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# QUASICONFORMAL MAPPINGS AND GRID GENERATION*

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## ABSTRACT

A finite difference scheme is developed for constructing quasiconformal mappings for arbitrary simply and doubly-connected regions. Computational grids are generated to reduce elliptic equations to canonical form. Examples of conformal mappings on surfaces are also included.

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## 1. Introduction

The motivation for constructing quasiconformal mappings lies in their application to the generation of curvilinear coordinate systems. Quasiconformal mappings may be used to reduce any second order linear elliptic partial differential equation to canonical form (i.e., the principal part of the differential operator reduces to the Laplacian). Consequently, if one is solving an elliptic boundary value problem, an appropriate quasiconformal mapping could be used to simultaneously fit the boundary contours with coordinate lines and simplify the original partial differential equation. The equation, in canonical form, could possibly be solved more efficiently or by methods which would not be applicable to the original equation in cartesian coordinates. A related application of quasiconformal mappings is in the construction of conformal mappings on surfaces. Although they have been studied extensively by complex analysis, very little work has been done on the numerical construction of quasiconformal mappings. Of the methods which have been proposed, that of Belinskii et al. [2] uses a fixed boundary correspondence which determines the mapping parameters, and the method of Mastin and Thompson [4] would be difficult to implement on arbitrary regions. A finite element version of the latter method developed by Weisel [9] appears promising. However, the class of mappings and the type of regions presented in the examples are very limited.

In recent years a finite-difference method for constructing conformal mappings developed by Allen [1] has been used by Mobley and Stewart [5], Pope [6], and Yen and Lee [8] in the construction of orthogonal coordinate systems. Although all of these authors use essentially the same numerical method, there are differences in the way the boundary values and the conformal module of the region are computed. This method is not as accurate or efficient as other conformal mapping methods using integral equations or series expansions, but it does have the advantage of simplicity since the module of the region, the boundary correspondence, and the interior grid points are determined in a single iterative procedure. Modeling this conformal mapping procedure, it will be shown how quasiconformal mappings can be constructed and applied to the reduction of elliptic equations to canonical form and the construction of conformal mappings on surfaces. Except for the method of Godunov and Prokopov [3], this appears to be the only conformal mapping method which can be easily adapted to handle the problem of constructing quasiconformal mappings.

## 2. Boundary Value Problem for Quasiconformal Mappings

Let D be a bounded simply-connected region in the xy-plane whose boundary C is a simple closed contour. Let $z_1$, $z_2$, $z_3$ and $z_4$ be distinct boundary points ordered by the orientation on C. There exists a unique quasiconformal mapping of D onto the interior of a rectangle such that the points $z_i$ map to the vertices which are also ordered by the orientation of the rectangle. The ratio of the length to the width of the rectangle is a quasiconformal invariant called the module of D and will be denoted by m. The quasiconformal mapping of the region D onto a rectangular region can be obtained by constructing the mapping of D onto a square region S which satisfies the linear system

$$\xi_x = m(c\eta_y + b\eta_x)$$

$$\xi_y = -m(a\eta_x + b\eta_y) \qquad (1)$$

where $ac - b^2 = 1$. On setting $(\mu,\nu) = (\xi, m\eta)$, it is obvious that $\mu$ and $\nu$, as functions of x and y, satisfy the Beltrami system and hence we arrive at the desired quasiconformal mapping (see [4] for further details).

It is easily shown that x and y, as functions of $\xi$ and $\eta$, satisfy

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = J^2(a_x + b_y)$$

$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = J^2(b_x + c_y) \qquad (2)$$

where

$$\alpha = ay_\eta^2 - 2bx_\eta y_\eta + cx_\eta^2$$

$$\beta = ay_\xi y_\eta - b(x_\xi y_\eta + x_\eta y_\xi) + cx_\xi x_\eta$$

$$\gamma = ay_\xi^2 - 2bx_\xi y_\xi + cx_\xi^2$$

$$J = x_\xi y_\eta - x_\eta y_\xi .$$

At each boundary point of D one of the functions, $\xi$ or $\eta$, is constant while the other satisfies an oblique derivative condition. This implies that the condition $\beta = 0$ must be satisfied by x and y at each boundary point of S (except for vertices). Note that in the solution of (1) we would have $\beta = 0$ throughout S. Thus the computed value of $\beta$ can serve as a test of the accuracy of our solution. It also follows from (1) that $m = \sqrt{\alpha/\gamma}$. Therefore, the equations (2) can be written as

$$m^2 x_{\xi\xi} + x_{\eta\eta} = mJ(a_x + b_y)$$

$$m^2 y_{\xi\xi} + y_{\eta\eta} = mJ(b_x + c_y). \qquad (3)$$

The same procedure is used for doubly-connected regions with a periodicity condition applied on two opposite sides of S.

The system (2) or (3) may be solved using an iterative procedure with (i) the right-hand sides of the equations, (ii) the boundary values of x and y, and (iii) either $\alpha$, $\beta$, $\gamma$, or an approximation of m, re-evaluated at each iteration. As in the previously discussed conformal mapping methods, both (2) and (3) have performed equally well in numerical examples.

## 3. Reduction of Elliptic Equations to Canonical Form

The application of quasiconformal mappings in the solution of elliptic equations is well-known. An elliptic equation of the form

$$au_{xx} + 2bu_{xy} + cu_{yy} = f(u,u_x,u_y), \; ac - b^2 = 1$$

transforms to an equation of the form

$$m^2u_{\xi\xi} + u_{\eta\eta} = g(u,u_\xi,u_\eta)$$

under the transformation defined by (1). This transformation will make the solution of many problems much easier. For example, if a, b, and c are constants and f = 0, then it can be shown that g = 0 and we need only solve $m^2u_{\xi\xi} + u_{\eta\eta} = 0$ on a square region. This is efficiently done by separation of variables or a direct numerical method. Computational grids for solving this problem on a simply and doubly-connected region are given in Figure 1.

The practicality of using quasiconformal mappings in solving elliptic equations will be further examined in the following example. The function $u = \cos(x - y)$ is a solution of the partial differential equation

$$u_{xx} + u_{xy} + u_{yy} + \cos(x - y) = 0. \tag{4}$$

We will solve this equation numerically for $0 \leq x, y \leq \pi$ with Dirichet boundary conditions prescribed by the known solution. In terms of curvilinear coordinates, this equation can also be written as

$$m^2 u_{\xi\xi} + u_{\eta\eta} + \frac{2Jm}{\sqrt{3}} \cos(x - y) = 0. \tag{5}$$

The quasiconformal grid for solving (5) is illustrated in Figure 2. For the purpose of assessing the influence of the error in the iterative solution of (2) on the error in the solution of (5), the iteration was stopped occasionally and the solution of (5) was computed. A comparison of a normalized value of $\beta$ with the error in the solution of (5) is plotted in Figure 3. Note that $\beta$ has been normalized so that for the construction of conformal mappings ($a = c = 1$, $b = 0$), the values along the abscissa would represent the degree of nonorthogonality. Equation (4) was also solved on a uniform rectangular mesh with the same number of grid points. The error in this solution serves as an approximation of the discretization error which would result in solving (5) with the exact quasiconformal mapping. It also serves as a test of our method against the traditional method for solving (4). Figure 3 indicates a nearly linear relation between the plotted variables. This is to be expected since the major part of the truncation error for larger $\beta$, is due to the omission of the mixed derivative term, which is a linear function of $\beta$.

A few remarks concerning the numerical solution of (2) and (5) will be made. The system (2) was solved using point SOR in the same way one would construct a conformal mapping (see [7]). For the condition $\beta = 0$ on the boundary, a form of one-side "upwind" differencing was necessary to maintain convergence for the value of $b = 1/\sqrt{3}$ in this example. The elliptic equation (5) was solved using a direct elliptic solver (see [7]). After 35 iterations of (2) the maximum error in the solution of (5) was within 125 per cent of what we estimated to be the maximum discretization error. At this point the value of $|\beta|$ was still decreasing, but at a very slow rate. The exact value for m in this example is 1 due to symmetry. The computed estimate, which was the root-mean-square value of $\sqrt{\alpha/\gamma}$, was 1.00006 after 35 iterations.

This example does not illustrate an efficient use of quasiconformal mappings. The method would be very efficient when one had to solve an elliptic equation with many different boundary conditions or inhomogeous terms. In that case the quasiconformal mapping would only have to be constructed once.

## 4. Conformal Mapping on Surfaces

A second area of application of quasiconformal mappings is in the construction of conformal (isothermal) coordinates on a surface. Let M be a smooth bounded surface in xyz - space which is defined by the parametric equations

$$x = x(\phi,\theta),\ y = y(\phi,\theta),\ z = z(\phi,\theta).$$

The parameter region in the $\phi\theta$ - plane may be an arbitrarily shaped simply or doubly-connected region but it is assumed that the boundary is composed of simple closed contours and the mapping from the parameter region to the surface has a nonvanishing Jacobian.

A conformal mapping of M onto a rectangular region can be constructed by constructing a mapping from a square region of the $\xi\eta$ - plane onto M which satifies

$$P_\xi \cdot P_\eta = 0 \text{ and } m|P_\xi| = |P_\eta|$$

where $P = (x,y,z)$ and $m$ is the module of the surface $M$. If these equations are written in terms of the parametric variables $\phi$ and $\theta$, we conclude that $\phi$ and $\theta$ satisfy

$$m\theta_\xi = b\theta_\eta - c\phi_\eta$$

$$m\phi_\xi = a\theta_\eta - b\phi_\eta$$

where

$$a = |P_\theta|^2/d$$

$$b = P_\phi \cdot P_\theta /d$$

$$c = |P_\phi|^2/d$$

$$d = |P_\phi \times P_\theta|$$

However, this is equivalent to (1) with $(x,y)$ replaced by $(\phi,\theta)$. In this case the quantity which corresponds to $\beta/\sqrt{\alpha\gamma}$ would be the cosine of the angle between a $\xi$ = constant and a $\eta$ = constant coordinate line on the surface $M$.

Conformal grids have been constructed for several simply connected surfaces. Three surfaces are listed below. In the first two cases, the parametric region was the projection of the surface onto the xy - plane.

i. Paraboloid: $z = 1 - x^2 - y^2,\ x^2 + y^2 \leq 1$

ii. Bicubic: $z = x^2y^3,\ -1 \leq x,y \leq 1$

iii. Torus: $x = (2 + \sin\phi)\cos\theta$

$$y = 1 - (2\phi/\pi)^2$$

$$z = (2 + \sin\phi)\sin\theta,$$

$$-\pi/2 \leq \phi \leq \pi/2,\ 0 \leq \theta \leq \pi$$

The plots of these surfaces appear in Figure 4. It is difficult to visualize the orthogonality from the plots, but the departure from orthogonality was less than one degree except near vertices on the boundary where the orthogonality condition was not imposed.

The advantages of conformal coordinates are well-known. Problems involving heat conduction, ideal fluid flow, and electric fields can be solved as easily on the surface as they can on a rectangular region in the cartesian plane.

## 6. Conclusions and Discussion

A finite difference method, which has been widely used for the construction of conformal mappings, has been generalized to construct quasiconformal mappings. This development will increase the class of problems which can be solved using the currently available fast elliptic solvers developed by Swarztrauber and Sweet [7]. Even when iterative methods are required, the absence of a mixed derivative and a rectangular region both would tend to give faster convergence especially when optimal iteration parameters are known.

We will conclude this report with an open problem. It is known that if $a = c = 1$ and $b = 0$, then the solution of (2) which satisfies $\beta = 0$ on the boundary of the square S will also satisfy the system (1), and hence determines a conformal mapping. This follows directly once it is noted that the quotient

$$\frac{\eta_y + i\eta_x}{\xi_y + i\xi_x}$$

is an analytic function. Here it has been assumed, and numerical results tend to verify, that the same result holds for arbitrary quasiconformal mappings. However, no proof has been found.

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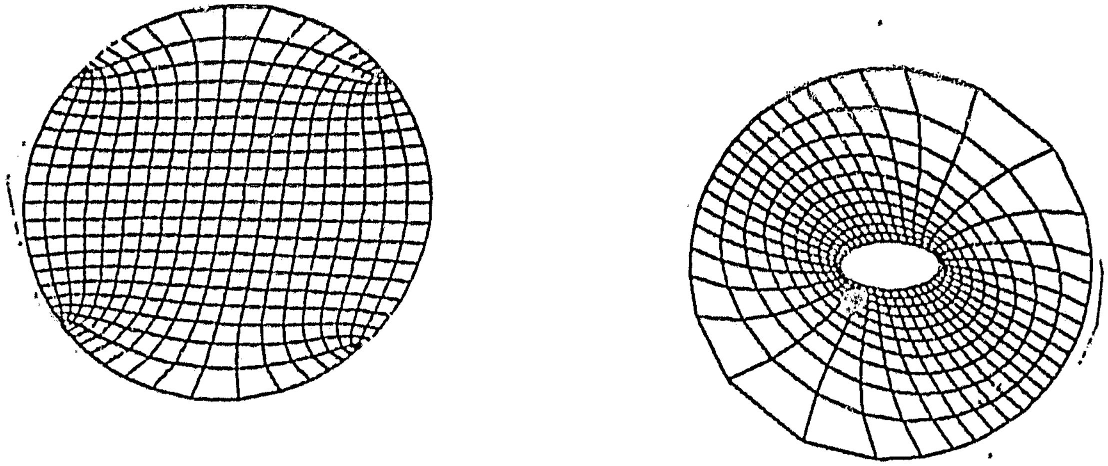


Figure 1. Quasiconformal grids for simply and doubly-connected regions. $a = 1$, $b = 1/2$, $c = 5/4$.

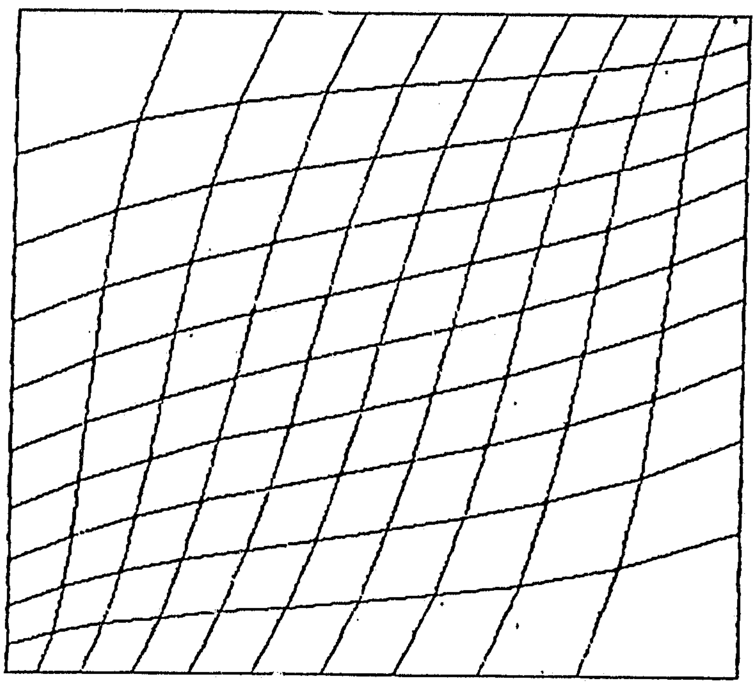


Figure 2. Quasiconformal grid for the solution of elliptic equation. $a = c = 2/\sqrt{3}$, $b = 1/\sqrt{3}$.

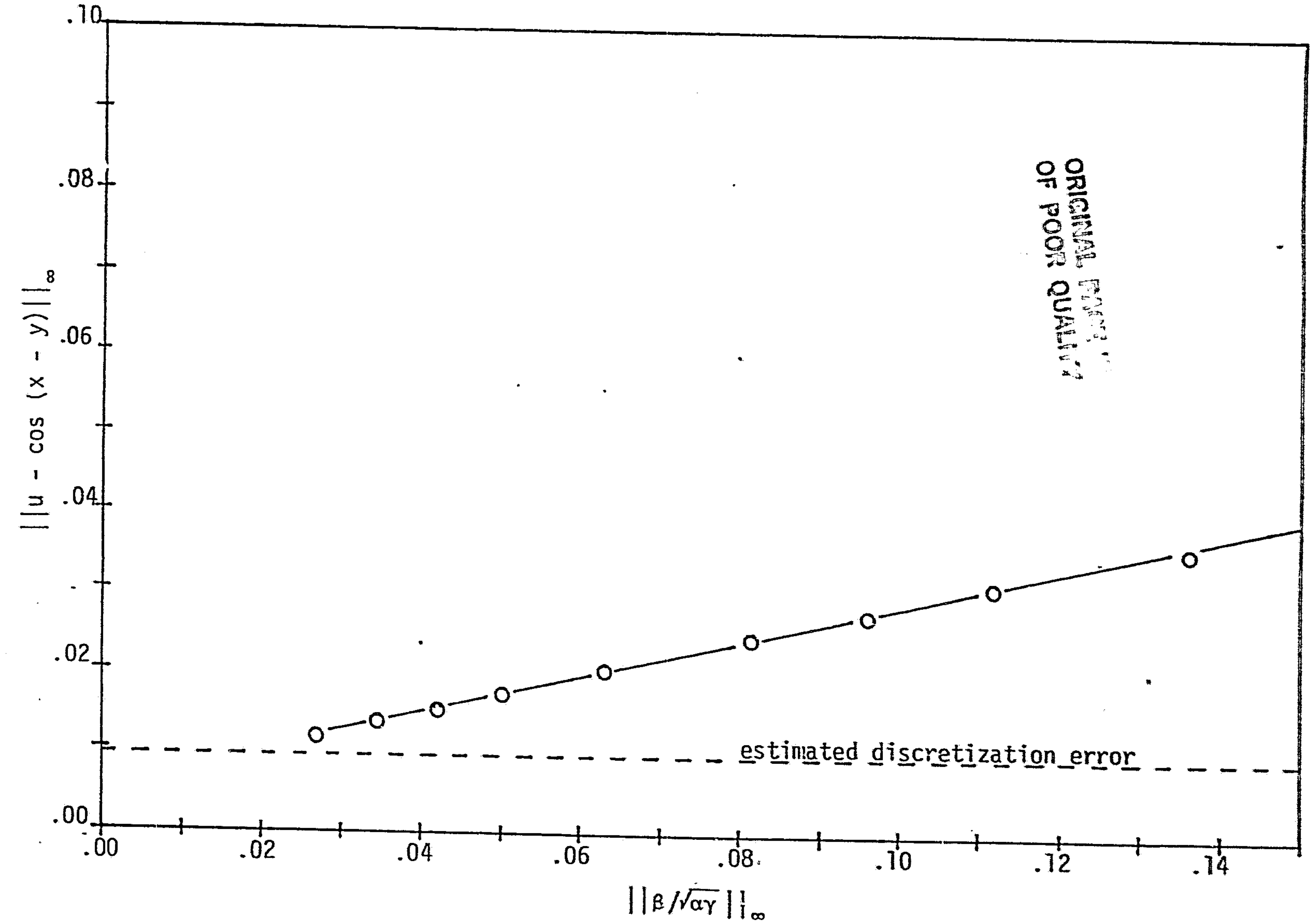
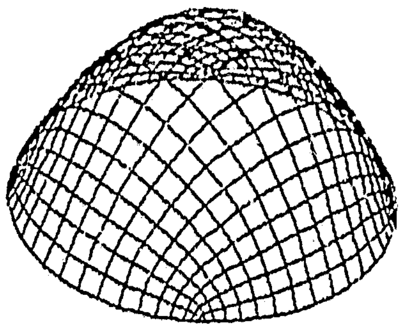
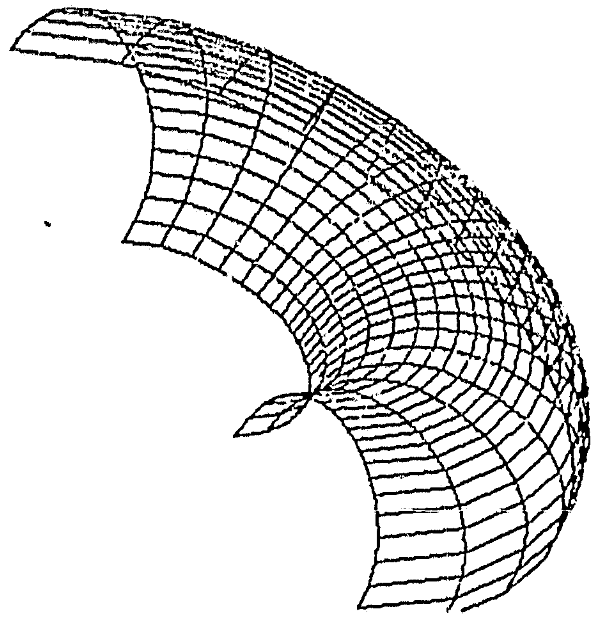





Figure 3. Effect of error in the quasiconformal mapping on error in the solution of the partial differential equation.





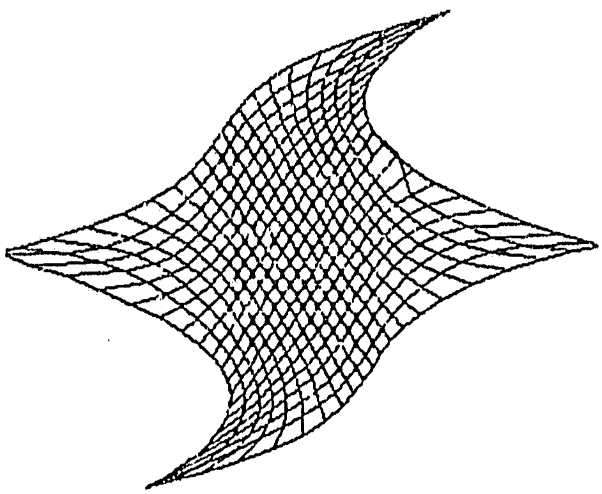


Figure 4. Conformal grids on subsets of a paraboloid, torus, and bicubic surface.