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CLASSICAL FREE-STREAMLINE FLOW OVER A POLYGONAL OBSTACLE

A. R. Elcrat

and

L. N. Trefethen

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NASA Langley Research Center, Hampton, Virginia 23665

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AUTH: A/ELCRAT, A. R.; B/TREFETHEN, L. N. PAA: A/(Wichita State Univ., Kansas); B/(Massachusetts Inst. of Tech.)

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ABA: Author

ABS: In classical Kirchhoff flow, an ideal incompressible fluid flows past an obstacle and around a motionless wake bounded by free streamlines. Since 1869 it has been known that in principle, the two-dimensional Kirchhoff flow over a polygonal obstacle can be determined by constructing a conformal map onto a polygon in the log-hodograph plane. In practice, however, this idea has rarely been put to use except for very simple obstacles, because the conformal mapping problem has been too difficult. This paper presents a practical method for computing flows over arbitrary

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Classical free-streamline flow over a polygonal obstacle

Alan R. Elcrat

Department of Mathematics
Wichita State University

*Lloyd N. Trefethen**

Department of Mathematics
Massachusetts Institute of Technology

Abstract

In classical Kirchhoff flow, an ideal incompressible fluid flows past an obstacle and around a motionless wake bounded by free streamlines. Since 1869 it has been known that in principle, the two-dimensional Kirchhoff flow over a polygonal obstacle can be determined by constructing a conformal map onto a polygon in the log-hodograph plane. In practice, however, this idea has rarely been put to use except for very simple obstacles, because the conformal mapping problem has been too difficult. This paper presents a practical method for computing flows over arbitrary polygonal obstacles to high accuracy in a few seconds of computer time. We achieve this high speed and flexibility by working with a modified Schwarz-Christoffel integral that maps onto the flow region directly rather than onto the log-hodograph polygon. This integral and its associated parameter problem are treated numerically by methods developed earlier by Trefethen for standard Schwarz-Christoffel maps.

AMS (MOS) Subject Classification: primary 76B10; secondary 30C30, 65E05

Key phrases: free estimate, Kirchhoff flow, hodograph, Schwarz-Christoffel, conformal mapping, jets, wakes, cavities

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Notation

z: physical space coordinate (region bounded by obstacle and free streamlines)

w: velocity potential (slit plane)

$x = \sqrt{2w/W} + x_*$ (upper half plane)

$\zeta = \frac{dw}{dz}$: hodograph or conjugate-velocity (gearlike region or Riemann surface)

$\Omega = -\log\zeta$ (polygonal region or Riemann surface)

1. Introduction

Figure 1 shows the geometry we are concerned with. An ideal incompressible fluid in the complex z plane undergoes irrotational flow rightward past a solid obstacle Γ.

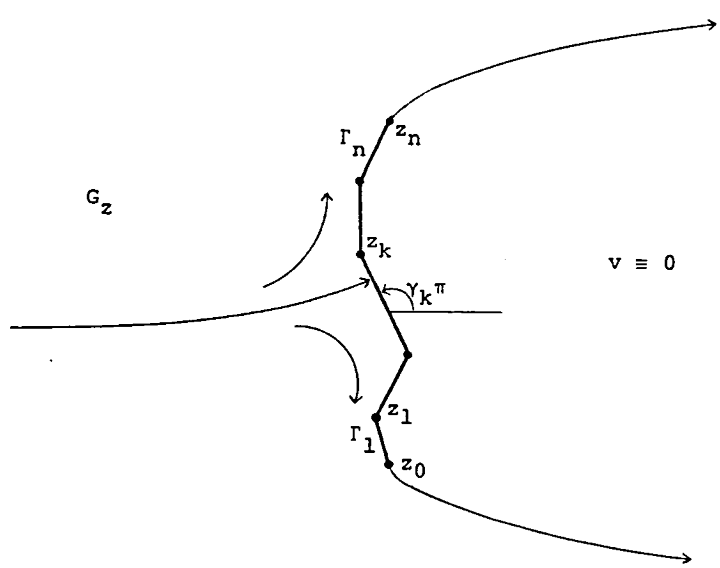


Figure 1. Geometry of the Kirchhoff flow problem.

The complex velocity is denoted by $v(z)$ and is normalized by $v(\infty)=1$. The obstacle consists of n solid line segments $\Gamma_k = (z_{k-1}, z_k)$, $1 \le k \le n$, bounded by vertices z_k, $0 \le k \le n$, and $\gamma_k\pi$ denotes the angle of Γ_k counterclockwise from the horizontal. At an unspecified stagnation point z_* along Γ, the flow divides between an upper part passing over z_n and a lower part passing under z_0. At z_0 and z_n, the fluid does not undergo infinite acceleration and turn 180° back on itself; rather, it continues smoothly towards $z = +\infty$ by flowing around a *wake* in which $v \equiv 0$. The curves of discontinuity separating the wake from the moving flow are two free streamlines with stream function equal to zero, which we label Γ_- and Γ_+. The shapes of Γ_- and Γ_+ are unknown a priori, but are determined implicitly by the condition that all along both of them, $|v(z)|$ must be constant and equal to 1. The physical origin of this condition is the requirement that the pressure should have the constant value p_∞ throughout the wake, and be continuous across Γ_- and Γ_+; $|v|=1$ then follows from Bernoulli's equation.

Here then is our *Kirchhoff flow problem:* given an obstacle Γ, calculate the free streamlines $\Gamma_\pm$, stagnation point z_*, and velocity field $v(z)$ for a flow of the type described, together with associated numerical quantities such as lift and drag.

Free-streamline flows of this kind have a long history. They were introduced by Helmholtz and Kirchhoff [8] in 1868-9 in an attempt to resolve *D'Alembert's paradox:* in ideal potential flow (with no wake), the pressure forces around an object balance exactly and so the drag is zero [13]. Similar ideas apply also in the study of *jets* and *cavities,* where the free streamlines typically separate a liquid from a gas. We will say very little about the physical aspects of our problem, which is obviously idealized; for surveys of the large mathematical and physical literature of wakes, jets, and cavities, see [2,5,7,9,10,13,19]. For some recent computational work in this area, see [1,3,14].

The Kirchhoff flow problem can be cast as a problem in complex analysis. Let ζ denote the *hodograph* variable, which is simply the complex conjugate of velocity:

$$\zeta(z) = \bar{v}(z). \tag{1}$$

Let G_z denote the region of moving flow bounded by the solid boundary Γ and the unknown free streamlines $\Gamma_\pm$. Since the fluid is incompressible and irrotational, v (if interpreted as a real vector) is the gradient of a real *velocity potential* $\phi(z)$ satisfying Laplace's equation. Equivalently, ζ is the complex derivative of a *complex velocity potential* $w(z) = \phi(z)+i\psi(z)$,

$$\zeta(z) = \frac{dw}{dz}(z), \tag{2}$$

where the *stream function* $\psi(z)$ is conjugate to ϕ. The function $w(z)$ is analytic in G_z, and maps G_z conformally onto a slit plane $G_w = w(G_z)$, shown in Figure 2a, where the slit begins at the point $w_* = w(z_*)$ at which the flow divides to go around the obstacle. Without loss of generality we can take $w_*=0$, so that G_w is the slit plane $\mathbf{C}-\mathbf{R}^+$.

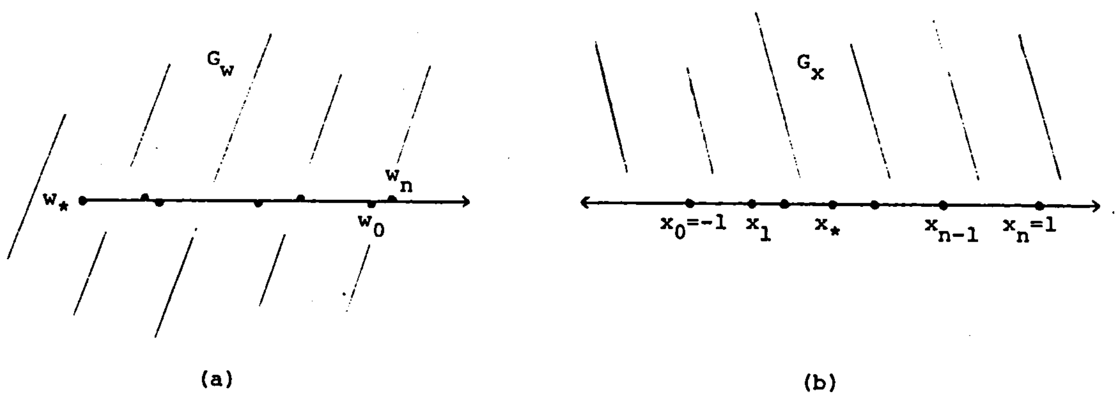


Figure 2. w **and** x **domains.**

To simplify subsequent manipulations, it is desirable to reduce G_w to a half-plane. Let x be a new complex variable related to w by

$$w = \frac{W}{2}(x-x_*)^2, \qquad x = \sqrt{2w/W}+x_*, \tag{3}$$

as indicated in Figure 2b, and write $x_k = x(w_k)$. Here $W \in \mathbf{R}$ and $x_* \in (-1,1)$ are constants that will be chosen so that $x_0 = -1$ and $x_n = 1$. Now $G_x = x(G_w)$ is the upper half plane, with $[-1,1]$ corresponding to Γ and with $(-\infty,-1)$ and $(1,\infty)$ corresponding to Γ_- and Γ_+, respectively. We write $X_k = x(\Gamma_k)$, $X_\pm = x(\Gamma_\pm)$.

The classical hodograph method of solution for Kirchhoff flows begins by calculating a conformal map of G_x onto the hodograph domain $G_\zeta = \zeta(G_z)$. What makes this possible is the fact that although G_z is unknown because of the free streamlines, G_ζ is (more or less) known. On the solid boundary, ζ has known argument, since the flow must be tangential. This argument is $\pi - \gamma_k\pi$ or $-\gamma_k\pi$, depending on whether the point on the boundary lies downstream of z_* in the direction of z_0 or z_n, respectively:

$$\arg\zeta(z) = \begin{cases} \pi-\gamma_k\pi & \text{for } z \in \Gamma_k,\ x<x_*, \\ -\gamma_k\pi & \text{for } z \in \Gamma_k,\ x>x_*. \end{cases} \tag{4a}$$

On the free streamlines, ζ has known modulus:

$$|\zeta(z)| = 1 \qquad \text{for } z \in \Gamma_\pm. \tag{4b}$$

Thus G_ζ is a "gearlike" region bounded by circular arcs and subsets of rays passing through the origin. If we introduce the log-hodograph variable

$$\Omega(z) = -\log\zeta(z), \tag{5}$$

then the corresponding domain G_Ω is bounded by vertical and horizontal line segments. Therefore a conformal map of G_x onto G_Ω can be written explicitly in the form of a Schwarz-Christoffel integral. The classical solution method first calculates this map, thereby obtaining the analytic relationship between ζ and w, and then integrates (2) to get w and ζ as functions of z.

In practice, the only flows that have ever been obtained in this way involve very simple obstacles Γ, such as the flate plate considered by Kirchhoff and certain wedges [2,7,13]. As the number of sides increases the conformal mapping problem rapidly becomes too difficult for analytical solutions. The following are the principal reasons why this is true, and why even numerical solutions have not been carried very far.

The first difficulty is that although the conformal map onto any polygon can be expressed by the Schwarz-Christoffel formula, this formula depends on *accessory parameters* or *prevertices* whose values must be determined numerically. This is the Schwarz-Christoffel *parameter problem*. In the past, researchers concerned with free-streamline flows have not had methods available for solving the parameter problem reliably. Our own interest in this project was motivated in part by the fact that the second author has recently developed a numerical method that does this, computing the map onto an arbitrary polygon typically in just a few seconds of computer time. This method has been implemented in a Fortran package called SCPACK [15,16].

A second difficulty is that even if the prevertices for the map $\Omega(x)$ are known, the further integration of (2) must be performed to recover quantities of physical interest. As a result, Kirchhoff flow calculations based in the traditional way on the log-hodograph domain may be time-consuming, requiring on the order of minutes of computer time.

The final and most serious difficulty is that except in the case of a very simple obstacle, G_Ω is generally a polygonal Riemann surface rather than just a polygon, and contains slits and/or branch points of unknown dimensions and even of unknown topology. The occurrence of a Riemann surface is in itself not a serious problem, for the Schwarz-Christoffel formula can readily be modified to handle such domains [4]. But the presence of unknown dimensions and topology is more serious. To overcome this problem, one can formulate and solve a *generalized Schwarz-Christoffel parameter problem* [17] in which some of the conditions that determine $\zeta(z)$ involve dimensions in G_z rather than G_Ω. We have successfully performed such calculations for certain geometries and hope to describe these results in a later paper. But it seems that this kind of calculation always requires careful attention to the details of the geometry of the hodograph domain.

As a result the construction of a single computer program to handle a wide range of obstacles by the hodograph method seems to be a difficult matter.

In this paper we combine the numerical ideas of SCPACK with an analytical trick that appears in §4.1 of the book by Monakhov [10] (see also [18]) to calculate Kirchhoff flows over arbitrary polygonal obstacles much more efficiently than the classical hodograph method permits. The key idea is that since our flow problem has only a single pair of free streamlines meeting at infinity, $z(w)$ can be written in the form of a modified Schwarz-Christoffel integral. By working with this integral directly, we dispense with all explicit consideration of hodograph domains, hence of Riemann surfaces and unknown slits and branch points, and we also avoid the need for a second integration. The result is a computer program that solves the problem of Figure 1 for an arbitrary obstacle to high accuracy typically in a matter of seconds, that is, at an expense comparable to that of finding the conformal map onto a closed polygon with the same number of vertices.

Section 2 describes our modified Schwarz-Christoffel map for $z(w)$, adapted from Monakhov. Section 3 outlines an efficient procedure for computing it numerically. Section 4 gives numerical results for some idealized problems. In later papers we will consider in greater detail certain particular configurations of physical interest, and we will also investigate models featuring a wake pressure $p_{\text{wake}} < p_{\infty}$, which is essential if one wants quantitative agreement with laboratory data.

Our Fortran package for Kirchhoff flows is a modification of SCPACK called KIRCH1. Machine-readable copies of SCPACK and KIRCH1 can be obtained by contacting the second author.

2. The modified Schwarz-Christoffel integral

To put the modified Schwarz-Christoffel formula in context, we begin with a description of the standard Schwarz-Christoffel problem. Suppose P is a polygonal region in the complex z-plane with vertices z_k, $1 \le k \le n$, and sides $\Gamma_k = (z_{k-1}, z_k)$ oriented at angles $\gamma_k \pi$ counterclockwise from the real axis, and define the external angle parameters β_k by

$$\beta_k = \gamma_{k+1} - \gamma_k, \quad 1 \le k \le n-1 \tag{6}$$

and $\beta_n = \gamma_1 - \gamma_n + 2$. (For convenience we write $z_0 = z_n$, $z_{n+1} = z_1$, etc.) Let $z = z(x)$ be a conformal map of the upper half complex x plane onto P, with $z(x=\infty) \in \Gamma_1$, and let $X_k = x(\Gamma_k)$ be the interval (x_{k-1}, x_k) bounded by the *prevertices* $x_{k-1} = x(z_{k-1})$ and $x_k = x(z_k)$, where $x_1 < x_2 < \cdots < x_n$. Then $\arg \frac{dz}{dx}(x)$ is a known function on all of $\mathbf{R}$ which has constant value $\gamma_k \pi$ on each X_k, and jumps by $\beta_k \pi$ at x_k:

$$\arg \frac{dz}{dx} = \gamma_1 \pi \qquad \text{for } x \in (x_n, \infty), \tag{7a}$$

$$\Delta \arg \frac{dz}{dx} = \beta_k \pi \qquad \text{at } x = x_k,\ 1 \le k \le n. \tag{7b}$$

The basis of the Schwarz-Christoffel formula is the fact that it is easy to write down the function $\frac{dz}{dx}$ determined by these conditions. Let g_k be defined by

$$g_k(x) = (x - x_k)^{-\beta_k}, \tag{8}$$

with the branch chosen so that $g_k(x)$ is positive for $x > x_k$. Then g_k has constant argument on $\mathbf{R}$ except for a jump by $\beta_k \pi$ at x_k. To be precise, it maps $\operatorname{Im} x > 0$ onto the wedge bounded by the rays $e^{-i\beta_k \pi}\mathbf{R}^+$ and $\mathbf{R}^+$, as shown in Figure 3a. It follows that $\frac{dz}{dx}$ can be written as a product of these wedge maps,

$$\frac{dz}{dx}(x) = A e^{i\gamma_1 \pi} \prod_{k=1}^{n} g_k(x) = A e^{i\gamma_1 \pi} \prod_{k=1}^{n} (x - x_k)^{-\beta_k}, \tag{9}$$

for some $A > 0$. The Schwarz-Christoffel formula is the integral of this,

$$z(x) = C + A e^{i\gamma_1 \pi} \int^{x} \prod_{k=1}^{n} (x' - x_k)^{-\beta_k} dx', \tag{10}$$

where C is a complex constant.

Returning to the Kirchhoff flow problem, consider now the function $z(w)$ that maps the slit plane G_w of Figure 2a onto the flow domain G_z of Figure 1. Instead of knowing $\arg \frac{dz}{dw}$ for all $x \in \mathbf{R}$, by (2)-(4) we now know $\arg \frac{dz}{dw}$ for $x \in [-1,1]$ and $\left|\frac{dz}{dw}\right|$ elsewhere.

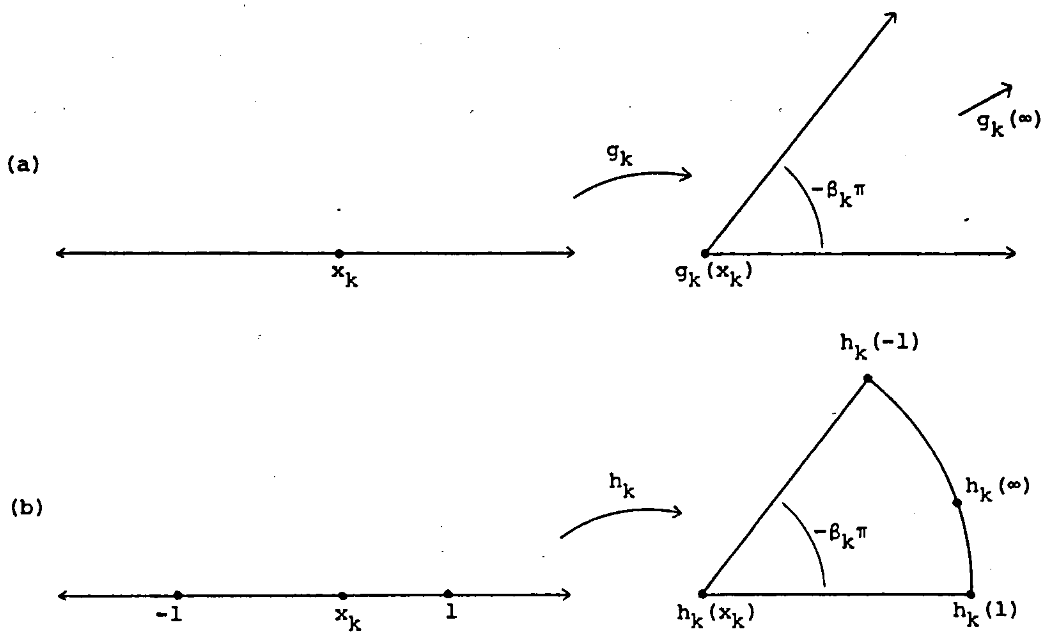


Figure 3. Mapping properties of the individual factors g_k and h_k in the Schwarz-Christoffel and modified Schwarz-Christoffel integrands, respectively.

Defining β_k again by (6) and setting $\beta_* = 1$, we can write

$$\arg \frac{dz}{dw} = \gamma_n \pi \qquad \text{for } x = x_n, \tag{11a}$$

$$\Delta \arg \frac{dz}{dw} = \beta_k \pi \qquad \text{at } x = x_k,\ 1 \le k \le n-1 \text{ and } k = \text{“}*\text{”}, \tag{11b}$$

$$\left| \frac{dz}{dw} \right| = 1 \qquad \text{for } x \in X_\pm, \text{ i.e. } |x| > 1, \tag{11c}$$

$$\arg \frac{dz}{dw} = 0 \qquad \text{at } x = \infty. \tag{11d}$$

Thus the Kirchhoff flow problem is a modification of the Schwarz-Christoffel problem (7) in which a constant-modulus condition rather than a constant-argument condition is applied over part of the boundary. As soon as one formulates the problem in these terms it becomes evident that here again, $\frac{dz}{dw}$ can be written as a product. Let h_k be defined by

$$h_k(x) = \left(\frac{x - x_k}{1 - x_k x + \sqrt{(1-x^2)(1-x_k^2)}} \right)^{-\beta_k}, \tag{12}$$

and h_* similarly, where the branch is taken so that $h_k(x)$ is positive for $x \in (x_k, 1)$. Obviously h_k has a singularity like that of g_k at x_k, plus additional singularities at $x = \pm 1$. In fact, h_k maps $\mathrm{Im}\, x > 0$ onto the pie-slice-shaped region bounded by the ray $e^{-i\beta_k \pi}\mathbf{R}^+$, the ray $\mathbf{R}^+$, and the circle $|z| = 1$, as shown in Figure 3b. (If $\beta_k > 0$, the region becomes an inverted pie slice with sides meeting at ∞.) Because every h_k has modulus 1 outside $[-1, 1]$, we can write $\frac{dz}{dw}$ as

$$\frac{dz}{dw} = \frac{1}{\zeta} = e^{i\gamma_n \pi} \prod_* h_k(x). \tag{13}$$

Here and below, $\prod_*$ denotes the product over $k = 1, 2, \ldots, n-1$ and $k = $ "$*$", and $\sum_*$ is the analogous sum.

By construction, the function on the right in (13) satisfies all of the conditions (11) except possibly (11d). We must choose x_* so that (11d) is satisfied too. To do this, note that by an easy computation

$$\arg h_k(\infty) = -\beta_k \cos^{-1}(-x_k),$$

with the analogous formula for h_*, and therefore

$$\arg\left(e^{i\gamma_n \pi} \prod_* h_k(\infty)\right) = \gamma_n \pi - \sum_* \beta_k \cos^{-1}(-x_k).$$

Condition (11d) therefore amounts to

$$\gamma_n \pi - \sum_* \beta_k \cos^{-1}(-x_k) = 0,$$

that is,

$$x_* = -\cos\left(\gamma_n \pi - \sum_{k=1}^{n-1} \beta_k \cos^{-1}(-x_k)\right). \tag{14}$$

The Kirchhoff flow can now be written as the integral of (13),

$$z(w) = C + e^{i\gamma_n \pi} \int^w \prod_* h_k(x')\, dw'. \tag{15}$$

For this formula to be usable we want to integrate with respect to x rather than w. By (3), dw' can be replaced by $W(x' - x_*)\, dx'$, and we get

$$\boxed{z(x) = C + W \int^x \frac{x' - x_*}{\zeta(x')}\, dx' = C + W e^{i\gamma_n \pi} \int^x (x' - x_*) \prod_* h_k(x')\, dx'} \tag{16}$$

in direct analogy to (10).

The factor $(x'-x_*)$ in (16) cancels another factor $(x'-x_*)^{-1}$ hidden there in h_*. Consequently $z(x)$ has singularities at x_k, $0 \le k \le n$, but not at x_*. We can write the integral out in full as follows:

$$z(x) = C + W e^{i\gamma_* \pi} \int^x \left(1 - x_* x + \sqrt{(1-x^2)(1-x_*^2)}\right) \prod_{k=1}^{n-1} \left(\frac{x - x_k}{1 - x_k x + \sqrt{(1-x^2)(1-x_k^2)}} \right)^{-\beta_k} dx'. \quad (17)$$

This is essentially eq. (5) of p. 185 of [10], except that Monakhov unjustifiably assumes $x_* = 0$ and hence $w_0 = w_n$.

3. Numerical solution procedure

The last section reduced the Kirchhoff flow problem of Figure 1 to the modified Schwarz-Christoffel integral (16). As always in Schwarz-Christoffel mapping, before one can make use of this integral one must solve a parameter problem in order that not only the angles but also the lengths of sides in G_z come out right. In this case the parameter problem involves $n-1$ unknown prevertices $\{x_k\}$:

$$n-1 \textit{ unknowns:} \quad x_1, x_2, \ldots, x_{n-1} \quad \text{satisfying} \quad -1 < x_1 < \cdots < x_{n-1} < 1. \tag{18}$$

The correct values for $\{x_k\}$ will be determined by an iterative process. Suppose that at some step of this iteration a set of estimates $\{x_k\}$ is available. Then we first calculate a corresponding value x_* from (14), and a value W from the side-length condition

$$\left|W \int_{x_0}^{x_1} (x'-x_*) \prod_* h_k(x')\,dx'\right| = |z_1 - z_0|$$

derived from (16). This leaves $n-1$ further side-length conditions to be satisfied:

$$n-1 \textit{ equations:} \quad \left|W \int_{x_k}^{x_{k+1}} (x'-x_*) \prod_* h_k(x')\,dx'\right| = |z_{k+1} - z_k|, \quad 1 \le k \le n-1. \tag{19}$$

Thus the count is right, and under suitable additional hypotheses, the existence of a unique solution to (18)-(19) can presumably be proved. For general $\{x_k\}$ these equalities will not hold, and the errors in them will be used to devise a new guess $\{x_k\}$ for the next iterate.

We now sketch how we carry out this process numerically in the Fortran package KIRCH1. Most of the ideas are adapted from SCPACK and are discussed more fully in [15].

Solution of nonlinear system of equations. There is little reason to write one's own program to solve (18)-(19); excellent robust programs for this purpose already exist in the public domain. Several of these are based on the "hybrid" methods developed by M. J. D. Powell, which combine a steepest descent algorithm in early stages with a quasi-Newton algorithm as the solution is approached. One well-known program of this kind is the HYBRD1 code in the MINPACK library from Argonne National Laboratory, U.S.A. All of our own work has used instead Powell's code NS01A [11], which can also be found in the Harwell Subroutine Library. Although one could in principle compute the Jacobian matrix for (18)-(19) exactly, NS01A achieves superlinear convergence while requiring function values only. Beginning with the trivial initial guess of equally spaced prevertices, KIRCH1 typically converges to machine accuracy in around $4n$ iterations.

Change of variables to eliminate constraints. The parameter problem as written is constrained by the prevertex ordering conditions (18). If these constraints are ignored, meaningless iterates will be generated and the correct solution will generally not be found. However, it is a simple matter to eliminate the constraints by adapting the change of variables introduced in SCPACK and also proposed by Reppe [12]. Define

$$y_k = \log \frac{x_k - x_{k-1}}{x_{k+1} - x_k}, \quad 1 \le k \le n-1. \tag{20}$$

Then there is a one-to-one correspondence between sets of unconstrained parameters $\{y_k\} \in \mathbf{R}^{n-1}$ and sets of constrained parameters $\{x_k\}$ satisfying (18). To work in these variables, one can simply write the function evaluation subroutine called by NS01A so that it takes as input $\{y_k\}$ instead of $\{x_k\}$. The effect of this is that in our experience so far, KIRCH1 has converged to the correct solution for every obstacle Γ attempted.

Compound Gauss-Jacobi integration. Obviously (16) must be evaluated numerically except in the most trivial cases. Like the standard Schwarz-Christoffel integrand of (10), the integrand in (16) has the form $h(x)(x-x_k)^{-\beta_k}$ for some analytic function h near each prevertex x_k, $1 \le k \le n-1$, and the numerical integration procedure must take this singularity into account or it will be hopelessly inefficient. In KIRCH1 we apply the appropriate Gauss-Jacobi quadrature formula, whose nodes and weights are determined by calling the program GAUSSQ by Golub and Welsch [6], a version of which is also available in the NAG Subroutine Library. However, in the development of SCPACK it was observed that Gauss-Jacobi quadrature alone is not enough to ensure accurate integrals, because the exponentially large crowding factors common in conformal mapping often lead to one prevertex x_j being so near another one x_k that the associated singularity strongly affects intervals of integration ending at x_k (see also [12]). The solution devised there was to divide the interval into subintervals on which Gauss-Jacobi or pure Gauss (-Legendre) rules are applied, with the lengths of these subintervals chosen dynamically in such a way that none is ever longer than the distance to the nearest singularity [15]. This is what is meant by *compound* Gauss-Jacobi quadrature. We recommend the same procedure for Kirchhoff flow computations. It has proved highly effective in KIRCH1, where we consistently obtain integrals accurate to around d digits when $2d$ is taken as the standard number of quadrature points per interval.

Numerical integration near the separation points. At the separation points x_0 and x_n, the integrand of (16) has a singularity that is not simply of Gauss-Jacobi type, for although the boundary of G_z does not turn a corner at these points, it changes abruptly from a straight segment to a smooth curve. Again the singularity must be treated properly if the integration is to be efficient. To determine its form, note that at x_0, say, the boundary of the log-hodograph domain G_Ω defined by (5) consists of an intersection of a vertical and a horizontal straight line segment. Therefore $\Omega(x)$ has the form $\Omega(x)=\Omega(x_0)+h(x)\sqrt{x-x_0}$ near x_0 with h analytic. It follows that the integrand of (16), say $H(x)$, has a singularity of the same type as $e^{\Omega(x)}$ at x_0, i.e. $H(x)=h(\sqrt{x-x_0})$ for some new analytic function h. To integrate this, introduce a new variable $y=\sqrt{x'-x_0}$. Then the integral becomes

$$\int_{x_0}^{x} H(x')\,dx \;=\; \int_{0}^{\sqrt{x-x_0}} 2yH(y^2+x_0)\,dy, \tag{21}$$

and the integrand on the right is analytic at $y=0$, so it can be treated by Gauss-Legendre quadrature. We apply this rule in the usual compound way.

Evaluation of the inverse map. Equation (16) gives z as a function of x, which is what is needed for solving the parameter problem or producing plots of streamlines and equipotential lines as in the next section. To determine the potential or velocity at a given point in space, however, one needs to evaluate the inverse map $x(z)$. The obvious approach to this is to solve the equation $z(x)=z$ iteratively for x by Newton's method. Since $\frac{dz}{dx}$ is known exactly as the integrand of (16), this iteration can be carried out very efficiently. In practice any reasonable initial guess for x typically leads to convergence in 3 or 4 iterations.

Computation of drag and lift. By Bernoulli's equation, the pressure at a point $z\in\Gamma$ acts in the normal direction pointing into the fluid and is equal to to $\frac{1}{2}|v(z)|^2=\frac{1}{2}\bar{\zeta}\zeta$, if we assume the fluid has density 1 and take the pressure for $v=0$ as zero. Therefore the total force on a segment of Γ bounded by z_a and z_b is

$$F_{a,b}^{\text{flow}} = \frac{i}{2}\int_{z_a}^{z_b}\bar{\zeta}\zeta\,dz = \frac{i}{2}\int_{w_a}^{w_b}\bar{\zeta}(w)\,dw = \frac{iW}{2}\int_{x_a}^{x_b}(x-x_*)\bar{\zeta}(x)\,dx = \frac{iW}{2}\int_{x_a}^{x_b}(x-x_*)\bar{\zeta}(\bar{x})\,dx. \tag{22}$$

Now as pointed out in sightly different formulations by Levi-Civita and by Schiffman (see [5], p. 370 and p. 350), this result has a remarkable interpretation. Since $\zeta(x)$ maps $(-\infty,-1)$ onto an arc of the unit circle, it can be analytically continued by reflection to a function $\hat{\zeta}$ in the lower half x-plane defined by the formula $\hat{\zeta}(x) = 1/\overline{\zeta}(\overline{x})$ for $\operatorname{Im} x \le 0$. Thus (22) is equivalent to

$$F_{a,b}^{\text{flow}} = \frac{iW}{2}\int_{x_a}^{x_b} \frac{(x-x_*)}{\hat{\zeta}(x)}\,dx. \tag{23}$$

By (16), if $\hat{\zeta}$ were ζ, this would be a formula for $\frac{i}{2}(z_b - z_a)$. In other words to determine $F_{a,b}^{\text{flow}}$, we have reflected Γ across the free streamline Γ_- into a new obstacle $\hat{\Gamma}$, and each force along Γ corresponds to a distance along $\hat{\Gamma}$:

$$F_{a,b}^{\text{flow}} = i(\hat{z}_b - \hat{z}_a). \tag{24}$$

This interpretation of the forces on Γ suggests immediately how to compute them numerically: one calculates the dimensions of $\hat{\Gamma}$ by the usual compound Gauss-Jacobi quadrature procedure, making use of the branch $\hat{\zeta}$ instead of ζ and of Gauss-Jacobi formulas based on exponents $\hat{\beta}_k = -\beta_k$.

In the wake, the pressure has the constant value corresponding to $|v_\infty| = 1$, so the forces are given by

$$F_{a,b}^{\text{wake}} = \frac{i}{2}\int_{z_a}^{z_b} dz = \frac{i}{2}\int_{w_a}^{w_b} \frac{1}{\zeta(w)}\,dw = \frac{iW}{2}\int_{x_a}^{x_b} \frac{(x-x_*)}{\zeta(x)}\,dx = \frac{i}{2}(z_b - z_a), \tag{25}$$

where we have assumed that the wake lies to the left as Γ is traversed from z_a to z_b. From (23)-(25) it follows that the total force on the obstacle from both sides is

$$F = F_{0,n}^{\text{flow}} + F_{n,0}^{\text{wake}} = \frac{i}{2}(\hat{z}_n - z_n), \tag{26}$$

since $z_0 = \hat{z}_0$. It is customary to resolve this number into its components,

$$F = F_{\text{drag}} + iF_{\text{lift}} = (C_D + C_L)L, \tag{27}$$

where the *drag* and *lift coefficients* C_D and C_L are defined in terms of some reference dimension L of Γ, such as its total height or length.

Together, ζ and $\hat{\zeta}$ define a single-valued analytic function ζ on $C \setminus [-1,1]$, and the force can equivalently be obtained by integrating (25) clockwise around any contour enclosing $[-1,1]$ in this domain:

$$F = \frac{iW}{2}\oint \frac{(x-x_*)}{\zeta(x)}\,dx. \tag{28}$$

Clearly this number is equal to $2\pi i$ times an appropriate residue at ∞, but in practice it is just as convenient to obtain F from (26) by computing integrals for $\hat{z}_n$ and z_n.

4. Computed results

To test the accuracy of KIRCH1, we have compared lift and drag coefficients computed by it with various values that are given in the literature. First we considered three geometries whose exact solutions are known: the flat plate due to Kirchhoff in 1869 [5, p. 329], the inclined plate due to Rayleigh in 1876 [5, p. 330], and the symmetric wedge due to Bobyleff in 1881 [9, p. 104]. In these cases KIRCH1 reproduces the exact values C_D and C_L to many digits. Then we considered certain geometries for which numerical solutions have been published: an inclined plate with separation from the back face studied by Chaplygin and Lavrentiev in 1933 and by Sekerzh-Zenkovich in 1934 [7], an asymmetrical wedge studied by C. C. Lin in 1960 [20], a symmetrical 4-piece wedge studied by Wu and Wang in 1964 [20], and a plate with spoiler studied previously by the first author [3]. (The values tablulated under the heading "LS" in [3] are in error, and were replaced for this comparison by Elcrat's previously announced corrections.) We also considered a circular arc of half-angle 55°, studied by Brodetsky in 1923 and Schmieden in 1929 [13], which we approximated by inscribed polygons. In all of these cases KIRCH1 reproduces the published values C_D and C_L up to small discrepancies which we attribute to the published sources, except that the numbers of Sekerzh-Zenkovich reported in [7] appear to be wrong.

On the basis of these tests and other evidence, we believe that KIRCH1 can reliably compute flows to arbitrary accuracy over arbitrary polygonal obstacles with up to one or two dozen vertices. The number of correct digits obtained increases roughly linearly with the number of Gauss-Jacobi quadrature points per interval, and therefore the accuracy can be doubled by roughly doubling the computation time.

We will now present some new Kirchhoff flow calculations of our own, summarized in Figures 4-9. Our purpose in presenting these examples is first, to demonstrate visually that the ideas described above really work, and second, to record some further numbers for comparison with future experiments. Examples of more direct physical interest will be considered in later papers.

Each figure shows the obstacle Γ together with a system of streamlines at invervals $\Delta\psi = .1$ and equipotential lines at intervals $\Delta\phi = .2$. These curves are obtained by mapping a rectilinear grid in G_w conformally into G_z. The first thing to notice in looking at the figures is that evidently the Kirchhoff flow problem has indeed been solved -- for along the free streamlines, the equipotential lines are in each case evenly spaced, signifying constant flow speed. Moreover, since the successive lines are separated by distances of .2, the speed has the correct value 1. By contrast, note that in the examples with large drags especially, the separation between adjacent streamlines or equipotential lines in the inflow end of the plots is visibly greater.

The additional shape shown inside each wake (in Figs. 6 and 8, below the wake) is the reflected boundary $\hat{\Gamma}$ described in the last section. Long sides in $\hat{\Gamma}$ correspond to high flow speeds. The segments of $\hat{\Gamma}$ that come out especially small are those along which the fluid is nearly stagnant.

Each figure lists the computed drag and lift coefficients and parameter W, all of them probably accurate to the six digits given. The number of steps in the iterative solution of the parameter problem by NS01A is also listed. The total amount of work for solving the parameter problem scales roughly as n^2 times this number of steps, and for drawing the plot, as n^2 times a large constant [15]. A convenient and machine-independent way to measure the computer times for these tasks is to count the total number of complex logarithms calculated in all products (17) during the computation, for these calculations turn out to dominate the total computer time. In each figure the first logarithm count listed corresponds to the solution of the parameter problem, and the second to the construction of the plot. The counts are approximate.

It remains to state the dimensions and reference lengths L of the various obstacles, and to make a few comments on each.

Figure 4: flat plate. This is the problem treated by Kirchhoff, whose solution can be found in several of our references, e.g. [5]. The plate has length $L = 1$. The exact values for C_D and W are $2\pi/(4+\pi)$ and $2/(4+\pi)$, respectively.

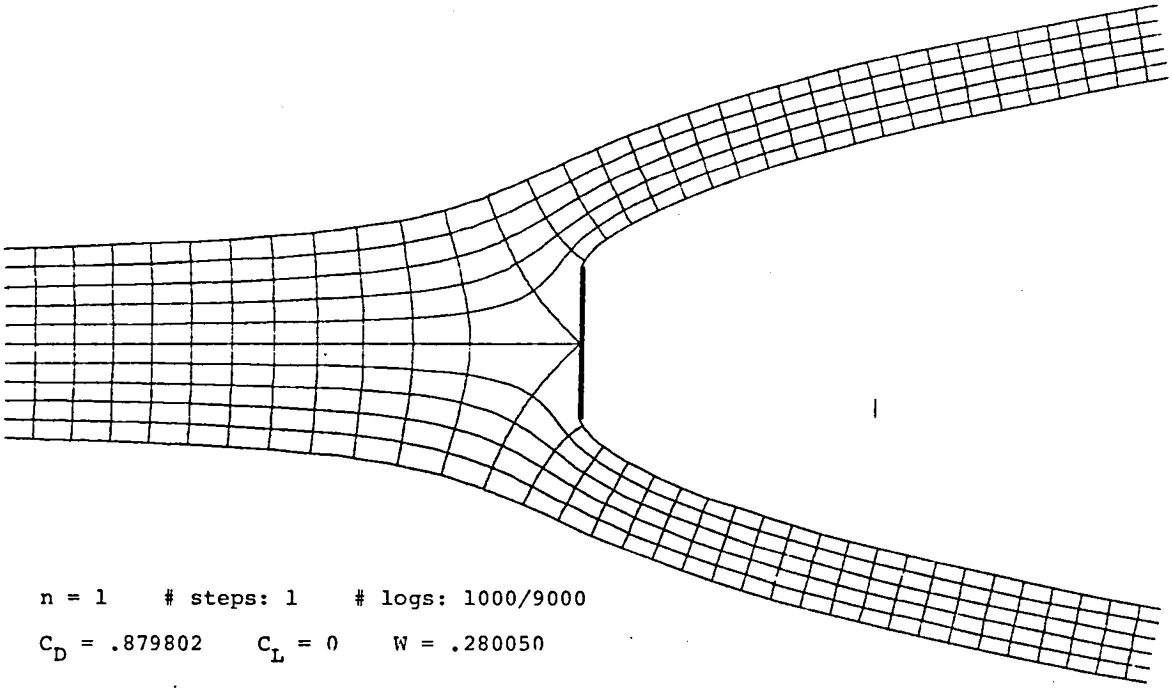
Figure 5: wedge. Here the plate still has length $L = 1$ but has been bent at the middle, with the lower half inclined at an angle 45° and the upper half at 30°. The computed stagnation point lies on the lower face at a distance .017348 from the vertex. Even simple wedge problems of this kind cannot be solved analytically.

Figure 6: plate with separation from rear face. This obstacle, motivated by the model proposed by Chaplygin and Lavrentiev mentioned above [7], consists of a plate of length $L = 1$ inclined at angle 30° that bends back 180° at the leading edge into another plate of length 1/2. In other words, it is an inclined plate with separation prescribed at the middle of the back face. Note that the sharp edge in Γ maps to a broad channel extending to ∞ in $\hat{\Gamma}$. The stagnation point lies at a distance .193308 from the leading edge.

Figure 7: plate with spoiler. Here a spoiler of length .2 at angle 45° has been added to the last obstacle, forming the geometry considered previously by the first author [3] and also by Bassanini [1]. The stagnation point moves up to a distance .090124 from the leading edge, and the singularity there weakens considerably.

Figure 8: equilateral triangle. This obstacle is a symmetrical equilateral triangle with side lengths $L = 1/2$, having four vertices all told since the two separation points are mathematically distinct. The flow crosses over itself, hence is nonphysical. We give this example to emphasize that nothing in our formlulation requires that the Kirchhoff flow, or even the obstacle itself, be embeddable in the plane. We have also computed some extremely nonphysical flows over various more exotic obstacles, but there is no space to present them here.

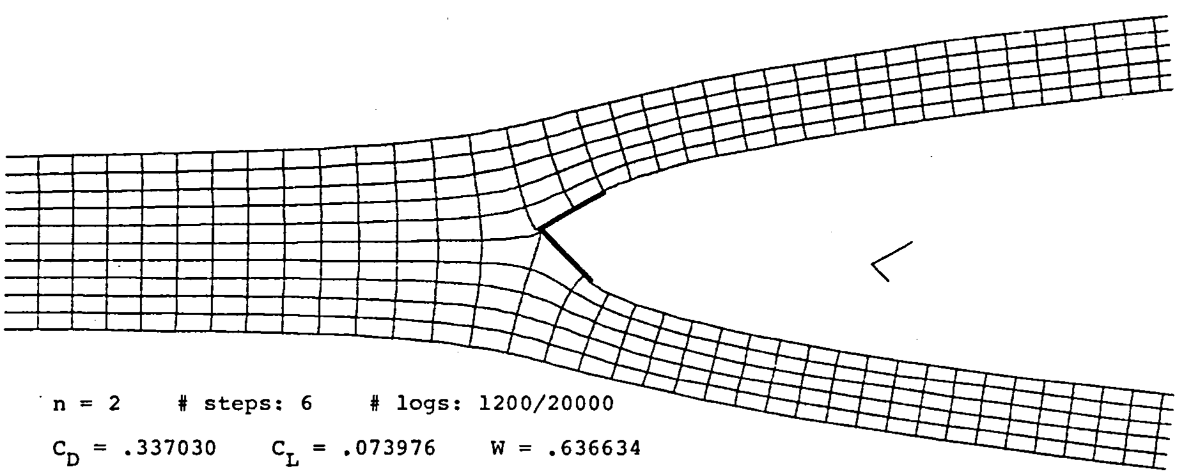
Figure 9: charm bracelet. Finally, we include this 13-gon, whose aerodynamic importance is limited, to emphasize that the methods described in this paper work for arbitrary polygons. The three straight legs, inclined at angles 45°, 90°, and 30°, each have length 1/2, and they meet each other at distances .15 and .1 above the square and the triangle, respectively, both of which have have side lengths .15. The reference length is $L = 1$. Note that the sections $\hat{z}_1$-$\hat{z}_3$ and $\hat{z}_6$-$\hat{z}_{10}$ of $\hat{\Gamma}$ are so small, since the fluid is nearly stagnant there, that they are scarcely visible in the plot.



n = 1 # steps: 1 # logs: 1000/9000

C_D = .879802 C_L = 0 W = .280050

Figure 4



n = 2 # steps: 6 # logs: 1200/20000

C_D = .337030 C_L = .073976 W = .636634

Figure 5

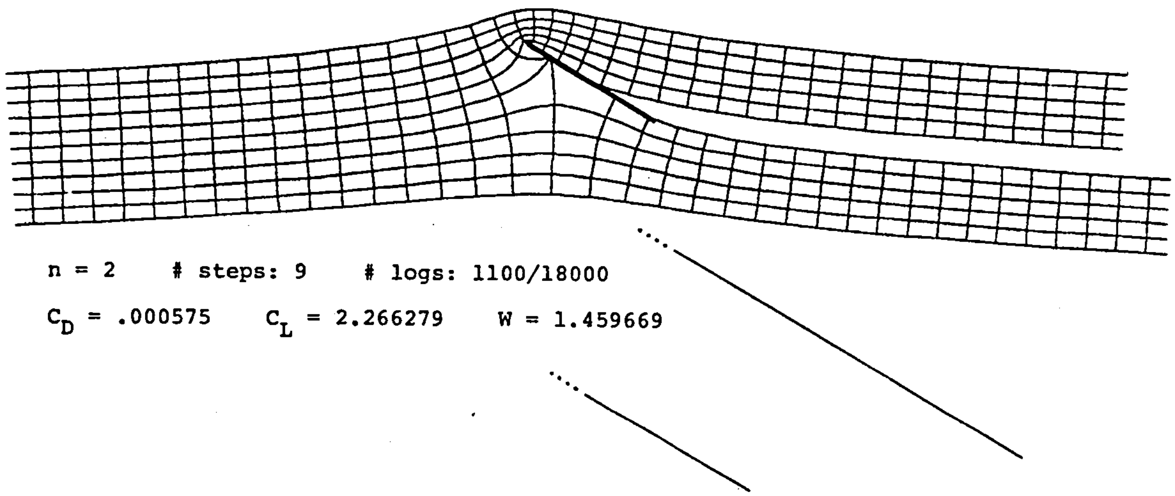


Figure 6

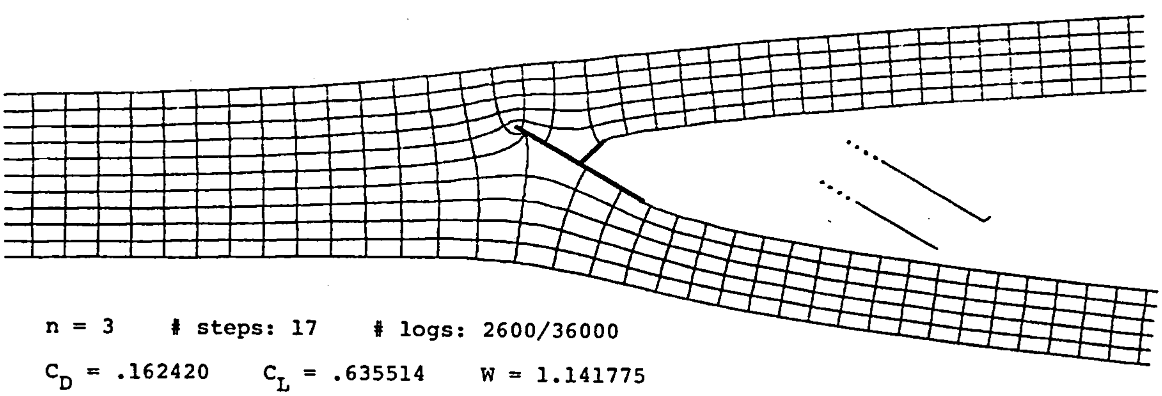
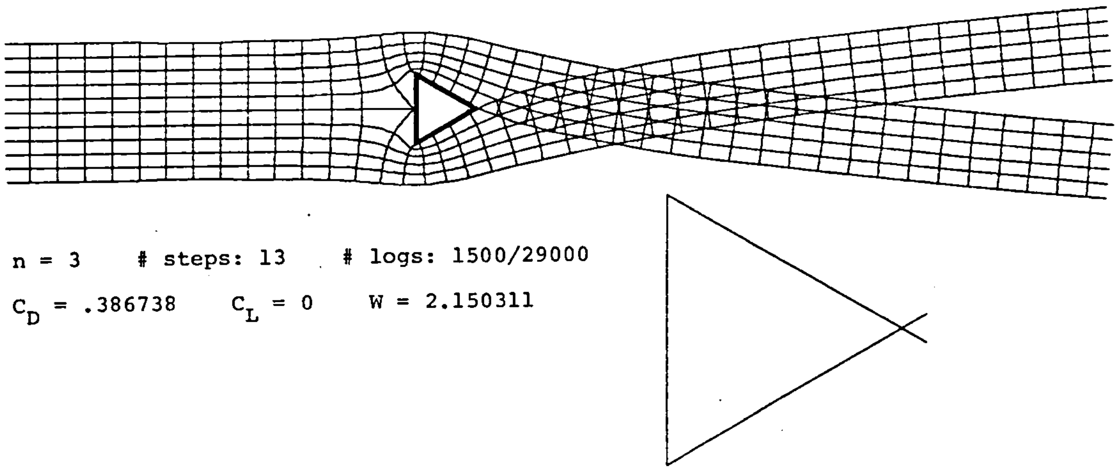


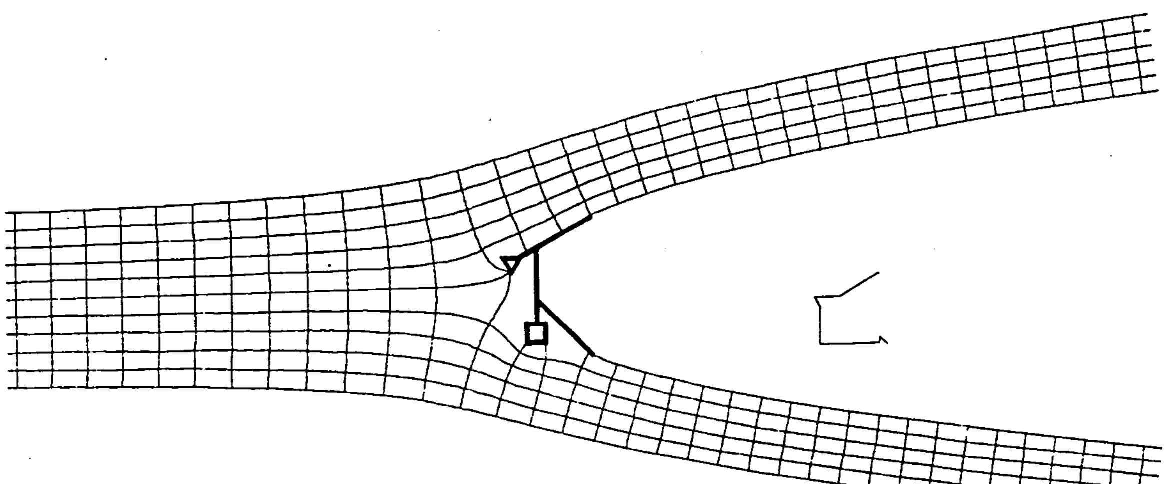
Figure 7



n = 3   # steps: 13   # logs: 1500/29000

C_D = .386738 C_L = 0   W = 2.150311

Figure 8



n = 13   # steps: 96   # logs: 319000/265000

C_D = .428060 C_L = -.044866   W = 1.159328

Figure 9

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References

[1] P. Bassanini and A. Guspini, Un programmadi calculo per un nodello dis fluso con scia per un profela con spoiler, *IAC Serie III* (1983), 1-17.

[2] G. Birkhoff and E. H. Zarantonello, *Jets, Wakes, and Cavities,* Academic Press, New York, 1957.

[3] A. R. Elcrat, Separated flow past a plate with spoiler, *SIAM J. Math. Anal. 13* (1982), 632-639.

[4] D. Gilbarg, A generalization of the Schwarz-Christoffel transformation, *Proc. Nat. Acad. Sci. 35* (1949), 609-611.

[5] D. Gilbarg, Jets and Cavities, *Handbuch der Physik 9* (1960), Springer, Berlin, 311-445.

[6] G. H. Golub and J. H. Welsch, Calculation of Gaussian quadrature rules, *Math. Comp. 23* (1969), 221-230.

[7] M. I. Gurevich, *Theory of Jets in Ideal Fluids,* Academic Press, New York, 1965.

[8] G. Kirchhoff, Zur Theorie freier Flüssigkeitsstrahlen, *J. Reine Angew. Math. 70* (1869), 289-298.

[9] H. Lamb, *Hydrodynamics,* Dover, New York, 1945.

[10] V. N. Monakhov, *Boundary-Value Problems with Free Boundaries for Elliptic Systems of Equations,* Amer. Math. Soc., Providence, R.I., 1983.

[11] M. J. D. Powell, A Fortran subroutine for solving systems of non-linear algebraic equations, Tech. Rep. AERE-R. 5947, Harwell, England, 1968.

[12] K. Reppe, Berechnung von Magnetfeldern mit Hilfe der konformen Abbildung durch numerische Integration der Abbildungsfunktion von Schwarz-Christoffel, *Siemens Forsch. u. Entwickl. Ber. 8* (1979), 190-195.

[13] J. A. Robertson, *Hydrodynamics in Theory and Application,* Prentice-Hall, Englewood Cliffs, N.J., 1965.

[14] P. G. Saffman and S. Tanveer, Vortex induced lift on two dimensional low speed wings, *Studies in Appl. Math.,* to appear.

[15] L. N. Trefethen, Numerical computation of the Schwarz-Christoffel transformation, *SIAM J. Sci. Stat. Comput. 1* (1980), 82-102.

[16] L. N. Trefethen, SCPACK Version 2 User's Guide, Int. Rep. 24, Inst. for Computer Applics. in Sci. and Engr., NASA Langley Res. Ctr., 1983.

[17] L. N. Trefethen, Analysis and design of polygonal resistors by conformal mapping, *Z. Angew. Math. Phys.,* to appear.

[18] A. Weinstein, Non-linear problems in the theory of fluid motion with free boundaries, in *Proc. Symp. Appl. Math.,* v. 1, Amer. Math. Soc., Providence, R.I., 1949.

[19] T. Y. Wu, Cavity and Wake Flows, *Annual Reviews in Fluid Mech.* (1972), 243-84.

[20] T. Y. Wu and D. P. Wang, A wake model for free-streamline flow theory II, *J. Fluid. Mech. 18* (1964), 65-93.

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16. Abstract

In classical Kirchhoff flow, an ideal incompressible fluid flows past an obstacle and around a motionless wake bounded by free streamlines. Since 1869 it has been known that in principle, the two-dimensional Kirchhoff flow over a polygonal obstacle can be determined by constructing a conformal map onto a polygon in the log-hodograph plane. In practice, however, this idea has rarely been put to use except for very simple obstacles, because the conformal mapping problem has been too difficult. This paper presents a practical method for computing flows over arbitrary polygonal obstacles to high accuracy in a few seconds of computer time. We achieve this high speed and flexibility by working with a modified Schwarz-Christoffel integral that maps onto the flow region directly rather than onto the log-hodograph polygon. This integral and its associated parameter problem are treated numerically by methods developed earlier by Trefethen for standard Schwarz-Christoffel maps.

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