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Geometric Interpretations of the Discrete Fourier Transform (DFT)

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GEOMETRIC INTERPRETATIONS OF THE DISCRETE FOURIER TRANSFORM (DFT)

I. BACKGROUND

Since the Fast Fourier Transform (FFT) implementation of the DFT was popularized by Cooley and Tukey [1], use of the FFT has spread to many fields. For the purpose of this report, the acronyms FFT and DFT will be used interchangeably. Examples of the use of the FFT include data processing and filtering, spectral analysis, image processing, pattern recognition, and Monte Carlo turbulence simulation. A recent trend in the literature seems to be toward the idea that the DFT is an end unto itself. The relation to the continuous Fourier transform seems to be ignored. This report focuses on the relationship between continuous and discrete Fourier transforms by means of various geometric interpretations. These interpretations are given for one-, two-, and three-dimensional transforms. Interpretations are simple and are easily extended to higher dimensions.

The DFT can be interpreted in one, two, or three dimensions as an approximation to the two-sided continuous Fourier transform. Because of the well known periodicity properties of the DFT, and the way that the FFT is stored, the DFT must be rearranged to obtain direct correspondence with the continuous Fourier transform. This rearrangement is examined for one-, two-, and three-dimensional cases.

The work described here was motivated in part by Monte Carlo turbulence simulation in three dimensions [2]. In this simulation work, one of two three-dimensional FFT's were avoided by generating random noise in the FFT frequency domain. After transforming back to the space domain, the turbulence should have real values. For this to be the case, a certain Hermitian symmetry must be observed. A simple geometric interpretation of this symmetry is possible and is presented in a later section.

II. PERIODICITIES OF THE DFT

The one-, two-, and three-dimensional DFTs are defined by equations (1) through (3), respectively.

$$X(k) = \sum_{n=0}^{N-1} x(n) \exp(-j2\pi nk/N) \tag{1}$$

$$X(k1,k2) = \sum_{n2=0}^{N_2-1} \sum_{n1=0}^{N_1-1} x(n1,n2) \exp[-j2\pi(n1k1/N1+n2k2/N2)] \tag{2}$$

$$X(k1,k2,k3) = \sum_{n3=0}^{N3-1} \sum_{n2=0}^{N2-1} \sum_{n1=0}^{N1-1} x(n1,n2,n3) \exp\left[-j2\pi(n1k1/N1+n2k2/N2+n3k3/N3)\right] \qquad (3)$$

Because of the periodicity of the exponential factor, the DFTs obey the periodicities given in Table 1. These results are well known and follow directly from the defining equations. For clarity, a logical progression from the simple one-dimensional case up through the more complex higher-dimensional cases will be followed.

TABLE 1. GENERAL IDENTITIES FOR THE DFT

(1)   X(k) = X(k+N) = X(k+2N) = . . .

(2)   X(k1,k2) = X(K1+N1,k2) = X(k1,k2+N2) = X(k1+N1,k2+N2) = . . .

(3)   X(k1,k2,k3) = X(k1+N1,k2,k3) = X(k1,k2+N2,k3) = X(k1,k2,k3+N3)

       = X(k1+N1,k2+N2,k3) = . . .

(4)   X(-k) = X(N-k)

(5)   X(-k1,k2) = X(N1-k1,k2)

(6)   X(k1,-k2) = X(k1,N2-k2)

(7)   X(-k1,-k2) = X(N1-k1,N2-k2)

(8)   X(-k1,k2,k3) = X(N1-k1,k2,k3)

(9)   X(k1,-k2,k3) = X(k1,N2-k2,k3)

(10) X(k1,k2,-k3) = X(k1,k2,N3-k3)

(11) X(-k1,-k2,k3) = X(N1-k1,N2-k2,k3)

(12) X(-k1,k2,-k3) = X(N1-k1,k2,N3-k3)

(13) X(k1,-k2,-k3) = X(k1,N2-k2,N3-k3)

(14) X(-k1,-k2,-k3) = X(N1-k1,N2-k2,N3-k3)

The one-dimensional FFT given by equation (1) may be related to the one-dimensional, two-sided, continuous Fourier transform. Elementary correspondences between continuous and discrete transforms are described in the Appendix. The geometrical correspondence is established by considering the identities given in Table 1. The discrete sequence given by X(k) is defined for all positive and negative integers, k. From Identity (1) in Table 1, the infinite sequence is periodic with period N. The sequence

X(k) for k=0,1,. . .N/2 corresponds to the nonnegative frequencies in the continuous Fourier transform. Notice that k=N/2 corresponds to the Nyquist sampling frequency. X(k) for k=N/2+1,N/2+2,...,N-1 corresponds to the continuous Fourier transform evaluated at negative frequencies. The sequence X(k) for N/2<k<N are translated to the negative frequency axis without reflection. The situation is illustrated in Figure 1. Figure 1 depicts the normal storage strategy for an eight point FFT. Points 1 through 4 correspond to the positive frequency values for the continuous Fourier transform. Points 5 through 7 correspond to the negative frequencies of the continuous Fourier transform. Because of the periodicity pattern shown in the figure, the three points (5-7) can be translated without rotation or reflection to the negative frequency axis.

NYQUIST FREQUENCY

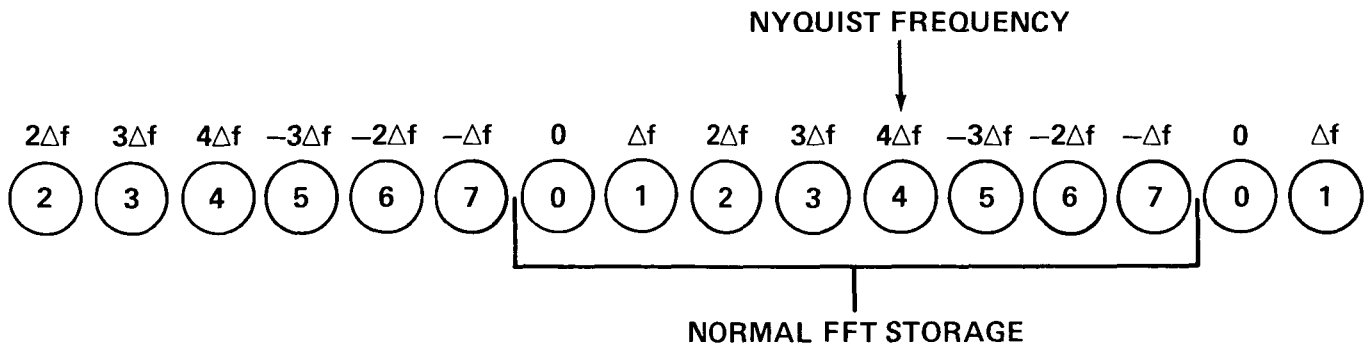| 2Δf	3Δf	4Δf	−3Δf	−2Δf	−Δf	0	Δf	2Δf	3Δf	4Δf	−3Δf	−2Δf	−Δf	0	Δf
(2)	(3)	(4)	(5)	(6)	(7)	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(0)	(1)

NORMAL FFT STORAGE

Figure 1. Correspondence of the one-dimensional DFT to the two-sided continuous transform.

For the two-dimensional transform, the periodicities and storage strategies are similar. Figure 2 depicts the situation for the DFT for a two-dimensional array. In this case the rearrangement is more complicated since the array contains points corresponding to negative frequencies in one coordinate direction with positive frequencies in the other. As a result four sections must be rearranged. The normal FFT storage arrangement is in the first quadrant as shown. The rearranged DFT is centered on the coordinate origin.
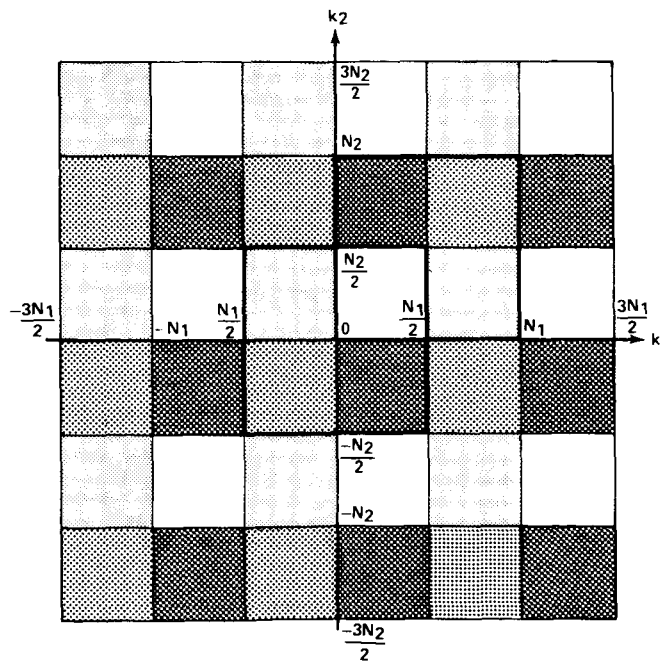
Figure 2. 2-D Fast Fourier Transform periodicity.

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In three dimensions the DFT must be broken into eight blocks and rearranged. Figure 3 indicates how the eight blocks can be rearranged so that the direct correspondence with the continuous Fourier transform is established. As was the case for one and two dimensions, the three-dimensional DFT can be considered to be defined for all values of k1, k2, and k3 from minus to plus infinity. Conceptually, Fourier space is filled with identical blocks stacked in the same orientation throughout. The same is true for one and two dimensions except that one- and two-dimensional Fourier space is filled with one- and two-dimensional blocks.
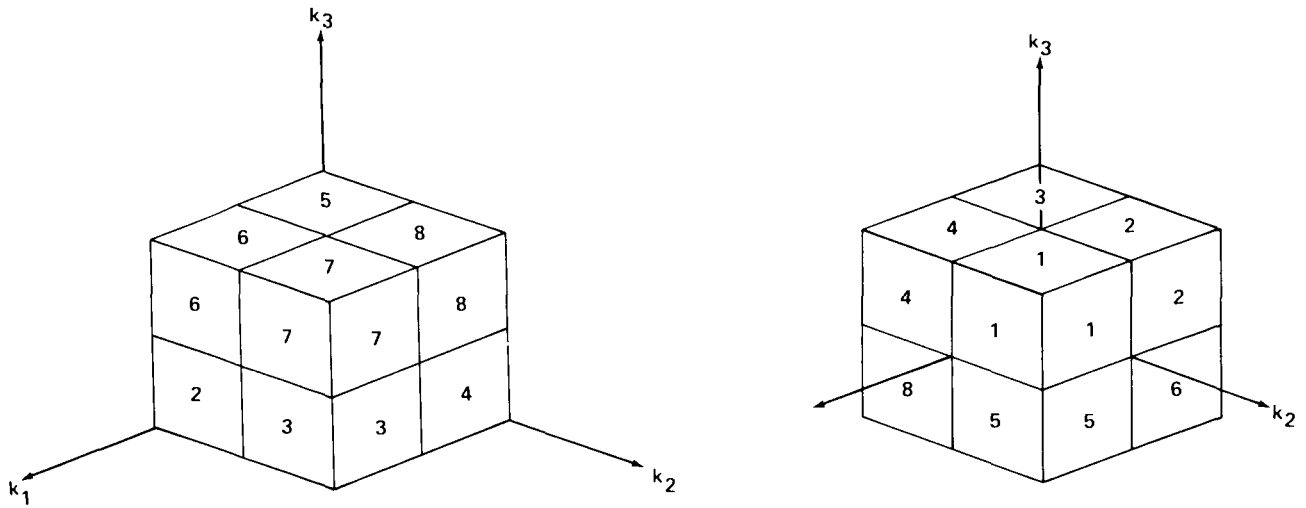
| a. Normal FFT storage. | b. Rearranged FFT storage for direct correspondence with the continuous transform. |

Figure 3. Three-dimensional DFT storage strategies.

III. HERMITIAN SYMMETRY OF DFTs OF REAL FUNCTIONS

The two-sided continuous Fourier transform defined by equation (4) has the very well known symmetry property given by equation (5).

$$X_c(f1,f2,f3) = \int\int\int_{-\infty}^{\infty} x_c(t1,t2,t3) \exp[-j2\pi(f1t1+f2t2+f3t3)] \, dt_1 \, dt_2 \, dt_3 \tag{4}$$

$$X_c(f1,f2,f3) = X_c^*(-f1,-f2,-f3) \tag{5}$$

Equations (4) and (5) are given for the three-dimensional case but apply for one and two dimensions as well.

The rearranged DFT described in the previous section has the same Hermitian symmetry. Unfortunately, in practice the rearranged DFT is not stored. Rather, the normal FFT storage strategy is used. In some applications the above symmetries must be observed in the frequency domain so that when the inverse DFT is performed, the result is a real function [2]. While the resulting symmetries are seemingly complex in the FFT storage strategy, a simple geometrical principle unravels the apparent complexity.

Figure 4 depicts the one-dimensional case. The symmetry identity depicted and others for higher dimensions are summarized in Table 2. Notice that in the figure the complex conjugate point is obtained by reflecting through the $N/2$nd point of the sequence. The $N/2$nd point reflects through itself, i.e., it is the complex conjugate of itself and is therefore a real point of the transform. This simple reflection principle will be found to hold for the two-, three-, and higher-dimensional cases.
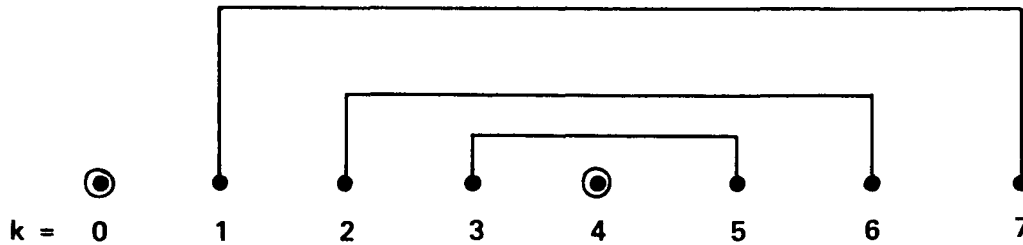
EXPRESSION 1, TABLE 2

Figure 4. Correspondence of complex conjugate pairs in the one-dimensional transform domain.

TABLE 2. SYMMETRY PROPERTIES OF THE DFTs OF REAL ONE-, TWO-, AND THREE-DIMENSIONAL FUNCTIONS

One-Dimensional Symmetry

Expression	Observation
1. $X(N-k) = X^*(k)$	$Im[X(N/2)] = 0$

Two-Dimensional Symmetry

2. $X(N1-k1,0) = X^*(k1,0)$	$Im[X(N1/2,0)] = 0$
3. $X(0,N2-k2) = X^*(0,k2)$	$Im[X(0,N2/2)] = 0$
4. $X(N1-k1,N2-k2) = X^*(k1,k2)$	$Im[X(N1/2,N2/2)] = 0$

Three-Dimensional Symmetry

5. $X(N1-k1,0,0) = X^*(k1,0,0)$	$Im[X(N1/2,0,0)] = 0$
6. $X(0,N2-k2,0) = X^*(0,k2,0)$	$Im[X(0,N2/2)] = 0$
7. $X(0,0,N3-k3) = X^*(0,0,k3)$	$Im[X(0,0,N3/2)] = 0$
8. $X(N1-k1,N2-k2,0) = X^*(k1,k2,0)$	$Im[X(N1/2,N2/2,0)] = 0$
9. $X(N1-k1,0,N3-k3) = X^*(k1,0,k3)$	$Im[X(N1/2,0,N3/2)] = 0$
10. $X(0,N2-k2,N3-k3) = X^*(0,k2,k3)$	$Im[X(0,N2/2,N3/2)] = 0$
11. $X(N1-k1,N2-k2,N3-k3) = X^*(k1,k2,k3)$	$Im[X(N1/2,N2/2,N3/2)] = 0$

The two-dimensional case contains a one-dimensional symmetry for each of the coordinate directions plus a two-dimensional symmetry. These symmetries are given respectively, by Expressions (2-4) of Table 2. Notice the previously described reflection which must now be observed for each coordinate axis. In addition, a new two-dimensional reflection about the interior real point of the transform must be observed. Figure 5 depicts all of these results.
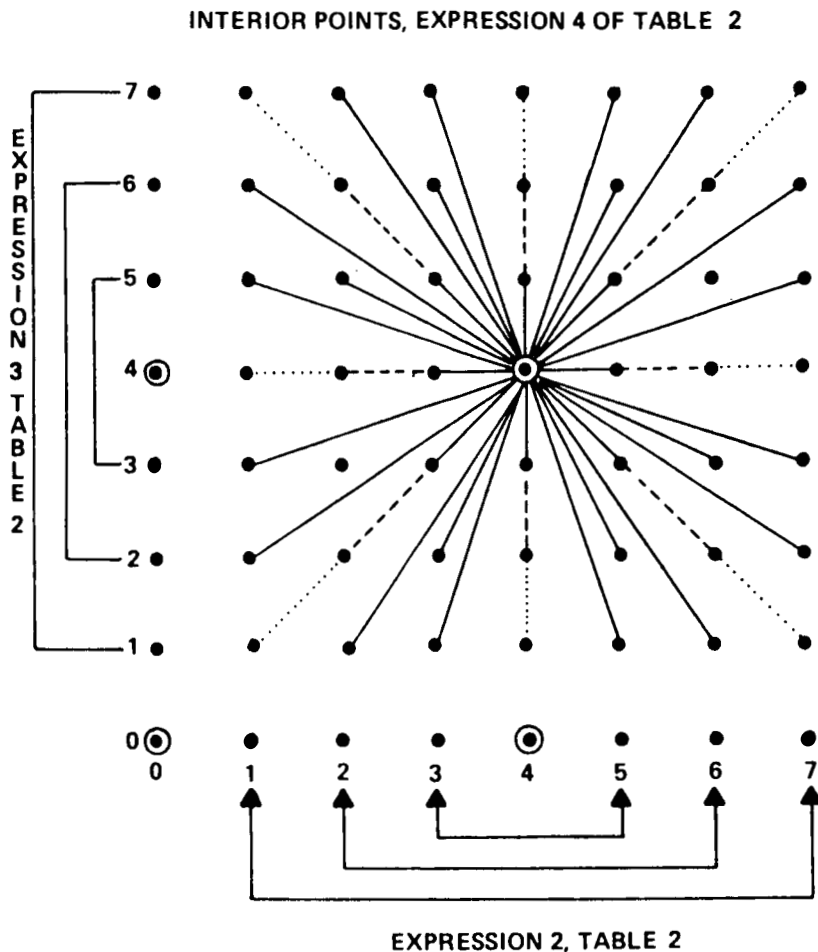

Figure 5. Correspondence of complex conjugate pairs in the
two-dimensional transform domain.

In the one-dimensional case, the transform has two real points, one at the origin and one at N/2. In the two-dimensional case, the transform has four real points, one at the origin, one on each coordinate axis, and one in the interior. Reflections are in effect about the midpoint of the coordinate axes in one dimension and two-dimensionally about the interior point.

In three dimensions the DFT of a real function is seen to have eight real transform values. Reflections occur about the midpoint of each of the three coordinate axes, about the midpoint of each of the three coordinate planes, and three-dimensionally about the interior real point. Figure 6 shows the real transform point locations for the three-dimensional case. In general for d dimensions the number of real points for the DFT of a real d-dimension function is 2^d.
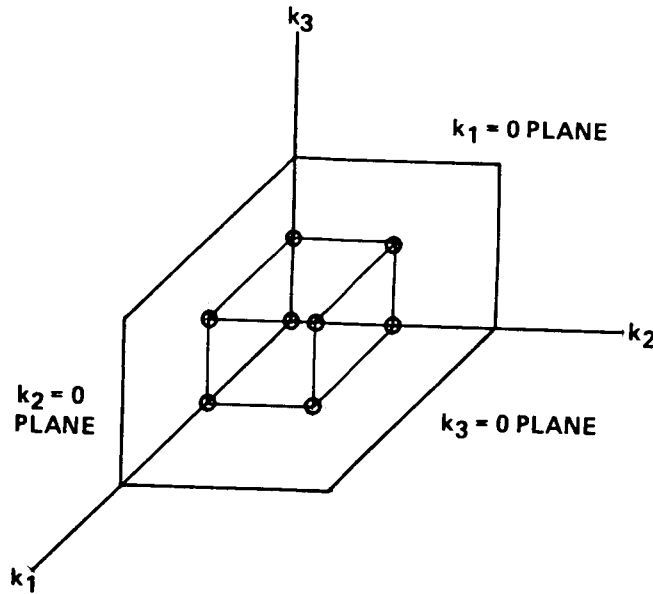
Figure 6. Points in the transform domain where the
transforms must be real.

IV. SUMMARY

The one-dimensional DFT has a period of length N. The two-dimensional DFT is periodic in the k1 direction with period N1, and in the k2 direction with period N2. The two-dimensional transform map shown in Figure 2 has the appearance of a chess board with four different colors. Each of the colors correspond to one of the quadrants of the N1 by N2 array. The first quadrant $(0 \leqslant k1 \leqslant N1/2)$ corresponds to the positive frequencies of the continuous two-sided Fourier transform $(f1 \geqslant 0, f2 \geqslant 0)$. The second quadrant $(N1/2+1 \leqslant k1 \leqslant N1-1, 0 \leqslant k2 \leqslant N1/2)$ corresponds to the frequencies $f1 < 0$ and $f2 \geqslant 0$. The third quadrant $(N1/2+1 \leqslant k1 \leqslant N1-1, N2/2+1 \leqslant k2 \leqslant N2-1)$ corresponds to the negative frequencies $f1,f2 < 0$. The fourth quadrant $(0 \leqslant k1 \leqslant N1/2, N2/2+1 \leqslant k2 \leqslant N2-1)$ corresponds to frequencies $f1 \geqslant 0$ and $f2 < 0$. All two-dimensional DFT space is filled with these four types of rectangles arranged in a regular pattern. To reach a square of the same type as a designated square move two squares in any direction horizontally, vertically, or diagonally.

A similar pattern holds for the three-dimensional case. For three dimensions, the block of FFT memory is divided up into eight sub-blocks. Frequency space is filled by stacking the sub-blocks in a regular pattern. To arrive at an identical sub-block from a given sub-block, move two sub-blocks in any direction. If the positive f1 axis points south, the positive f2 axis points east, and the positive f3 axis points up, then move two sub-blocks in any direction, i.e., north, northwest, northup, eastdown, southwestup, etc. The only restriction is that the movement must be in a straight line.

The one-, two-, three-, and d-dimensional DFT of a real function has a kind of Hermitian symmetry which is entirely equivalent to the symmetry shown by the two-sided continuous Fourier transform of a real continuous function. Because of the storage strategy usually employed in FFT implementations of the DFT, the symmetry takes an apparently complex form. Using the reflection principle described earlier unravels the apparent complexity of the symmetry.

REFERENCES

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2. Campbell, C. Warren: A Spatial Model of Wind Shear and Turbulence. Ph.D. Dissertation, Colorado State University, Spring, 1984.

MATHEMATICS OF THE CORRESPONDENCE BETWEEN THE DFT
AND CONTINUOUS FOURIER TRANSFORM

The purpose of this appendix is to describe the correspondence between the DFT and the two-sided continuous Fourier transform. The one-, two-, and three-dimensional DFTs and their inverses are defined by equations (6) through (11).

$$X(k) = \sum_{n=0}^{N-1} x(n) \exp[-j2\pi nk/N] \tag{6}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp[j2\pi nk/N] \tag{7}$$

$$X(k1,k2) = \sum_{n1=0}^{N1-1} \sum_{n2=0}^{N2-1} x(n1,n2) \exp[-j2\pi(n1k1/N1 + n2k2/N2)] \tag{8}$$

$$x(n1,n2) = \frac{1}{N1N2} \sum_{k1=0}^{N1-1} \sum_{k2=0}^{N2-1} X(k1,k2) \exp[j2\pi(n1k1/N1 + n2k2/N2)] \tag{9}$$

$$X(k1,k2,k3) = \sum_{n1=0}^{N1-1} \sum_{n2=0}^{N2-1} \sum_{n3=0}^{N3-1} x(n1,n2,n3) \exp[-j2\pi(n1k1/N1 + n2k2/N2 + n3k3/N3)] \tag{10}$$

$$x(n1,n2,n3) = \frac{1}{N1N2N3} \sum_{k1=0}^{N1-1} \sum_{k2=0}^{N2-1} \sum_{k3=0}^{N3-1} X(k1,k2,k3) \exp[j2\pi(n1k1/N1 + n2k2/N2 + n3k3/N3)] \tag{11}$$

The corresponding continuous Fourier transforms are given by equations (12) through (17).

$$X_c(f) = \int_{-\infty}^{\infty} x_c(t) \exp(-j2\pi ft) \, dt \tag{12}$$

$$x_c(t) = \int_{-\infty}^{\infty} X_c(f) \exp(j2\pi ft) \, df \tag{13}$$

$$X_c(f1,f2) = \int_{-\infty}^{\infty} x_c(t1,t2) \exp[-j2\pi(f1t1 + f2t2)] \, dt1 \, dt2 \tag{14}$$

$$x_c(t1,t2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_c(f1,f2) \exp[j2\pi(f1t1 + f2t2)] \, df1 \, df2 \tag{15}$$

$$X_c(f1,f2,f3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_c(t1,t2,t3) \exp[-j2\pi(f1t1 + f2t2 + f3t3)] \, dt1 \, dt2 \, dt3 \tag{16}$$

$$x_c(t1,t2,t3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_c(f1,f2,f3) \exp[j2\pi(f1t1 + f2t2 + f3t3)] \, df1 \, df2 \, df3 \tag{17}$$

The subscript c in these equations refers to the fact that the functions are continuous. The relationships between the continuous and digital functions are given by equations (18) and (19).

$$x_c(n\Delta t) = x(n) \tag{18}$$

$$X_c(k\Delta f) = X(k) \tag{19}$$

Extensions to higher dimensions are obvious. The relationships between sampling frequency, f_s, sampling period Δt, and number of points are given as follows.

$$\Delta t = 1/f_s \tag{20}$$

$$\Delta f = f_s/N \tag{21}$$

Again extensions to higher dimensions are obvious.

By inspection of the above continuous and digital relationships the approximations are given as follows.

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$$X_c(k\Delta f) \approx DFT[x(n)] \; \Delta t \tag{22}$$

$$x_c(n\Delta t) \approx DFT^{-1}[X(k)] \; N\Delta f \tag{23}$$

In the interpretation presented in the body of the paper, k in equation (22) lies between $-N/2+1$ and $N/2$. The negative values of k correspond to the negative frequencies of the continuous transform. Most implementations of the DFT store frequency values between $k=0$ and $k=N-1$. Even though the DFT can be defined for all values of k from $-\infty$ to $+\infty$ all of these values are known by storing any contiguous N values of the DFT. This is a direct result of the periodicity of the DFT.

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15. SUPPLEMENTARY NOTES

16. ABSTRACT

A recent tendency in technical literature has been to ignore the relationship of the DFT to the real world. Rather the DFT has become an end unto itself. This attitude is somewhat surprising since the DFT's reason for existence is its relationship to the real, i.e., continuous, world. One-, two-, and three-dimensional DFTs and geometric interpretations of their periodicities are presented. These operators are examined in light of their relationship with the two-sided, continuous Fourier transform. Discrete or continuous transforms of real functions have certain symmetry properties. These symmetries are examined in detail for the one-, two-, and three-dimensional cases. Extension to higher dimensions is straight-forward.

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