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A NUMERICAL SCHEME FOR THE IDENTIFICATION  
OF HYBRID SYSTEMS DESCRIBING THE VIBRATION  
OF FLEXIBLE BEAMS WITH TIP BODIES

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1. The first part of the document discusses the importance of maintaining accurate records of all transactions and activities. It emphasizes the need for transparency and accountability in financial reporting.

2. The second part of the document outlines the various methods and techniques used to collect and analyze data. It includes a detailed description of the experimental procedures and the statistical tools employed.

3. The third part of the document presents the results of the study, including a comparison of the different methods and a discussion of the implications of the findings.

4. The fourth part of the document discusses the limitations of the study and suggests areas for future research. It also provides a conclusion and a list of references.

5. The final part of the document is a list of references, which includes a variety of sources used in the study.

A Numerical Scheme for the Identification  
of Hybrid Systems Describing the Vibration  
of Flexible Beams with Tip Bodies\*

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Abstract

A cubic spline based Galerkin-like method is developed for the identification of a class of hybrid systems which describe the transverse vibration of flexible beams with attached tip bodies. The identification problem is formulated as a least squares fit to data subject to the system dynamics given by a coupled system of ordinary and partial differential equations recast as an abstract evolution equation (AEE) in an appropriate infinite dimensional Hilbert space. Projecting the AEE into spline-based subspaces leads naturally to a sequence of approximating finite dimensional identification problems. The solutions to these problems are shown to exist, are relatively easily computed, and are shown to, in some sense, converge to solutions to the original identification problem. Numerical results for a variety of examples are discussed.

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## 1. INTRODUCTION

In this paper we develop an approximation scheme for the identification of systems describing the planar transverse vibration of beams with attached tip bodies. Standard models from the theory of elasticity for the vibration of structures of this type involve hybrid systems of coupled partial and ordinary differential equations which describe the motion of the beam and tip bodies respectively. The approximation scheme is based upon the formulation of the identification problem as a least squares fit to data subject to the dynamical equations recast as an abstract evolution equation in an infinite dimensional Hilbert space. Using a cubic spline based Galerkin method, a sequence of successively higher (but finite) dimensional state approximations are constructed. This leads naturally to a sequence of approximating identification problems, each of which is shown to have a solution that can readily be computed using standard numerical techniques. Results from linear semigroup theory and the theory of evolution operators are used to demonstrate convergence of the state approximation. This in turn is used to argue that solutions to the finite dimensional identification problems, in some sense, approximate solutions to the original identification problem. Our effort here is similar in spirit to the approach taken in [1], [2], [3], [7], and [11] wherein approximation schemes for the estimation of parameters in beam equations with standard boundary conditions (i.e., clamped, simply supported, free, etc.) are developed. Our work is based to a large extent on the ideas suggested in the short note by Burns and Cliff [5].

Although our general approach is applicable to a broad class of problems (see Section 4), to illustrate our method we consider a beam, clamped at one end and cantilevered at the other with an attached tip body. In Section 2 the derivation of the equations of motion for the beam/tip body system is outlined, the equivalent abstract evolution equation is derived and the identification problem is formulated. In Section 3 the approximation scheme is constructed and convergence results are discussed. Numerical results for several examples are presented in Section 4.

Our notation is, for the most part, standard. The usual Sobolev spaces of real-valued functions on the interval  $[a, b]$  whose  $k^{\text{th}}$  derivatives are  $L^2$  are denoted by  $H^k(a, b)$ . These spaces are assumed to be endowed with the

usual Sobolev inner products  $\langle \cdot, \cdot \rangle_k$  and their induced norms  $\|\cdot\|_k$ . For  $Z$  a normed linear space with norm  $\|\cdot\|_Z$  and  $f : [0, T] \rightarrow Z$  we say that  $f \in L^2([0, T], Z)$  if  $\int_0^T \|f(t)\|_Z^2 dt < \infty$ . Similarly  $f$  will be said to be an element of  $C([0, T], Z)$  if the map  $t \rightarrow f(t)$  from  $[0, T]$  into  $Z$  is  $\ell$  time continuously differentiable on  $(0, T)$ . Finally for a function of one or more real variables, the symbol  $D_\theta f$  ( $D_\theta^k f$ ) will be used to denote the 1<sup>st</sup> ( $k^{\text{th}}$ ) derivative of  $f$  with respect to the independent variable  $\theta$ . If  $f$  is a function of a single variable only, the subscript may be omitted. On occasion, the short-hand notation  $D_\theta f(\theta_0)$  or  $Df(\theta_0)$  will be used in place of  $D_\theta f|_{\theta_0}$  or  $Df|_{\theta_0}$  to denote the derivative of  $f$  evaluated at  $\theta_0$ .

2. THE PARTIAL DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS, THEIR ABSTRACT FORMULATION, AND THE IDENTIFICATION PROBLEM

We consider (see Fig. 2.1) an inextensible beam of length  $\ell$ , having spatially dependent linear mass density  $\rho$  and flexural stiffness  $EI$ . The tip body is assumed to be of mass  $m$ , have mass center at a distance  $c$  from the end of the beam directed at an angle  $\delta$  measured from the extension of the longitudinal axis of the beam and having moment of inertia  $J$  about its center of mass.

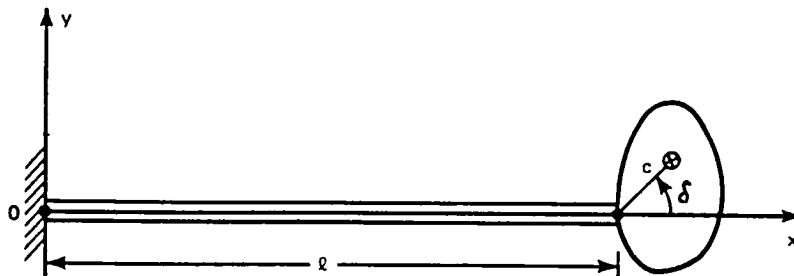


Figure 2.1.

Assuming small deformations ( $|u(t, x)| \ll \ell$ ), using the standard Euler-Bernoulli theory (neglecting rotatory inertia and shear deformations) and elementary Newtonian mechanics the equation describing the verticle displacement  $u(t, x)$  of the beam at position  $x \in [0, \ell]$  at time  $t > 0$

$$\rho D_t^2 u = -D_x^2 E I D_x^2 u + D_x \tau D_x u + f \quad (2.1)$$

is obtained where  $\tau(t, x)$  is the internal tension resulting from loads directed parallel to the beam's longitudinal axis and  $f(t, x)$  describes effects due to lateral or transverse loading and/or rigid-body rotations (see [6], [14], [17]).

If we let  $S$  denote the shear force and  $M$  the bending moment then using the standard moment equilibrium equation for a beam under tension

$$S = \tau D_x u - D_x M,$$

the basic bending moment-curvature relationship from the Euler-Bernoulli theory

$$M = E I D_x^2 u,$$

and the equations for the translational motion of the tip body we obtain the first boundary condition at  $x = \ell$

$$m D_t^2 u(t, \ell) + m c \cos \delta D_t^2 D_x u(t, \ell) = g_1(t) + D_x E I(\ell) D_x^2 u(t, \ell) - \tau(t, \ell) D_x u(t, \ell), \quad (2.2)$$

where  $g_1$  describes the net translational effects on the tip body's center of mass which result from externally applied lateral loads and moments (see [18]). The second boundary conditon at  $x = \ell$ , derived from the equations for the rotational motion of the tip body is given by

$$J D_t^2 D_x u(t, \ell) = -c \cos \delta D_x E I(\ell) D_x^2 u(t, \ell) - E I(\ell) D_x^2 u(t, \ell) - c \sin \delta \tau(t, \ell), + g_2(t), \quad (2.3)$$

where  $g_2$  is defined analogously to  $g_1$  with regard to rotational effects (see [18]).

The boundary conditions at the clamped end,  $x = 0$ , are of course, given by

$$u(t, 0) = 0 \quad D_x u(t, 0) = 0, \quad (2.4)$$

while the temporal boundary conditions (initial conditions) are of the form

$$u(0, x) = \phi(x) \quad D_t u(0, x) = \psi(x). \quad (2.5)$$

The equations (2.1) and (2.2) as they are written above are, in fact, nonlinear. Indeed, the internal tension  $\tau(t, x)$  is the sum of any externally applied loads  $\sigma(t, x)$  which are directed parallel to the longitudinal axis and the axially directed force  $mc \sin \delta D_t^2 D_x u(t, l)$  which results from the angular acceleration of the tip body (see [18]). Discarding the nonlinear terms in (2.1) and (2.2) as second order effects and choosing  $w_1(t) = D_t u(t, l)$ ,  $w_2(t) = D_t D_x u(t, l)$ ,  $w_3(t, x) = D_x^2 u(t, x)$ , and  $w_4(t, x) = D_t u(t, x)$  we rewrite (2.1), (2.2), (2.3), and (2.5) in state space form as

$$D_t^+ \vec{w}(t, x) = \begin{bmatrix} \alpha D_x EI(l)w_3(t, l) + \beta EI(l)w_3(t, l) - \frac{1}{m} \sigma(t, l) \int_0^l w_3(t, \theta) d\theta \\ -\beta D_x EI(l)w_3(t, l) - \gamma EI(l)w_3(t, l) \\ D_x^2 w_4(t, x) \\ -\frac{1}{\rho} D_x^2 EIw_3(t, x) + \frac{1}{\rho} D_x \sigma(t, x) \int_0^x w_3(t, \theta) d\theta \end{bmatrix} + \begin{bmatrix} \mu \sigma(t, l) + \frac{1}{m} g_1(t) - \beta g_2(t) \\ \lambda \sigma(t, l) + \gamma g_2(t) \\ 0 \\ \frac{1}{\rho} f(t, x) \end{bmatrix} \quad (2.6)$$



$$\vec{w}(0, x) = \begin{bmatrix} \psi(l) \\ D_x \psi(l) \\ D_x^2 \phi(x) \\ \psi(x) \end{bmatrix} \quad (2.7)$$

where

$$\alpha = \frac{J + mc^2}{mJ + m^2 c^2 \sin^2 \delta},$$

$$\beta = \frac{c \cos \delta}{J + mc^2 \sin^2 \delta},$$

$$\gamma = \frac{1}{J + mc^2 \sin^2 \delta},$$

$$\mu = -\frac{mc^2 \cos \delta \sin \delta}{J + mc^2 \sin^2 \delta},$$

$$\lambda = -\frac{c \sin \delta}{J + mc^2 \sin^2 \delta},$$

and

$$\vec{w}(t, x) = (w_1(t), w_2(t), w_3(t, x), w_4(t, x))^T.$$

Recalling (2.4) displacement,  $u(t, x)$  is recovered from  $\vec{w}(t, x)$  by

$$u(t, x) = \int_0^x \int_0^\theta w_3(t, \tau) d\tau d\theta.$$

The identification problem which we shall consider involves the estimation of the flexural stiffness  $EI$ , the mass density  $\rho$ , the externally applied forces

and moments in the form of  $\sigma$ ,  $f$ ,  $g_1$ ,  $g_2$ , and the initial conditions  $\phi$  and  $\psi$ . Although (laying identifiability questions aside) our approximation and convergence results would be applicable to inverse problems involving the estimation of any or all of the parameters in (2.6) and (2.7), for ease of exposition, we assume that the rigid-body mass properties  $m$ ,  $J$ ,  $c$ , and  $\delta$  of the tip body are known a priori. The identification problem is formulated as a least-squares fit to data. Our approach is based upon recasting (2.6) and (2.7) in terms of an abstract evolution equation.

Let  $Q$  be a subset of  $R^L$  and assume that the unknown temporally and/or spatially varying functions  $EI$ ,  $\rho$ ,  $\sigma$ ,  $f$ ,  $g_1$ ,  $g_2$ ,  $\phi$ , and  $\psi$  appearing in (2.6) and (2.7) which are to be identified have been parameterized by  $q \in Q$  (i.e.,  $EI(x) = EI(x; q)$ ,  $\rho(x) = \rho(x; q)$ ,  $\sigma(t, x) = \sigma(t, x; q)$ , etc.). We require and assume throughout that the following assumptions hold:

- A1:  $Q$  is a compact subset of  $R^L$ .
- A2: The mappings  $q \rightarrow EI(q)$  and  $q \rightarrow \rho(q)$  are continuous from  $Q$  into  $H^2(0, \ell)$  and  $Q$  into  $H^1(0, \ell)$ , respectively, and there exist positive constants  $m_{EI}$ ,  $m_\rho$ ,  $M_{EI}$ ,  $M_\rho$  such that  $m_{EI} \leq EI(q) \leq M_{EI}$ ,  $m_\rho < \rho(q) < M_\rho$  for all  $q \in Q$ .
- A3: The mappings  $q \rightarrow \phi(q)$  and  $q \rightarrow \psi(q)$  are continuous from  $Q$  into  $H^2(0, \ell)$  and  $Q$  into  $H^0(0, \ell)$ , respectively.
- A4: There exists a  $T > 0$  such that the mapping  $t \rightarrow \sigma(t, \cdot; q)$  is an element of  $C^1([0, T], H^1(0, \ell))$  and the mapping  $q \rightarrow \sigma(t, \cdot; q)$  is continuous from  $Q$  into  $H^1(0, \ell)$  for each  $t \in [0, T]$ .
- A5: The function  $f$  satisfies:
- (i) The mapping  $t \rightarrow f(t, \cdot; q)$  is an element of  $L^2([0, T], H^0(0, \ell))$  for each  $q \in Q$ .
  - (ii) The mapping  $q \rightarrow f(t, \cdot; q)$  is continuous from  $Q$  into  $H^0(0, \ell)$  for each  $t \in [0, T]$ .
  - (iii) There exists  $K_f \in L^2(0, T)$  independent of  $q \in Q$  for which  $|f(t, \cdot; q)|_0 \leq K_f(t)$  for all  $q \in Q$  and  $t \in [0, T]$ .

A6: The functions  $g_i$ ,  $i = 1, 2$  satisfy

- (i)  $g_i \in L^2(0, T)$  for each  $q \in Q$ .
- (ii) The mappings  $q \rightarrow g_i(t; q)$  are continuous from  $Q$  into  $R$  for each  $t \in [0, T]$ .
- (iii) There exist  $K_{g_i} \in L^2(0, T)$  independent of  $q \in Q$  for which  $|g_i(t; q)| \leq K_{g_i}(t)$  for all  $q \in Q$  and  $t \in [0, T]$ .

Let  $Z = R^2 \times H^0(0, l) \times H^0(0, l)$  and for each  $q \in Q$  let  $Z_q$  denote the Hilbert space  $\{Z, \langle \cdot, \cdot \rangle_q\}$  where

$$\langle (r_1, u_1, v_1), (r_2, u_2, v_2) \rangle_q = r_1^T W r_2 + \int_0^l EI(q) u_1 u_2 + \int_0^l \rho(q) v_1 v_2$$

with

$$W = \begin{bmatrix} m & mc \cos \delta \\ mc \cos \delta & J + mc^2 \end{bmatrix}.$$

The definition of  $\langle \cdot, \cdot \rangle_q$  is motivated by an energy expression. Indeed the sum of the kinetic and strain energies for the system described by (2.6) and (2.7) is given by

$$\frac{1}{2} [w_1(t), w_2(t)] W \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + \frac{1}{2} \int_0^l \rho w_4(t, x)^2 dx + \frac{1}{2} \int_0^l EI w_3(t, x)^2 dx$$

(see [18]).

Define  $A_0(q) : D_0 \subset Z_q \rightarrow Z_q$  by

$$D_0 = \{ (r, u, v) \in Z : u, v \in H^2(0, l), v(0) = Dv(0) = 0, r = (v(l), Dv(l))^T \},$$

$$\begin{aligned} A_0(q) \left( (v(l), Dv(l))^T, u, v \right) = & \left( (\alpha DEI(l; q) u(l) + \beta EI(l; q) u(l), \right. \\ & - \beta DEI(l; q) u(l) - \gamma EI(l; q) u(l) \left. \right)^T, D^2 v, \\ & - \frac{1}{\rho(q)} D^2 EI(q) u \left. \right). \end{aligned} \tag{2.8}$$

For each  $t \in [0, T]$  define  $B(t; q): Z_q \rightarrow Z_q$  by

$$B(t; q)(r, u, v) = \left( \left( -\frac{1}{m} \sigma(t, \ell; q) \int_0^\ell u, 0 \right)^T, 0, \frac{1}{\rho(q)} D_x \sigma(q) \int_0^\cdot u \right)$$

and  $A(t; q): D_0 \hookrightarrow Z_q \rightarrow Z_q$  by

$$A(t; q) = A_0(q) + B(t; q).$$

For each  $q \in Q$  it can be argued that the operator  $A_0(q)$  is densely defined and dissipative (in fact conservative, i.e.  $\langle A_0(q) z, z \rangle_q = 0, z \in D_0$ ). Moreover, it can be shown that it is skew self adjoint (i.e.,  $A_0(q)^* = -A_0(q)$ ) and therefore that it is closed and maximal dissipative. This in turn implies that  $A_0(q)$  is the infinitesimal generator of a  $C_0$  semigroup of contractions  $\{S_0(t; q) : t \geq 0\}$  on  $Z_q$  (see [10], Theorems 4.4 and 4.5). It is in fact the case that Stone's theorem ([20], pg. 345) implies that  $S_0(t; q)$  is defined for  $t < 0$  and that  $\{S_0(t; q) : -\infty < t < \infty\}$  is a  $C_0$  group of unitary operators on  $Z_q$ .

The operators  $B(t; q)$  are bounded (uniformly in  $t$  and  $q$  for  $t \in (0, T)$  and  $q \in Q$ ) from which it follows that  $\{A(t; q) : t \in [0, T]\}$  is a stable family of infinitesimal generators of  $C_0$  semigroups  $\{S_t(\tau; q) : \tau \geq 0\}$  on  $Z_q$  with stability constants 1 and  $K = \sup_{\substack{t \in [0, T] \\ q \in Q}} |B(t; q)|_q$  (see [12], Section 5.2). Since  $D_0$

is independent of  $t$  and  $t \rightarrow \sigma(t, \cdot, q) \in C^1((0, T), H^1(0, \ell))$  for each  $q \in Q$ , the homogeneous initial value problem

$$D_t z(t) = A(t; q) z(t) \quad 0 \leq s < t \leq T \tag{2.9}$$

$$z(s) = z_0 \tag{2.10}$$

with  $z_0 \in Z$  has a unique evolution system  $\{U(t, s; q) : 0 \leq s \leq t \leq T\}$  associated with it which satisfies

$$(i) \quad |U(t, s; q)|_q \leq e^{K(t-s)}$$

$$(ii) \quad U(t, s; q) D_0 \hookrightarrow D_0$$

(iii)  $U(t, s; q)z$  is strongly continuously differentiable in  $Z_q$  for all  $z \in D_0$  with

$$D_t U(t, s; q)z = A(t; q)U(t, s; q)z$$

and

$$D_s U(t, s; q)z = -U(t, s; q)A(s; q)z$$

for  $0 \leq s \leq t \leq T$ .

If  $z_0 \in D_0$ ,  $z(t) = U(t, s; q)z_0$  is the unique solution to (2.9) and (2.10).

For each  $t \in [0, T]$  and  $q \in Q$  let  $F(t; q) \in Z$  and  $z_0(q) \in Z$  be given by

$$F(t; q) = \left( (\mu\sigma(t, \ell; q) + \frac{1}{m} g_1(t; q) - \beta g_2(t; q), \lambda\sigma(t, \ell; q) + \gamma g_2(t; q))^T, 0, \frac{1}{\rho(q)} f(t, \cdot; q) \right)$$

and

$$z_0(q) = (r(q), D^2\phi(q), \psi(q))$$

respectively where  $r(q) \in R^2$  and consider

$$D_t z(t) = A(t; q)z(t) + F(t; q) \tag{2.11}$$

$$z(0) = z_0(q). \tag{2.12}$$

Writing formally

$$z(t; q) = U(t, 0; q)z_0(q) + \int_0^t U(t, \tau; q)F(\tau; q) d\tau, \tag{2.13}$$

assumptions A3-A6 imply that the function  $t \rightarrow z(t; q)$  is well defined and continuous from  $[0, T]$  into  $Z_q$ . If, in addition,  $t \rightarrow F(t; q) \in C^1([0, T], Z_q)$ ,  $\psi(q) \in H^2(0, \ell)$ ,  $\psi(0; q) = D\psi(0, q) = 0$  and  $r(q) = (\psi(\ell; q), D\psi(\ell; q))^T$  (i.e.,  $z_0(q) \in D_0$ ) then  $z(t; q)$  as given by (2.13) is the unique classical solution to (2.11), (2.12) in the sense that  $z(\cdot; q) \in C^1([0, T], Z_q)$ ,  $z(t; q) \in D_0$ ,  $0 \leq t \leq T$  and (2.11), (2.12) is satisfied in  $Z_q$  (see [12], Section 5.5). Under assumptions A3-A6 only, however, a classical solution to (2.11), (2.12) does not, in general, exist. In this case,  $z(t; q)$  as given by (2.13) is known as a mild or generalized solution to (2.11), (2.12) in that it

is the limit of classical solutions to sequences of problems of the form (2.11), (2.12) for which a unique classical solution does exist. (See [12]).

In light of the above remarks we use (2.13) to formulate the identification problem. For each  $x \in [0, \ell]$  define the operators  $C(x): Z \rightarrow R$  by

$$C(x)(r, u, v) = \int_0^x \int_0^T u(\sigma) d\sigma d\tau. \quad (2.14)$$

We assume that we have been provided with displacement measurements,  $\{u(t_i, x_j)\}_{i=1, \nu, j=1, \mu}$ ,  $t_i \in [0, T]$ ,  $i = 1, 2, \dots, \nu$ ,  $x_j \in [0, \ell]$   $j = 1, 2, \dots, \mu$ , taken from the actual system and state the identification problem as

(ID) Find  $q \in Q$  which minimizes

$$J(q) = \sum_{i=1}^{\nu} \sum_{j=1}^{\mu} \left| C(x_j)z(t_i; q) - u(t_i, x_j) \right|^2$$

where  $z(t; q)$  is given by (2.13).

The infinite dimensionality of the constraints, (2.13), of course necessitates the use of some form of approximation in solving problem (ID). We develop one such scheme in the next section.

### 3. APPROXIMATION AND CONVERGENCE RESULTS

Our approximation scheme is based upon the formulation of a sequence of approximating identification problems in which the underlying state equations are finite dimensional semi-discrete approximations to (2.13). The approximating evolution equations are constructed using a standard cubic spline based Galerkin approach to effect the spatial discretization. It will be shown that each of the approximating identification problems has a solution, and via convergence of the states, that the resulting sequence of solutions admits a subsequence which converges to a solution to problem (ID).

Working abstractly at first, for each  $N = 1, 2, \dots$  and each  $q \in Q$  let  $Z_q^N$  be a finite dimensional subspace of  $Z_q$  which is contained in  $D_0$ . Let  $P_q^N$  denote

the orthogonal projection of  $Z_q$  onto  $Z_q^N$  with respect to the  $\langle \cdot, \cdot \rangle_q$  inner product. Define the linear operators  $A_0^N(q): Z_q^N \rightarrow Z_q^N$ ,  $B^N(t; q): Z_q^N \rightarrow Z_q^N$ , and  $A^N(t; q): Z_q^N \rightarrow Z_q^N$  by  $A_0^N(q) = P_q^N A_0(q)$ ,  $B^N(t; q) = P_q^N B(t; q)$ , and  $A^N(t; q) = P_q^N A(t; q) = A_0^N(q) + B^N(t; q)$ , respectively. The finite dimensionality of  $Z_q^N$  implies of course, that each of these operators is bounded, although not necessarily uniformly in  $N$ .

Since  $A_0(q)$  is conservative, the  $B(t; q)$  are bounded uniformly for  $t \in [0, T]$  and  $q \in Q$  and the  $P_q^N$  are orthogonal projections, it follows that the  $A_0^N(q)$  are conservative and that the  $B^N(t; q)$  are bounded uniformly in  $N$  as well. Indeed for  $z^N \in Z_q^N$  we have

$$\langle A_0^N(q)z^N, z^N \rangle_q = \langle P_q^N A_0(q)z^N, z^N \rangle_q = \langle A_0(q)z^N, z^N \rangle_q = 0$$

and

$$\left| B^N(t; q)z^N \right|_q = \left| P_q^N B(t; q)z^N \right|_q \leq \left| B(t; q)z^N \right|_q \leq K \left| z^N \right|_q.$$

This in turn implies that the  $A_0^N(q)$  are infinitesimal generators of  $C_0$  semi-groups of contractions,  $\{S_0^N(t; q): t \geq 0\}$  on  $Z_q^N$  and that the initial value problems

$$D_t z^N(t) = A^N(t; q)z^N(t) \tag{3.1}$$

$$z^N(0) = z_0^N \tag{3.2}$$

have unique evolution systems  $\{U^N(t, s; q): 0 \leq s \leq t \leq T\}$  associated with them which satisfy

$$(i) \quad \left| U^N(t, s; q) \right|_q \leq e^{K(t-s)}$$

$$(ii) \quad D_t U^N(t, s; q)z^N = A^N(t; q)U^N(t, s; q)z^N \text{ for all } z^N \in Z_q^N, \\ 0 \leq s \leq t \leq T.$$

The two parameter families  $U^N(t, s; q)$  are the solution operators for the initial value problems (3.1) and (3.2). We note that since for each  $N$ ,  $Z_q^N$  is finite dimensional, once a suitable basis has been chosen, the initial value problem (3.1) and (3.2) can be written in matrix form with  $U^N(t, s; q)$  then being represented by the corresponding principal fundamental matrix solution.

For each  $q \in Q$  and  $N = 1, 2, \dots$  we define the function  $z^N(\cdot; q): [0, T] \rightarrow Z_q^N$  by

$$z^N(t; q) = U^N(t, 0; q) P_q^N z_0(q) + \int_0^t U^N(t, \tau; q) P_q^N F(\tau; q) d\tau \quad (3.3)$$

and state the approximating identification problem as

(IDN) Find  $q \in Q$  which minimizes

$$J^N(q) = \sum_{i=1}^{\nu} \sum_{j=1}^{\mu} \left| C(x_j) z^N(t_i; q) - u(t_i, x_j) \right|^2$$

where  $z^N(t; q)$  is given by (3.3) and the operators  $C(x)$  are as they were defined in (2.14).

Once a basis for  $Z_q^N$  has been chosen, problem (IDN) takes the form of a least squares minimization problem subject to a linear non-autonomous, non-homogeneous matrix ordinary differential equation which can be solved (assuming for the moment that a solution exists) using standard techniques and readily available software.

In terms of the abstract formulation above, our general convergence results are summarized in the following two theorems.

Theorem 3.1

Suppose  $\{q^N\}$  is a sequence in  $Q$  with  $q^N \rightarrow q^* \in Q$  as  $N \rightarrow \infty$ . Suppose further that



- (1)  $P_q^N \rightarrow I$  strongly in  $Z_q$  uniformly in  $q$  for  $q \in Q$  as  $N \rightarrow \infty$ .
- (2)  $\lim_{N \rightarrow \infty} \left| U^N(t, s; q^N) P_{q^N}^N z - P_{q^N}^N U(t, s; q^*) z \right|_{q^N} = 0$  uniformly in  $t, s$  for  $0 \leq s \leq t \leq T$ .

Then  $\lim_{N \rightarrow \infty} \left| z^N(t; q^N) - z(t; q^*) \right|_{q^N} \rightarrow 0$  for each  $t \in [0, T]$ .

Proof

$$\begin{aligned} \left| z^N(t; q^N) - z(t; q^*) \right|_{q^N} &\leq \left| U^N(t, 0; q^N) P_{q^N}^N (z_0(q^N) - z_0(q^*)) \right|_{q^N} \\ &+ \left| U^N(t, 0; q^N) P_{q^N}^N z_0(q^*) - P_{q^N}^N U(t, 0; q^*) z_0(q^*) \right|_{q^N} \\ &+ \left| (P_{q^N}^N - I) U(t, 0; q^*) z_0(q^*) \right|_{q^N} \\ &+ \int_0^t \left| U^N(t, \tau; q^N) P_{q^N}^N \right|_{q^N} \left| F(\tau; q^N) - F(\tau; q^*) \right|_{q^N} d\tau \\ &+ \int_0^t \left| (U^N(t, \tau; q^N) P_{q^N}^N - P_{q^N}^N U(t, \tau; q^*)) F(\tau; q^*) \right|_{q^N} d\tau \\ &+ \int_0^t \left| (P_{q^N}^N - I) U(t, \tau; q^*) F(\tau; q^*) \right|_{q^N} d\tau \end{aligned}$$

The properties of  $U^N(t, s; q)$ , the fact that  $P_q^N$  is an orthogonal projection and assumption A3 imply that the first term above tends to zero as  $N \rightarrow \infty$ . Hypotheses (1) and (2) in the statement of the theorem imply that the third and second terms respectively tend to zero as  $N \rightarrow \infty$ . Similar arguments and the Lebesgue dominated convergence theorem can be used to argue that the last three terms tend to zero as well and the theorem is proven.

Theorem 3.2

Suppose that hypotheses (1) and (2) of Theorem 3.1 hold. Suppose further that for each  $N = 1, 2, \dots$  problem (IDN) has a solution denoted by  $\overline{q}^N$ . Then

the sequence  $\{\bar{q}^N\}$  has a convergent subsequence,  $\{\bar{q}^{N_k}\}$  with  $\bar{q}^{N_k} \rightarrow \bar{q} \in Q$   $k \rightarrow \infty$ . Moreover,  $\bar{q}$  is a solution to problem (ID).

Proof

The existence of the convergent subsequence  $\{\bar{q}^{N_k}\}$  is an immediate consequence of Assumption A1. Theorem 3.1, therefore, implies that  $\left| z^N(t; \bar{q}^{N_k}) - z(t; \bar{q}) \right|_{\bar{q}^{N_k}} \rightarrow 0$  as  $k \rightarrow \infty$  for each  $t \in [0, T]$ . This in turn implies that  $\left| C(x)z^{N_k}(t; \bar{q}^{N_k}) - C(x)z(t; \bar{q}) \right| \rightarrow 0$  as  $k \rightarrow \infty$  for each  $x \in [0, \ell]$  and each  $t \in [0, T]$ . It then follows that for any  $q \in Q$

$$J(\bar{q}) = \lim_{k \rightarrow \infty} J^{N_k}(\bar{q}^{N_k}) \leq \lim_{k \rightarrow \infty} J^{N_k}(q) = J(q)$$

and consequently that  $\bar{q}$  is a solution to problem (ID). The final equality in the expression above follows from an application of Theorem 3.1 with the constant sequence  $\{q\}$ .

Remark

In the identification problem (ID) as stated in the previous section, the fit is based upon spatially sampled displacement measurements. We note, however, that the convergence results given in Theorems 3.1 and 3.2 remain valid for identification based upon spatially sampled slope measurements, spatially distributed displacement, slope or velocity observations, velocity data at  $x = \ell$  or any combination thereof.

We next describe a particular realization involving cubic spline functions of the abstract ideas presented above and show that the resulting approximation system satisfies the hypotheses of Theorems 3.1 and 3.2

For each  $N = 1, 2, \dots$  let  $\{B_j^N\}_{j=-1}^{N+1}$  denote the standard cubic B-splines on the interval  $[0, \ell]$  corresponding to the partition  $\Delta^N \equiv \{0, \frac{\ell}{N}, \frac{2\ell}{N}, \dots, \ell\}$

(see [13]) and let  $\{\hat{B}_j^N\}_{j=1}^{N+1}$  denote the modified cubic B-splines which satisfy  $\hat{B}_j^N(0) = DB_j^N(0) = 0$ ,  $j = 1, 2, \dots, N+1$ . The  $\{\hat{B}_j^N\}_{j=1}^{N+1}$  are given by

$$\hat{B}_1^N(x) = B_0^N(x) - 2B_1^N(x) - 2B_{-1}^N(x)$$

$$\hat{B}_j^N(x) = B_j^N(x) \quad j = 2, 3, \dots, N+1.$$

Let  $S^3(\Delta^N) = \text{SPAN} \{B_j^N\}_{j=-1}^{N+1}$ ,  $\hat{S}^3(\Delta^N) = \text{SPAN} \{\hat{B}_j^N\}_{j=1}^{N+1}$ ,  $Z^N = \{(\hat{s}(\ell), D\hat{s}(\ell))^T, s, \hat{s}\} \in Z: s \in S^3(\Delta^N), \hat{s} \in \hat{S}^3(\Delta^N)\}$ , and  $Z_q^N = \{Z^N, \langle \cdot, \cdot \rangle_q\}$ . Defining

$$\phi_j^N = ((0, 0)^T, B_j^N, 0) \quad j = -1, 2, \dots, N+1 \tag{3.4}$$

$$\psi_j^N = ((\hat{B}_j^N(\ell), D\hat{B}_j^N(\ell))^T, 0, \hat{B}_j^N) \quad j = 1, 2, \dots, N+1$$

we have that  $Z^N = \text{SPAN} \{\phi_j^N\}_{j=-1}^{N+1} + \text{SPAN} \{\psi_j^N\}_{j=1}^{N+1}$ ,  $\{\phi_j^N\}_{j=-1}^{N+1} \cup \{\psi_j^N\}_{j=1}^{N+1}$ ,

is a basis for  $Z^N$ , and since  $\hat{S}^3(\Delta^N) \subset S^3(\Delta^N) \subset C^2(0, \ell)$ , that  $Z_q^N$  is a  $2N + 4$  dimensional subspace of  $Z_q$  which is contained in  $D_0$ .

The vector representation  $\zeta^N$  with respect to the basis (3.4) for  $P_q^N z$ , where  $z$  is an arbitrary element in  $Z_q$  can be computed using the standard normal equation characterization for  $P_q^N$ :

$$\langle P_q^N z - z, z^N \rangle_q = 0 \quad z^N \in Z_q^N. \tag{3.5}$$

For  $z = (r, u, v) \in Z_q$  we find that

$$\zeta^N = [M_q^N]^{-1} H_q^N(z)$$

where

$$M_q^N = \left[ \begin{array}{c|c} C_q^N & 0 \\ \hline 0 & D_q^N \end{array} \right]$$

and

$$H_q^N(z) = \begin{bmatrix} u_q^N \\ v_q^N \end{bmatrix}$$

with

$$[C_q^N]_{i+2, j+2} = \int_0^\ell EI(q) B_i^N B_j^N \quad i, j = -1, 0, 1, \dots, N+1$$

$$[D_q^N]_{i, j} = [B_i^N(\ell), DB_i^N(\ell)] W \begin{bmatrix} \hat{B}_j^N(\ell) \\ DB_j^N(\ell) \end{bmatrix} + \int_0^\ell \rho(q) \hat{B}_i^N \hat{B}_j^N \quad i, j = 1, 2, \dots, N+1$$

$$[u_q^N]_{i+2} = \int_0^\ell EI(q) u B_i^N \quad i = -1, 0, 1, \dots, N+1$$

and

$$[v_q^N]_i = r^T W \begin{bmatrix} \hat{B}_i^N(\ell) \\ \hat{DB}_i^N(\ell) \end{bmatrix} + \int_0^\ell \rho(q) v B_i^N \quad i = 1, 2, \dots, N+1.$$

The matrix representation  $\bar{A}_0^N(q)$  for the operator  $A_0^N(q)$  can be computed using (3.5) with  $z = A_0(q) \tilde{z}^N$ ,  $\tilde{z}^N$  an arbitrary element in  $Z_q^N$ . We find

$$\bar{A}_0^N(q) = [M_q^N]^{-1} K_q^N$$

where

$$K_q^N = \left[ \begin{array}{c|c} 0 & E_q^N \\ \hline -[E_q^N]^T & 0 \end{array} \right]$$

with

$$[E_q^N]_{i+2,j} = \int_0^\ell EI(q) B_i^N D^2 B_j^N \quad i = -1, 0, 1, 2, \dots, N+1, \quad j = 1, 2, \dots, N+1.$$

Similarly the matrix representation  $\bar{B}^N(t; q)$  for the operator  $B^N(t; q)$  is found to be

$$\bar{B}^N(t; q) = [M_q^N]^{-1} L_q^N(t)$$

where

$$L_q^N(t) = \left[ \begin{array}{c|c} 0 & 0 \\ \hline G_q^N(t) & 0 \end{array} \right]$$

with

$$[G_q^N(t)]_{i,j+2} = - \int_0^\ell \sigma(t, \cdot; q) (DB_i^N) \int_0^\cdot B_j^N \quad \begin{array}{l} i = 1, 2, \dots, N+1 \\ j = -1, 0, 1, \dots, N+1 \end{array}$$

It then immediately follows that the matrix representation for  $A^N(t; q)$  is given by

$$\bar{A}^N(t; q) = \bar{A}_0^N(q) + \bar{B}^N(t; q) = \left[ \begin{array}{c|c} 0 & [C_q^N]^{-1} E_q^N \\ \hline [D_q^N]^{-1} [-[E_q^N]^T + G_q^N(t)] & 0 \end{array} \right].$$

If for each  $t \in [0, T]$  and  $q \in Q$  we set  $F_N^N(t; q) = [M_q^N]^{-1} H_q^N(F(t; q))$  and  $w_0^N(q) = [M_q^N]^{-1} H_q^N(z_0(q))$ , with respect to the cubic splines, the evolution equation (3.3), in differentiated form, is given by the following  $2N + 4$  dimensional initial value problem

$$D_t w^N(t) = \bar{A}^N(t; q) w^N(t) + F^N(t; q) \quad (3.6)$$

$$w^N(0) = w_0^N(q). \quad (3.7)$$

The approximating identification problems take the form

Find  $q \in Q$  which minimizes

$$J^N(q) = \sum_{i=1}^{\nu} \sum_{j=1}^{\mu} \left| \sum_{k=-1}^{N+1} w_{k+2}^N(t_i; q) s_k^N(x_j) - \hat{u}(t_i, x_j) \right|^2$$

where  $w^N(t; q)$  is the solution to (3.6), (3.7) corresponding to  $q \in Q$  and

$$s_j^N(x) = C(x) \phi_j^N = \int_0^x \int_0^\tau B_j^N(\sigma) d\sigma d\tau \quad j = -1, 0, 1, \dots, N+1.$$

In order to demonstrate that the scheme described above satisfies the hypotheses of Theorems 3.1 and 3.2 the following approximation theoretic results for cubic splines will be required (see [15], Chapter 4). Let  $I^N$  denote the standard cubic spline interpolation operator on  $[0, \ell]$  corresponding to the partition  $\Delta^N$ . That is, for  $\phi$  a function defined on the interval  $[0, \ell]$ ,  $I^N \phi$  is defined to be that element in  $S^3(\Delta^N)$  which satisfies  $(I^N \phi)(\frac{j\ell}{N}) = \phi(\frac{j\ell}{N})$ ,  $j = 0, 1, 2, \dots, N$  and  $D(I^N \phi)(\frac{j\ell}{N}) = D\phi(\frac{j\ell}{N})$   $j = 0$  and  $N$ .

Proposition 3.1 For  $\phi \in H^2(0, \ell)$

$$\left| D^k (I^N \phi - \phi) \right|_0 \leq C_k^1 N^{-2+k} \left| D^2 \phi \right|_0 \quad k = 0, 1$$

where  $C_k^1$  is independent of  $\phi$  and  $N$ .

Proposition 3.2 For  $\phi \in H^4(0, \ell)$

$$\left| D^k (I^N \phi - \phi) \right|_0 \leq C_k^2 N^{-4+k} \left| D^4 \phi \right|_0 \quad k = 0, 1, 2$$

where  $C_k^2$  is independent of  $\phi$  and  $N$ .

Let

$$V = R^2 \times H^0(0, \ell) \text{ and let } v_q = \{v, \langle\langle \cdot, \cdot \rangle\rangle_q\},$$

$$\langle\langle (r_1, v_1), (r_2, v_2) \rangle\rangle_q = r_1^T w r_2 + \langle v_1, v_2 \rangle_{\rho(q)}$$

where

$$\langle \phi, \psi \rangle_{\rho(q)} = \int_0^\ell \rho(q) \phi \psi.$$

Let

$$V^N = \{(\eta, \phi) \in V: \phi \in \hat{S}^3(\Delta^N), \eta = (\phi(\ell), D\phi(\ell))^T\}$$

and

$$v_q^N = \{v^N, \langle\langle \cdot, \cdot \rangle\rangle_q\}.$$

Let  $P_1^N(q)$  denote the orthogonal projection of  $H^0(0, \ell)$  onto  $S^3(\Delta^N)$  with respect to the inner product  $\langle \phi, \psi \rangle_{EI(q)} = \int_0^\ell EI(q) \phi \psi$  and  $P_2^N(q)$  denote the orthogonal projection of  $v_q$  on to  $v_q^N$ . Adopting the convention that for  $z = (r, u, v) \in Z_q$ ,  $z^N = P_q^N z$  will be denoted by  $(r^N, u^N, v^N)$ , it is easily seen that  $u^N = P_1^N(q)u$  and  $(r^N, v^N) = P_2^N(q)(r, v)$ .

Lemma 1 Let  $z = (r, u, v) \in D_0$ . Then

$$(1) \quad |u^N - u|_2 \rightarrow 0 \text{ as } N \rightarrow \infty \text{ uniformly in } q \text{ for } q \in Q.$$

$$(2) \quad |v^N - v|_2 \rightarrow 0 \text{ as } N \rightarrow \infty \text{ uniformly in } q \text{ for } q \in Q.$$

Proof

To verify (1) we show that  $|D^k(u^N - u)|_0 \rightarrow 0$  as  $N \rightarrow \infty$ ,  $k = 0, 1, 2$  uniformly in  $q$ . Recalling assumption A2, the fact that  $z \in D_0$  implies  $u \in H^2(0, \ell)$  and Proposition 3.1 above we have

$$\begin{aligned}
 |u^N - u|_0 &= |P_1^N(q)u - u|_0 \leq \frac{1}{m_{EI}} |P_1^N(q)u - u|_{EI(q)} \\
 &\leq \frac{1}{m_{EI}} |I^N u - u|_{EI(q)} \leq \frac{M_{EI}}{m_{EI}} |I^N u - u|_0 \\
 &\leq \frac{M_{EI}}{m_{EI}} C_0^1 N^{-2} |D^2 u|_0 \rightarrow 0 \quad \text{as } N \rightarrow \infty.
 \end{aligned}$$

The convergence of the first derivative is argued using the Schmidt inequality ([15] Theorem 1.5),

$$\begin{aligned}
 |D(u^N - u)|_0 &\leq |D(u^N - I^N u)|_0 + |D(I^N u - u)|_0 \\
 &\leq \tilde{K}N |u^N - I^N u|_0 + C_1^1 N^{-1} |D^2 u|_0 \\
 &\leq \tilde{K}N |u^N - u|_0 + \tilde{K}N |I^N u - u|_0 + C_1^1 N^{-1} |D^2 u|_0 \\
 &\leq \frac{\tilde{K}M_{EI}}{m_{EI}} C_0^1 N^{-1} |D^2 u|_0 + \tilde{K} C_0^1 N^{-1} |D^2 u|_0 + C_1^1 N^{-1} |D^2 u|_0 \rightarrow 0 \text{ as } N \rightarrow \infty.
 \end{aligned}$$

For the second derivative, we first note that for  $u \in H^2(0, \ell)$ , the Schmidt inequality and the first integral relation ([15] pg. 52) imply

$$\begin{aligned}
 |D^2 P_1^N(q)u|_0^2 &\leq 2 |D^2 (P_1^N(q)u - I^N u)|_0^2 + 2 |D^2 I^N u|_0^2 \\
 &\leq 2\tilde{K}N^4 |P_1^N(q)u - I^N u|_0^2 + 2 |D^2 u|_0^2 - 2 |D^2 (u - I^N u)|_0^2 \\
 &\leq 2\tilde{K}N^4 |P_1^N(q)u - u|_0^2 + 2\tilde{K}N^4 |I^N u - u|_0^2 + 2 |D^2 u|_0^2 \\
 &\leq \hat{K}^2 |D^2 u|_0^2
 \end{aligned} \tag{3.8}$$

where  $\hat{K}$  is independent of  $u$ ,  $N$ , and  $q \in Q$ . Now for  $w \in H^4(0, \ell)$  with  $w^N = P_1^N(q)w$ ,



Proposition 3.2 together with the Schmidt inequality imply that  $\left| D^k(w^N - w) \right|_0 = O(N^{-4+k})$ ,  $k = 0, 1, 2$ . Therefore

$$\begin{aligned} \left| D^2(u^N - u) \right|_0 &\leq \left| D^2(u^N - w^N) \right|_0 + \left| D^2(w^N - w) \right|_0 + \left| D^2(w - u) \right|_0 \\ &\leq \left| D^2(w^N - w) \right| + (1 + \hat{K}) \left| D^2(w - u) \right|_0 \\ &\leq O(N^{-2}) + (1 + \hat{K}) \left| D^2(w - u) \right|_0 \end{aligned}$$

where we have used (3.8) to bound  $\left| D^2(u^N - w^N) \right|_0$ . Since  $H^4(0, \ell)$  is dense in  $H^2(0, \ell)$  we can choose  $w$  and then  $N$  (since the  $O(N^{-2})$  term depends upon  $\left| D^4 w \right|_0$ ) to make the right hand side of the last inequality above arbitrarily small.

Turning next to statement (2) and recalling that  $z \in D_0$  implies that  $v \in H^2(0, \ell)$ ,  $v(0) = Dv(0) = 0$  and  $r = (v(\ell), Dv(\ell))^T$  we have

$$\begin{aligned} \left| v^N - v \right|_0^2 &\leq \frac{1}{m_\rho} \left\| (r^N, v^N) - (r, v) \right\|_q^2 \\ &\leq \frac{1}{m_\rho} \left\| ((\hat{I}^N v(\ell), D\hat{I}^N v(\ell))^T, \hat{I}^N v) - (v(\ell), Dv(\ell))^T, v \right\|_q^2 \\ &= \frac{1}{m_\rho} \left| \hat{I}^N v - v \right|_{\rho(q)}^2 = \frac{1}{m_\rho} \left| I^N v - v \right|_{\rho(q)}^2 \\ &\leq \frac{M_\rho}{m_\rho} \left| I^N v - v \right|_0^2 \leq \frac{M_\rho}{m_\rho} (C_0^1)^2 N^{-4} \left| D^2 v \right|_0^2 + 0 \end{aligned}$$

as  $N \rightarrow \infty$  where for  $\phi$  a function defined on  $[0, \ell]$ ,  $\hat{I}^N \phi$  denotes that element in  $\hat{S}^3(\Delta^N)$  which satisfies  $\hat{I} \phi \left( \frac{j\ell}{N} \right) = \phi \left( \frac{j\ell}{N} \right)$ ,  $j = 1, 2, \dots, N$  and  $D\hat{I}^N \phi(\ell) = D\phi(\ell)$ .

The convergence of the derivatives is verified in essentially the same manner as it was in the proof of Statement 1.

### Theorem 3.3

For the cubic spline based scheme described above, hypothesis (1) of Theorem 3.1 is satisfied.

Proof

Let  $z = (r, u, v) \in D_0$  and let  $z^N = P_q^N z = (r^N, u^N, v^N)$ . Then  $r = (v(\ell), Dv(\ell))^T$ ,  $r^N = (v^N(\ell), Dv^N(\ell))^T$ , and

$$\begin{aligned} |P_q^N z - z|_q^2 &= |z^N - z|_q^2 \\ &= (r^N - r)^T W (r^N - r) + |u^N - u|_{EI(q)}^2 + |v^N - v|_{\rho(q)}^2 \\ &\leq \|W\|_2 |v^N(\ell) - v(\ell)|^2 + \|W\|_2 |Dv^N(\ell) - Dv(\ell)|^2 \\ &\quad + M_{EI} |u^N - u|_0^2 + M_\rho |v^N - v|_0^2 \\ &\leq \tilde{K}_1 |u^N - u|_2^2 + \tilde{K}_2 |v^N - v|_2^2 \end{aligned}$$

where  $\|W\|_2$  is the Euclidean matrix norm of  $W$  and  $\tilde{K}_1$  and  $\tilde{K}_2$  are constants which are independent of  $N$ ,  $z$ , and  $q \in Q$ . Lemma 3.1 implies that the right hand side of the final inequality above tends toward zero as  $N \rightarrow \infty$  and consequently  $P_q^N z \rightarrow z$  as  $N \rightarrow \infty$  for all  $z \in D_0$ . However,  $D_0$  dense in  $Z_q$  (uniformly with respect to  $q \in Q$ ) and the  $P_q^N$  uniformly bounded in  $N$  (being orthogonal projections) imply that  $P_q^N z \rightarrow z$  as  $N \rightarrow \infty$  for all  $z \in Z_q$  uniformly in  $q$  for  $q \in Q$  and the theorem is proven.

In order to verify hypothesis (2) of Theorem 3.1 we require the following lemma.

Lemma 3.2

Let  $\{q^N\} \subset Q$  with  $q^N \rightarrow q^* \in Q$  as  $N \rightarrow \infty$ . Let  $Z_q^N$  be the cubic spline based subspaces of  $Z_q$  defined above,  $P_q^N$  the orthogonal projections of  $Z_q$  onto  $Z_q^N$ ,  $A_0^N(q) = P_q^N A_0(q)$ , and  $\{S_0^N(t; q) : t \geq 0\}$  the  $C_0$  semigroups of contractions generated by the  $A_0^N(q)$ . Then

$$\left| S_0^N(t; q^N) P_{q^N}^N z - P_{q^N}^N S_0(t; q^*) z \right|_{q^N} \rightarrow 0$$

as  $N \rightarrow \infty$  for each  $z \in Z$ , uniformly in  $t$  for  $t \in [0, T]$ .

Proof

Using a variation of the well known Trotter approximation theorem (see [4], Theorem 6.2) the desired conclusion will follow if we can show that

$$\left| R(\lambda, A_0^N(q^N)) P_{q^N}^N z - P_{q^N}^N R(\lambda, A_0(q^*)) z \right|_{q^N} \rightarrow 0$$

as  $N \rightarrow \infty$  for each  $z \in Z$  for some  $\lambda > 0$  where  $R(\lambda, A) = (\lambda - A)^{-1}$ . However  $\left| S^N(t; q) \right|_q \leq 1$  implies (see [9])

$$\begin{aligned} & \left| R(\lambda, A_0^N(q^N)) P_{q^N}^N z - P_{q^N}^N R(\lambda, A_0(q^*)) z \right|_{q^N} \\ &= \left| R(\lambda, A_0^N(q^N)) (A_0^N(q^N) P_{q^N}^N - P_{q^N}^N A_0(q^*)) R(\lambda, A_0(q^*)) z \right|_{q^N} \\ &\leq \frac{1}{\lambda} \left| (A_0^N(q^N) P_{q^N}^N - P_{q^N}^N A_0(q^*)) R(\lambda, A_0(q^*)) z \right|_{q^N} \\ &= \left| (A_0^N(q^N) P_{q^N}^N - P_{q^N}^N A_0(q^*)) y \right|_{q^N} \end{aligned}$$

where we have chosen  $\lambda = 1$  and  $y = R(\lambda, A_0(q^*)) z \in D_0$ . Now

$$\begin{aligned} & \left| (A_0^N(q^N) P_{q^N}^N - P_{q^N}^N A_0(q^*)) y \right|_{q^N} \\ &= \left| (P_{q^N}^N (A_0^N(q^N) P_{q^N}^N - P_{q^N}^N A_0(q^*)) y \right|_{q^N} \\ &\leq \left| (A_0^N(q^N) - A_0(q^*)) P_{q^N}^N y \right|_{q^N} + \left| A_0(q^*) (P_{q^N}^N - I) y \right|_{q^N} \\ &= T_1^N + T_2^N. \end{aligned}$$

Recalling the definition of the operator  $A_0(q)$ , (2.8), the fact that  $y \in D_0$  and the estimates given in Lemma 3.1, standard estimates yield  $T_2^N \rightarrow 0$  as  $N \rightarrow \infty$

while  $q^N \rightarrow q^*$  as  $N \rightarrow \infty$  and assumption A2 imply  $T_1^N \rightarrow 0$  as  $N \rightarrow \infty$ .

Theorem 3.4

Hypothesis (2) of Theorem 3.1 is satisfied by the cubic spline scheme.

Proof

Since for  $z \in Z$  and  $0 \leq s \leq t \leq T$  we have

$$\begin{aligned} D_t U(t, s; q)z &= A(t; q)U(t, s; q)z \\ &= A_0(q)U(t, s; q)z + B(t; q)U(t, s; q)z \end{aligned}$$

it follows that

$$U(t, s; q)z = S_0(t - s; q)z + \int_s^t S_0(t - \tau; q)B(\tau; q)U(\tau, s; q)z \, d\tau.$$

Similarly

$$U^N(t, s; q)P_q^N z = S_0^N(t - s; q)P_q^N z + \int_s^t S_0^N(t - \tau; q)B^N(\tau; q)U^N(\tau, s; q)P_q^N z \, d\tau.$$

Therefore, letting

$$\Delta^N(t, s) = \left| U^N(t, s; q^N)P_{q^N}^N z - P_{q^N}^N U(t, s; q^*)z \right|_{q^N},$$

we have

$$\begin{aligned} \Delta^N(t, s) &\leq \left| S_0^N(t - s; q^N)P_{q^N}^N z - P_{q^N}^N S_0(t - s; q^*)z \right|_{q^N} \\ &\quad + \int_s^t \left| S_0^N(t - \tau; q^N) \right|_{q^N} \left| B^N(\tau; q^N) \right|_{q^N} \Delta^N(\tau, s) \, d\tau \end{aligned}$$

$$\begin{aligned}
 & + \int_s^t \left| S_0^N(t - \tau; q^N) \right|_{q^N} \left| B^N(\tau; q^N) \right|_{q^N} \left| (P_{q^N}^N - I)U(\tau, s; q^*)z \right|_{q^N} d\tau \\
 & + \int_s^t \left| S_0^N(t - \tau; q^N) \right|_{q^N} \left| P_{q^N}^N \right|_{q^N} \left| (B(\tau; q^N) - B(\tau; q^*))U(\tau, s; q^*)z \right|_{q^N} d\tau \\
 & + \int_s^t \left| (S_0^N(t - \tau; q^N) P_{q^N}^N - P_{q^N}^N S_0^N(t - \tau; q^*))B(\tau; q^*)U(\tau, s; q^*)z \right|_{q^N} d\tau \\
 & + \int_s^t \left| (P_{q^N}^N - I)S_0^N(t - \tau; q^*)B(\tau; q^*)U(\tau, s; q^*)z \right|_{q^N} d\tau
 \end{aligned}$$

or

$$\begin{aligned}
 \Delta^N(t, s) & \leq \left| S_0^N(t - s; q^N) P_{q^N}^N z - P_{q^N}^N S_0^N(t - s; q^*)z \right|_{q^N} \\
 & + K \int_s^t \Delta^N(\tau, s) d\tau + \int_s^t h^N(\tau; t, s) d\tau \tag{3.9}
 \end{aligned}$$

where  $K = \sup \left| B(t; q) \right|_q$ . Lemma 3.2 implies that the first term on the right hand side of (3.9) tends toward zero as  $N \rightarrow \infty$  uniformly in  $s$  and  $t$  for  $0 \leq s \leq t \leq T$ , while Lemmas 3.1 and 3.2,  $q^N \rightarrow q^*$  as  $N \rightarrow \infty$ , assumption A2 and the boundedness and strong continuity of the operators imply  $h^N(\tau; t, s) \rightarrow 0$  as  $N \rightarrow \infty$  uniforming in  $\tau, t, s$  for  $0 \leq s \leq \tau \leq t \leq T$ . Therefore (3.9) can be written as

$$\Delta^N(t, s) \leq \varepsilon^N + K \int_s^t \Delta^N(\tau, s) d\tau$$

with  $\varepsilon^N \rightarrow 0$  as  $N \rightarrow \infty$  uniformly in  $t$  and  $s$ . An application of the Gronwall inequality yields

$$\Delta^N(t, s) \leq \varepsilon^N e^{K(t - s)}$$

from which the theorem immediately follows.

Finally, for the cubic spline scheme, under our general assumptions, using either standard continuous dependence results for ordinary differential equations or the Trotter approximation theorem it can be argued that for each  $N = 1, 2, \dots$  and  $t \in [0, T]$ ,  $z^N(t; q)$  given by (3.3) is continuous in  $q$ . Consequently  $J^N(q)$  is continuous in  $q$ , which together with assumption A1 implies that problem (IDN) has a solution.

Having now demonstrated that the hypotheses of Theorems 3.1 and 3.2 are satisfied by the cubic spline scheme, we turn next to a discussion of examples and numerical results which provides an indication of how well the scheme performs in practice.

#### 4. NUMERICAL RESULTS AND CONCLUDING REMARKS

In this section we present numerical results that were obtained by applying variations of the cubic spline based scheme discussed in the previous section to the identification of a variety of hybrid systems involving the vibration of beams with attached tip bodies. Since our primary objective was to demonstrate the feasibility of our scheme, we considered only relatively simple examples.

In the first example we consider a cantilevered beam with a tip (point) mass. The second example involves a free-free beam with an attached tip body at each end. A cantilevered beam with a tip body subject to an axially directed base acceleration is considered in the final example. Strictly speaking, the theory developed in the previous two sections applies directly only to the last example. However, the relatively minor modifications which are required to make our general method and the corresponding convergence results applicable to the other two examples should be immediately clear.

In general the observational data upon which the fits were based was obtained by generating solutions using fixed (or so called "true") values of the parameters with a Galerkin method and a finite number of the unforced system's natural mode shapes. Computing the modal frequencies and corresponding mode shapes for systems of the type considered here, in general requires the locating of zeros of transcendental equations involving the various beam and tip body parameters which appear in the problem. The resulting modal equations tend to be stiff and must be integrated using an appropriate method if a valid solution is to be obtained.

The approximating identification problems (IDN) are solved using the IMSL (see [8]) routine ZXSSQ. This routine is an iterative Levenberg-Marquardt Newton's Method/Steepest descent hybrid algorithm for the minimization of the sum of squares of a system of functions of several variables. The required gradients and entries in the Jacobian matrix are computed numerically using finite difference approximations. The method required that we supply initial start-up values for the unknown parameters and a subroutine which evaluates  $J^N(q)$  for a given value of  $q$ . The latter requirement necessitates the integration of the initial value problem (3.6) and (3.7). This is accomplished

using the IMSL routine DGEAR, a variable order ADAMS predictor-corrector method. It is interesting to note that unlike the modal approximations, the spline equations did not require the use of the stiff option. This of course makes the spline schemes attractive from a computational point of view. We note also that due to the narrow support of the B-splines, the matrices which appear in the resulting Galerkin equations tend to be banded, thus facilitating efficient integration of the system of differential equations. The inner products which determine the entries in the mass and stiffness matrices, as well as the generalized Fourier coefficients required to project the non-homogeneous term and the initial conditions were computed using a composite two point Gauss-Legendre quadrature rule.

Example 4.1

We consider a cantilevered beam of length  $\ell$ , constant stiffness  $EI$  and mass density  $\rho$  with a tip mass of mass  $m$  (see Figure 4.1). The differential equations and boundary conditions which describe the system are given by

$$\rho D_t^2 u = -EI D_x^4 u + f$$

$$m D_t^2 u(t, \ell) = EI D_x^3 u(t, \ell) + g$$

$$u(t, 0) = D_x u(t, 0) = D_x^2 u(t, \ell) = 0$$

$$u(0, x) = \phi(x) \quad D_t u(0, x) = \psi(x)$$

where we have assumed that no axially directed loading is present.

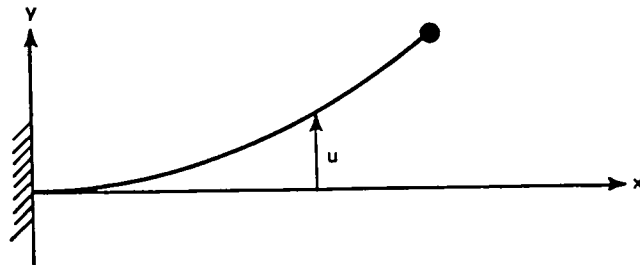


Figure 4.1.

We set  $l = 1.0$ ,  $EI = 1.0$ ,  $\rho = 3.0$ ,  $m = 1.5$ , assumed that the system was initially at rest (i.e.,  $\phi = \psi = 0$ ) and excited the structure with an impulsive lateral force at the end of the beam at time  $t = 0$ . The input disturbance was modeled as  $f(t, x) = 20e^{-2t}e^{-20(1-x)}$  and  $g(t) = 0$ . Using the first three normal modes of the unforced system to generate observations in the form of displacement measurements at positions  $x_j = 0.125(j+3)$ ,  $j = 1, 2, \dots, 5$ , and times  $t_j = 0.2 j$ ,  $j = 1, 2, \dots, 10$  we identified  $EI$  and  $\rho$ . The start-up values for the Levenberg-Marquardt routine were taken to be  $EI_0 = 0.7$  and  $\rho_0 = 2.7$ . The final converged values for  $\overline{EI}^N$ ,  $\overline{\rho}^N$  and the residual sum of squares  $\overline{J}^N$  are given in Table 4.1 below.

Table 4.1.

N	$\overline{EI}^N$	$\overline{\rho}^N$	$\overline{J}^N$
2	0.9976	3.0262	$0.39 \times 10^{-5}$
3	0.9994	3.0382	$0.46 \times 10^{-5}$
4	0.9951	3.0544	$0.48 \times 10^{-5}$
5	0.9961	3.0409	$0.40 \times 10^{-5}$
6	0.9995	2.9976	$0.35 \times 10^{-5}$

True Values:            1.0            3.0

It is clear that relatively accurate estimates of the parameters can be obtained using small values of  $N$ .

Based upon the scheme's performance on examples for which exact solutions were available, we feel that the somewhat erratic convergence exhibited in Table 4.1 is most likely a consequence of using approximate solutions to generate observations.



Example 4.2

In this example we identify the stiffness and mass density of a free-free beam of length  $l$  which has a tip body (having different mass properties) attached to each end (see Fig. 4.2). The system was assumed to be initially at rest and then excited by a time varying, spatially distributed transverse load given by  $f(t, x) = 10 \sin(2\pi t)e^x$ .

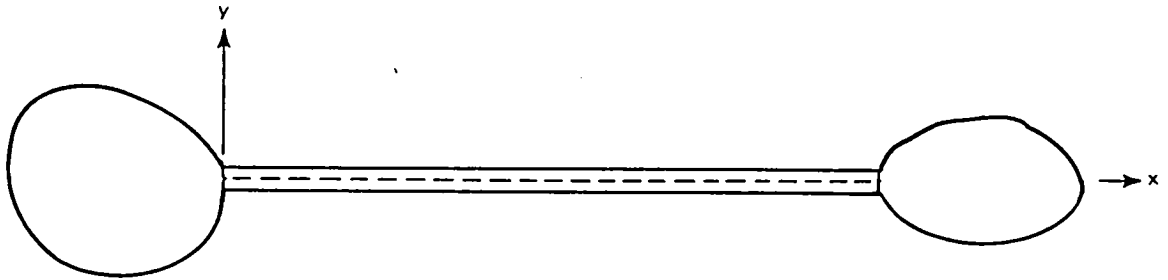


Figure 4.2.

The vibrations of the beam are described by the following partial differential equation and boundary conditions (see [16])

$$\begin{aligned}
 \rho D_t^2 u &= -E I D_x^4 u + f & t > 0 & \quad x \in (0, l) \\
 D_t^2 u &= -\alpha_1 E I D_x^3 u + \beta_1 E I D_x^2 u & t > 0 & \quad x = 0 \\
 D_t^2 D_x u &= -\beta_1 E I D_x^3 u + \gamma_1 E I D_x^2 u & t > 0 & \quad x = 0 \\
 D_t^2 u &= \alpha_2 E I D_x^3 u + \beta_2 E I D_x^2 u & t > 0 & \quad x = l \\
 D_t^2 D_x u &= -\beta_2 E I D_x^3 u - \gamma_2 E I D_x^2 u & t > 0 & \quad x = l
 \end{aligned} \tag{4.1}$$

$$u = 0 \quad D_t u = 0 \quad t = 0 \quad x \in [0, \ell]$$

with

$$\alpha_i = \frac{J_i + m_i c_i^2}{m_i (J_i + m_i c_i^2 \sin^2 \delta_i)},$$

$$\beta_i = \frac{c_i \cos \delta_i}{J_i + m_i c_i^2 \sin^2 \delta_i}$$

and

$$\gamma_i = \frac{1}{J_i + m_i c_i^2 \sin^2 \delta_i} \quad i = 1, 2$$

where the mass properties for the tip bodies,  $m_i$ ,  $J_i$ ,  $c_i$ , and  $\delta_i$  are as they were defined for the single tip body problem in Section 2.

Settling  $EI = 1.0$ ,  $\rho = 3.0$ ,

$$m_1 = 0.75, \quad J_1 = 0.6, \quad c_1 = 0.1, \quad \delta_1 = \pi/6,$$

$$m_2 = 1.5, \quad J_2 = 0.4, \quad c_2 = 0.2, \quad \delta_2 = \pi/3,$$

and  $\ell = 1.0$  we based our fit on velocity data at the ends of the beam only generated at times  $t_j = 0.2 j$ ,  $j = 1, 2, \dots, 10$  using the first six natural modes including the two which correspond to rigid body translation and rotation (see the remark following the proof of Theorem 3.2). We note that the system in (4.1) is free with respect to the inertial frame in which displacement is measured. A formula analogous to (2.14) therefore can not be used to recover displacement from the solutions to either the abstract or the approximating evolution equations.

The start-up values used were  $EI_0 = 0.7$  and  $\rho_0 = 2.7$ . Our results are summarized in Table 4.2

Table 4.2

N	$\frac{N}{EI}$	$\frac{N}{\rho}$	$\frac{N}{J}$
2	0.9957	3.0052	$0.26 \times 10^{-3}$
3	0.9963	3.0065	$0.18 \times 10^{-3}$
4	0.9982	3.0037	$0.10 \times 10^{-3}$
5	1.0018	3.0008	$0.71 \times 10^{-4}$

True Value:            1.0            3.0

Example 4.3

For our final example, we consider a problem of the form discussed in Sections 2 and 3. The parameters to be estimated are the spatially invariant stiffness EI and mass density  $\rho$ . We assume that the system is initially at rest and is then acted upon by an impulsive lateral force at the end of the beam at time  $t = 0$  and a piecewise constant base acceleration described by

$$f(t, x) = 20 e^{-2t} e^{-20(1-x)}$$

and

$$a_0(t) = \begin{matrix} 1.0 & 0 \leq t \leq 1.5 \\ 0 & 1.5 < t < 3.0 \\ 1.0 & 3.0 \leq t < 4.0 \\ 0 & 4.0 \leq t \end{matrix}$$

respectively.

The externally applied axial load  $\sigma(t, x)$  can be related to the base acceleration  $a_0(t)$  as follows. Recalling that  $\tau$  denotes the internal tension in the beam, equilibrium in the x direction yields  $D_x \tau = \rho a_0$ , or

$$\tau(t, x) = -\rho(\ell - x) a_0(t) + \tau(t, \ell). \tag{4.2}$$

For the tip body we have

$$-\tau(t, \ell) = m a_0(t) - m c \sin \delta D_t^2 D_x u(t, \ell). \tag{4.3}$$

Combining (4.2) and (4.3) we obtain

$$\tau(t, x) = -\rho(\ell - x)a_0(t) - ma_0(t) + mc \sin \delta \cdot D_t^2 D_x^2 u(t, \ell),$$

from which we immediately conclude that  $\sigma(t, x) = -a_0(t)(\rho(\ell - x) + m)$ . Taking  $EI = 1.0$ ,  $\rho = 2.0$ ,  $m = 4.0$ ,  $J = 0.4$ ,  $c = 0.2$ ,  $\delta = \pi/3$  and  $\ell = 1.0$ , displacement data at positions  $x_j = 0.75, 0.87, 1.0$ , at times  $t_j = 4.0, 4.5, 5.0$  was generated using the first 3 natural modes for the unforced, fixed base beam/tip body system. The start-up values were chosen as  $EI_0 = 0.7$  and  $\rho_0 = 2.5$ . The results are given in Table 4.3.

Table 4.3.

N	$\overline{EI}^N$	$\overline{\rho}^N$	$\overline{J}^N$
2	1.0003	2.0635	$0.35 \times 10^{-3}$
3	1.0014	2.0668	$0.22 \times 10^{-3}$
4	1.0007	2.0404	$0.26 \times 10^{-3}$
5	.9994	2.0274	$0.35 \times 10^{-3}$

True Values:                    1.0            2.0

Once again the erratic convergence is most probably attributable to using an approximate solution to generate observations.

We note that strictly speaking, our theory requires that  $a_0 \in C^1$ . However, as is evidenced by the example above the scheme appears to perform satisfactorily on problems involving  $a_0$  which are discontinuous.

Although it was not considered in the present effort, it is possible to include terms which model viscous damping in the partial differential equations and boundary conditions. Furthermore, it is possible to extend our approximation theory and corresponding convergence results to include the ability to estimate parameters associated with the damping effects (see [1], [7], and [11]).

It is also possible to develop an approximation theory which is similar in spirit to the one presented here for the identification of models based upon the use of the higher-order Timoshenko theory to model the vibration of the beam. The Timoshenko theory includes effects due to shear deformation and rotatory inertia (see [2], [6], [7], and [19]).

Finally we note that while the scheme described in Section 3 is cubic spline based, because of the choice of state variables, it in fact relies upon quintics to represent displacement. We are currently developing a method using a somewhat more direct approach than the abstract operator formulation employed here which does use cubic splines to represent displacement. Starting with a weak/variational form of the underlying hybrid system, a cubic spline based Galerkin approach is used to construct a sequence of approximating identification problems wherein the constraints are given by a finite dimensional linear second order ordinary differential equation in the approximate displacement. Based upon preliminary results this scheme promises to be computationally more efficient than and to perform as well as the one which was presented above. Moreover, it will permit the identification of spatially varying  $EI$  and  $\rho$  under for less stringent smoothness and continuity hypotheses than the ones given in assumption A2. This work will be discussed in a forthcoming paper.

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16. Abstract  A cubic spline based Galerkin-like method is developed for the identification of a class of hybrid systems which describe the transverse vibration of flexible beams with attached tip bodies. The identification problem is formulated as a least squares fit to data subject to the system dynamics given by a coupled system of ordinary and partial differential equations recast as an abstract evolution equation (AEE) in an appropriate infinite dimensional Hilbertspace. Projecting the AEE into spline-based subspaces leads naturally to a sequence of approximating finite dimensional identification problems. The solutions to these problems are shown to exist, are relatively easily computed, and are shown to, in some sense, converge to solutions to the original identification problem. Numerical results for a variety of examples are discussed.					
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