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A SIMPLE RECTANGULAR ELEMENT FOR TWO-DIMENSIONAL ANALYSIS OF LAMINATED COMPOSITES

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ABSTRACT

A simple rectangular finite element was developed for two-dimensional analysis of laminated composite materials. The rectangular laminated composite (RLC) element eliminates the need to add elements to a model simply to account for the material properties of various laminae. This is particularly advantageous for thick laminates with many lamina. Explicit integration in terms of generalized displacements minimizes the algebraic effort required to derive the element stiffnesses and the thermal load vector. A substitute shape function technique is used to improve the performance of the element in modeling bending type deformation. Results for several example problems are discussed.

INTRODUCTION

The development of an appropriate finite element mesh is a key step in successful finite element analysis. For homogeneous materials the mesh refinement is dictated by geometrical considerations. The shape of a structure should be faithfully modeled. Also, extra mesh refinement is required in regions with strong strain gradients caused by holes, cracks, or boundary conditions. For laminated materials the analyst must also account for the different material properties of the various laminae. These ideas are illustrated by the laminated composite beam shown in figure 1. Geometrical considerations require very few elements except close to the point where the load is applied, where strain gradients are large. But since standard finite elements cannot account for stacking sequence effects such elements should not span across lamina boundaries. Hence, because of the laminated character of the material, the mesh should be highly refined even where the strain gradients are small.

The expense of modeling each lamina individually rapidly becomes intolerable as the number of laminae increases. Reference 1 presents an approximate technique to reduce costs. Laminate theory is used to obtain effective extensional moduli for a group of laminae. Then the group of laminae rather than the individual lamina are modeled using finite elements. This approach ignores stacking sequence effects within the lamina group. Therefore, the flexural and flexural-extension coupling properties of the lamina group cannot be faithfully modeled. Reference 2 presents a hybrid analysis for thick laminates. In this analysis (which is not a finite element analysis) the laminate is divided into global and local regions. The terms global and local refer to the detail with which the individual lamina is modeled; the local region is modeled with much greater detail than the global region. Conceptually, this is similar to using a finite element model with smaller- or higher-order elements in one region than in another. However, the analysis in reference 2 does not offer the inherent flexibility of the finite element method for modeling complicated geometries and for performing convergence checks. The objective of this paper is to introduce a new type of two-dimensional (i.e., plane stress or plane strain) finite element for analysis of laminated composites.

The element is a four-node, bilinear, rectangular element. An ordinary bilinear rectangle performs poorly in modeling bending-type deformation. The performance can be improved by using reduced numerical integration or substitute shape functions (ref. 3). Because of the multiple laminae within the element, numerical integration is not appropriate. Therefore, substitute shape functions are used to improve the performance. Explicit integration of the element stiffness matrix in terms of generalized displacements minimizes the algebraic effort required to account for the various laminae within a single element.

After describing the theoretical aspects of the element, results from analyses of several simple configurations are discussed.

NOMENCLATURE

A	area
$A_{11},A_{12},A_{22},A_{33},B_{11},B_{12},D_{11}$	coefficients related to laminae material properties
$a,b,c,d,\bar{a},\bar{b},\bar{c},\bar{d},e,f,g,\bar{e},\bar{f},\bar{g},h$	generalized degrees of freedom
C_{ij}	plane stress or plane strain material stiffness coefficients $i,j = 1,3$
F	force vector
$\bar{F}$	transformed force vector
F_n	force corresponding to degree of freedom n
F_N	subvector of force vector related to normal strains
F_S	subvector of force vector related to shear strain
H	matrix used in calculation of transformation matrix
K	element stiffness matrix
$\bar{K}$	transformed element stiffness matrix
K_{nm}	stiffness term in K, $n,m = 1$, number of degrees of freedom
K_N	submatrix of stiffness matrix related to normal strains
K_S	submatrix of stiffness matrix related to shear strain
ℓ_x	half-length of element in x-direction
ℓ_y	half-length of element in y-direction
N	number of plies
T	transformation matrix
T_N	transformation matrix for normal strain related terms in generalized stiffness matrix and force vector
T_S	transformation matrix for shear strain related terms in generalized stiffness matrix and force vector
t	thickness in z-direction
U	strain energy

u	displacement in x-direction
u_1, u_2, u_3, u_4	nodal displacements in x-direction
v	displacement in y-direction
v_1, v_2, v_3, v_4	nodal displacements in y-direction
W	potential energy of external loads
x, y	rectangular Cartesian coordinates
y_i	y-coordinate of bottom surface of i^{th} ply. Bottom ply in element is ply 1.
Δ	vector of generalized displacements
Δ_n	generalized degree of freedom n
δ	vector of nodal displacements
ε_i	strains ($\varepsilon_1 \equiv \varepsilon_x$, $\varepsilon_2 \equiv \varepsilon_y$, $\varepsilon_3 \equiv \varepsilon_{xy}$)
$\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$	strains
Π	total potential energy

Definitions

generalized displacements	parameters related to translation, rotation, and deformation of an element but not associated with a node
generalized forces	force associated with generalized displacement
generalized stiffness matrix	stiffness of an element in terms of generalized displacements

THEORY

The RLC element is a rectangular element with four nodes. All laminae in the element are assumed to be orthotropic, oriented parallel to the x-axis, and to extend across the element width (fig. 2). The origin of the x-y coordinate system is at the element centroid.

The following sections derive the RLC element stiffness matrix and the equivalent nodal load vector for initial thermal strains. The derivation begins with the presentation of general expressions for element forces and stiffnesses for an arbitrary finite element. Next, the particular shape functions used to approximate the displacements and strains are discussed. Then explicit expressions for the stiffness matrix and element forces due to thermal strains are derived. These expressions are in terms of generalized displacements. The final section describes how to transform the stiffness matrix and forces from a system of generalized displacements to one of nodal displacements.

Cartesian tensor notation is used herein to express several of the complicated equations in compact form. In these compact equations the strains ε_1, ε_2, and ε_3 refer to ε_x, ε_y, and ε_{xy}, respectively. Also, some parameters refer to an entire vector or matrix when there is no subscript (eg. F) and to a single value when there is a subscript (eg. F_n).

General Expressions

The total potential energy, Π , is given by equation (1) (ref. 1):

$$\Pi = U + W = \frac{t}{2}\int C_{ij}\varepsilon_j\varepsilon_i \, dA - F_n \Delta_n \tag{1}$$

where U is the strain energy and W is the potential energy of the applied loads. Minimization of Π with respect to the generalized degrees of freedom (d.o.f.), Δ_n, yields the generalized force F_n associated with each d.o.f..

$$F_n = \frac{\partial U}{\partial \Delta_n} = t\int C_{ij}\varepsilon_j \frac{\partial \varepsilon_i}{\partial \Delta_n} \, dA \tag{2}$$

The terms in the element stiffness matrix are calculated by differentiating the generalized forces with respect to the d.o.f., equation (3).

$$K_{nm} = \frac{\partial F_n}{\partial \Delta_m} = \frac{\partial^2 U}{\partial \Delta_n \, \partial \Delta_m} = t\int C_{ij} \frac{\partial \varepsilon_j}{\partial \Delta_m} \frac{\partial \varepsilon_i}{\partial \Delta_n} \, dA + t\int C_{ij}\varepsilon_j \frac{\partial^2 \varepsilon_i}{\partial \Delta_n \, \partial \Delta_m} \, dA \tag{3}$$

Since linear strain-displacement relations are used in this paper, the term $\frac{\partial^2 \varepsilon_i}{\partial \Delta_n \, \partial \Delta_m}$ is zero. Therefore, the terms in the element stiffness matrix are

$$K_{nm} = t\int C_{ij} \frac{\partial \varepsilon_j}{\partial \Delta_m} \frac{\partial \varepsilon_i}{\partial \Delta_n} \, dA \tag{4}$$

Shape Functions and Strain Expressions

The technique of substitute shape functions was used to improve the performance of the RLC element in modeling bending type deformation (ref. 3). This technique involves using different shape functions for terms related to normal strains and for those related to shear strains.

The shape functions used in calculating terms related to normal strains are given by equations (5):

$$
\begin{aligned}
u &= a + bx + cy + dxy \\
v &= \bar{a} + \bar{b}x + \bar{c}y + \bar{d}xy
\end{aligned}
\tag{5}
$$

The normal strains, ε_x and ε_y, are therefore

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = b + dy \\
\varepsilon_y &= \frac{\partial v}{\partial y} = \bar{c} + \bar{d}x
\end{aligned}
\tag{6}
$$

The shape functions used in calculating terms related to the shear strain are given in equations (7)

$$
\begin{aligned}
u &= e + fx + gy \\
v &= \bar{e} + \bar{f}x + \bar{g}y
\end{aligned}
\tag{7}
$$

The shear strain, ε_{xy}, is therefore

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = g + \bar{f} \tag{8}$$

Equation (8) shows that the shear strain is constant for the RLC element. This constant value is defined to be "h" in equation (9).

$$\varepsilon_{xy} = h = g + \bar{f} \tag{9}$$

Element Stiffness Matrix

Identification of the relevant d.o.f. is the first step in the calculation of the element stiffness matrix. Equation (4) shows that K is only a function of those d.o.f. used to define the strains. Hence, the relevant d.o.f. are

$$\Delta = \begin{bmatrix} b \\ d \\ \bar{c} \\ \bar{d} \\ h \end{bmatrix} \tag{10}$$

The 5 × 5 element stiffness matrix can now be calculated using equations (4), (6), (8), (9), and (10). The non-zero terms in the stiffness matrix are

$$\begin{aligned}
K_{11} &= 2\ell_x t\ A_{11} & K_{44} &= \tfrac{2}{3}\ell_x^3 t\ A_{22} \\
K_{12} &= 2\ell_x t\ B_{11} & K_{55} &= 2\ell_x t\ A_{33} \\
K_{13} &= 2\ell_x t\ A_{12} & K_{21} &= K_{12} \\
K_{22} &= 2\ell_x t\ D_{11} & K_{31} &= K_{13} \\
K_{23} &= 2\ell_x t\ B_{12} & K_{32} &= K_{23} \\
K_{33} &= 2\ell_x t\ A_{22}
\end{aligned} \tag{11}$$

The remaining K_{nm} are zero.

where

$$A_{11} = \sum_{i=1}^{N} C_{11}^{i}(y_{i+1} - y_i)$$

$$A_{12} = \sum_{i=1}^{N} C_{12}^{i}(y_{i+1} - y_i)$$

$$A_{22} = \sum_{i=1}^{N} C_{22}^{i}(y_{i+1} - y_i)$$

$$A_{33} = \sum_{i=1}^{N} C_{33}^{i}(y_{i+1} - y_i) \tag{12}$$

$$B_{11} = \frac{1}{2} \sum_{i=1}^{N} C_{11}^{i}\left(y_{i+1}^{2} - y_i^{2}\right)$$

$$B_{12} = \frac{1}{2} \sum_{i=1}^{N} C_{12}^{i}\left(y_{i+1}^{2} - y_i^{2}\right)$$

$$D_{11} = \frac{1}{3} \sum_{i=1}^{N} C_{11}^{i}\left(y_{i+1}^{3} - y_i^{3}\right)$$

where

t = element thickness

ℓ_x = half-width of element (see fig. 2)

ℓ_y = half-height of element (see fig. 2)

N = number of laminae

C_{ij} = plane stress or plane strain stiffness coefficients, $(i,j = 1,3)$

y_i = y coordinate of bottom surface of ply number "i", $(i = 1,N)$

Note that in equations (11), the only nonzero term in K related to the shear strain is K_{55}.

Thermal Load Vector

Equation (2) gives the general expression for element forces. As was the case for the stiffness matrix, the relevant d.o.f. for thermal loads are given by equation (10). The strain ε_j to be used in equation (2) is

$$\varepsilon_j = \alpha_j \, \Delta T \tag{13}$$

Because the material is orthotropic, $\alpha_3 = 0$. The element forces corresponding to each d.o.f. can now be calculated using equations (2), (6), (8), (9), (10), and (13). These forces are

$$\begin{aligned}
F_1 &= 2t\ell_x \sum_{i=1}^{N} (C_{11}^i \alpha_1^i \, \Delta T^i + C_{12}^i \alpha_2^i \, \Delta T^i)(y_{i+1} - y_i) \\
F_2 &= t\ell_x \sum_{i=1}^{N} (C_{11}^i \alpha_1^i \, \Delta T^i + C_{12}^i \alpha_2^i \, \Delta T^i)(y_{i+1}^2 - y_i^2) \\
F_3 &= 2t\ell_x \sum_{i=1}^{N} (C_{12}^i \alpha_1^i \, \Delta T^i + C_{22}^i \alpha_2^i \, \Delta T^i)(y_{i+1} - y_i) \\
F_4 &= 0 \\
F_5 &= 0
\end{aligned} \tag{14}$$

Note that the force corresponding to the shear strain related d.o.f., F_5, is zero.

Transformation of Element Stiffnesses and Forces

The preceding sections give expressions for element forces and stiffnesses for a system with generalized d.o.f. Δ. However, to assemble element stiffnesses and forces into a global system of equations requires that the d.o.f. be nodal displacements, not generalized displacements.

Equations (15) give the general procedure for transforming the stiffness matrix and force vector from one set of d.o.f. (Δ) to another set (δ):

$$\begin{aligned} K' &= T^T K T \\ F' &= T^T F \end{aligned} \tag{15}$$

where T is defined by the equation:

$$\Delta = T\delta$$

In equations (15), K and F are in terms of generalized displacements Δ and K' and F' are in terms of nodal displacements δ. The matrix T is the transformation matrix.

The first step is to calculate the transformation matrix T. Since the displacements u and v are approximated by different shape functions for terms related to normal and shear strains, the transformation matrices for these two types of terms must be calculated independently.

The transformation matrix for the terms related to normal strains is calculated first. Equations expressing the nodal displacements u_i, v_i, $i = 1,4$ in terms of all the element's generalized displacements are given in equations (16).

$$\underset{\uparrow\,\delta}{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}} = \underset{\uparrow\,H}{\begin{bmatrix} 1 & -\ell_x & -\ell_y & \ell_x\ell_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\ell_x & -\ell_y & \ell_x\ell_y \\ 1 & \ell_x & -\ell_y & -\ell_x\ell_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \ell_x & -\ell_y & -\ell_x\ell_y \\ 1 & \ell_x & \ell_y & \ell_x\ell_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \ell_x & \ell_y & \ell_x\ell_y \\ 1 & -\ell_x & \ell_y & -\ell_x\ell_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\ell_x & \ell_y & -\ell_x\ell_y \end{bmatrix}} \underset{\uparrow\,\Delta}{\begin{bmatrix} a \\ b \\ c \\ d \\ \bar{a} \\ \bar{b} \\ \bar{c} \\ \bar{d} \end{bmatrix}} \tag{16}$$

Equations (16) are formed using the expressions in equations (5). Equations (16) can be solved for Δ.

$$\underset{\uparrow\,\Delta}{\begin{bmatrix} a \\ b \\ c \\ d \\ \bar{a} \\ \bar{b} \\ \bar{c} \\ \bar{d} \end{bmatrix}} = \frac{1}{4\,\ell_x\ell_y} \underset{\uparrow\,4\ell_x\ell_y\,H^{-1}}{\begin{bmatrix} \ell_x\ell_y & 0 & \ell_x\ell_y & 0 & \ell_x\ell_y & 0 & \ell_x\ell_y & 0 \\ -\ell_y & 0 & \ell_y & 0 & \ell_y & 0 & -\ell_y & 0 \\ -\ell_x & 0 & -\ell_x & 0 & \ell_x & 0 & \ell_x & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & \ell_x\ell_y & 0 & \ell_x\ell_y & 0 & \ell_x\ell_y & 0 & \ell_x\ell_y \\ 0 & -\ell_y & 0 & \ell_y & 0 & \ell_y & 0 & -\ell_y \\ 0 & -\ell_x & 0 & -\ell_x & 0 & \ell_x & 0 & \ell_x \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \end{bmatrix}} \underset{\uparrow\,\delta}{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}} \tag{17}$$

Comparison of H^{-1} in equation (17) and T in equation (15) shows that $T = H^{-1}$. Since the generalized forces F_n and stiffnesses K_{nm} involve only the d.o.f. b, d, $\bar{c}$, and $\bar{d}$, only rows 2, 4, 7, and 8 of H^{-1} are required for the transformation matrix. Hence, the transformation matrix for terms related to normal strains is

$$T_N = \frac{1}{4\ell_x \ell_y} \begin{bmatrix} -\ell_y & 0 & \ell_y & 0 & \ell_y & 0 & -\ell_y & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -\ell_x & 0 & -\ell_x & 0 & \ell_x & 0 & \ell_x \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \end{bmatrix} \tag{18}$$

Now the transformation matrix for the terms related to shear strains is derived. As before, the first step is to express the nodal displacements in terms of the generalized displacements; in this case equations (7) are used.

$$\underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}}_{\delta} = \underbrace{\begin{bmatrix} 1 & -\ell_x & -\ell_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\ell_x & -\ell_y \\ 1 & \ell_x & -\ell_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \ell_x & -\ell_y \\ 1 & \ell_x & \ell_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \ell_x & \ell_y \\ 1 & -\ell_x & \ell_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\ell_x & \ell_y \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} e \\ f \\ g \\ \bar{e} \\ \bar{f} \\ \bar{g} \end{bmatrix}}_{\Delta} \tag{19}$$

Since the number of equations exceeds the number of unknowns, equations (19) cannot be solved for Δ in terms of δ. Equations (20) give a new set of equations which are equivalent to equations (19) in a least squares sense (ref. 3).

$$H^T \delta = H^T H \ \Delta \tag{20}$$

Solution of equations (20) yields:

$$\underbrace{\begin{bmatrix} \\ e \\ f \\ g \\ \bar{e} \\ \bar{f} \\ \bar{g} \\ \\ \end{bmatrix}}_{\Delta} = \frac{1}{4\ell_x\ell_y} \underbrace{\begin{bmatrix} \ell_x\ell_y & 0 & \ell_x\ell_y & 0 & \ell_x\ell_y & 0 & \ell_x\ell_y & 0 \\ -\ell_y & 0 & \ell_y & 0 & \ell_y & 0 & -\ell_y & 0 \\ -\ell_x & 0 & -\ell_x & 0 & \ell_x & 0 & \ell_x & 0 \\ 0 & \ell_x\ell_y & 0 & \ell_x\ell_y & 0 & \ell_x\ell_y & 0 & \ell_x\ell_y \\ 0 & -\ell_y & 0 & \ell_y & 0 & \ell_y & 0 & -\ell_y \\ 0 & -\ell_x & 0 & -\ell_x & 0 & \ell_x & 0 & \ell_x \\ & & & & & & & \\ & & & & & & & \end{bmatrix}}_{4\ell_x\ell_y\,[H^TH]^{-1}H^T} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}}_{\delta} \tag{21}$$

The parameters g and $\bar{f}$ are the only relevant d.o.f. These two are combined to form h (see eq. (9)). Therefore the transformation matrix for the shear related terms is obtained by adding rows 3 and 5 of $[H^TH]^{-1}H^T$.

$$T_S = \frac{1}{4\ell_x\ell_y}\left[-\ell_x \quad -\ell_y \quad -\ell_x \quad \ell_y \quad \ell_x \quad \ell_y \quad \ell_x \quad -\ell_y\right] \tag{22}$$

The transformation matrices T_N and T_S are combined to form the total transformation matrix, as illustrated in equations (23).

$$K' = \left[T_N{}^T \;\vdots\; T_S{}^T \right] \left[\begin{array}{cccc|c} K_{11} & K_{12} & K_{13} & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 \\ K_{13} & K_{23} & K_{33} & 0 & 0 \\ 0 & 0 & 0 & K_{44} & 0 \\ \hline 0 & 0 & 0 & 0 & K_{55} \end{array} \right] \left[\begin{array}{c} T_N \\ \hline T_S \end{array} \right]$$

$$F' = \left[T_N{}^T \;\vdots\; T_S{}^T \right] \left[\begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \hline F_5 \end{array} \right] \qquad (23)$$

The stiffness matrix K' in equation (23) was derived assuming there were multiple laminae within the element. If there is only one lamina (or all the lamina are identical), the stiffness matrix is identical to that for an ordinary bilinear rectangular element with reduced integration.

EXAMPLE PROBLEMS

This section discusses results from analyses of several simple problems using the RLC element. All the configurations consist of a laminated cantilevered beam (fig. 3) with either mechanical or thermal load. The cantilevered beam configuration was chosen because it is a severe test for 2-D plane stress (or plane strain) elements. Several combinations of lamina properties and stacking sequence were examined. Material properties for the various laminae are given in Table 1. All laminae were assumed to be isotropic. Lamina types H, I, and J have the same Young's modulus and shear modulus--only the thermal expansion coefficients are different. Lamina type S has 10% of the stiffness of the other lamina types. Figure 4 shows the five finite element meshes used in the convergence study. Results for mechanical and thermal loads are discussed separately in the following sections.

Mechanical Load

Three laminates were examined: (H/H/H/H), (H/S/S/H), and (S/H/H/S). The first two laminates have about the same flexural stiffness. The third laminate is much more flexible, since the outer plies are soft.

The loading consisted of a single point load at the end of the beam. For a long, thin cantilevered beam (such as that in fig. 3), the tip deflection calculated using strength of materials, is given as:

$$\Delta_R = \frac{P\ell^3}{3D} \tag{24}$$

where P = applied load

ℓ = beam length

D = flexural stiffness

Δ_R = tip deflection calculated using strength of materials

Assuming equation (24) gives the correct solution, figure 5 shows the error in the calculated tip deflection for the three laminates using the five meshes in figure 4. The open symbols show the results obtained using the RLC element described in the preceding sections. The solid symbols are for a modified RLC element and will be discussed later in this section. The error reduces rapidly with increased mesh refinement. The laminate (H/H/H/H) has no lamination effects--since all the layers are the same. As pointed out earlier, the element stiffness matrix is therefore identical to that for an ordinary bilinear rectangular element with reduced integration. Figure 5 shows that the error for the nonhomogeneous laminates (ie. (H/S/S/H) and (S/H/H/S)) is comparable to that for the (H/H/H/H) laminate. Therefore, additional errors due to lamination within an element appear to be small.

Much of the error in the results is due to the assumption that ε_y within an element does not vary in the y-direction (see eq. (6)). Because of the Poisson effect, the upper part of the beam (which has positive ε_x) should have a negative ε_y. The lower part of the beam (which has a negative ε_x) should have a positive ε_y. But within a single element ε_y is constant in the y-direction. Therefore, if there is only one element through the thickness, ε_y is calculated to be zero. This results in an overly stiff element. The magnitude of the error for a homogeneous isotropic can be estimated by examining the constitutive equations. Assuming plane stress conditions for an isotropic material, the stresses can be expressed as

$$\sigma_x = \frac{E}{1 - \nu^2} (\varepsilon_x + \nu\varepsilon_y)$$

$$\sigma_y = \frac{E}{1 - \nu^2} (\nu\varepsilon_x + \varepsilon_y) \qquad (25)$$

For a long, thin beam, σ_y should be negligible; therefore, $\varepsilon_y = -\nu\varepsilon_x$ and $\sigma_x = E\varepsilon_x$. But if ε_y is constrained to be zero, then

$$\sigma_x = \frac{E}{1 - \nu^2} \varepsilon_x \tag{26}$$

This results in an effective modulus of $E/(1 - \nu^2)$. For $\nu = 0.3$ this produces a modulus which is 10% too large. Since the flexural stiffness is linearly related to the modulus, the deflection is inversely and linearly related to the modulus (see eq. (24)). Therefore, a minimum of about 10% error is expected when 1 element is used through the thickness of the beam. Figure 5 agrees quite well with this prediction. Of course, with two elements through the thickness, the spurious stiffening is less (see fig. 5). Spurious stiffening due to Poisson's effect can be eliminated by artificially setting $\nu_{xy} = 0$. When ν_{xy} is artificially set to zero, the element will be referred to as the modified RLC element. The solid symbols in figure 5 show results obtained with the modified RLC element. The convergence is seen to be extremely rapid. Further testing of the modified RLC element is needed to determine when the artificial prescription of $\nu_{xy} = 0$ may lead to problems.

Thermal Load

Two laminates were examined: (H/H/I/I) and (H/H/J/J). In the first laminate, the initial thermal strains in the top two laminae are simply the negative of the initial thermal strains in the bottom two laminae. The second laminate is like the first except that the initial thermal strain ε_y is the same for all four laminae. Using elementary beam theory, the end displacement is independent of the initial strain ε_y. Hence, both beams should have the same end displacement. The technique for solving this problem using strength of materials is outlined in reference 4 and will not be discussed here.

Figure 6 shows the accuracy of the RLC element in calculating end displacement for the (H/H/I/I) laminate. The strength of materials solution Δ_R is assumed to be correct. The open and solid symbols are results for the standard and modified RLC elements, respectively. Since there is no gradient in the stresses in the x-direction, the accuracy is independent of the refinement in the x-direction.

As was the case for mechanical load, the assumption of constant ε_y within an element in the y-direction severely stiffens the system when only one element is used through the thickness. This is because the initial ε_y in the upper two lamina is of opposite sign to the initial ε_y in the lower two lamina. Imposing constant final ε_y in the y-direction results in a calculated value of $\varepsilon_y = 0$, even though physically there is nothing in the beam to restrict the expansion or contraction in the y-direction. Figure 6 shows that using the modified RLC element eliminates this problem entirely. Note that very accurate results are also obtained with just two unmodified RLC elements through the thickness.

The other laminate examined was (H/H/J/J). This laminate is like the previous one, except the initial ε_y is the same in all four plies. For this case the assumption of constant final ε_y in the y-direction is not a problem, since all four laminae are supposed to have nearly the same ε_y. Consequently, even with only an unmodified RLC element through the thickness, the error was less than 0.1%.

CONCLUSIONS

A simple two-dimensional (2-D) element was developed for analysis of laminated composite materials. The rectangular laminated composite (RLC) element eliminates the need to add elements to a model simply to account for the material properties of various laminae. Explicit integration in terms of generalized displacements minimizes the algebraic effort required to derive the element stiffnesses and the thermal load vector. A substitute shape function technique was used to avoid the excessive bending stiffness of ordinary bilinear rectangular elements.

Several sample problems were analyzed using the RLC element. Results from these analyses demonstrated that the RLC element accurately accounts for the presence of multiple lamina within a single element. Use of the RLC element will reduce the number of elements required to analyze many linear 2-D laminated composite problems. The basic technique described herein also should be applicable for deriving elements for linear three-dimensional (3-D) and geometrically nonlinear 2-D and 3-D problems.

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Table 1 Lamina Properties

Lamina Type	E_x	E_y	ν_{xy}	G_{xy}	α_x	α_y
H	1.0	1.0	.3	0.385	1.E-6	1.E-6
I	1.0	1.0	.3	.385	-1.E-6	-1.E-6
J	1.0	1.0	.3	.385	-1.E-6	1.E-6
S	.1	.1	.3	.0385	-	-

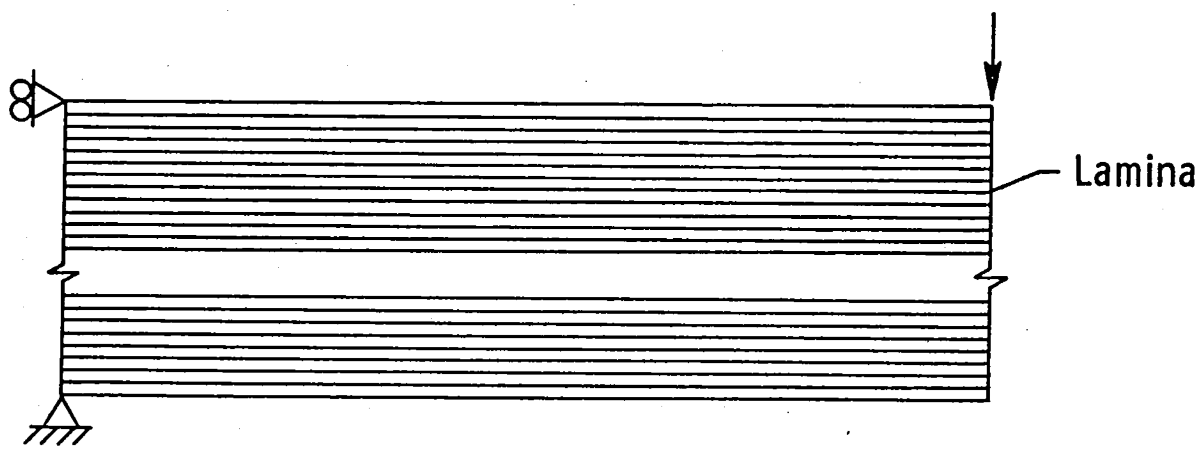

Fig. 1. - Beam with many lamina.
(Not all lamina are shown)

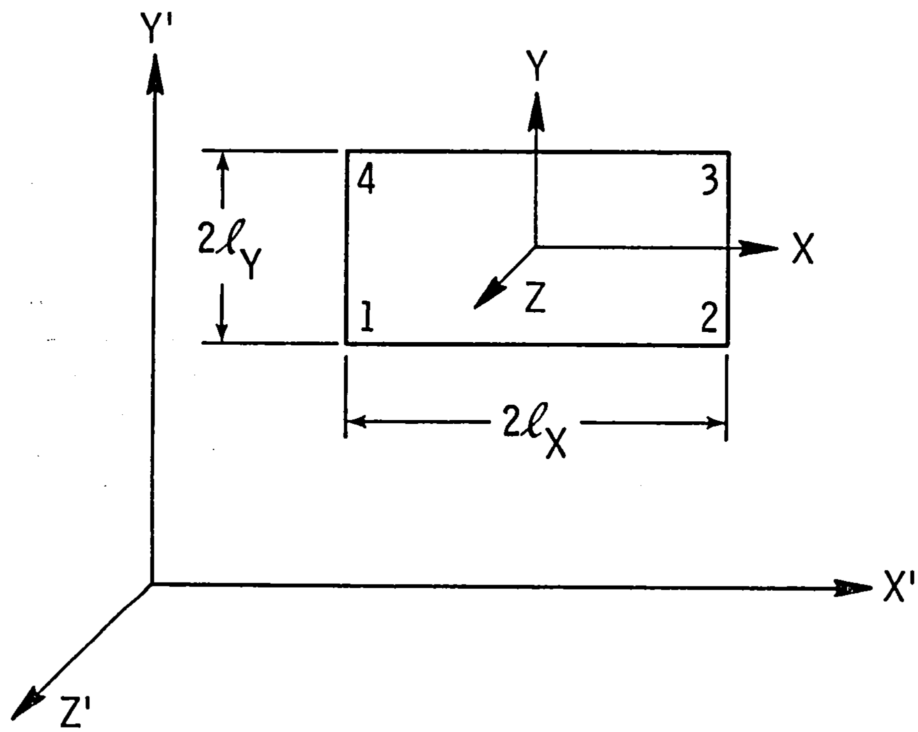

Fig. 2. - Element configuration.

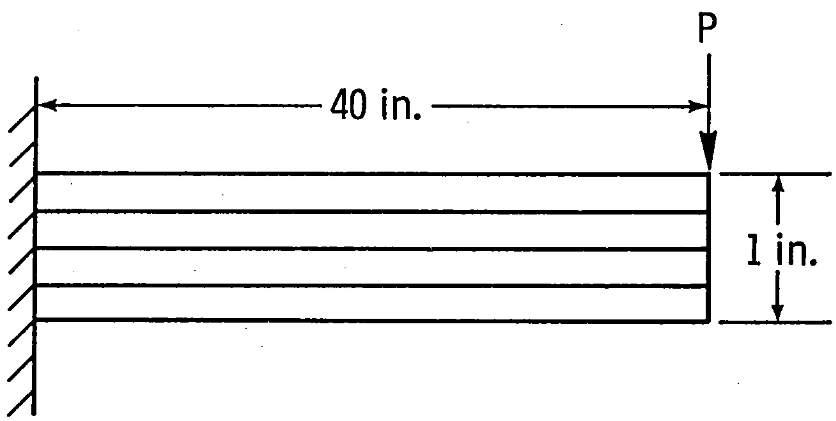

Fig. 3. - Cantilevered laminated beam.

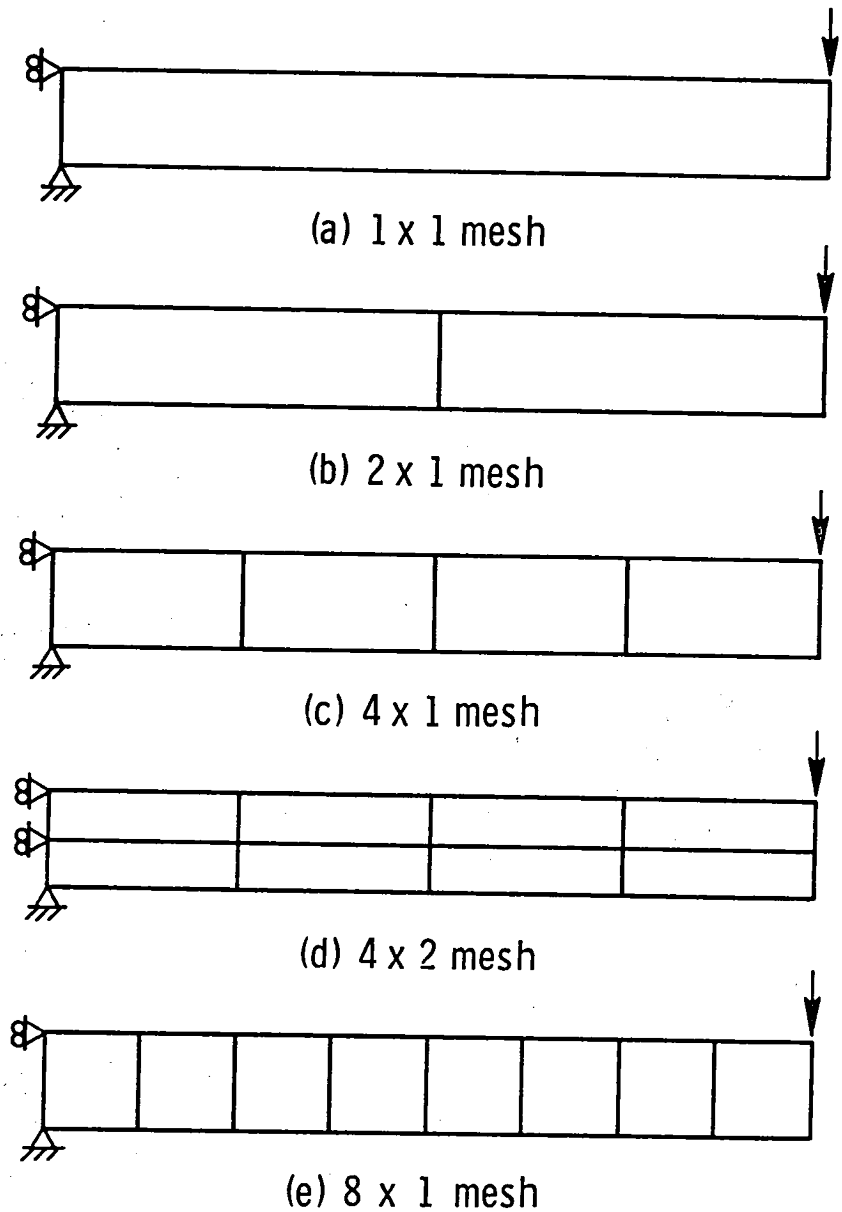

Fig. 4. - Finite element meshes for cantilevered beam problem.

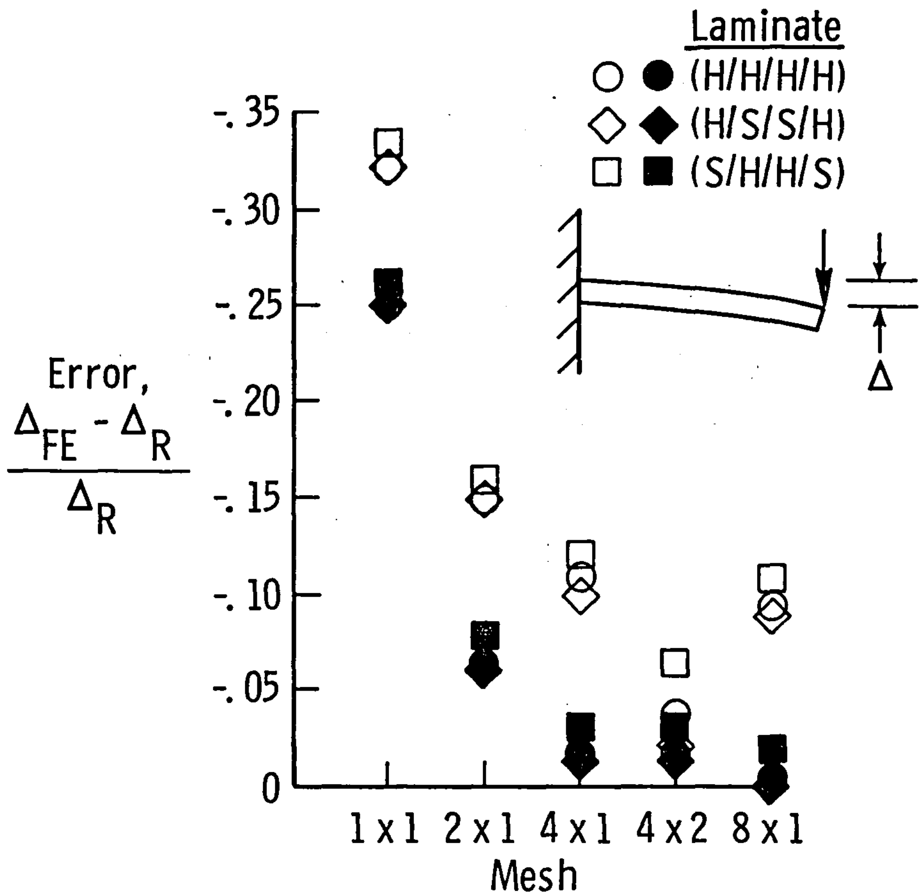

Fig. 5. - Accuracy of calculated tip displacement for tip loaded cantilevered beam. Open symbols are for unmodified RLC element. Solid symbols are for modified RLC element.

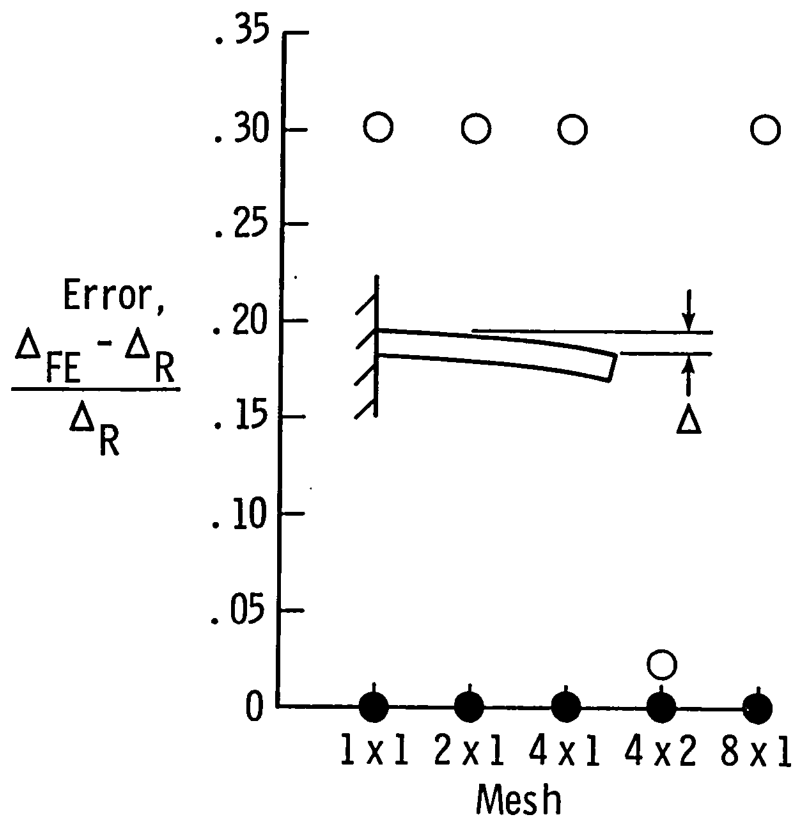

Fig. 6. - Accuracy of calculated tip displacement for thermal loading of unsymmetric cantilevered beam. Open symbols are for unmodified RLC element. Solid symbols are for modified RLC element. (Laminate = (H/H/I/I))

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15. Supplementary Notes

16. Abstract

A simple rectangular finite element was developed for two-dimensional analysis of laminated composite materials. The rectangular laminated composite (RLC) element eliminates the need to add elements to a model simply to account for the material properties of various laminae. This is particularly advantageous for thick laminates with many lamina. Explicit integration in terms of generalized displacements minimizes the algebraic effort required to derive the element stiffnesses and the thermal load vector. A substitute shape function technique is used to improve the performance of the element in modeling bending type deformation. Results for several example problems are discussed.

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