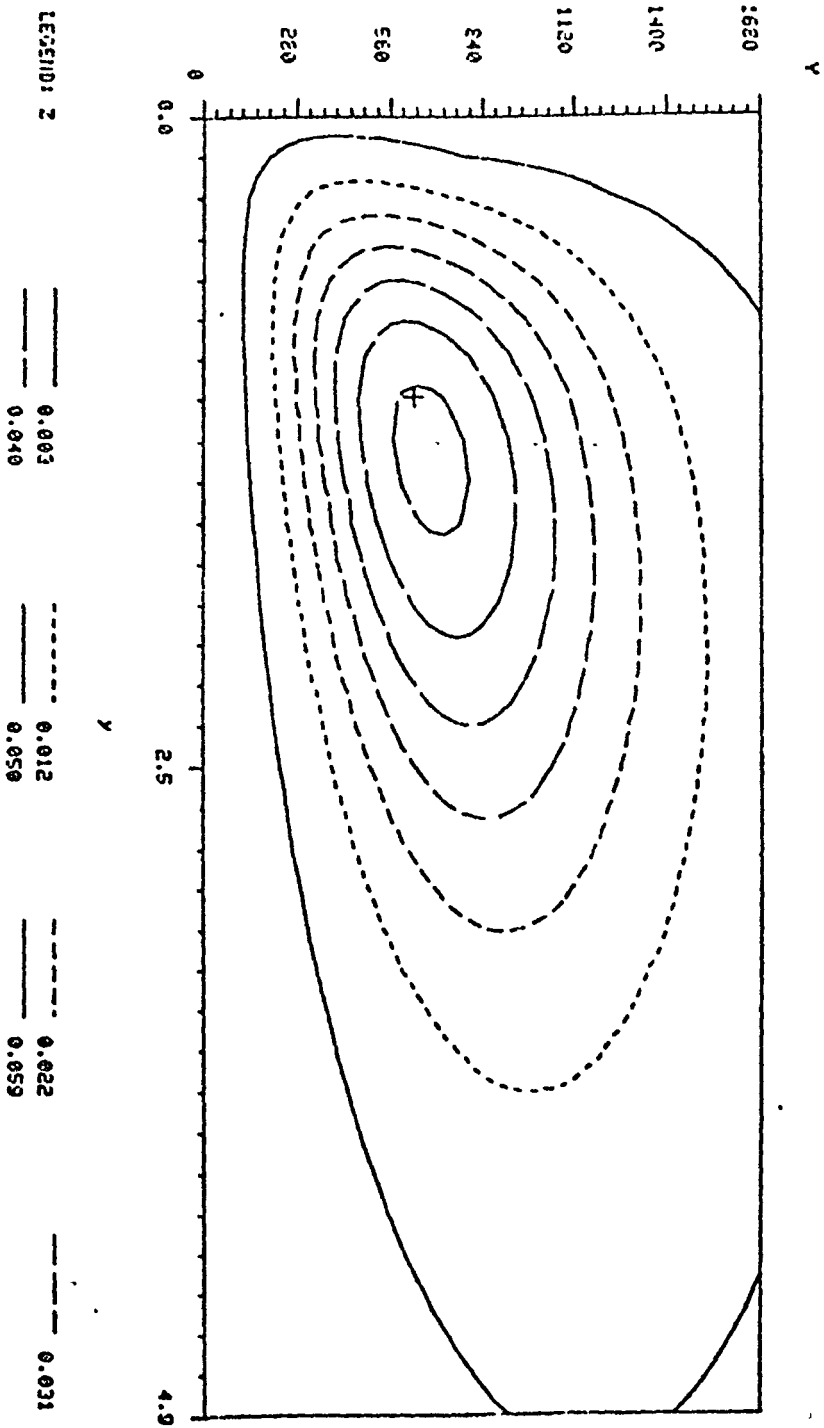
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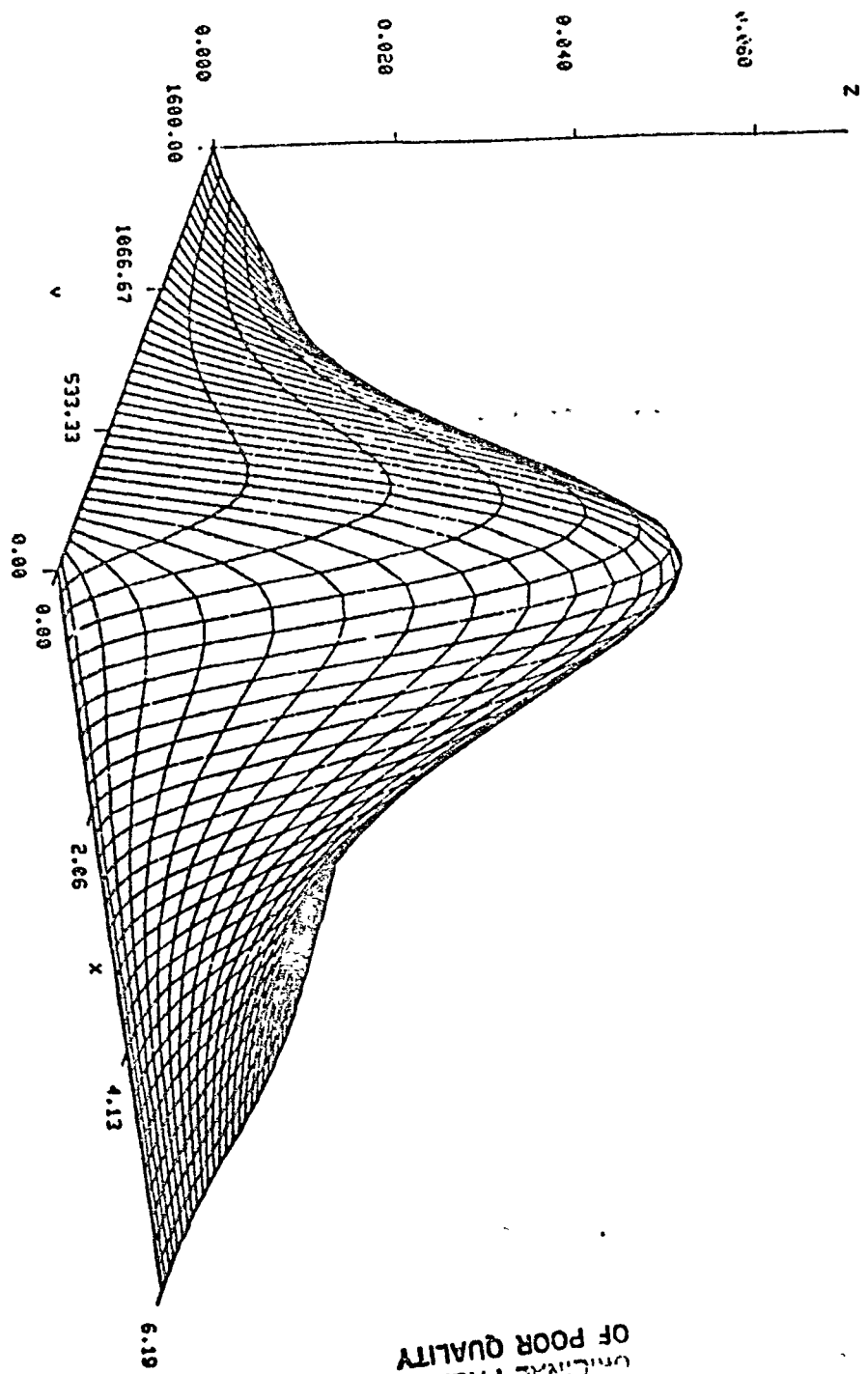
CONTOUR PLOTS

VARIABLES (AV,LY)
ALTITUDE = 10000



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BIVARIATE GAMMA DENSITY
VARIABLES (AU,LU)
ALTITUDE = 12000

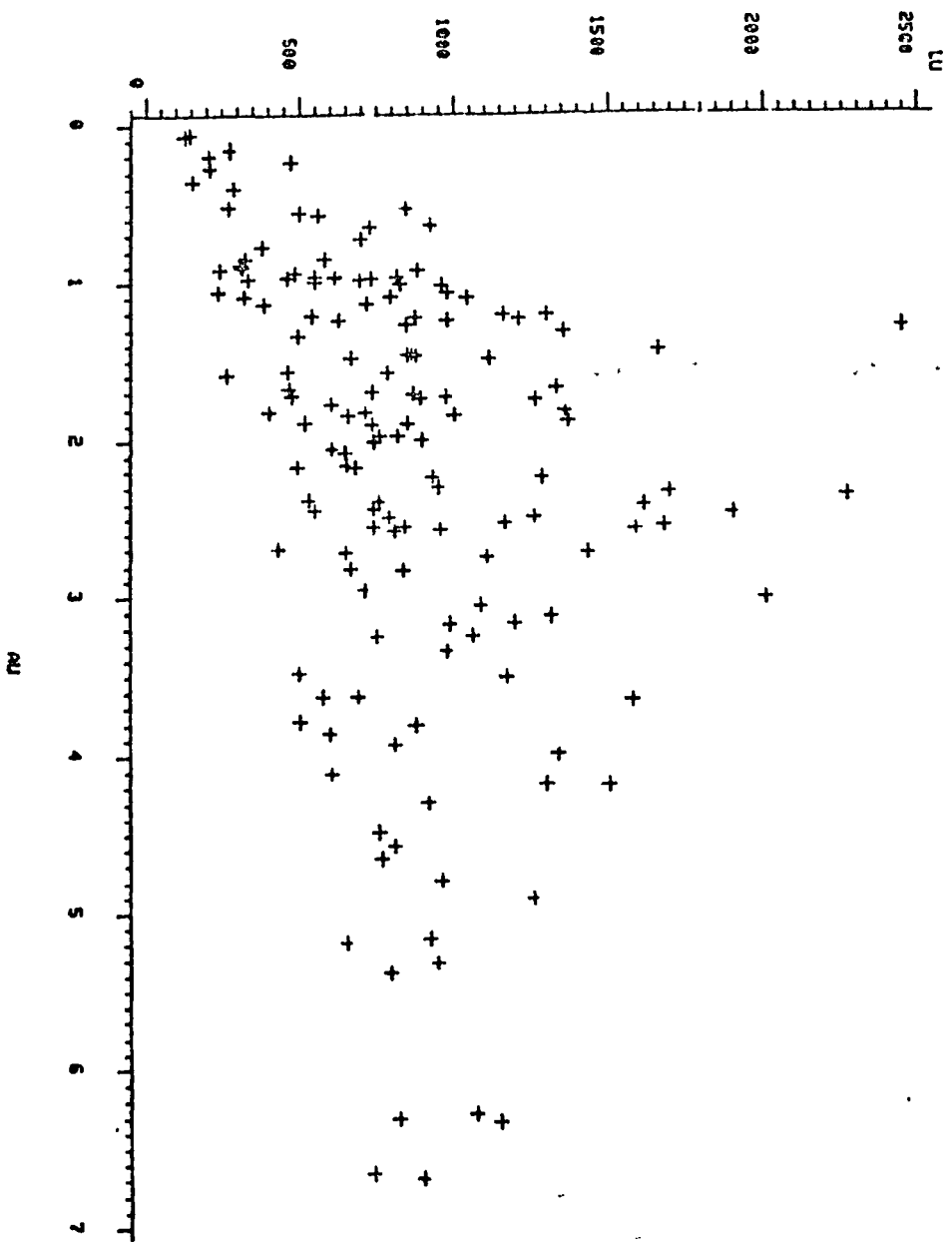


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Handwritten notes and markings at the top of the page, including a large 'M' and other illegible scribbles.

SCATTER PLOT OF RAW DATA

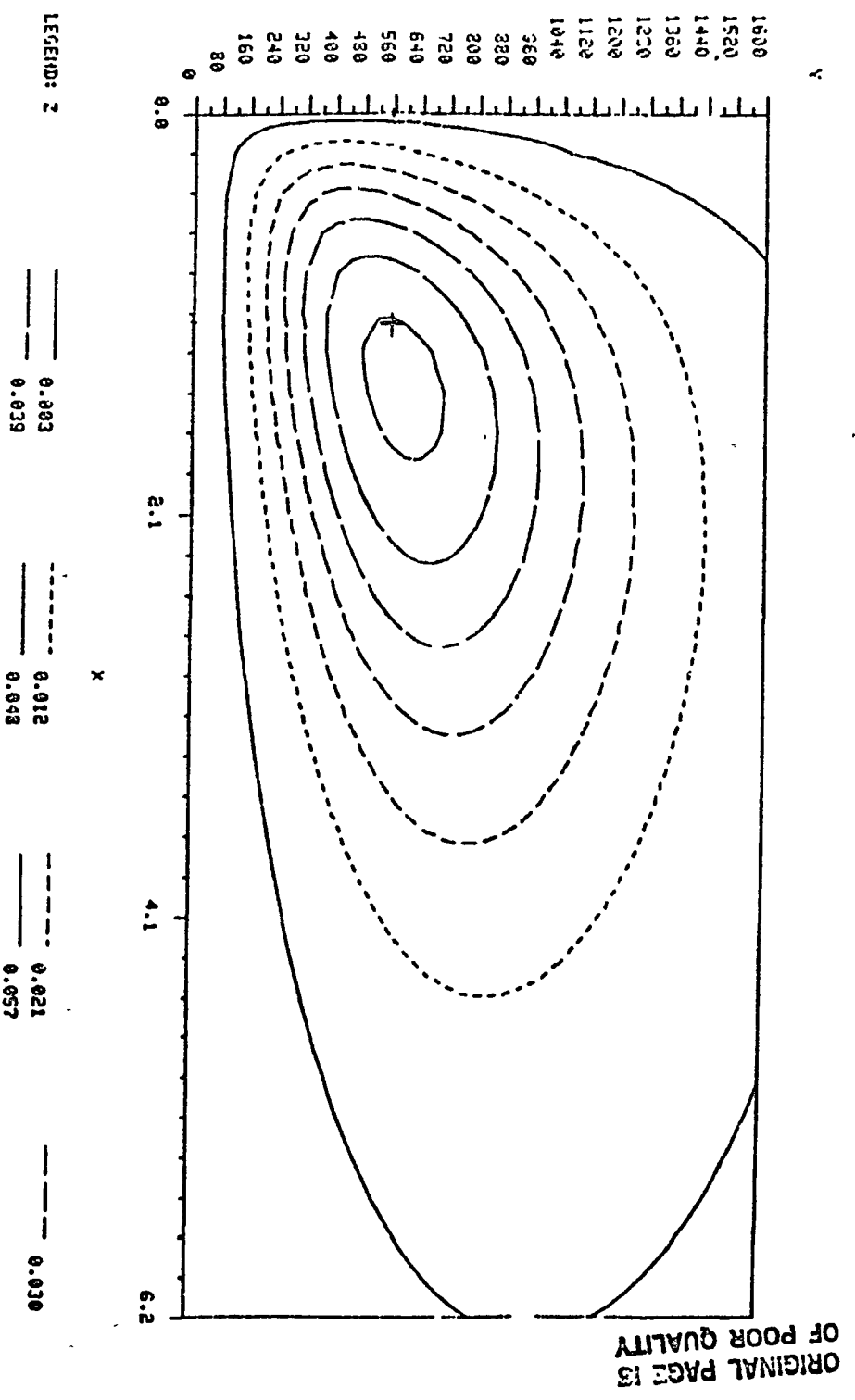
REFERENCE ALTITUDE = 12000



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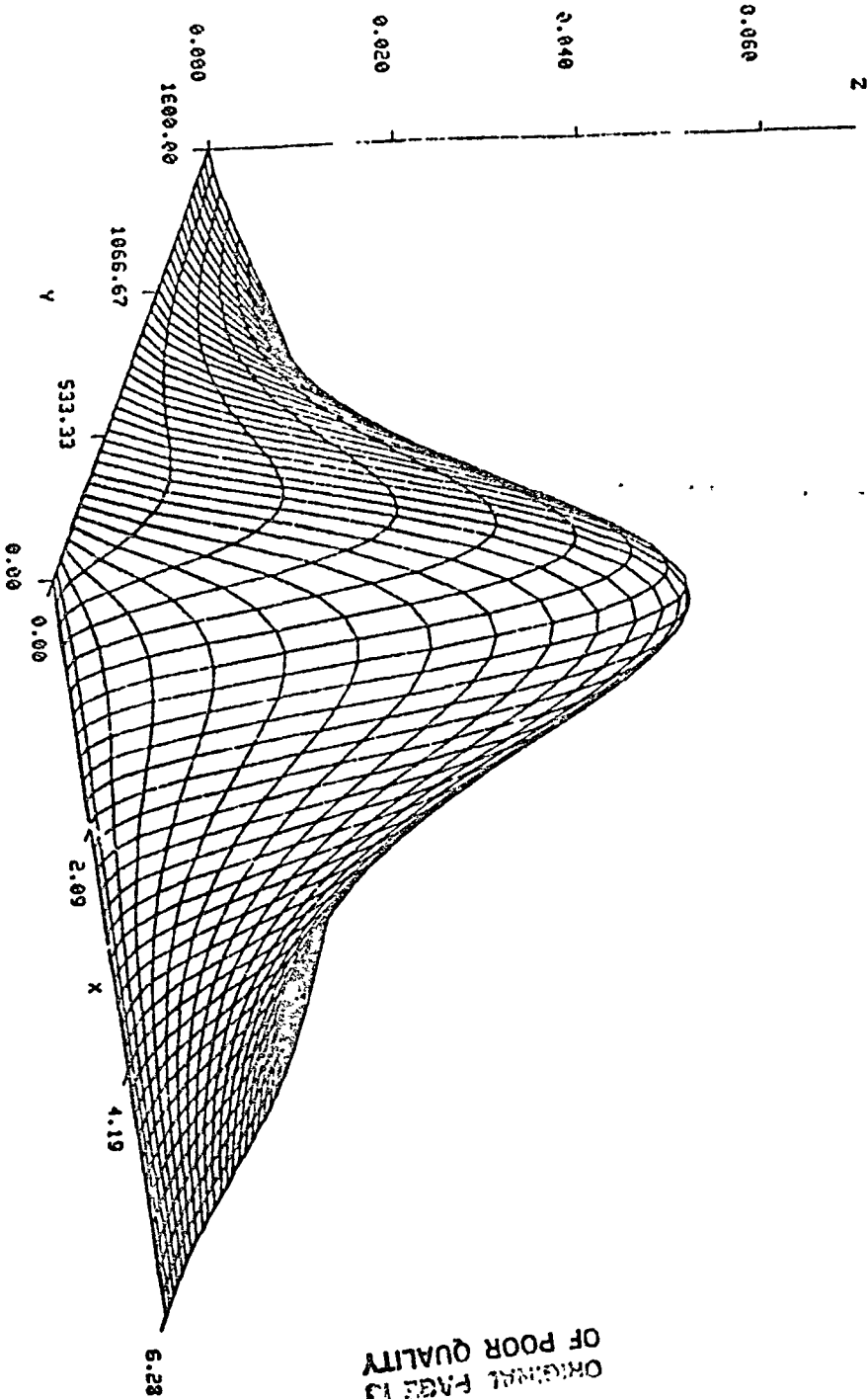
CONTOUR PLOTS

VARIABLES (AU,LU)
ALTITUDE = 12000



BIVARIATE GAMMA DENSITY

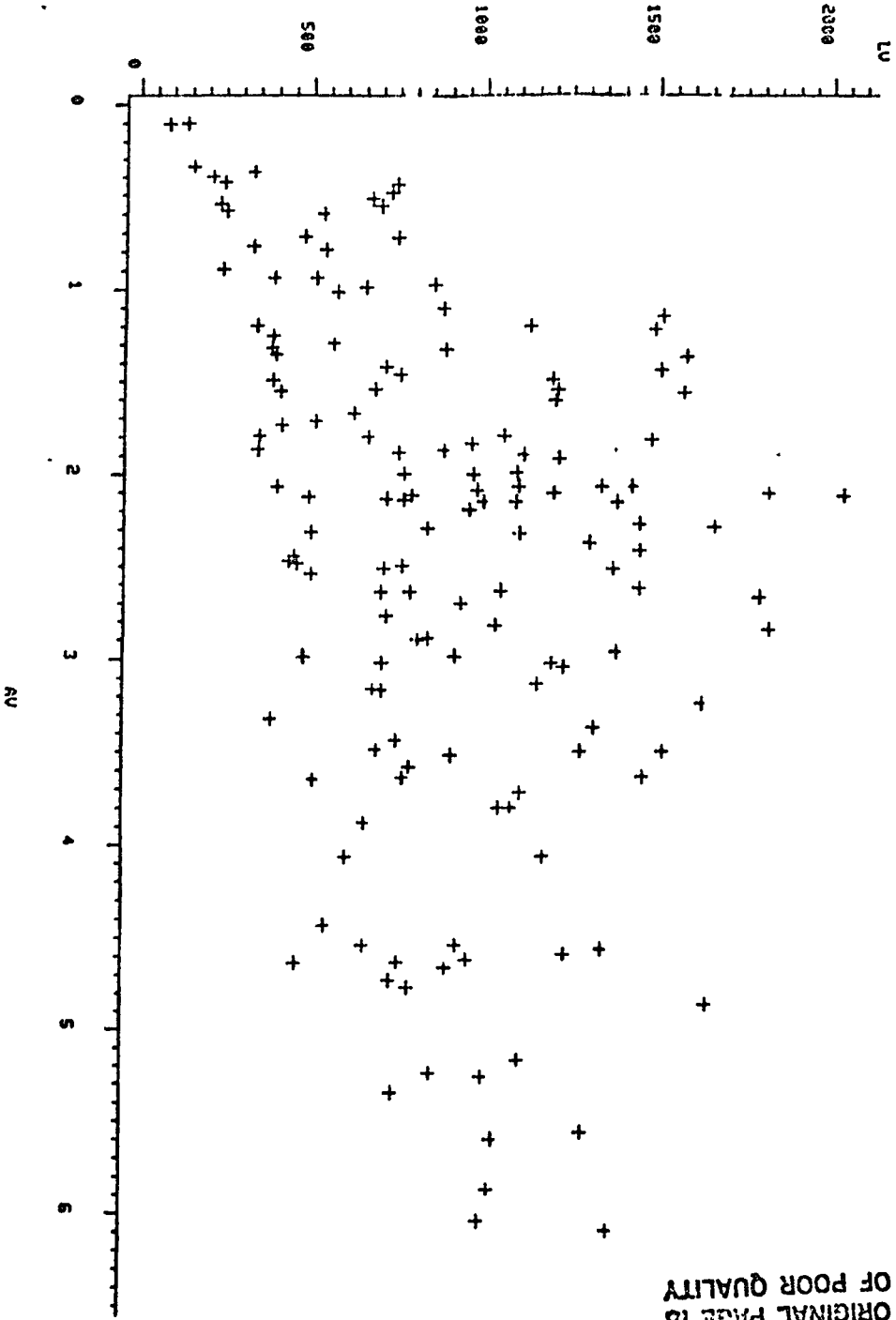
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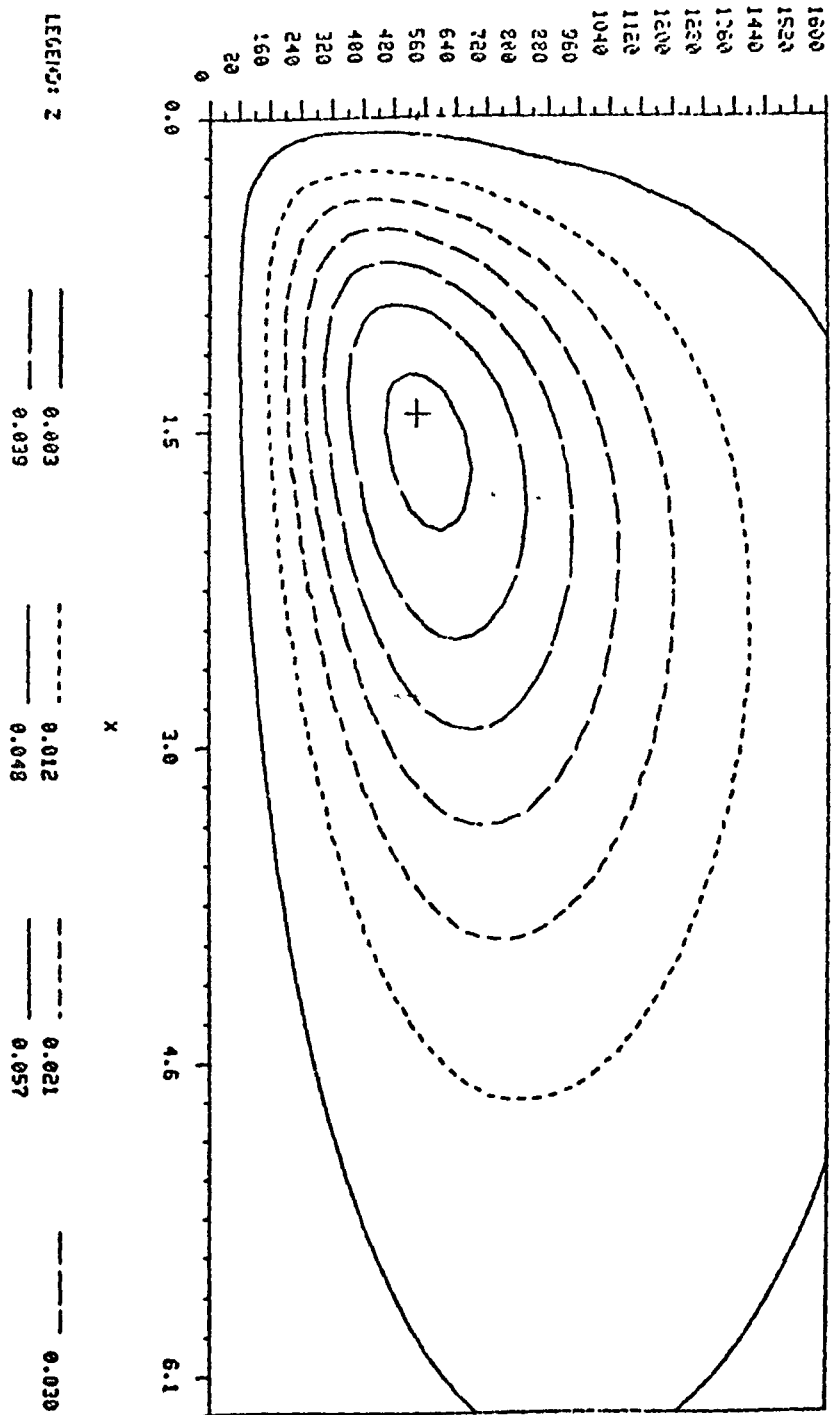
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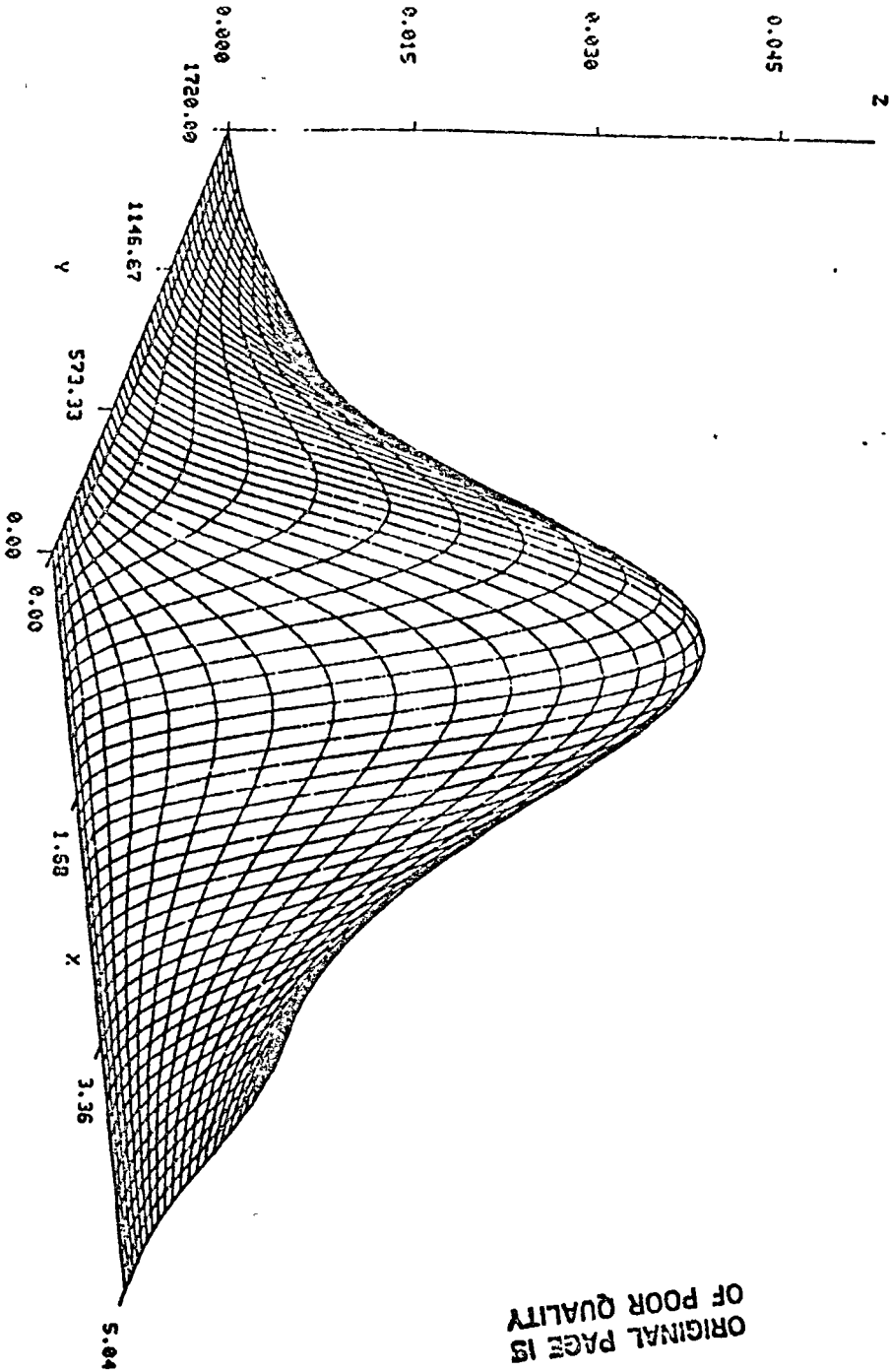
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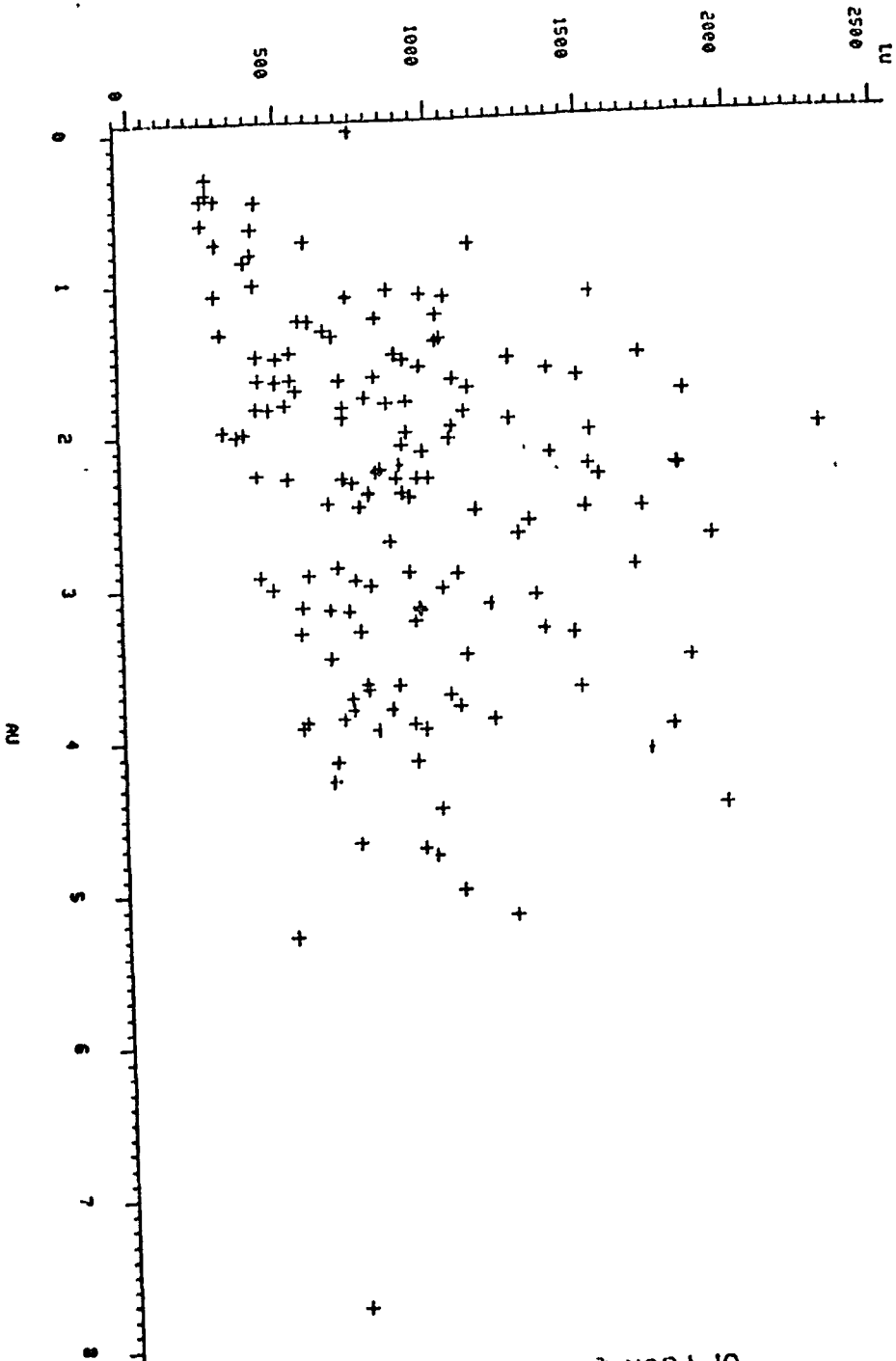
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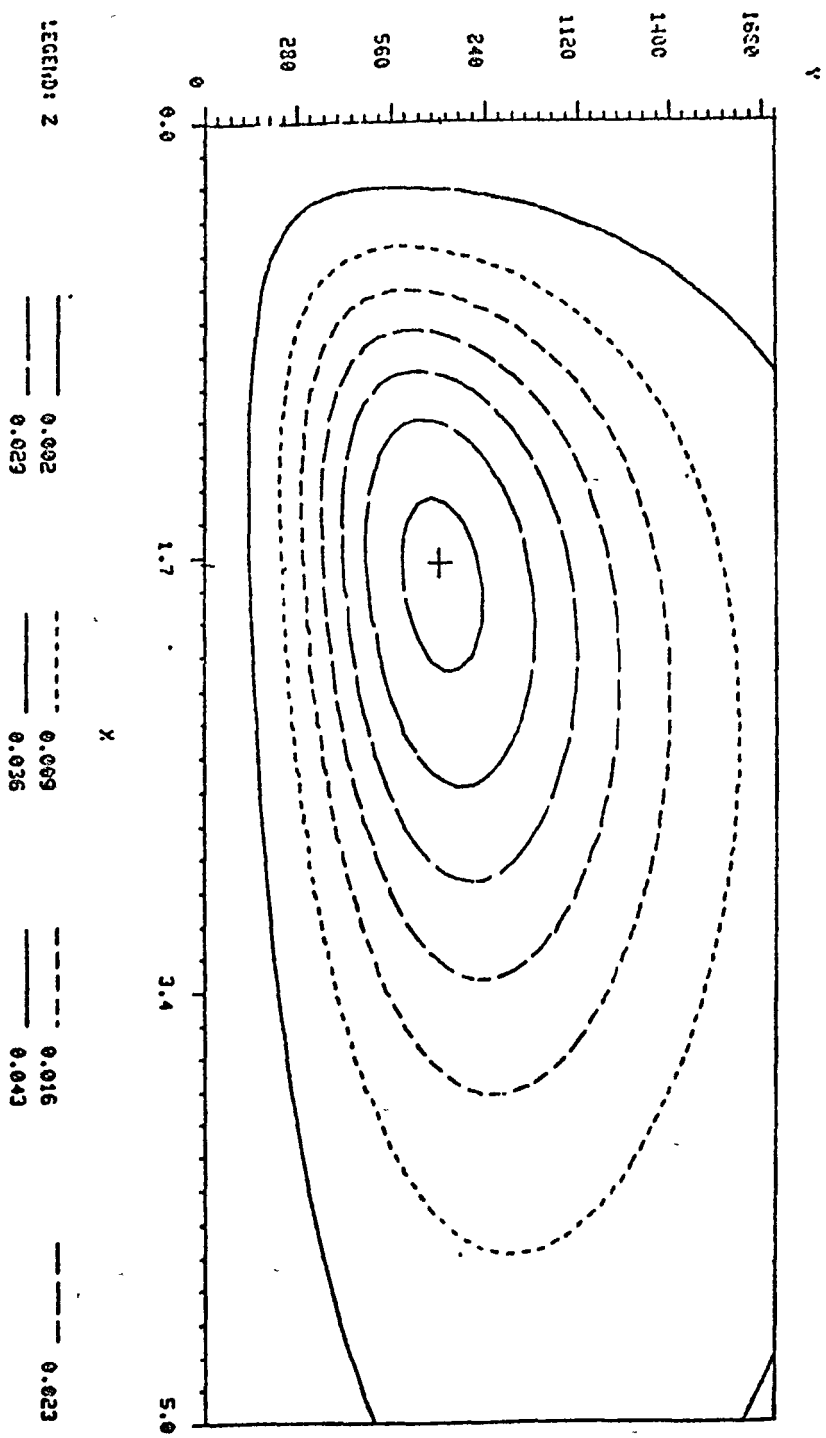
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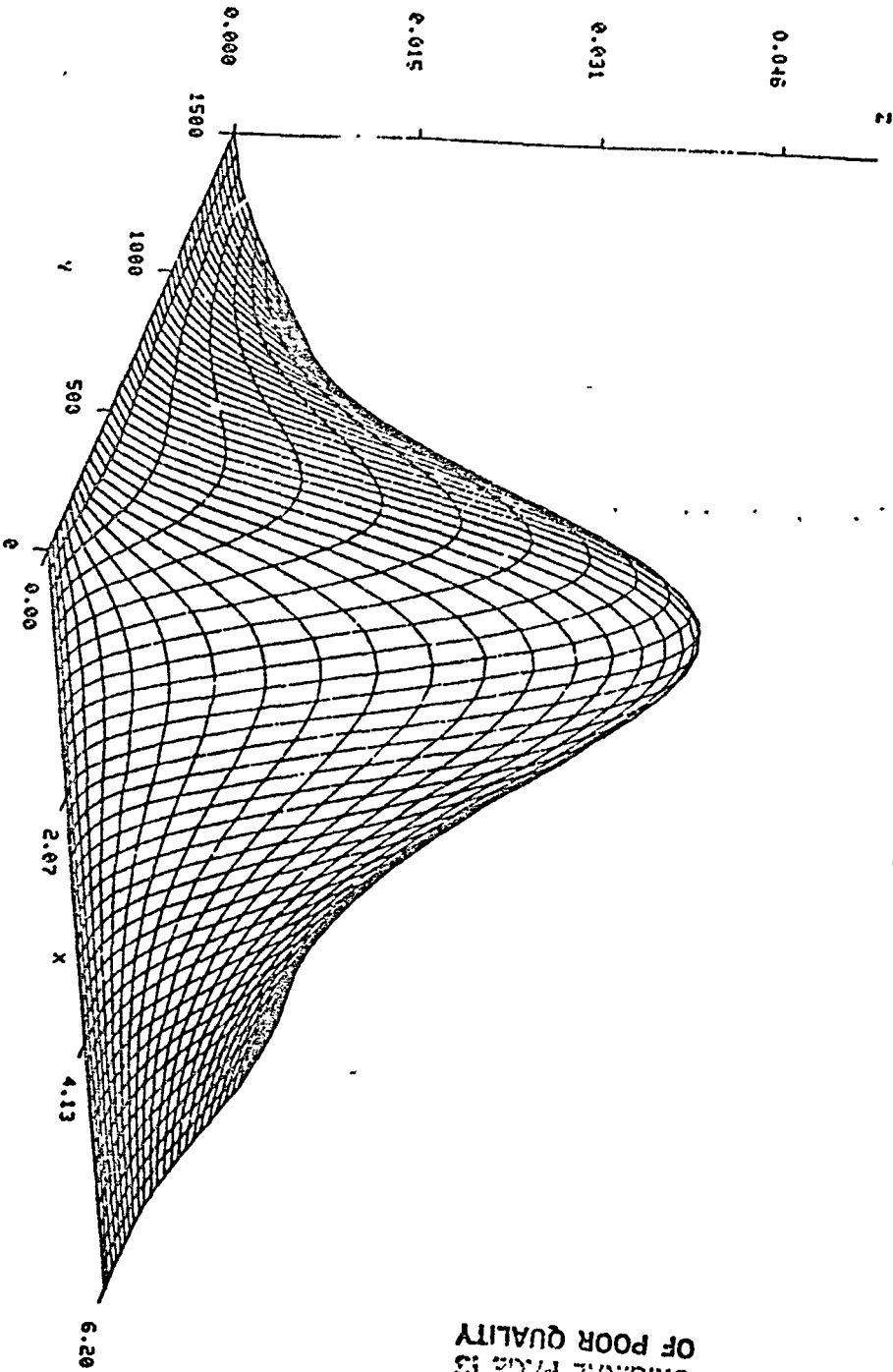
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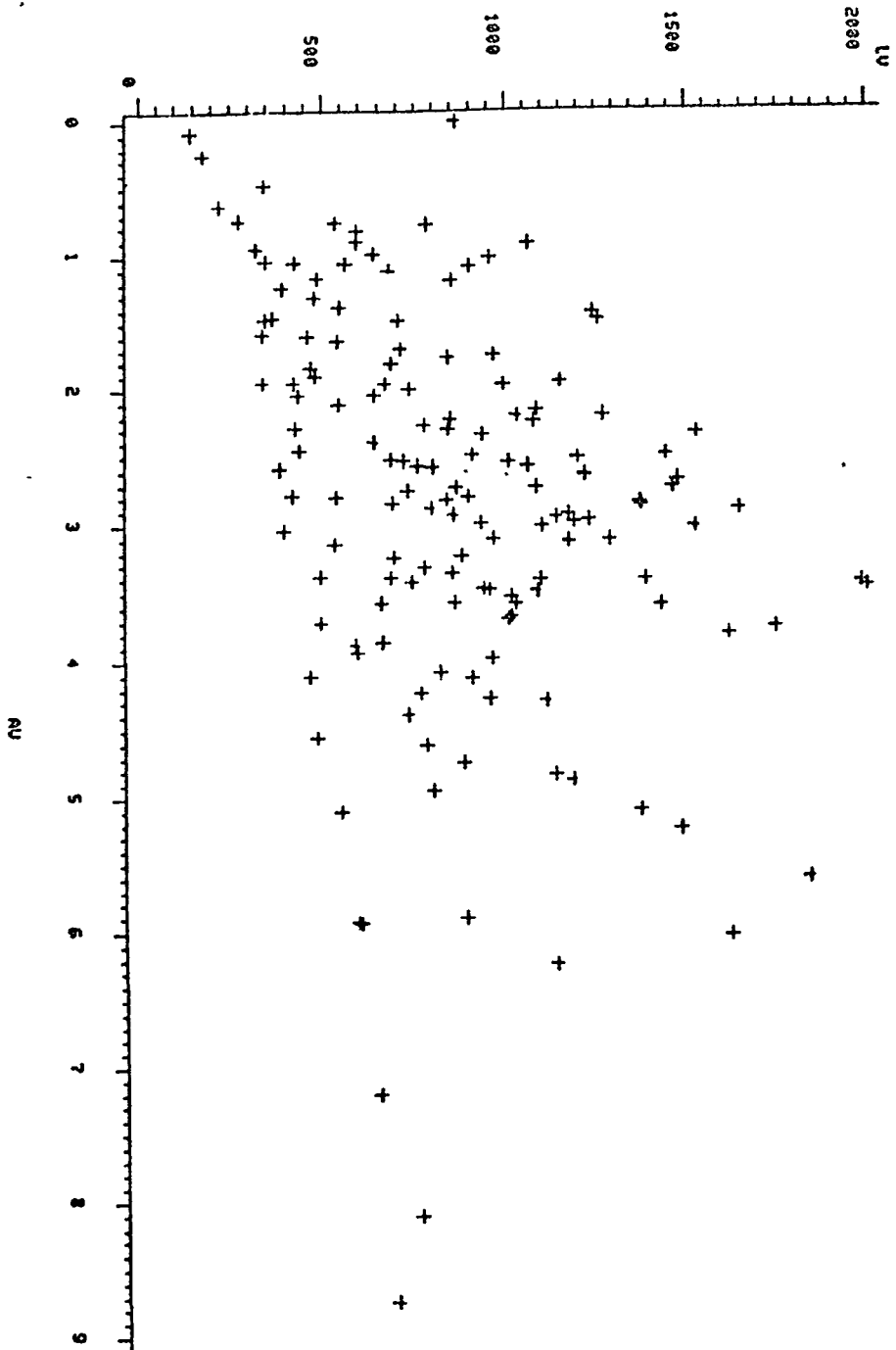
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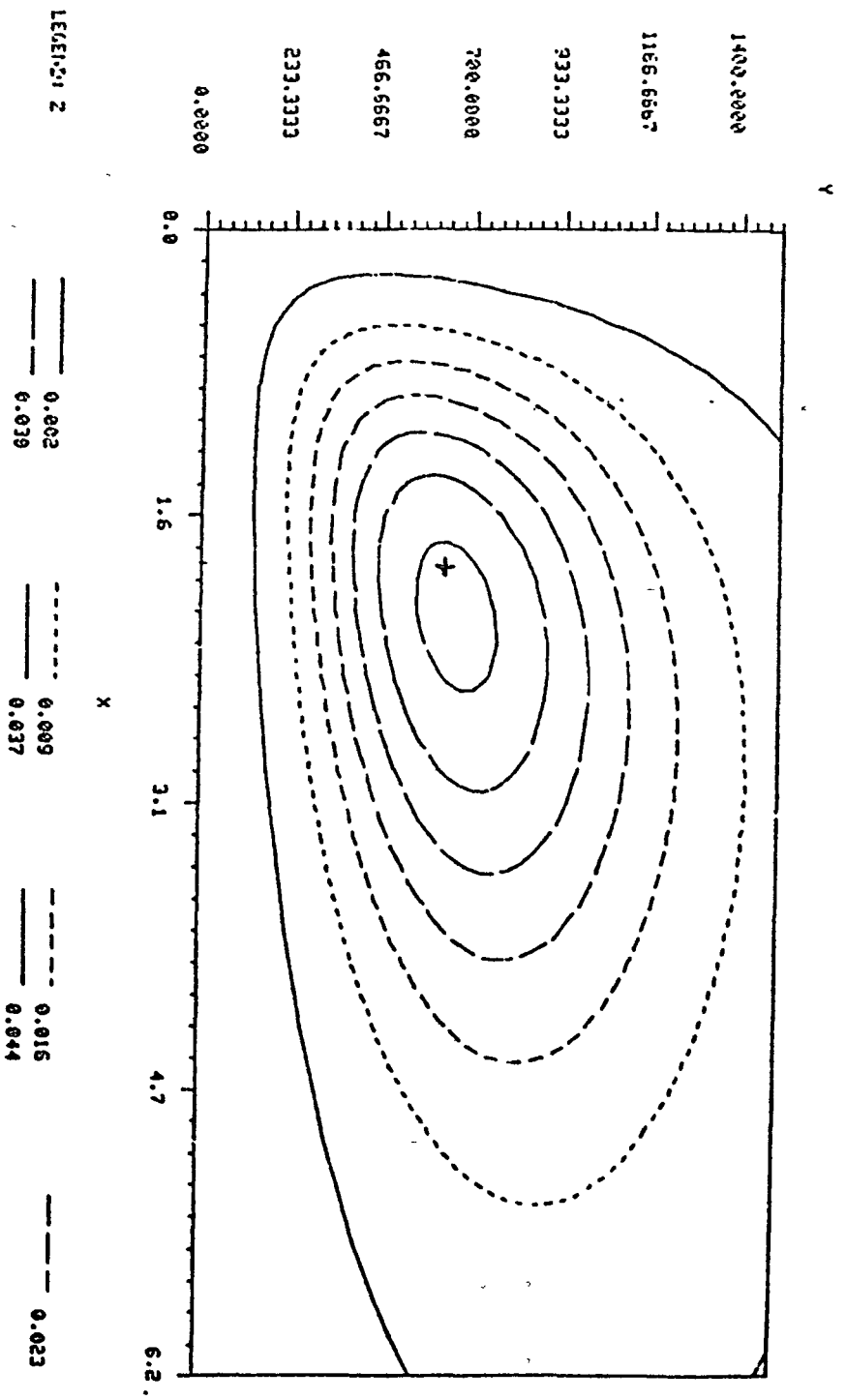
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