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NOMENCLATURE

$^{N}\vec{a}^{B^*}$	acceleration of helicopter c.g. in reference frame N
$^{N}\vec{a}^{E^*}$	acceleration of load c.g. in reference frame N
a	longitudinal distance between helicopter cable attachment point and helicopter c.g.
a'	longitudinal distance between load cable attachment point and load c.g.
B	helicopter (rigid body)
B^*	helicopter c.g.
$B_x, B_y, B_z, B_{xy}, B_{xz}, B_{yz}$	helicopter body reference frame moments and products of inertia
C	rear cable (rigid link)
$\vec{c}_1, \vec{c}_2, \vec{c}_3$	mutually orthogonal unit vectors fixed in C
$\vec{c}_1', \vec{c}_2', \vec{c}_3'$	mutually orthogonal unit vectors fixed in an intermediate reference frame between B and C
E	slung load (rigid body)
E^*	c.g. of slung load
E_1, E_2, E_3, E_{13}	load body reference frame moments and product of inertia
$\vec{e}_1, \vec{e}_2, \vec{e}_3$	mutually orthogonal unit vectors fixed in E
$\vec{e}_1', \vec{e}_2', \vec{e}_3', \vec{e}_1'', \vec{e}_2'', \vec{e}_3''$	mutually orthogonal unit vectors fixed in an intermediate reference frame
F	forward cable (rigid link)
$\vec{F}^B$	external force applied to helicopter
$\vec{F}^E$	external force applied to load
F_s	generalized active force for system
F_s^*	generalized inertial force for system
F_s^B	generalized active force for helicopter
F_s^{*B}	generalized inertial force for helicopter
F_s^E	generalized active force for load

F_s^{*E}	generalized inertial force for load
g	acceleration of gravity
h	distance along $\vec{e}_3$ between load c.g. and $\vec{e}_1$ axis
$\vec{i},\vec{j},\vec{k}$	mutually orthogonal unit vectors fixed in B
$\vec{I}^B,\vec{I}^E$	inertial dyadic for helicopter and load, respectively
L_B,L_E	aerodynamic rolling moments applied to helicopter and load, respectively
ℓ	rear cable length
ℓ'	forward cable length
M	helicopter mass
M_B,M_E	aerodynamic pitching moments applied to helicopter and load, respectively
m	slung load mass
N	inertial reference frame (fixed at Earth's surface)
N_B,N_E	aerodynamic yawing moments applied to helicopter and load, respectively
$\vec{n}_1,\vec{n}_2,\vec{n}_3$	mutually orthogonal unit vectors fixed in N
P,P_1	rear cable attachment points to helicopter and load, respectively
p,q,r	angular velocities about helicopter body axes (roll, pitch, and yaw rates)
Q,Q_1	forward cable attachment points to helicopter and load, respectively
R	distance along $\vec{k}$ axis between helicopter c.g. and helicopter load attachment points
R^{*B}	inertial force for helicopter
R^{*E}	inertial force for load
s	index variable for generalized speeds
$\vec{T}^B$	external torque applied to helicopter
$\vec{T}^E$	external torque applied to load
$\vec{T}^{*B}$	inertial torque for helicopter
$\vec{T}^{*E}$	inertial torque for load
u,v,w	translational velocities along helicopter body axes

$^{N}\vec{v}^{B^*}$	translational velocity of helicopter c.g. in reference frame N
$^{N}\vec{v}^{E^*}$	translational velocity of load c.g. in reference frame N
$\vec{v}_s$	$\frac{\partial \vec{v}}{\partial s}$
X_B, X_E	longitudinal aerodynamic force of helicopter and load, respectively
Y_B, Y_E	lateral aerodynamic force of helicopter and load, respectively
Z_B, Z_E	normal aerodynamic force of helicopter and load, respectively
α_E	load angle of attack
$^{N}\vec{\alpha}^{B}$	helicopter angular acceleration in reference frame N
$^{N}\vec{\alpha}^{E}$	load angular acceleration in reference frame N
β_E	load sideslip angle
δ	load angular displacement about $\vec{e}_1$ axis
ζ	total load angular displacement about $\vec{j}$ axis = $(\sigma + \theta_E)$
η	rear cable longitudinal sway angle
θ	helicopter pitch attitude
θ_E	incidence of load relative to helicopter
ν	load angular displacement about $\vec{e}_3'$ axis
ξ	rear cable lateral sway angle
σ	variable load angular displacement about $\vec{j}$ axis
ϕ	helicopter roll attitude
ψ	helicopter yaw attitude
$^{N}\vec{\omega}^{B}$	helicopter angular velocity in reference frame N
$^{B}\vec{\omega}^{C}$	rear cable angular velocity in reference frame B
$^{B}\vec{\omega}^{E}$	load angular velocity in reference frame B
$^{N}\vec{\omega}^{E}$	load angular velocity in reference frame N
$\vec{\omega}_s$	$\frac{\partial \vec{\omega}}{\partial s}$

A LAGRANGE-D'ALEMBERT FORMULATION OF THE EQUATIONS OF MOTION OF A HELICOPTER CARRYING AN EXTERNALLY SUSPENDED LOAD

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ABSTRACT

The exact nonlinear equations of motion are derived for a helicopter with an external load suspended by fore and aft, rigid-link cables. Lagrange's form of D'Alembert's principle is used. Ten degrees of freedom are necessary to represent the motion of this system in an inertial reference frame: six for the helicopter relative to inertial space and four for the load relative to the helicopter.

INTRODUCTION

Civilian and military use of helicopters to carry externally suspended loads has resulted in dynamic instabilities being encountered at certain flight conditions. Previous research aimed at alleviating these instabilities has led to the development of cable angle feedback control techniques (refs. 1-3) and hydraulically powered mechanization of the suspension cable to fuselage attachments (ref. 4). Because of the mathematical complexity of the system, the kinematic models used therein have been developed either with the motion of the load relative to the helicopter linearized about a fixed operating point or with assumptions incorporated to simplify the relative load motion in one or more axes.

It is desirable to examine the quality and validity of these simplifications and also to provide a reference from which additional simplified models may be developed for the purpose of further control law design and real-time piloted simulation. This report documents the derivation of this reference: a mathematical model of the helicopter and slung load with as much fidelity to the physical system as possible. Specifically, the model consists of the ten-degree-of-freedom nonlinear equations of motion for a typical two-cable slung loac developed using Lagrange's form of D'Alembert's principle (ref. 5). All six rigid-body degrees of freedom (three translational and three rotational) have been allowed for the helicopter. The load is suspended from the fuselage by two cables. Four rotational degrees of freedom relative to the helicopter are allowed for the suspension/load system, considered in the analysis to be a nonholonomic system. The derivation is exact except for the assumption that the cables are massless, rigid links.

The balance of this report is organized as follows: the main text contains a model description, a definition of applied forces and moments, a discussion of basic assumptions, the method of analysis, and finally the actual formulation of the equations of motion. Several appendixes contain the appropriate details of each step of the analysis and the formulation.

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MODEL DESCRIPTION

Figure 1 illustrates the system to be modeled mathematically. Comprised of a helicopter B, load E, and suspension cables C and F, the system moves in an inertial (Earth-fixed) reference frame N, in which the orthogonal set of unit vectors, $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$, are fixed.

Helicopter B is a rigid body of mass M, in which orthogonal unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are fixed. The orientation of reference frame B relative to N may be given in terms of the conventional Euler angles of flight mechanics: successive rotations through angles ψ, θ, and ϕ (appendix A). The center of mass of B is B*, at which central inertia quantities B_x, B_y, B_z, B_{xy}, B_{xz}, and B_{yz} are defined. The reference frame B has six degrees of freedom relative to N. Equations (1) and (2) express ${}^N\vec{\omega}^B$, the angular velocity of B relative to N, and ${}^N\vec{v}^{B^*}$, the translational velocity of B* in N.

$$ {}^N\vec{\omega}^B = p\vec{i} + q\vec{j} + r\vec{k} \tag{1} $$

$$ {}^N\vec{v}^{B^*} = u\vec{i} + v\vec{j} + w\vec{k} \tag{2} $$

where the scalars are the helicopter body axis velocity components typically used in flight mechanics analysis.

Figure 2 shows a more detailed geometrical representation of the system at rest. The incidence of the load relative to the helicopter is θ_E, and is defined as the angle between $\vec{e}_1$ and $\vec{i}$ in the $\vec{i}$-$\vec{k}$ plane. Since it is fixed when the load is attached, it has been considered constant in this derivation. Constant lengths a, a', ℓ, ℓ', and R are defined as shown. P_1, P, Q_1, and Q locate cable attachment points.

Four degrees of freedom are necessary to describe the motion of the suspension/load system (C, F, and E) relative to B. The method of defining these angular relationships (discussed in ref. 6) is particularly efficient and has been used here. Reference frame C, in which $\vec{c}_1$, $\vec{c}_2$, and $\vec{c}_3$ are fixed, is oriented relative to B with two successive rotations: η and ξ, as shown in figure 3 and described in detail in appendix A. The reference frame C moves with two degrees of freedom relative to B such that the angular velocity of C relative to B may be defined in equation (3):

$$ {}^B\vec{\omega}^C = \dot{\eta}\vec{j} + \dot{\xi}\vec{c}_1 \tag{3} $$

Load E, consisting of a massless spreader bar and cargo, has mass center E*. It is a rigid body in which $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ are fixed. It has mass m and central inertia properties E_1, E_2, E_3, and E_{13} (E_{12} and E_{23} have been assumed negligible by symmetry). The orientation of E relative to B (fig. 4) results from three successive rotations through angles ζ, ν, and δ and is described in detail in appendix A. The angle, ζ, has been defined to represent the absolute angular displacement about $\vec{j}$:

$$ \zeta \equiv c + \theta_E \tag{4} $$

Load E moves relative to B as in equation (5):

$$\vec{\omega}^{B\to E} = \dot{\sigma}\vec{j} + \dot{\nu}\vec{e}_3' + \dot{\delta}\vec{e}_1 \tag{5}$$

where

$$\dot{\sigma} = \dot{\zeta} \tag{6}$$

Although the motion of E relative to B may be expressed in terms of three angular rates, only two are independent. From geometry (ref. 6) a nonholonomic constraint may be found for $\dot{\nu}$ such that:

$$\dot{\nu} = \dot{\nu}(\dot{\eta},\dot{\xi},\dot{\sigma},\eta,\xi,\sigma) \tag{7}$$

APPLIED FORCES AND MOMENTS

External inputs to the helicopter/slung load system consist of aerodynamic and gravitational forces acting at the mass center of each body, together with aerodynamic moments.

Using conventional flight mechanics notation, $\vec{F}^B$ and $\vec{T}^B$, the forces and moments applied to B are:

$$\vec{F}^B = X_B\vec{i} + Y_B\vec{j} + Z_B\vec{k} + Mg\vec{n}_3 \tag{8}$$

$$\vec{T}^B = L_B\vec{i} + M_B\vec{j} + N_B\vec{k} \tag{9}$$

Using equation (A4) to resolve $\vec{F}^B$ into the helicopter body reference frame, equation (8) becomes:

$$\vec{F}^B = (X - Mg\sin\theta)\vec{i} + (Y + Mg\cos\theta\sin\phi)\vec{j} + (Z + Mg\cos\theta\cos\phi)\vec{k} \tag{10}$$

Applied to E, the forces and moments are:

$$\vec{F}^E = X_E\vec{e}_1 + Y_E\vec{e}_2 + Z_E\vec{e}_3 + Mg\vec{n}_3 \tag{11}$$

$$\vec{T}^E = L_E\vec{e}_1 + M_E\vec{e}_2 + N_E\vec{e}_3 \tag{12}$$

Using equation (A4), $\vec{F}^E$ may be resolved into the helicopter body reference frame:

$$\begin{aligned}\vec{F}^E = {}&\{c_\nu c_\zeta X_E + (s_\delta s_\zeta - c_\delta c_\zeta s_\nu)Y_E + (c_\delta s_\zeta + c_\zeta s_\delta s_\nu)Z_E - s_\theta mg\}\vec{i} + \{s_\nu X_E + c_\delta c_\nu Y_E \\ &- s_\delta c_\nu Z_E + c_\theta s_\phi mg\}\vec{j} + \{-c_\nu s_\zeta X_E + (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)Y_E + (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta)Z_E \\ &+ c_\theta c_\phi mg\}\vec{k} \text{ (Footnote } a\text{)}\end{aligned} \tag{13}$$

[a] $s_\gamma = \sin\gamma$, $c_\gamma = \cos\gamma$.

Wind-tunnel data for load forces and moments are often given as a function of the angle of attack and sideslip angles of the load itself: α_E and β_E. In the Formulation of Equations of Motion section of this report, these angles have been calculated using the load fixed-reference-frame components of the load translational velocity, ${}^N\vec{v}^{E^*}$.

BASIC ASSUMPTIONS

Several kinematic and geometric assumptions have been made in the problem statement. To summarize, they are:

1. The rotation of the Earth may be neglected, thus making an Earth-fixed reference frame the inertial frame.

2. Suspension cables C and F are massless and rigid.

3. The cables joining the spreader bar and the load are rigidly attached, each to the spreader bar and to the load.

4. Load products of inertia E_{12} and E_{23} are zero ($\vec{e}_1 - \vec{e}_3$ defines a plane of symmetry).

5. Load suspension attachment points are equidistant from B^* along $\vec{x}$, and equidistant from E^* along $\vec{e}_1$.

METHOD OF ANALYSIS

The equations of motion of this system may be written using Lagrange's form of D'Alembert's principle as developed by Kane in reference 5. With F_s and F_s^* defined as the generalized active and generalized inertia forces, respectively, this principle (eq. 14) holds for a nonholonomic system moving with $n - m$ degrees of freedom in an inertial reference frame.

$$F_s + F_s^* = 0 \quad (s = 1, \ldots, n - m) \tag{14}$$

where n is the number of system-generalized coordinates (or generalized speeds) and m is the number of nonholonomic constraints.

For a rigid body, the generalized active force F_s is (ref. 5):

$$F_s = \vec{F} \cdot \vec{v}_s + \vec{T} \cdot \vec{\omega}_s \tag{15}$$

where $\vec{F}$ and $\vec{T}$ are, respectively, the external forces and torques applied to the body,

$\vec{v}_s$ is the partial derivative of the velocity of the point of application of $\vec{F}$ with respect to generalized speed $s = \partial\vec{V}/\partial s\big|_{\text{of point of application of } \vec{F}}/\partial s$,

and $\vec{\omega}_s$ is the partial derivative of the angular velocity of the body to which $\vec{T}$ is applied with respect to generalized speed $s = \partial\vec{\omega}\big|_{\text{of body to which } \vec{T} \text{ is applied}}$.

Likewise, the generalized inertia force F_s^* is:

$$F_s^* = \vec{R}^* \cdot \vec{v}_s + \vec{T}^* \cdot \vec{\omega}_s \tag{16}$$

such that for the inertia force $\vec{R}^*$

$$\vec{R}^* = -m\vec{a}^*$$

where m is the mass of the rigid body, $\vec{a}^*$ is the acceleration of the mass center of the body, and for the inertial torque $\vec{T}^*$

$$\vec{T}^* = -\vec{\vec{I}} \cdot \vec{\alpha} + (\vec{\vec{I}} \cdot \vec{\omega}) \times \vec{\omega} \tag{17}$$

where $\vec{\omega}$ and $\vec{\alpha}$ are, respectively, the angular velocity and angular acceleration of the rigid body.

Finally, $\vec{\vec{I}}$ is the inertial dyadic of the body

$$\vec{\vec{I}} = \sum_{j=1}^{3} \sum_{k=1}^{3} \vec{b}_j I_{jk} \vec{b}_k$$

where $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$ are unit vectors along lines about which the I_{jk} are taken.

Specifically, for the helicopter/slung load system with two rigid bodies B and E:

$$F_s + F_s^* = \underbrace{F_s^B + F_s^{*B}}_{\text{helicopter}} + \underbrace{F_s^E + F_s^{*E}}_{\text{load}} = 0 \qquad (s = 1, \ldots, n - m) \tag{18}$$

To further expand equation (18), consider that since the external forces applied to B and E act at the center of mass of each body, the partial translational velocities, $\vec{v}_s$, are taken at points B^* and E^*:

$$F_s + F_s^* = \underbrace{\vec{F}^B \cdot {}^N\vec{v}_s^{B^*} + \vec{T}^B \cdot {}^N\vec{\omega}_s^B + \vec{R}^{*B} \cdot {}^N\vec{v}_s^{B^*} + \vec{T}^{*B} \cdot {}^N\vec{\omega}_s^B}_{\text{helicopter}}$$

$$+ \underbrace{\vec{F}^E \cdot {}^N\vec{v}_s^{E^*} + \vec{T}^E \cdot {}^N\vec{\omega}_s^E + \vec{R}^{*E} \cdot {}^N\vec{v}_s^{E^*} + \vec{T}^{*E} \cdot {}^N\vec{\omega}_s^E}_{\text{load}} = 0$$

$$(s = 1, \ldots, n - m) \tag{19}$$

FORMULATION OF EQUATIONS OF MOTION

Eleven generalized speeds (translational and angular velocity components) have been used to describe the motion of this system:

$$s = u, v, w, p, q, r, \dot{\eta}, \dot{\xi}, \dot{\sigma}, \dot{\nu}, \dot{\delta} \tag{20}$$

One nonholonomic constraint,

$$\dot{\nu} = \dot{\nu}(\dot{\eta},\dot{\xi},\dot{\sigma},\eta,\xi,\sigma) \tag{5}$$

may be found, resulting in a system with ten degrees of freedom.

In this derivation the system (although holonomic) has been treated as being nonholonomic, in which the (nonholonomic) constraint (eq. (5)) has been substituted for $\dot{\nu}$ prior to the kinematic analysis. To write the ten equations of motion (eq. (19)), it is necessary to determine, for each rigid body, the angular and translational velocities, partial velocities, angular and translational accelerations, and finally the inertial forces and torques.

Nonholonomic Constraint Equation

To express $\dot{\nu}$ in terms of the independent variables: $\dot{\eta}$, $\dot{\xi}$, $\dot{\sigma}$, η, ξ, and σ, consider the physical constraint that the forward cable, F, is a rigid link of length ℓ'. This may be expressed mathematically by saying that the magnitude of the distance between Q_1 and Q is always ℓ':

$$|Q_1 - Q| = |-R\vec{k} - a\vec{i} + \vec{R}\vec{k} + \ell\vec{c}_3 + a'\vec{e}_1| = \ell' \tag{21}$$

Resolving the expression to reference frame B, and squaring each side of the equation gives:

$$\frac{a^2}{\ell^2} + \frac{a'^2}{\ell^2} + 1 + \frac{2a}{\ell} c_\xi s_\eta - \frac{2aa'}{\ell^2} c_\nu c_\zeta + \frac{2a'}{\ell}(c_\nu c_\zeta c_\xi s_\eta - s_\xi s_\nu - c_\xi c_\eta c_\nu s_\zeta) = \frac{\ell'^2}{\ell^2} \tag{22}$$

Differentiating equation (22) and solving for $\dot{\nu}$ gives the necessary constraint equation (ref. 6):

$$\dot{\nu} = Z_1\dot{\eta} + Z_2\dot{\xi} + Z_3\dot{\sigma} \tag{23}$$

where

$$Z_1 = \frac{\frac{-2a}{\ell} c_\xi c_\eta + \frac{2a'}{\ell}(c_\nu c_\zeta c_\xi c_\eta + c_\xi s_\eta c_\nu s_\zeta)}{\frac{-2aa'}{\ell^2} s_\nu c_\zeta + \frac{2a'}{\ell}(s_\nu c_\zeta c_\xi s_\eta + s_\xi c_\nu - c_\xi c_\eta s_\nu s_\zeta)}$$

$$Z_2 = \frac{\frac{2a}{\ell} s_\xi s_\eta + \frac{2a'}{\ell}(-c_\nu c_\zeta s_\xi s_\eta - c_\xi s_\nu + s_\xi c_\eta c_\nu s_\zeta)}{\frac{-2aa'}{\ell^2} s_\nu c_\zeta + \frac{2a'}{\ell}(s_\nu c_\zeta c_\xi s_\eta + s_\xi c_\nu - c_\xi c_\eta s_\nu s_\zeta)}$$

$$Z_3 = \frac{\frac{2aa'}{\ell^2} c_\nu s_\zeta + \frac{2a'}{\ell}(-c_\nu s_\zeta c_\xi s_\eta - c_\xi c_\eta c_\nu c_\zeta)}{\frac{-2aa'}{\ell^2} s_\nu c_\zeta + \frac{2a'}{\ell}(s_\nu c_\zeta c_\xi s_\eta + s_\xi c_\nu - c_\xi c_\eta s_\nu s_\zeta)}$$

(Details of the solution for this expression are given in appendix B.)

Angular Velocity

Helicopter relative to inertial space- The angular velocity of the helicopter in the inertial reference frame was given in equation (1).

Load relative to inertial space- Relative to the helicopter body frame, the angular velocity of the load was defined in equation (5) or, after substitution for the nonholonomic constraint:

$$\overset{B\to E}{\omega} = \dot{\sigma}\vec{j} + (Z_1\dot{\eta} + Z_2\dot{\zeta} + Z_3\dot{\sigma})\vec{e}_3' + \dot{\delta}\vec{e}_1 \tag{24}$$

Since $\overset{N\to B}{\omega}$ is known, the angular velocity of the load relative to inertial space may be found:

$$\overset{N\to E}{\omega} = \overset{N\to B}{\omega} + \overset{B\to E}{\omega} \tag{25}$$

$$= p\vec{i} + (q + \dot{\sigma})\vec{j} + r\vec{k} + (Z_1\dot{\eta} + Z_2\dot{\zeta} + Z_3\dot{\sigma})\vec{e}_3' + \dot{\delta}\vec{e}_1 \tag{26}$$

and resolving to reference frame E with equations (A9), (A10) and (A11):

$$\begin{aligned}\overset{N\to E}{\omega} = {} & \{c_\nu c_\zeta p + s_\nu q - c_\nu s_\zeta r + s_\nu\dot{\sigma} + \dot{\delta}\}\vec{e}_1 + \{(s_\delta s_\zeta - c_\delta c_\zeta s_\nu)p + (c_\delta c_\nu)q \\ & + (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)r + (s_\delta Z_1)\dot{\eta} + (s_\delta Z_2)\dot{\zeta} + (c_\delta c_\nu + s_\delta Z_3)\dot{\sigma}\}\vec{e}_2 \\ & + \{(c_\delta s_\zeta + c_\zeta s_\delta s_\nu)p + (-s_\delta c_\nu)q + (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta)r \\ & + (c_\delta Z_1)\dot{\eta} + (c_\delta Z_2)\dot{\zeta} + (-s_\delta c_\nu + c_\delta Z_3)\dot{\sigma}\}\vec{e}_3\end{aligned} \tag{27}$$

Translational Velocity

Helicopter mass center- Equation (2) defines the translational velocity of the helicopter mass center in the inertial reference frame.

Load mass center- Using the kinematic law which relates the translational velocities of two points on a rotating rigid body, expressions may be written for the translational velocities, relative to reference frame N, of cable attachment points P_1 and P, and load mass center E^*:

$$\overset{N\to P_1}{V} = \overset{N\to B^*}{V} + \overset{N\to B}{\omega} \times \vec{p}^{\,B^*-P_1} \tag{28}$$

where $\vec{p}^{\,B^*-P_1}$ is the position vector from B^* to P_1 $= -a/2\vec{i} + R\vec{k}$

$$\overset{N\to P}{V} = \overset{N\to P_1}{V} + \overset{N\to C}{\omega} \times \vec{p}^{\,P_1-P} \tag{29}$$

where $\vec{p}^{\,P_1-P} = \ell\vec{c}_3$

$$\overset{N\to E^*}{V} = \overset{N\to P}{V} + \overset{N\to E}{\omega} \times \vec{p}^{\,P-E^*} \tag{30}$$

where ${}^{\vec{p}}{}^{P-E^*} = (a'/2)\vec{e}_1 + h\vec{e}_3$. Substituting expressions for ${}^{N}\vec{v}^{P_1}$ and ${}^{N}\vec{v}^{P}$ into equation (30) gives:

$$
{}^{N}\vec{v}^{E^*} = {}^{N}\vec{v}^{B^*} + {}^{N}\vec{\omega}^{B} \times \vec{p}^{\,B^*-P_1} + {}^{N}\vec{\omega}^{C} \times \vec{p}^{\,P_1-P} + {}^{N}\vec{\omega}^{E} \times \vec{p}^{\,P-E^*} \tag{31}
$$

which, after expanding cross products and collecting terms (appendix C), given equation (32) when expressed in the helicopter body reference frame:

$$
\begin{aligned}
{}^{N}\vec{v}^{E^*}(\vec{i},\vec{j},\vec{k}) = \Bigg\{ & u + \left(R + \ell c_{\zeta}c_{\eta} - \frac{a'}{2} c_{\nu}s_{\zeta} - hs_{\zeta}s_{\delta}s_{\nu} + hc_{\delta}c_{\zeta}\right)q + \left(\ell s_{\zeta} - \frac{a'}{2} s_{\nu} + hs_{\delta}c_{\nu}\right)r \\
& + \left[\ell c_{\zeta}c_{\eta} + \left(\frac{a'}{2} s_{\nu} + hs_{\delta}c_{\nu}\right)Z_1 c_{\zeta}\right]\dot{\eta} + \left[-\ell s_{\zeta}s_{\eta} + \left(-\frac{a'}{2} s_{\nu} + hs_{\delta}c_{\nu}\right)Z_2 c_{\zeta}\right]\dot{\xi} \\
& + \left[-\frac{a'}{2} c_{\nu}s_{\zeta} - hs_{\zeta}s_{\delta}s_{\nu} + hc_{\delta}c_{\zeta} + \left(-\frac{a'}{2} s_{\nu} + hs_{\delta}c_{\nu}\right)Z_3 c_{\zeta}\right]\dot{\sigma} \\
& + \left(hc_{\delta}c_{\zeta}s_{\nu} - hs_{\delta}s_{\zeta}\right)\dot{\delta}\Bigg\}\vec{i} + \Bigg\{v + \left(-R - \ell c_{\xi}c_{\eta} + \frac{a'}{2} c_{\nu}s_{\zeta} + hs_{\zeta}s_{\delta}s_{\nu} - hc_{\delta}c_{\zeta}\right)p \\
& + \left(-\frac{a}{2} + \ell c_{\xi}s_{\eta} + \frac{a'}{2} c_{\nu}c_{\zeta} + hc_{\delta}s_{\zeta} + hc_{\zeta}s_{\delta}s_{\nu}\right)r + \left(\frac{a'}{2} c_{\nu} + hs_{\delta}s_{\nu}\right)Z_1\dot{\eta} \\
& + \left[-\ell c_{\xi} + \left(\frac{a'}{2} c_{\nu} + hs_{\delta}s_{\nu}\right)Z_2\right]\dot{\xi} + \left(\frac{a'}{2} c_{\nu} + hs_{\delta}s_{\nu}\right)Z_3\dot{\sigma} + \left(-hc_{\delta}c_{\nu}\right)\dot{\delta}\Bigg\}\vec{j} \\
& + \Bigg\{w + \left(-\ell s_{\xi} + \frac{a'}{2} s_{\nu} - hs_{\delta}c_{\nu}\right)p + \Bigg(\frac{a}{2} - \ell c_{\xi}s_{\eta} - \frac{a'}{2} c_{\nu}c_{\zeta} - hc_{\delta}s_{\zeta} \\
& - hc_{\zeta}s_{\delta}s_{\nu}\Bigg)q + \left[-\ell c_{\xi}s_{\eta} + \left(\frac{a'}{2} s_{\nu} - hs_{\delta}c_{\nu}\right)s_{\zeta}Z_1\right]\dot{\eta} + \Bigg[-\ell s_{\xi}c_{\eta} \\
& + \left(\frac{a'}{2} s_{\nu} - hs_{\delta}c_{\nu}\right)s_{\zeta}Z_2\Bigg]\dot{\xi} + \Bigg[\left(\frac{a}{2} s_{\nu} - hs_{\delta}c_{\nu}\right)s_{\zeta}Z_3 - \frac{a'}{2} c_{\nu}c_{\zeta} - hc_{\delta}s_{\zeta} \\
& - hc_{\zeta}s_{\delta}s_{\nu}\Bigg]\dot{\sigma} + \left(-hs_{\delta}c_{\zeta} - hs_{\nu}c_{\delta}s_{\zeta}\right)\dot{\delta}\Bigg\}\vec{k}
\end{aligned} \tag{32}
$$

Using the transformation matrix E_B from equation (A12), ${}^{N}\vec{v}^{E^*}$ may be expressed in the load-fixed reference frame:

$$
{}^{N}\vec{v}^{E^*}(\vec{e}_1,\vec{e}_2,\vec{e}_3) = E_B\,{}^{N}\vec{v}^{B^*}(\vec{i},\vec{j},\vec{k}) \tag{33}
$$

Load angle of attack and sideslip angle may then be defined:

$$
\alpha_E = \tan^{-1}\left(\frac{{}^{N}\vec{v}^{E^*} \cdot \vec{e}_3}{{}^{N}\vec{v}^{E^*} \cdot \vec{e}_1}\right) \tag{34}
$$

$$
\beta_E = \sin^{-1}\left(\frac{{}^{N}\vec{v}^{E^*} \cdot \vec{e}_2}{\left|{}^{N}\vec{v}^{E^*}\right|}\right) \tag{35}
$$

Partial Derivatives of Velocities

The partial derivatives of angular and translational velocities for the two rigid bodies are given in equations (36)-(75).

$$\vec{\omega}_u^{N\to B} = \frac{\partial \vec{\omega}^{N\to B}}{\partial u} = 0 \qquad (36)$$

$$\vec{\omega}_v^{N\to B} = 0 \qquad (37)$$

$$\vec{\omega}_w^{N\to B} = 0 \qquad (38)$$

$$\vec{\omega}_p^{N\to B} = \vec{i} \qquad (39)$$

$$\vec{\omega}_q^{N\to B} = \vec{j} \qquad (40)$$

$$\vec{\omega}_r^{N\to B} = \vec{k} \qquad (41)$$

$$\vec{\omega}_{\dot{\eta}}^{N\to B} = 0 \qquad (42)$$

$$\vec{\omega}_{\dot{\xi}}^{N\to B} = 0 \qquad (43)$$

$$\vec{\omega}_{\dot{\delta}}^{N\to B} = 0 \qquad (44)$$

$$\vec{\omega}_{\dot{\sigma}}^{N\to B} = 0 \qquad (45)$$

$$\vec{v}_u^{N\to B^*} = \frac{\partial \vec{v}^{N\to B^*}}{\partial u} \qquad (46)$$

$$\vec{v}_v^{N\to B^*} = \vec{j} \qquad (47)$$

$$\vec{v}_w^{N\to B^*} = \vec{k} \qquad (48)$$

$$\vec{v}_p^{N\to B^*} = 0 \qquad (49)$$

$$\vec{v}_q^{N\to B^*} = 0 \qquad (50)$$

$$\vec{v}_r^{N\to B^*} = 0 \qquad (51)$$

$$\vec{v}_{\dot{\eta}}^{N\to B^*} = 0 \qquad (52)$$

$$\vec{v}_{\dot{\xi}}^{N\to B^*} = 0 \qquad (53)$$

$$\vec{v}_{\dot{\delta}}^{N\to B^*} = 0 \qquad (54)$$

$$\vec{v}_{\dot{\sigma}}^{N\to B^*} = 0 \qquad (55)$$

$$\vec{\omega}_u^{N\to L} = 0 \qquad (56)$$

$$\vec{\omega}_v^{N\to L} = 0 \qquad (57)$$

$$\vec{\omega}_w^{N\to L} = 0 \qquad (58)$$

$$\vec{\omega}_p^{N\to L} = (c_\nu c_\zeta)\vec{e}_1 + (s_\delta s_\zeta - c_\delta c_\zeta s_\nu)\vec{e}_2 + (c_\delta s_\zeta + c_\zeta s_\delta s_\nu)\vec{e}_3 \qquad (59)$$

$$\vec{\omega}_q^{N\to L} = (s_\nu)\vec{e}_1 + (c_\delta c_\nu)\vec{e}_2 + (-s_\delta c_\nu)\vec{e}_3 \qquad (60)$$

$$\vec{\omega}_r^{N\to L} = (-c_\nu s_\zeta)\vec{e}_1 + (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)\vec{e}_2 + (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta)\vec{e}_3 \qquad (61)$$

$$\vec{\omega}_{\dot{\eta}}^{N\to L} = (s_\delta Z_1)\vec{e}_2 + (c_\delta Z_1)\vec{e}_3 \qquad (62)$$

$$ {}^{N\to L}\omega_{\dot{\xi}} = (s_\delta Z_2)\vec{e}_2 + (c_\delta Z_2)\vec{e}_3 \tag{63} $$

$$ {}^{N\to L}\omega_{\dot{\delta}} = \vec{e}_1 \tag{64} $$

$$ {}^{N\to L}\omega_{\dot{\sigma}} = s_\nu \vec{e}_1 + (c_\delta c_\nu + s_\delta Z_3)\vec{e}_2 + (-s_\delta c_\nu + c_\delta Z_3)\vec{e}_3 \tag{65} $$

$$ {}^{N\to L^*}v_u = \vec{i} \tag{66} $$

$$ {}^{N\to L^*}v_v = \vec{j} \tag{67} $$

$$ {}^{N\to L^*}v_w = \vec{k} \tag{68} $$

$$ {}^{N\to L^*}v_p = \left(-R - \ell c_\xi c_\eta + \frac{a'}{2} c_\nu s_\zeta + h s_\zeta s_\delta s_\nu - h c_\delta c_\zeta\right)\vec{j} + \left(-\ell s_\xi + \frac{a'}{2} s_\nu - h s_\delta c_\nu\right) \tag{69} $$

$$ {}^{N\to L^*}v_q = \left(R + \ell c_\xi c_\eta - \frac{a'}{2} c_\nu s_\zeta - h s_\zeta s_\delta s_\nu + h c_\delta c_\zeta\right)\vec{i} + \left(\frac{a}{2} - \ell c_\xi s_\eta - \frac{a'}{2} c_\nu c_\zeta - h c_\delta s_\zeta - h c_\zeta s_\delta s_\nu\right)\vec{k} \tag{70} $$

$$ {}^{N\to L^*}v_r = \left(\ell s_\xi - \frac{a'}{2} s_\nu + h s_\delta c_\nu\right)\vec{i} + \left(-\frac{a}{2} + \ell c_\xi s_\eta + \frac{a'}{2} c_\nu c_\zeta + h c_\delta s_\zeta + h c_\zeta s_\delta s_\nu\right)\vec{j} \tag{71} $$

$$ {}^{N\to L^*}v_{\dot{\eta}} = \left[\ell c_\xi c_\eta - \left(\frac{a'}{2} s_\nu c_\zeta - h s_\delta c_\nu c_\zeta\right)Z_1\right]\vec{i} + \left[\left(\frac{a'}{2} c_\nu + h s_\delta s_\nu\right)Z_1\right]\vec{j} + \left[-\ell c_\xi s_\eta + \left(\frac{a'}{2} s_\nu s_\zeta - h s_\delta c_\nu s_\zeta\right)Z_1\right]\vec{k} \tag{72} $$

$$ {}^{N\to L^*}v_{\dot{\xi}} = \left[-\ell s_\xi s_\eta - \left(\frac{a'}{2} s_\nu c_\zeta - h s_\delta c_\nu c_\zeta\right)Z_2\right]\vec{i} + \left[-\ell c_\xi + \left(\frac{a'}{2} c_\nu + h s_\delta s_\nu\right)Z_2\right]\vec{j} + \left[-\ell s_\xi c_\eta + \left(\frac{a'}{2} s_\nu s_\zeta - h s_\delta c_\nu s_\zeta\right)Z_2\right]\vec{k} \tag{73} $$

$$ {}^{N\to L^*}v_{\dot{\delta}} = (-h s_\zeta s_\delta + h s_\nu c_\delta c_\zeta)\vec{i} + (-h c_\delta c_\nu)\vec{j} + (-h s_\delta c_\zeta - h c_\delta s_\zeta s_\nu)\vec{k} \tag{74} $$

$$ {}^{N\to L^*}v_{\dot{\sigma}} = \left[-\frac{a'}{2} c_\nu s_\zeta - h s_\zeta s_\delta s_\nu + h c_\delta c_\zeta - \left(\frac{a'}{2} s_\nu c_\zeta - h s_\delta c_\nu c_\zeta\right)Z_3\right]\vec{i} + \left[\left(\frac{a'}{2} c_\nu + h s_\delta s_\nu\right)Z_3\right]\vec{j} + \left[-\frac{a'}{2} c_\nu c_\zeta - h c_\delta s_\zeta - h c_\zeta s_\delta s_\nu + \left(\frac{a'}{2} s_\nu s_\zeta - h s_\delta c_\nu s_\zeta\right)Z_3\right]\vec{k} \tag{75} $$

Angular Acceleration

Helicopter relative to inertial space- To obtain ${}^{N}\vec{\alpha}^{B}$, the angular acceleration of rigid body B in reference frame N, ${}^{N}\vec{\omega}^{B}$ is differentiated with respect to time[b]:

$$\frac{{}^{N}d\,{}^{N}\vec{\omega}^{B}}{dt} = \frac{{}^{B}d\,{}^{N}\vec{\omega}^{B}}{dt} + \cancelto{0}{{}^{N}\vec{\omega}^{B} \times {}^{N}\vec{\omega}^{B}} \tag{76}$$

$$= \dot{p}\vec{i} + \dot{q}\vec{j} + \dot{r}\vec{k} \tag{77}$$

Load relative to inertial space- In a similar manner, ${}^{N}\vec{\alpha}^{L}$ is obtained by differentiation of ${}^{N}\vec{\omega}^{L}$:

$${}^{N}\vec{\alpha}^{E} = \frac{{}^{N}d\,{}^{N}\vec{\omega}^{E}}{dt} = \frac{{}^{E}d\,{}^{N}\vec{\omega}^{E}}{dt} + \cancelto{0}{{}^{N}\vec{\omega}^{E} \times {}^{N}\vec{\omega}^{E}} \tag{78}$$

Expanding the above gives equation (79), where the Z_i are functions of ℓ, a, and a', and trigonometric functions of η, ξ, ζ, and ν. (See appendix D for an explicit definition of the Z_i.)

$$\begin{aligned}
{}^{N}\vec{\alpha}^{E} = \{ & Z_{21}\dot{p} + Z_{22}p\dot{\eta} + Z_{23}p\dot{\xi} + Z_{24}p\dot{\sigma} + Z_{25}\dot{q} + Z_{26}q\dot{\eta} + Z_{27}q\dot{\xi} + Z_{28}q\dot{\sigma} + Z_{29}\dot{r} + Z_{30}r\dot{\eta} \\
& + Z_{31}r\dot{\xi} + Z_{32}r\dot{\sigma} + Z_{33}\ddot{\sigma} + Z_{34}\dot{\eta}\dot{\sigma} + Z_{35}\dot{\xi}\dot{\sigma} + Z_{36}\dot{\sigma}^2 + Z_{37}\ddot{\delta}\}\vec{e}_1 + \{Z_{38}\dot{p} + Z_{39}p\dot{\eta} \\
& + Z_{40}p\dot{\xi} + Z_{41}p\dot{\sigma} + Z_{42}p\dot{\delta} + Z_{43}\dot{q} + Z_{44}q\dot{\eta} + Z_{45}q\dot{\xi} + Z_{46}q\dot{\sigma} + Z_{47}q\dot{\delta} + Z_{48}\dot{r} + Z_{49}r\dot{\eta} \\
& + Z_{50}r\dot{\xi} + Z_{51}r\dot{\sigma} + Z_{52}r\dot{\delta} + Z_{53}\dot{\eta}\dot{\xi} + Z_{54}\dot{\eta}^2 + Z_{55}\dot{\eta}\dot{\sigma} + Z_{56}\ddot{\eta} + Z_{57}\dot{\xi}^2 + Z_{50}\dot{\xi}\dot{\sigma} + Z_{59}\ddot{\xi} \\
& + Z_{60}\dot{\sigma}^2 + Z_{61}\ddot{\sigma} + Z_{62}\dot{\eta}\dot{\delta} + Z_{63}\dot{\xi}\dot{\delta} + Z_{64}\dot{\sigma}\dot{\delta}\}\vec{e}_2 + \{Z_{65}\dot{p} + Z_{66}p\dot{\eta} + Z_{67}p\dot{\xi} + Z_{68}p\dot{\sigma} \\
& + Z_{69}p\dot{\delta} + Z_{70}\dot{q} + Z_{71}q\dot{\eta} + Z_{72}q\dot{\xi} + Z_{73}q\dot{\sigma} + Z_{74}q\dot{\delta} + Z_{75}\dot{r} + Z_{76}r\dot{\eta} + Z_{77}r\dot{\xi} + Z_{78}r\dot{\sigma} \\
& + Z_{79}r\dot{\delta} + Z_{80}\dot{\eta}\dot{\xi} + Z_{81}\dot{\eta}^2 + Z_{82}\dot{\eta}\dot{\sigma} + Z_{83}\ddot{\eta} + Z_{84}\dot{\xi}^2 + Z_{85}\dot{\xi}\dot{\sigma} + Z_{86}\ddot{\xi} + Z_{87}\dot{\sigma}^2 + Z_{88}\ddot{\sigma} \\
& + Z_{89}\dot{\eta}\dot{\delta} + Z_{90}\dot{\xi}\dot{\delta} + Z_{91}\dot{\sigma}\dot{\delta}\}\vec{e}_3
\end{aligned} \tag{79}$$

Translational Acceleration

Helicopter relative to inertial space- ${}^{N}\vec{a}^{B^*}$, the translational acceleration of B^* in N, may be found by differentiating ${}^{N}\vec{v}^{B^*}$ with respect to time:

$${}^{N}\vec{a}^{B^*} = \frac{{}^{N}d}{dt}\,{}^{N}\vec{v}^{B^*} = \frac{{}^{B}d}{dt}\,{}^{N}\vec{v}^{B^*} + {}^{N}\vec{\omega}^{B} \times {}^{N}\vec{v}^{B^*} \tag{80}$$

---

[b] ${}^{X}d\,{}^{Y}\vec{\omega}^{Z}/dt$ denotes time differentiation in reference frame X, of ${}^{Y}\vec{\omega}^{Z}$.

$$ {}^{N}\vec{a}^{B^*} = (\dot{u} - rv + qw)\vec{i} + (\dot{v} + ru - pw)\vec{j} + (\dot{w} - qu + pv)\vec{k} \tag{81} $$

Load relative to inertial space- The acceleration of the load mass center may be found by differentiating ${}^{N}\vec{v}^{E^*}$:

$$ {}^{N}\vec{a}^{E^*} = \frac{{}^{N}d}{dt}\,{}^{N}\vec{v}^{E^*} = \frac{{}^{B}d}{dt}\,{}^{N}\vec{v}^{E^*} + {}^{N}\vec{\omega}^{B} \times {}^{N}\vec{v}^{E^*} \tag{82} $$

Expanding equation (82) gives (83).

$$ \begin{aligned} {}^{N}\vec{a}^{L^*} = \{&\dot{u} - rv + qw + Z_{92}\dot{q} + Z_{93}\dot{r} + Z_{94}pq + Z_{95}pr + Z_{96}q^2 + Z_{97}r^2 + Z_{98}q\dot{\eta} + Z_{99}q\dot{\xi} \\ &+ Z_{100}q\dot{\sigma} + Z_{101}q\dot{\delta} + Z_{102}r\dot{\eta} + Z_{103}r\dot{\xi} + Z_{104}r\dot{\sigma} + Z_{105}r\dot{\delta} + Z_{106}\ddot{\eta} + Z_{107}\dot{\eta}^2 + Z_{108}\dot{\eta}\dot{\xi} \\ &+ Z_{109}\ddot{\xi} + Z_{110}\dot{\xi}^2 + Z_{111}\dot{\eta}\dot{\sigma} + Z_{112}\dot{\xi}\dot{\sigma} + Z_{113}\ddot{\sigma} + Z_{114}\dot{\sigma}^2 + Z_{115}\dot{\eta}\dot{\delta} + Z_{116}\dot{\xi}\dot{\delta} + Z_{117}\dot{\sigma}\dot{\delta} \\ &+ Z_{118}\dot{\delta}^2 + Z_{119}\ddot{\delta}\}\vec{i} + \{\dot{v} + ru - pw + Z_{120}\dot{p} + Z_{121}\dot{r} + Z_{122}pq + Z_{123}qr + Z_{124}p^2 \\ &+ Z_{125}r^2 + Z_{126}p\dot{\eta} + Z_{127}p\dot{\xi} + Z_{128}p\dot{\sigma} + Z_{129}p\dot{\delta} + Z_{130}r\dot{\eta} + Z_{131}r\dot{\xi} + Z_{132}r\dot{\sigma} \\ &+ Z_{133}r\dot{\delta} + Z_{134}\ddot{\eta} + Z_{135}\dot{\eta}^2 + Z_{136}\dot{\eta}\dot{\xi} + Z_{137}\ddot{\xi} + Z_{138}\dot{\xi}^2 + Z_{139}\dot{\eta}\dot{\sigma} + Z_{140}\dot{\xi}\dot{\sigma} + Z_{141}\ddot{\sigma} \\ &+ Z_{142}\dot{\sigma}^2 + Z_{143}\dot{\eta}\dot{\delta} + Z_{144}\dot{\xi}\dot{\delta} + Z_{145}\dot{\sigma}\dot{\delta} + Z_{146}\dot{\delta}^2 + Z_{147}\ddot{\delta}\}\vec{j} + \{\dot{w} - qu + pv + Z_{148}\dot{p} \\ &+ Z_{149}\dot{q} + Z_{150}pr + Z_{151}qr + Z_{152}p^2 + Z_{153}q^2 + Z_{154}p\dot{\eta} + Z_{155}p\dot{\xi} + Z_{156}p\dot{\sigma} \\ &+ Z_{157}p\dot{\delta} + Z_{158}q\dot{\eta} + Z_{159}q\dot{\xi} + Z_{160}q\dot{\sigma} + Z_{161}q\dot{\delta} + Z_{162}\ddot{\eta} + Z_{163}\dot{\eta}^2 + Z_{164}\dot{\eta}\dot{\xi} \\ &+ Z_{165}\ddot{\xi} + Z_{166}\dot{\xi}^2 + Z_{167}\dot{\eta}\dot{\sigma} + Z_{168}\dot{\xi}\dot{\sigma} + Z_{169}\ddot{\sigma} + Z_{170}\dot{\sigma}^2 + Z_{171}\dot{\eta}\dot{\delta} + Z_{172}\dot{\xi}\dot{\delta} \\ &+ Z_{173}\dot{\sigma}\dot{\delta} + Z_{174}\dot{\delta}^2 + Z_{175}\ddot{\delta}\}\vec{k} \end{aligned} \tag{83} $$

Inertia Forces

$\vec{R}^{*B}$ and $\vec{R}^{*E}$, the inertia forces for the helicopter and load, respectively, can be defined in the following equations:

$$ \vec{R}^{*B} = -M\,{}^{N}\vec{a}^{B^*} \tag{84} $$

$$ \vec{R}^{*E} = -m\,{}^{N}\vec{a}^{E^*} \tag{85} $$

Inertia Torques

The inertia torque, $\vec{T}^*$, was defined originally as:

$$ \vec{T}^* = -\vec{I} \cdot \alpha + (\vec{I} \cdot \vec{\omega}) \times \vec{\omega} \tag{17} $$

Helicopter- The helicopter inertia dyadic, $\vec{\vec{I}}^B$, may be defined as follows:

$$\vec{\vec{I}}^B = \sum_{j=1}^{3} \sum_{k=1}^{3} \vec{b}_j I_{jk} \vec{b}_k \tag{86}$$

$$= (\vec{i}B_x - \vec{k}B_{xz})\vec{i} + (\vec{j}B_y)\vec{j} + (\vec{k}B_z - \vec{i}B_{xz})\vec{k} \tag{87}$$

since $B_{xy} = B_{yz} = 0$. Also, in this formulation:

$$B_{xz} = -\sum_{i=1}^{N} \rho_i y_i z_i \tag{88}$$

(ref. 5) which is the negative of the standard definition of a product of inertia.

The helicopter inertia torque, $\vec{T}^{*B}$, is calculated as follows:

$$\vec{T}^{*B} = -\vec{\vec{I}}^B \cdot {}^N\vec{\alpha}^B + (\vec{\vec{I}}^B \cdot {}^N\vec{\omega}^B) \times {}^N\vec{\omega}^B \tag{89}$$

$$\vec{T}^{*B} = \{-B_x\dot{p} + B_{xz}\dot{r} + B_y qr + B_{xz}pq - B_z qr\}\vec{i} + \{-B_y\dot{q} - B_x pr - B_{xz}(p^2 - r^2) + B_z pr\}\vec{j}$$
$$+ \{B_{xz}\dot{p} - B_z\dot{r} + B_x pq - B_y pq - B_{xz}qr\}\vec{k} \tag{90}$$

Load- The load inertia dyadic, $\vec{\vec{I}}^E$, is given in equation (92):

$$\vec{\vec{I}}^E = (\vec{e}_1 E_1 - \vec{e}_3 E_{13})\vec{e}_1 + (\vec{e}_2 E_2)\vec{e}_2 + (\vec{e}_3 E_3 - \vec{e}_1 E_{13})\vec{e}_3 \tag{91}$$

and

$$\vec{T}^{*E} = -\vec{\vec{I}}^E \cdot {}^N\vec{\alpha}^E + (\vec{\vec{I}}^E \cdot {}^N\vec{\omega}^E) \times {}^N\vec{\omega}^E \tag{92}$$

$$\vec{T}^{*E} = \{Z_{278}\dot{p} + Z_{279}p^2 + Z_{280}pq + Z_{281}pr + Z_{282}p\dot{\eta} + Z_{283}p\dot{\xi} + Z_{284}p\dot{\sigma} + Z_{285}p\dot{\delta} + Z_{286}\dot{q}$$
$$+ Z_{287}q^2 + Z_{288}qr + Z_{289}q\dot{\eta} + Z_{290}q\dot{\xi} + Z_{291}q\dot{\sigma} + Z_{292}q\dot{\delta} + Z_{293}\dot{r} + Z_{294}r^2 + Z_{295}r\dot{\eta}$$
$$+ Z_{296}r\dot{\xi} + Z_{297}r\dot{\sigma} + Z_{298}r\dot{\delta} + Z_{299}\dot{\eta}\dot{\xi} + Z_{300}\dot{\eta}\dot{\sigma} + Z_{301}\dot{\eta}^2 + Z_{302}\ddot{\eta} + Z_{303}\dot{\xi}^2 + Z_{304}\dot{\xi}\dot{\sigma}$$
$$+ Z_{305}\ddot{\xi} + Z_{306}\dot{\sigma}^2 + Z_{307}\ddot{\sigma} + Z_{308}\dot{\eta}\dot{\delta} + Z_{309}\dot{\xi}\dot{\delta} + Z_{310}\dot{\sigma}\dot{\delta} + Z_{311}\ddot{\delta}\}\vec{e}_1 + \{Z_{312}\dot{p}$$
$$+ Z_{313}p^2 + Z_{314}pq + Z_{315}pr + Z_{316}p\dot{\eta} + Z_{317}p\dot{\xi} + Z_{318}p\dot{\sigma} + Z_{319}p\dot{\delta} + Z_{320}\dot{q} + Z_{321}q^2$$
$$+ Z_{322}qr + Z_{323}q\dot{\eta} + Z_{324}q\dot{\xi} + Z_{325}q\dot{\sigma} + Z_{326}q\dot{\delta} + Z_{327}\dot{r} + Z_{328}r^2 + Z_{329}r\dot{\eta} + Z_{330}r\dot{\xi}$$
$$+ Z_{331}r\dot{\sigma} + Z_{332}r\dot{\delta} + Z_{333}\dot{\eta}\dot{\xi} + Z_{334}\dot{\eta}\dot{\sigma} + Z_{335}\dot{\eta}^2 + Z_{336}\ddot{\eta} + Z_{337}\dot{\xi}^2 + Z_{338}\dot{\xi}\dot{\sigma} + Z_{339}\ddot{\xi}$$
$$+ Z_{340}\dot{\sigma}^2 + Z_{341}\ddot{\sigma} + Z_{342}\dot{\eta}\dot{\delta} + Z_{343}\dot{\xi}\dot{\delta} + Z_{344}\dot{\sigma}\dot{\delta} + Z_{345}\dot{\delta}^2\}\vec{e}_2 + \{Z_{346}\dot{p} + Z_{347}p^2 \tag{93}$$

(Eq. (93) continued on pg. 14)

$$+ Z_{348}pq + Z_{349}pr + Z_{350}p\dot{\eta} + Z_{351}p\dot{\xi} + Z_{352}p\dot{\sigma} + Z_{353}p\dot{\delta} + Z_{354}\dot{q} + Z_{355}q^2 + Z_{356}qr$$

$$+ Z_{357}q\dot{\eta} + Z_{358}q\dot{\xi} + Z_{359}q\dot{\sigma} + Z_{360}q\dot{\delta} + Z_{361}\dot{r} + Z_{362}r^2 + Z_{363}r\dot{\eta} + Z_{364}r\dot{\xi} + Z_{365}r\dot{\sigma}$$

$$+ Z_{366}r\dot{\delta} + Z_{367}\dot{\eta}\dot{\xi} + Z_{368}\dot{\eta}\dot{\sigma} + Z_{369}\dot{\eta}^2 + Z_{370}\ddot{\eta} + Z_{371}\dot{\xi}^2 + Z_{372}\dot{\xi}\dot{\sigma} + Z_{373}\ddot{\xi} + Z_{374}\dot{\sigma}^2$$

$$+ Z_{375}\ddot{\sigma} + Z_{376}\dot{\eta}\dot{\delta} + Z_{377}\dot{\xi}\dot{\delta} + Z_{378}\dot{\sigma}\dot{\delta} + Z_{379}\ddot{\delta}\}\vec{e}_3 \qquad (93)$$

Concluded

Equations of Motion

The ten equations of motion may be written using Lagrange's form of D'Alembert's principle, equation (14). Expanding equation (14) for this system gives equation (19), where the ten generalized speeds were chosen as in equation (20). The final ten equations are given here, with the Z_i defined in appendix D:

$$\dot{u} = \frac{X_B}{(m + M)} + \frac{X_E}{(m + M)} c_\nu c_\zeta + \frac{Y_E}{(m + M)} (s_\delta s_\zeta - c_\delta c_\zeta s_\nu) + \frac{Z_E}{(m + M)} (c_\delta s_\zeta + c_\zeta s_\delta s_\nu)$$

$$- gs_\theta + rv - qw - \frac{m}{(m + M)} [Z_{92}\dot{q} + Z_{93}\dot{r} + Z_{94}pq + Z_{95}pr + Z_{96}q^2 + Z_{97}r^2 + Z_{98}q\dot{\eta}$$

$$+ Z_{99}q\dot{\xi} + Z_{100}q\dot{\sigma} + Z_{101}q\dot{\delta} + Z_{102}r\dot{\eta} + Z_{103}r\dot{\xi} + Z_{104}r\dot{\sigma} + Z_{105}r\dot{\delta} + Z_{106}\ddot{\eta} + Z_{107}\dot{\eta}^2$$

$$+ Z_{108}\dot{\eta}\dot{\xi} + Z_{109}\ddot{\xi} + Z_{110}\dot{\xi}^2 + Z_{111}\dot{\eta}\dot{\sigma} + Z_{112}\dot{\xi}\dot{\sigma} + Z_{113}\ddot{\sigma} + Z_{114}\dot{\sigma}^2 + Z_{115}\dot{\eta}\dot{\delta} + Z_{116}\dot{\xi}\dot{\delta}$$

$$+ Z_{117}\dot{\sigma}\dot{\delta} + Z_{118}\dot{\delta}^2 + Z_{119}\ddot{\delta}] \qquad (94)$$

$$\dot{v} = \frac{Y_B}{(m + M)} + \frac{X_E}{(m + M)} s_\nu + \frac{Y_E}{(m + M)} c_\delta c_\nu - \frac{Z_E}{(m + M)} s_\delta c_\nu + gc_\theta s_\phi - ru + pw - \frac{m}{(m + M)}$$

$$[Z_{120}\dot{p} + Z_{121}\dot{r} + Z_{122}pq + Z_{123}qr + Z_{124}p^2 + Z_{125}r^2 + Z_{126}p\dot{\eta} + Z_{127}p\dot{\xi} + Z_{128}p\dot{\sigma}$$

$$+ Z_{129}p\dot{\delta} + Z_{130}r\dot{\eta} + Z_{131}r\dot{\xi} + Z_{132}r\dot{\sigma} + Z_{133}r\dot{\delta} + Z_{134}\ddot{\eta} + Z_{135}\dot{\eta}^2 + Z_{136}\dot{\eta}\dot{\xi} + Z_{137}\ddot{\xi}$$

$$+ Z_{138}\dot{\xi}^2 + Z_{139}\dot{\eta}\dot{\xi} + Z_{140}\dot{\xi}\dot{\sigma} + Z_{141}\ddot{\sigma} + Z_{142}\dot{\sigma}^2 + Z_{143}\dot{\eta}\dot{\delta} + Z_{144}\dot{\xi}\dot{\delta} + Z_{145}\dot{\sigma}\dot{\delta}$$

$$+ Z_{146}\dot{\delta}^2 + Z_{147}\ddot{\delta}] \qquad (95)$$

$$\begin{aligned}
\dot{w} = {} & \frac{Z_B}{(m + M)} - \frac{X_E}{(m + M)} c_\nu s_\zeta + \frac{Y_E}{(m + M)} (s_\zeta c_\delta s_\nu + s_\delta c_\zeta) + \frac{Z_E}{(m + M)} (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta) \\
& + qu - pv + g c_\theta c_\phi - \frac{m}{(m + M)} [Z_{148}\dot{p} + Z_{149}\dot{q} + Z_{150}pr + Z_{151}qr + Z_{152}p^2 + Z_{153}q^2 \\
& + Z_{154}p\dot{\eta} + Z_{155}p\dot{\xi} + Z_{156}p\dot{\sigma} + Z_{157}p\dot{\delta} + Z_{158}q\dot{\eta} + Z_{159}q\dot{\xi} + Z_{160}q\dot{\sigma} + Z_{161}q\dot{\delta} + Z_{162}\ddot{\eta} \\
& + Z_{163}\dot{\eta}^2 + Z_{164}\dot{\eta}\dot{\xi} + Z_{165}\ddot{\xi} + Z_{166}\dot{\xi}^2 + Z_{167}\dot{\eta}\dot{\sigma} + Z_{168}\dot{\xi}\dot{\sigma} + Z_{169}\ddot{\sigma} + Z_{170}\dot{\sigma}^2 + Z_{171}\dot{\eta}\dot{\delta} \\
& + Z_{172}\dot{\xi}\dot{\delta} + Z_{173}\dot{\sigma}\dot{\delta} + Z_{174}\dot{\delta}^2 + Z_{175}\ddot{\delta}] \qquad (96)
\end{aligned}$$

$$\begin{aligned}
B_x\dot{p} = {} & L_B + B_{xz}\dot{r} + (B_y - B_z)qr + B_{xz}pq + Z_{382}X_E + Z_{383}Y_E + Z_{384}Z_E + Z_{385}mg + Z_{386}L_E \\
& + Z_{387}M_E + Z_{388}N_E + Z_{389}\dot{u} + Z_{390}\dot{v} + Z_{391}\dot{w} + Z_{392}rv + Z_{393}qw + Z_{394}ru + Z_{395}pw \\
& + Z_{396}qu + Z_{397}pv + Z_{398}\dot{p} + Z_{399}\dot{q} + Z_{400}\dot{r} + Z_{401}pq + Z_{402}pr + Z_{403}qr + Z_{404}p^2 \\
& + Z_{405}q^2 + Z_{406}r^2 + Z_{407}p\dot{\eta} + Z_{408}p\dot{\xi} + Z_{409}p\dot{\sigma} + Z_{410}p\dot{\delta} + Z_{411}q\dot{\eta} + Z_{412}q\dot{\xi} \\
& + Z_{413}q\dot{\sigma} + Z_{414}q\dot{\delta} + Z_{415}r\dot{\eta} + Z_{416}r\dot{\xi} + Z_{417}r\dot{\sigma} + Z_{418}r\dot{\delta} + Z_{419}\ddot{\eta} + Z_{420}\dot{\eta}^2 + Z_{421}\dot{\eta}\dot{\xi} \\
& + Z_{422}\dot{\eta}\dot{\sigma} + Z_{423}\dot{\eta}\dot{\delta} + Z_{424}\ddot{\xi} + Z_{415}\dot{\xi}^2 + Z_{426}\dot{\xi}\dot{\sigma} + Z_{427}\dot{\xi}\dot{\delta} + Z_{428}\ddot{\sigma} + Z_{429}\dot{\sigma}^2 + Z_{430}\dot{\sigma}\dot{\delta} \\
& + Z_{431}\ddot{\delta} + Z_{432}\dot{\delta}^2 \qquad (97)
\end{aligned}$$

$$\begin{aligned}
B_y\dot{q} = {} & M_B - B_xpr - B_{xz}(p^2 - r^2) + B_zpr + Z_{439}X_E + Z_{440}Y_E + Z_{441}Z_E + Z_{442}mg + Z_{443}L_E \\
& + Z_{444}M_E + Z_{445}N_E + Z_{446}\dot{u} + Z_{447}\dot{v} + Z_{448}\dot{w} + Z_{449}rv + Z_{450}qw + Z_{451}ru + Z_{452}pw \\
& + Z_{453}qu + Z_{454}pv + Z_{455}\dot{p} + Z_{456}\dot{q} + Z_{457}\dot{r} + Z_{458}pq + Z_{459}pr + Z_{460}qr + Z_{461}p^2 \\
& + Z_{462}q^2 + Z_{463}r^2 + Z_{464}p\dot{\eta} + Z_{465}p\dot{\xi} + Z_{466}p\dot{\sigma} + Z_{467}p\dot{\delta} + Z_{468}q\dot{\eta} + Z_{469}q\dot{\xi} \\
& + Z_{470}q\dot{\sigma} + Z_{471}q\dot{\delta} + Z_{472}r\dot{\eta} + Z_{473}r\dot{\xi} + Z_{474}r\dot{\sigma} + Z_{475}r\dot{\delta} + Z_{476}\ddot{\eta} + Z_{477}\dot{\eta}^2 + Z_{478}\dot{\eta}\dot{\xi} \\
& + Z_{479}\dot{\eta}\dot{\sigma} + Z_{480}\dot{\eta}\dot{\delta} + Z_{481}\ddot{\xi} + Z_{482}\dot{\xi}^2 + Z_{483}\dot{\xi}\dot{\sigma} + Z_{484}\dot{\xi}\dot{\delta} + Z_{485}\ddot{\sigma} + Z_{486}\dot{\sigma}^2 + Z_{487}\dot{\sigma}\dot{\delta} \\
& + Z_{488}\ddot{\delta} + Z_{489}\dot{\delta}^2 \qquad (98)
\end{aligned}$$

$$
\begin{aligned}
B_z\dot{r} = {} & N_B + B_{xz}\dot{p} + (B_x - B_y)pq - B_{xz}qr + Z_{490}X_E + Z_{491}Y_E + Z_{492}Z_E + Z_{493}mg + Z_{494}L_E \\
& + Z_{495}M_E + Z_{496}N_E + Z_{497}\dot{u} + Z_{498}\dot{v} + Z_{499}\dot{w} + Z_{500}rv + Z_{501}qw + Z_{502}ru + Z_{503}pw \\
& + Z_{504}qu + Z_{505}pv + Z_{506}\dot{p} + Z_{507}\dot{q} + Z_{508}\dot{r} + Z_{509}pq + Z_{510}pr + Z_{511}qr + Z_{512}p^2 \\
& + Z_{513}q^2 + Z_{514}r^2 + Z_{515}p\dot{\eta} + Z_{516}p\dot{\xi} + Z_{517}p\dot{\sigma} + Z_{518}p\dot{\delta} + Z_{519}q\dot{\eta} + Z_{520}q\dot{\xi} \\
& + Z_{521}q\dot{\sigma} + Z_{522}q\dot{\delta} + Z_{523}r\dot{\eta} + Z_{524}r\dot{\xi} + Z_{525}r\dot{\sigma} + Z_{526}r\dot{\delta} + Z_{527}\ddot{\eta} + Z_{528}\dot{\eta}^2 \\
& + Z_{529}\dot{\eta}\dot{\xi} + Z_{530}\dot{\eta}\dot{\sigma} + Z_{531}\dot{\eta}\dot{\delta} + Z_{532}\ddot{\xi} + Z_{533}\dot{\xi}^2 + Z_{534}\dot{\xi}\dot{\sigma} + Z_{535}\dot{\xi}\dot{\delta} + Z_{536}\ddot{\sigma} + Z_{537}\dot{\sigma}^2 \\
& + Z_{538}\dot{\sigma}\dot{\delta} + Z_{539}\ddot{\delta} + Z_{540}\dot{\delta}^2
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
Z_{547}\ddot{\eta} = {} & Z_{548}X_E + Z_{549}Y_E + Z_{550}Z_E + Z_{551}mg + Z_{552}M_E + Z_{553}N_E + Z_{554}\dot{u} + Z_{555}\dot{v} + Z_{556}\dot{w} \\
& + Z_{557}rv + Z_{558}qw + Z_{559}ru + Z_{560}pw + Z_{561}qu + Z_{562}pv + Z_{563}\dot{p} + Z_{564}\dot{q} \\
& + Z_{565}\dot{r} + Z_{566}pq + Z_{567}pr + Z_{568}qr + Z_{569}p^2 + Z_{570}q^2 + Z_{571}r^2 + Z_{572}p\dot{\eta} \\
& + Z_{573}p\dot{\xi} + Z_{574}p\dot{\sigma} + Z_{575}p\dot{\delta} + Z_{576}q\dot{\eta} + Z_{577}q\dot{\xi} + Z_{578}q\dot{\sigma} + Z_{579}q\dot{\delta} + Z_{580}r\dot{\eta} \\
& + Z_{581}r\dot{\xi} + Z_{582}r\dot{\sigma} + Z_{583}r\dot{\delta} + Z_{584}\dot{\eta}^2 + Z_{585}\dot{\eta}\dot{\xi} + Z_{586}\dot{\eta}\dot{\sigma} + Z_{587}\dot{\eta}\dot{\delta} + Z_{588}\ddot{\xi} \\
& + Z_{589}\dot{\xi}^2 + Z_{590}\dot{\xi}\dot{\sigma} + Z_{591}\dot{\xi}\dot{\delta} + Z_{592}\ddot{\sigma} + Z_{593}\dot{\sigma}^2 + Z_{594}\dot{\sigma}\dot{\delta} + Z_{595}\ddot{\delta} + Z_{596}\dot{\delta}^2
\end{aligned}
\tag{100}
$$

$$
\begin{aligned}
Z_{600}\ddot{\xi} = {} & Z_{601}X_E + Z_{602}Y_E + Z_{603}Z_E + Z_{604}mg + Z_{605}M_E + Z_{606}N_E + Z_{607}\dot{u} + Z_{608}\dot{v} + Z_{609}\dot{w} \\
& + Z_{610}rv + Z_{611}qw + Z_{612}ru + Z_{613}pw + Z_{614}qu + Z_{615}pv + Z_{616}\dot{p} + Z_{617}\dot{q} \\
& + Z_{618}\dot{r} + Z_{619}pq + Z_{620}pr + Z_{621}qr + Z_{622}p^2 + Z_{623}q^2 + Z_{624}r^2 + Z_{625}p\dot{\eta} \\
& + Z_{626}p\dot{\xi} + Z_{627}p\dot{\sigma} + Z_{628}p\dot{\delta} + Z_{629}q\dot{\eta} + Z_{630}q\dot{\xi} + Z_{631}q\dot{\sigma} + Z_{632}q\dot{\delta} + Z_{633}r\dot{\eta} \\
& + Z_{634}r\dot{\xi} + Z_{635}r\dot{\sigma} + Z_{636}r\dot{\delta} + Z_{637}\ddot{\eta} + Z_{638}\dot{\eta}^2 + Z_{639}\dot{\eta}\dot{\xi} + Z_{640}\dot{\eta}\dot{\sigma} + Z_{641}\dot{\eta}\dot{\delta} \\
& + Z_{642}\dot{\xi}^2 + Z_{643}\dot{\xi}\dot{\sigma} + Z_{644}\dot{\xi}\dot{\delta} + Z_{645}\ddot{\sigma} + Z_{646}\dot{\sigma}^2 + Z_{647}\dot{\sigma}\dot{\delta} + Z_{648}\ddot{\delta} + Z_{649}\dot{\delta}^2
\end{aligned}
\tag{101}
$$

$$
\begin{aligned}
Z_{653}\ddot{\delta} = {} & Z_{654}X_E + Z_{655}Y_E + Z_{656}Z_E + Z_{657}mg + Z_{658}L_E + Z_{659}\dot{u} + Z_{660}\dot{v} + Z_{661}\dot{w} + Z_{662}rv \\
& + Z_{663}qw + Z_{664}ru + Z_{665}pw + Z_{666}qu + Z_{667}pv + Z_{668}\dot{p} + Z_{669}\dot{q} + Z_{670}\dot{r} + Z_{671}pq \\
& + Z_{672}pr + Z_{673}qr + Z_{674}p^2 + Z_{675}q^2 + Z_{676}r^2 + Z_{677}p\dot{\eta} + Z_{678}p\dot{\xi} + Z_{679}p\dot{\sigma} \\
& + Z_{680}p\dot{\delta} + Z_{681}q\dot{\eta} + Z_{682}q\dot{\xi} + Z_{683}q\dot{\sigma} + Z_{684}q\dot{\delta} + Z_{685}r\dot{\eta} + Z_{686}r\dot{\xi} + Z_{687}r\dot{\sigma} \\
& + Z_{688}r\dot{\delta} + Z_{689}\ddot{\eta} + Z_{690}\dot{\eta}^2 + Z_{691}\dot{\eta}\dot{\xi} + Z_{692}\dot{\eta}\dot{\sigma} + Z_{693}\dot{\eta}\dot{\delta} + Z_{694}\ddot{\xi} + Z_{695}\dot{\xi}^2 + Z_{696}\dot{\xi}\dot{\sigma} \\
& + Z_{697}\dot{\xi}\dot{\delta} + Z_{698}\ddot{\sigma} + Z_{699}\dot{\sigma}^2 + Z_{700}\dot{\sigma}\dot{\delta} + Z_{701}\dot{\delta}^2 \qquad (102)
\end{aligned}
$$

$$
\begin{aligned}
Z_{708}\ddot{\sigma} = {} & Z_{709}X_E + Z_{710}Y_E + Z_{711}Z_E + Z_{712}mg + Z_{713}L_E + Z_{714}M_E + Z_{715}N_E + Z_{716}\dot{u} + Z_{717}\dot{v} \\
& + Z_{718}\dot{w} + Z_{719}rv + Z_{720}qw + Z_{721}ru + Z_{722}pw + Z_{723}qu + Z_{724}pv + Z_{725}\dot{p} \\
& + Z_{726}\dot{q} + Z_{727}\dot{r} + Z_{728}pq + Z_{729}pr + Z_{730}qr + Z_{731}p^2 + Z_{732}q^2 + Z_{733}r^2 \\
& + Z_{734}p\dot{\eta} + Z_{735}p\dot{\xi} + Z_{736}p\dot{\sigma} + Z_{737}p\dot{\delta} + Z_{738}q\dot{\eta} + Z_{739}q\dot{\xi} + Z_{740}q\dot{\sigma} + Z_{741}q\dot{\delta} \\
& + Z_{742}r\dot{\eta} + Z_{743}r\dot{\xi} + Z_{744}r\dot{\sigma} + Z_{745}r\dot{\delta} + Z_{746}\ddot{\eta} + Z_{747}\dot{\eta}^2 + Z_{748}\dot{\eta}\dot{\xi} + Z_{749}\dot{\eta}\dot{\sigma} \\
& + Z_{750}\dot{\eta}\dot{\delta} + Z_{751}\ddot{\xi} + Z_{752}\dot{\xi}^2 + Z_{753}\dot{\xi}\dot{\sigma} + Z_{754}\dot{\xi}\dot{\delta} + Z_{755}\dot{\sigma}^2 + Z_{756}\dot{\sigma}\dot{\delta} + Z_{757}\dot{\delta}^2 \\
& + Z_{758}\ddot{\delta} \qquad (103)
\end{aligned}
$$

CONCLUDING REMARKS

Using Lagrange's form of D'Alembert's principle, a mathematical representation has been developed of a helicopter carrying an externally suspended load. Ten degrees of freedom are necessary to represent the system's motion in an inertial reference frame: six degrees of freedom for the helicopter relative to inertial space and four degrees of freedom for the motion of the load relative to the helicopter.

In this analysis, the suspension cables have been considered to be massless, rigid links. An extension of this work might be the development of a twelve-degree-of-freedom model in which the suspension cables are permitted to be slack or to have elastic properties at appropriate flight conditions. This consideration would be particularly realistic when modeling a load which develops its own lift.

APPENDIX A

REFERENCE FRAMES

Helicopter B is oriented relative to N with three successive rotations through Euler angles ψ, θ, and ϕ. Using intermediate reference frames B" and B' in which $\vec{i}$, $\vec{j}$, $\vec{k}$ and $\vec{i}'$, $\vec{j}'$, $\vec{k}'$ are fixed, these rotations can be shown individually in figures A1-A3 and represented in equations (A1)-(A4).

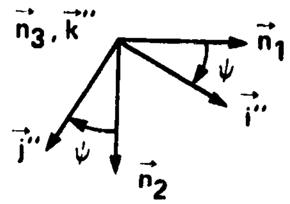


Figure A1.- Orientation of reference frame B" relative to N.

$$\begin{Bmatrix} \vec{i}'' \\ \vec{j}'' \\ \vec{k}'' \end{Bmatrix} = \begin{bmatrix} c_\psi & s_\psi & 0 \\ -s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{n}_1 \\ \vec{n}_2 \\ \vec{n}_3 \end{Bmatrix} \qquad \text{(A1)}$$

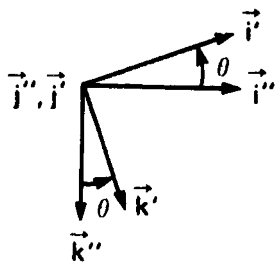


Figure A2.- Orientation of reference frame B' relative to B".

$$\begin{Bmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{Bmatrix} = \begin{bmatrix} c_\theta & 0 & -s_\theta \\ 0 & 1 & 0 \\ s_\theta & 0 & c_\theta \end{bmatrix} \begin{Bmatrix} \vec{i}'' \\ \vec{j}'' \\ \vec{k}'' \end{Bmatrix} \qquad \text{(A2)}$$

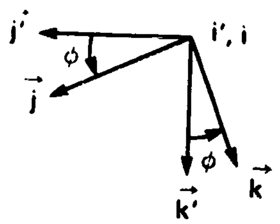
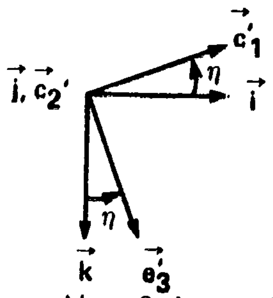


Figure A3.- Orientation of reference frame B relative to B'.

$$\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & s_\phi \\ 0 & -s_\phi & c_\phi \end{bmatrix} \begin{Bmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{Bmatrix} \qquad \text{(A3)}$$

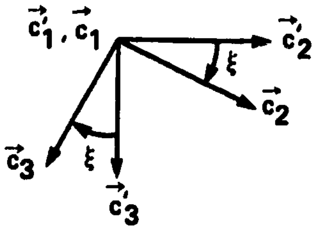
$$\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{bmatrix} c_\psi c_\theta & s_\psi c_\theta & -s_\theta \\ (c_\psi s_\theta s_\phi - s_\psi c_\phi) & (s_\psi s_\theta s_\phi - c_\psi c_\phi) & c_\theta s_\phi \\ (c_\psi s_\theta c_\phi + s_\psi c_\phi) & (s_\psi s_\theta c_\phi - c_\psi s_\phi) & c_\theta c_\phi \end{bmatrix} \begin{Bmatrix} \vec{n}_1 \\ \vec{n}_2 \\ \vec{n}_3 \end{Bmatrix} \qquad \text{(A4)}$$

Reference frame C, fixed in the aft cable, is located relative to B by two successive rotations. These are illustrated in figures A4 and A5 and in equations (A5)-(A7).



$$\begin{Bmatrix} \vec{c}_1' \\ \vec{c}_2' \\ \vec{c}_3' \end{Bmatrix} = \begin{bmatrix} c_\eta & 0 & -s_\eta \\ 0 & 1 & 0 \\ s_\eta & 0 & c_\eta \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \tag{A5}$$

Figure A4.- Orientation of reference frame C' relative to B.

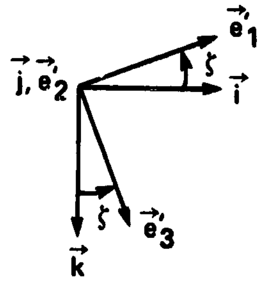


$$\begin{Bmatrix} \vec{c}_1 \\ \vec{c}_2 \\ \vec{c}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\xi & s_\xi \\ 0 & -s_\xi & c_\xi \end{bmatrix} \begin{Bmatrix} \vec{c}_1' \\ \vec{c}_2' \\ \vec{c}_3' \end{Bmatrix} \tag{A6}$$

Figure A5.- Orientation of reference frame C relative to C'.

$$\begin{Bmatrix} \vec{c}_1 \\ \vec{c}_2 \\ \vec{c}_3 \end{Bmatrix} = \begin{bmatrix} c_\eta & 0 & -s_\eta \\ s_\eta s_\xi & c_\xi & c_\eta s_\xi \\ s_\eta c_\xi & -s_\xi & c_\eta c_\xi \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \tag{A7}$$

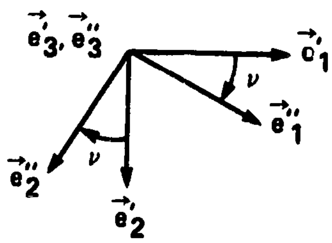
Rotations through three successive angles ζ, ν, and δ define the orientation of reference frame E relative to B. This is shown in figures A6-A8 and in equations (A8)-(A12).



$$\begin{Bmatrix} \vec{e}_1' \\ \vec{e}_2' \\ \vec{e}_3' \end{Bmatrix} = \begin{bmatrix} c_\zeta & 0 & -s_\zeta \\ 0 & 1 & 0 \\ s_\zeta & 0 & c_\zeta \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \tag{A8}$$

Figure A6.- Orientation of reference frame E' relative to B.

where $\zeta \equiv \sigma + \theta_E$.



$$\begin{Bmatrix} \vec{e}_1'' \\ \vec{e}_2'' \\ \vec{e}_3'' \end{Bmatrix} = \begin{bmatrix} c_\nu & s_\nu & 0 \\ -s_\nu & c_\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{e}_1' \\ \vec{e}_2' \\ \vec{e}_3' \end{Bmatrix} \tag{A9}$$

Figure A7.- Orientation of reference frame E'' relative to E'.

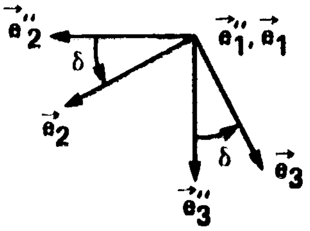


Figure A8.- Orientation of reference frame E relative to E".

$$\begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\delta & s_\delta \\ 0 & -s_\delta & c_\delta \end{bmatrix} \begin{Bmatrix} \vec{e}''_1 \\ \vec{e}''_2 \\ \vec{e}''_3 \end{Bmatrix} \tag{A10}$$

$$\begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix} = \begin{bmatrix} c_\zeta c_\nu & s_\nu & -s_\zeta c_\nu \\ (s_\zeta s_\delta - c_\zeta s_\nu c_\delta) & c_\nu c_\delta & (s_\zeta s_\nu c_\delta + c_\zeta s_\delta) \\ (s_\zeta c_\delta + c_\zeta s_\nu s_\delta) & -c_\nu s_\delta & (-s_\zeta s_\nu s_\delta + c_\zeta c_\delta) \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \tag{A11}$$

Likewise, using the transformation matrix E_B:

$$\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{bmatrix} c_\zeta c_\nu & (s_\zeta s_\delta - c_\zeta s_\nu c_\delta) & (s_\zeta c_\delta + c_\zeta s_\nu s_\delta) \\ s_\nu & c_\nu c_\delta & -c_\nu s_\delta \\ -s_\zeta c_\nu & (s_\zeta s_\nu c_\delta + c_\zeta s_\delta) & (-s_\zeta s_\nu s_\delta + c_\zeta c_\delta) \end{bmatrix} \begin{Bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{Bmatrix} \tag{A12}$$

APPENDIX B

SOLUTION FOR NONHOLONOMIC CONSTRAINT EQUATION

To solve for $\dot{\nu}$ in terms of $\dot{\eta}$, $\dot{\xi}$, $\dot{\sigma}$, η, ξ, and σ, express the forward cable as a rigid link of length ℓ'.

$$\left|-R\vec{k} - a\vec{I} + R\vec{k} + \ell\vec{c}_s + a'\vec{e}_1\right| = |Q_1Q| = \ell' \tag{B1}$$

$$\left|-a\vec{I} + \ell(c_\xi s_\eta \vec{I} - s_\xi \vec{J} + c_\xi c_\eta \vec{k}) + a'(c_\nu c_\zeta \vec{I} + s_\nu \vec{J} - c_\nu s_\zeta \vec{k})\right| = \ell' \tag{B2}$$

$$\left|(-a + \ell c_\xi s_\eta + a'c_\nu c_\zeta)\vec{I} + (-\ell s_\xi + a's_\nu)\vec{J} + (\ell c_\xi c_\eta - a'c_\nu s_\zeta)\vec{k}\right| = \ell' \tag{B3}$$

Squaring both sides yields:

$$a^2 + \ell^2 c_\xi^2 s_\eta^2 + a'^2 c_\nu^2 c_\zeta^2 - 2a\ell c_\xi s_\eta - 2aa'c_\nu c_\zeta + 2a'\ell c_\nu c_\zeta c_\xi s_\eta + \ell^2 s_\xi^2 - 2a'\ell s_\xi s_\nu$$
$$+ a'^2 s_\nu^2 + \ell^2 c_\xi^2 c_\eta^2 - 2a'\ell c_\xi c_\eta c_\nu s_\zeta + a'^2 c_\nu^2 s_\zeta^2 = \ell'^2 \tag{B4}$$

$$a^2 + a'^2 + \ell^2 - 2a\ell c_\xi s_\eta - 2aa'c_\nu c_\zeta + 2a'\ell(c_\nu c_\zeta c_\xi s_\eta - s_\xi s_\nu - c_\xi c_\eta c_\nu s_\zeta) = \ell'^2 \tag{B5}$$

$$\frac{a^2}{\ell^2} + \frac{a'^2}{\ell^2} + 1 - \frac{2a}{\ell} c_\xi s_\eta - \frac{2aa'}{\ell^2} c_\nu c_\zeta + \frac{2a'}{\ell}(c_\nu c_\zeta c_\xi s_\eta - s_\xi s_\nu - c_\xi c_\eta c_\nu s_\zeta) = \frac{\ell'^2}{\ell^2} \tag{B6}$$

Differentiating with respect to time will give a nonholonomic constraint for $\dot{\nu}$:

$$\frac{-2a}{\ell}\left[-(s_\xi)\dot{\xi}s_\eta + c_\xi \dot{\eta} c_\eta\right] - \frac{2aa'}{\ell^2}\left[(-s_\nu)\dot{\nu}c_\zeta + c_\nu\dot{\sigma}(-s_\zeta)\right] + \frac{2a'}{\ell}\left[\dot{\nu}(-s_\nu)c_\zeta c_\xi s_\eta + c_\nu\dot{\sigma}(-s_\zeta)c_\xi s_\eta\right.$$
$$+ c_\nu c_\zeta \dot{\xi}(-s_\xi)s_\eta + c_\nu c_\zeta c_\xi \dot{\eta} c_\eta - \dot{\xi}c_\xi s_\nu - s_\xi \dot{\nu} c_\nu - \dot{\xi}(-s_\xi)c_\eta c_\nu s_\zeta - c_\xi \dot{\eta}(-s_\eta)c_\nu s_\zeta$$
$$\left.- c_\xi c_\eta \dot{\nu}(-s_\nu)s_\zeta - c_\xi c_\eta c_\nu \dot{\sigma} c_\zeta\right] = 0 \tag{B7}$$

$$\dot{\nu} = Z_1\dot{\eta} + Z_2\dot{\xi} + Z_3\dot{\sigma} \tag{B8}$$

where:

$$Z_1 = \frac{\frac{-2a}{\ell} c_\xi c_\eta + \frac{2a'}{\ell}(c_\nu c_\zeta c_\xi c_\eta + c_\xi s_\eta c_\nu s_\zeta)}{\frac{-2aa'}{\ell^2} s_\nu c_\zeta + \frac{2a'}{\ell}(s_\nu c_\zeta c_\xi s_\eta + s_\xi c_\nu - c_\xi c_\eta s_\nu s_\zeta)} \tag{B9}$$

$$Z_2 = \frac{\frac{2a}{\ell} s_\xi s_\eta + \frac{2a'}{\ell}(-c_\nu c_\zeta s_\xi s_\eta - c_\xi s_\nu + s_\xi c_\eta c_\nu s_\zeta)}{\frac{-2aa'}{\ell^2} s_\nu c_\zeta + \frac{2a'}{\ell}(s_\nu c_\zeta c_\xi s_\eta + s_\xi c_\nu - c_\xi c_\eta s_\nu s_\zeta)} \tag{B10}$$

$$Z_3 = \frac{\frac{2aa'}{\ell^2} c_\nu s_\zeta + \frac{2a'}{\ell} (-c_\nu s_\zeta c_\xi s_\eta - c_\xi c_\eta c_\nu c_\zeta)}{\frac{-2aa'}{\ell^2} s_\nu c_\zeta + \frac{2a'}{\ell} (s_\nu c_\zeta c_\xi s_\eta + s_\xi c_\nu - c_\xi c_\eta s_\nu s_\zeta)} \tag{B11}$$

APPENDIX C

CALCULATION OF ${}^{N}\vec{v}^{E^*}$

To calculate ${}^{N}\vec{v}^{E^*}$, consider the relationships among the inertial velocities of points P_1, P, and E^*:

$${}^{N}\vec{v}^{P_1} = {}^{N}\vec{v}^{B^*} + {}^{N}\vec{\omega}^{B} \times \vec{p}^{B^*-P_1} \tag{28}$$

$${}^{N}\vec{v}^{P} = {}^{N}\vec{v}^{P_1} + {}^{N}\vec{\omega}^{C} \times \vec{p}^{P_1-P} \tag{29}$$

$${}^{N}\vec{v}^{E^*} = {}^{N}\vec{v}^{P} + {}^{N}\vec{\omega}^{E} \times \vec{p}^{P-E^*} \tag{30}$$

and substituting (C1) and (C2) into (C3):

$${}^{N}\vec{v}^{E^*} = {}^{N}\vec{v}^{B^*} + {}^{N}\vec{\omega}^{B} \times \vec{p}^{B^*-P_1} + {}^{N}\vec{\omega}^{C} \times \vec{p}^{P_1-P} + {}^{N}\vec{\omega}^{E} \times \vec{p}^{P-E^*} \tag{31}$$

where:

$${}^{N}\vec{v}^{B^*} = u\vec{i} + v\vec{j} + w\vec{k} \tag{2}$$

$${}^{N}\vec{\omega}^{B} = p\vec{i} + q\vec{j} + r\vec{k} \tag{1}$$

$$\vec{p}^{B^*-P_1} = -\frac{a}{2}\vec{i} + R\vec{k} \tag{C1}$$

$${}^{N}\vec{\omega}^{C} = {}^{N}\vec{\omega}^{B} + {}^{B}\vec{\omega}^{C} \tag{C2}$$

$$= p\vec{i} + q\vec{j} + r\vec{k} + \dot{\eta}\vec{j} + \dot{\xi}\vec{c}_1 \tag{C3}$$

$$= (p + \dot{\xi}c_\eta)\vec{i} + (q + \dot{\eta})\vec{j} + (r - \dot{\xi}s_\eta)\vec{k} \tag{C4}$$

$$\vec{p}^{P_1-P} = \ell\vec{c}_3 \tag{C5}$$

$$= \ell c_\xi s_\eta\vec{i} - \ell s_\xi\vec{j} + \ell c_\xi c_\eta\vec{k} \tag{C6}$$

$${}^{N}\vec{\omega}^{E} = {}^{N}\vec{\omega}^{B} + {}^{B}\vec{\omega}^{E} \tag{25}$$

$$= p\vec{i} + (q + \dot{\sigma})\vec{j} + r\vec{k} + (Z_1\dot{\eta} + Z_2\dot{\xi} + Z_3\dot{\sigma})\vec{e}_3' + \dot{\delta}\vec{e}_1 \tag{26}$$

$$= (p + Z_1s_\zeta\dot{\eta} + Z_2s_\zeta\dot{\xi} + Z_3s_\zeta\dot{\sigma} + c_\nu c_\zeta\dot{\delta})\vec{i} + (q + \dot{\sigma} + s_\nu\dot{\delta})\vec{j}$$

$$+ (r + Z_1c_\zeta\dot{\eta} + Z_2c_\zeta\dot{\xi} + Z_3c_\zeta\dot{\sigma} - c_\nu s_\zeta\dot{\delta})\vec{k} \tag{C7}$$

$$\vec{p}^{P_1-E^*} = \frac{a'}{2}\vec{e}_1 + h\vec{e}_3 \tag{C8}$$

$$= \left(\frac{a'}{2} c_\nu c_\zeta + hc_\delta s_\zeta + hc_\zeta s_\delta s_\nu\right)\vec{i} + \left(\frac{a'}{2} s_\nu - hs_\delta c_\nu\right)\vec{j} + \left(-\frac{a'}{2} c_\nu s_\zeta - hs_\zeta s_\delta s_\nu + hc_\delta c_\zeta\right)\vec{k} \tag{C9}$$

Expanding the individual cross products in equation (31) and collecting terms gives ${}^N\vec{v}^{E^*}$:

$${}^N\vec{\omega}^B \times \vec{p}^{B^*-P_1} = (p\vec{i} + q\vec{j} + r\vec{k}) \times \left(-\frac{a}{2}\vec{i} + R\vec{k}\right) \tag{C10}$$

$$= -Rq\vec{i} + \left(-Rp - \frac{a}{2} r\right)\vec{j} + \frac{a}{2} q\vec{k} \tag{C11}$$

$${}^N\vec{\omega}^C \times \vec{p}^{P_1-P} = [(p + c_\eta\dot{\xi})\vec{i} + (q + \dot{\eta})\vec{j} + (r - s_\eta\dot{\xi})\vec{k}] \times [\ell c_\xi s_\eta\vec{i} - \ell s_\xi\vec{j} + \ell c_\xi c_\eta\vec{k}] \tag{C12}$$

$$= [\ell c_\xi c_\eta q + \ell s_\xi r + \ell c_\xi c_\eta\dot{\eta} - \ell s_\xi s_\eta\dot{\eta}]\vec{i} + [-\ell c_\xi c_\eta p + \ell c_\xi s_\eta r - \ell c_\xi\dot{\xi}]\vec{j} + [-\ell s_\xi p - \ell c_\xi s_\eta q - \ell c_\xi s_\eta\dot{\eta} - \ell s_\xi c_\eta\dot{\xi}]\vec{k} \tag{C13}$$

$${}^N\vec{\omega}^E \times \vec{p}^{P-E^*} = [(p + Z_1 s_\zeta\dot{\eta} + Z_2 s_\zeta\dot{\xi} + Z_3 s_\zeta\dot{\sigma} + c_\nu c_\zeta\dot{\delta})\vec{i} + (q + \dot{\sigma} + s_\nu\dot{\delta})\vec{j} + (r + Z_1 c_\zeta\dot{\eta} + Z_2 c_\zeta\dot{\xi} + Z_3 c_\zeta\dot{\sigma} - c_\nu s_\zeta\dot{\delta})\vec{k}] \times \left[\left(\frac{a'}{2} c_\nu c_\zeta + hc_\delta s_\zeta + hc_\zeta s_\delta s_\nu\right)\vec{i} + \left(\frac{a'}{2} s_\nu - hs_\delta c_\nu\right)\vec{j} + \left(-\frac{a'}{2} c_\nu s_\zeta - hs_\zeta s_\delta s_\nu + hc_\delta c_\zeta\right)\vec{k}\right] \tag{C14}$$

$$\begin{aligned}
= &\left\{\left(-\frac{a'}{2} c_\nu s_\zeta - hs_\zeta s_\delta s_\nu + hc_\delta c_\zeta\right)q + \left(-\frac{a'}{2} s_\nu + hs_\delta c_\nu\right)r \right.\\
&+ \left(-\frac{a'}{2} s_\nu + hs_\delta c_\nu\right)Z_1 c_\zeta\dot{\eta} + \left(-\frac{a}{2} s_\nu + hs_\delta c_\nu\right)c_\zeta Z_2\dot{\xi} + \left[-\frac{a'}{2} c_\nu s_\zeta\right.\\
&\left.\left. - hs_\zeta s_\delta s_\nu + hc_\delta c_\zeta + \left(-\frac{a'}{2} s_\nu + hs_\delta c_\nu\right)c_\zeta Z_3\right]\dot{\sigma} + \left(hc_\delta c_\zeta s_\nu - hs_\delta s_\zeta\right)\dot{\delta}\right\}\vec{i}\\
&+ \left\{\left(\frac{a'}{2} c_\nu s_\zeta + hs_\zeta s_\delta s_\nu - hc_\delta c_\zeta\right)p + \left(\frac{a'}{2} c_\nu c_\zeta + hc_\delta s_\zeta + hc_\zeta s_\delta s_\nu\right)r\right.\\
&+ \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)Z_1\dot{\eta} + \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)Z_2\dot{\xi} + \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)Z_3\dot{\sigma}\\
&\left. + (-hc_\delta c_\nu)\dot{\delta}\right\}\vec{j} + \left\{\left(\frac{a'}{2} s_\nu - hs_\delta c_\nu\right)p + \left(-\frac{a'}{2} c_\nu c_\zeta - hc_\delta s_\zeta - hc_\zeta s_\delta s_\nu\right)q\right.\\
&+ \left(\frac{a'}{2} s_\nu - hs_\delta c_\nu\right)s_\zeta Z_1\dot{\eta} + \left(\frac{a'}{2} s_\nu - hs_\delta c_\nu\right)s_\zeta Z_2\dot{\xi} + \left[\left(\frac{a'}{2} s_\nu - hs_\delta c_\nu\right)s_\zeta Z_3\right.\\
&\left.\left. - \frac{a'}{2} c_\nu c_\zeta - hc_\delta s_\zeta - hc_\zeta s_\delta s_\nu\right]\dot{\sigma} + \left(-hs_\delta c_\zeta - hs_\nu c_\delta s_\zeta\right)\dot{\delta}\right\}\vec{k}
\end{aligned} \tag{C15}$$

$$
\begin{aligned}
{}^{N}\vec{v}^{E^*} = \Big\{ & u + \Big(R + \ell c_\xi c_\eta - \frac{a'}{2} c_\nu s_\zeta - h s_\zeta s_\delta s_\nu + h c_\delta c_\zeta\Big) q + \Big(\ell s_\xi - \frac{a'}{2} s_\nu + h s_\delta c_\nu\Big) r \\
& + \Big[\ell c_\xi c_\eta + \Big(-\frac{a'}{2} s_\nu + h s_\delta c_\nu\Big) Z_1 c_\zeta\Big]\dot{\eta} + \Big[-\ell s_\xi s_\eta + \Big(-\frac{a'}{2} s_\nu + h s_\delta c_\nu\Big) c_\zeta Z_2\Big]\dot{\xi} \\
& + \Big[-\frac{a'}{2} c_\nu s_\zeta - h s_\zeta s_\delta s_\nu + h c_\delta c_\zeta + \Big(-\frac{a'}{2} s_\nu + h s_\delta c_\nu\Big) c_\zeta Z_3\Big]\dot{\sigma} + [h c_\delta c_\zeta s_\nu - h s_\delta s_\zeta]\dot{\delta}\Big\}\vec{i} \\
& + \Big\{v + \Big(-R - \ell c_\xi c_\eta + \frac{a'}{2} c_\nu s_\zeta + h s_\zeta s_\delta s_\nu - h c_\delta c_\zeta\Big) p + \Big(-\frac{a}{2} + \ell c_\xi s_\eta + \frac{a'}{2} c_\nu c_\zeta \\
& + h c_\delta s_\zeta + h c_\zeta s_\delta s_\nu\Big) r + \Big(\frac{a'}{2} c_\nu + h s_\delta s_\nu\Big) Z_1 \dot{\eta} + \Big[-\ell c_\xi + \Big(\frac{a'}{2} c_\nu + h s_\delta s_\nu\Big) Z_2\Big]\dot{\xi} \\
& + \Big(\frac{a'}{2} c_\nu + h s_\delta s_\nu\Big) Z_3 \dot{\sigma} + (-h c_\delta c_\nu)\dot{\delta}\Big\}\vec{j} + \Big\{w + \Big(-\ell s_\xi + \frac{a'}{2} s_\nu - h s_\delta c_\nu\Big) p \\
& + \Big(\frac{a}{2} - \ell c_\xi s_\eta - \frac{a'}{2} c_\nu c_\zeta - h c_\delta s_\zeta - h c_\zeta s_\delta s_\nu\Big) q + \Big[-\ell c_\xi s_\eta + \Big(\frac{a'}{2} s_\nu - h s_\delta c_\nu\Big) s_\zeta Z_1\Big]\dot{\eta} \\
& + \Big[-\ell s_\xi c_\eta + \Big(\frac{a'}{2} s_\nu - h s_\delta c_\nu\Big) s_\zeta Z_2\Big]\dot{\xi} + \Big[\Big(\frac{a'}{2} s_\nu - h s_\delta c_\nu\Big) s_\zeta Z_3 - \frac{a'}{2} c_\nu c_\zeta - h c_\delta s_\zeta \\
& - h c_\zeta s_\delta s_\nu\Big]\dot{\sigma} + (-h s_\delta c_\zeta - h s_\nu c_\delta s_\zeta)\dot{\delta}\Big\}\vec{k} \qquad \text{(C16)}
\end{aligned}
$$

APPENDIX D

DEFINITION OF Z_i PARAMETERS DEVELOPED IN DERIVATION

$$Z_1 = \frac{-\frac{2a}{\ell} c_\xi c_\eta + \frac{2a'}{\ell}\left(c_\nu c_\zeta c_\xi c_\eta + c_\xi s_\eta c_\nu s_\zeta\right)}{-\frac{2aa'}{\ell^2} s_\nu c_\zeta + \frac{2a'}{\ell}\left(s_\nu c_\zeta c_\xi s_\eta + s_\xi c_\nu - c_\xi c_\eta s_\nu s_\zeta\right)}$$

$$Z_2 = \frac{\frac{2a}{\ell} s_\xi s_\eta + \frac{2a'}{\ell}\left(-c_\nu c_\zeta s_\xi s_\eta - c_\xi s_\nu + s_\xi c_\eta c_\nu s_\zeta\right)}{-\frac{2aa'}{\ell^2} s_\nu c_\zeta + \frac{2a'}{\ell}\left(s_\nu c_\zeta c_\xi s_\eta + s_\xi c_\nu - c_\xi c_\eta s_\nu s_\zeta\right)}$$

$$Z_3 = \frac{\frac{2aa'}{\ell^2} c_\nu s_\zeta + \frac{2a'}{\ell}\left(-c_\nu s_\zeta c_\xi s_\eta - c_\xi c_\eta c_\nu c_\zeta\right)}{-\frac{2aa'}{\ell^2} s_\nu c_\zeta + \frac{2a'}{\ell}\left(s_\nu c_\zeta c_\xi s_\eta + s_\xi c_\nu - c_\xi c_\eta s_\nu s_\zeta\right)}$$

$$Z_4 = -\frac{2aa'}{\ell^2} s_\nu c_\zeta + \frac{2a'}{\ell}\left(s_\nu c_\zeta c_\xi s_\eta + s_\xi c_\nu - c_\xi c_\eta s_\nu s_\zeta\right)$$

$$Z_5 = -\frac{2a}{\ell} c_\xi c_\eta + \frac{2a'}{\ell}\left(c_\nu c_\zeta c_\xi c_\eta + c_\xi s_\eta c_\nu s_\zeta\right)$$

$$Z_6 = \frac{2a}{\ell} s_\xi s_\eta + \frac{2a'}{\ell}\left(-c_\nu c_\zeta s_\xi s_\eta - c_\xi s_\nu + s_\xi c_\eta c_\nu s_\zeta\right)$$

$$Z_7 = \frac{2aa'}{\ell^2} c_\nu s_\zeta + \frac{2a'}{\ell}\left(-c_\nu s_\zeta c_\xi s_\eta - c_\xi c_\eta c_\nu c_\zeta\right)$$

$$Z_8 = Z_4^2$$

$$Z_9 = \frac{1}{Z_4}\left(\frac{4a}{\ell} s_\xi c_\eta - \frac{4a'}{\ell} c_\nu c_\zeta s_\xi c_\eta - \frac{4a'}{\ell} s_\xi s_\eta c_\nu s_\zeta\right) - \frac{Z_5}{Z_8}\left(-\frac{2a'}{\ell} s_\nu c_\zeta s_\xi s_\eta + \frac{2a'}{\ell} c_\xi c_\nu + \frac{2a'}{\ell} s_\xi c_\eta s_\nu s_\zeta\right) - \frac{Z_6}{Z_8}\left(\frac{2a'}{\ell} s_\nu c_\zeta c_\xi c_\eta + \frac{2a'}{\ell} c_\xi s_\eta s_\nu s_\zeta\right)$$

$$Z_{10} = \frac{1}{Z_4}\left(\frac{2a}{\ell} c_\xi s_\eta - \frac{2a'}{\ell} c_\nu c_\zeta c_\xi s_\eta + \frac{2a'}{\ell} c_\xi c_\eta c_\nu s_\zeta\right) - \frac{Z_5}{Z_8}\left(\frac{2a'}{\ell} s_\nu c_\zeta c_\xi c_\eta + \frac{2a'}{\ell} c_\xi s_\eta s_\nu s_\zeta\right)$$

$$Z_{11} = \frac{1}{Z_4}\left(-\frac{2a'}{\ell} s_\nu c_\zeta c_\xi c_\eta - \frac{2a'}{\ell} c_\xi s_\eta s_\nu s_\zeta\right) - \frac{Z_5}{Z_8}\left(-\frac{2aa'}{\ell^2} c_\nu c_\zeta + \frac{2a'}{\ell} c_\nu c_\zeta c_\xi s_\eta - \frac{2a'}{\ell} s_\xi s_\nu - \frac{2a'}{\ell} c_\xi c_\eta c_\nu s_\zeta\right)$$

$$Z_{12} = \frac{1}{Z_4}\left(-\frac{4a'}{\ell} c_\nu s_\zeta c_\xi c_\eta + \frac{4a'}{\ell} c_\xi s_\eta c_\nu c_\zeta\right) - \frac{Z_5}{Z_8}\left(\frac{2aa'}{\ell^2} s_\nu s_\zeta - \frac{2a'}{\ell} s_\nu s_\zeta c_\xi s_\eta - \frac{2a'}{\ell} c_\xi c_\eta s_\nu c_\zeta\right) - \frac{Z_7}{Z_8}\left(\frac{2a'}{\ell} s_\nu c_\zeta c_\xi c_\eta + \frac{2a'}{\ell} c_\xi s_\eta s_\nu s_\zeta\right)$$

$$Z_{13} = \frac{1}{Z_4}\left[\frac{4a'}{\ell}(c_\nu s_\zeta s_\xi s_\eta + s_\xi c_\eta c_\nu c_\zeta)\right] - \frac{Z_6}{Z_8}\left[\frac{2aa'}{\ell^2} s_\nu s_\zeta + \frac{2a'}{\ell}(-s_\nu s_\zeta c_\xi s_\eta - c_\xi c_\eta s_\nu c_\zeta)\right]$$

$$-\frac{Z_7}{Z_8}\left[\frac{2a'}{\ell}(-s_\nu c_\zeta s_\xi s_\eta + c_\xi c_\nu + s_\xi c_\eta s_\nu s_\zeta)\right]$$

$$Z_{14} = Z_1$$

$$Z_{15} = \frac{1}{Z_4}\left[\frac{2a}{\ell} c_\xi s_\eta + \frac{2a'}{\ell}(-c_\nu c_\zeta c_\xi s_\eta + s_\xi s_\nu + c_\xi c_\eta c_\nu s_\zeta)\right] - \frac{Z_6}{Z_8}\left[\frac{2a'}{\ell}(-s_\nu c_\zeta s_\xi s_\eta\right.$$

$$\left. + c_\xi c_\nu + s_\xi c_\eta s_\nu s_\zeta)\right]$$

$$Z_{16} = \frac{1}{Z_4}\left[\frac{2a'}{\ell}(s_\nu c_\zeta s_\xi s_\eta - c_\xi c_\nu - s_\xi c_\eta s_\nu s_\zeta)\right] - \frac{Z_6}{Z_8}\left[-\frac{2aa'}{\ell^2} c_\nu c_\zeta + \frac{2a'}{\ell}(c_\nu c_\zeta c_\xi s_\eta\right.$$

$$\left. - s_\xi s_\nu - c_\xi c_\eta c_\nu s_\zeta)\right]$$

$$Z_{17} = Z_2$$

$$Z_{18} = \frac{1}{Z_4}\left[-\frac{2aa'}{\ell^2} s_\nu s_\zeta + \frac{2a'}{\ell}(s_\nu s_\zeta c_\xi s_\eta + c_\xi c_\eta s_\nu c_\zeta)\right] - \frac{Z_7}{Z_8}\left[-\frac{2aa'}{\ell^2} c_\nu c_\zeta + \frac{2a'}{\ell}(c_\nu c_\zeta c_\xi s_\eta\right.$$

$$\left. - s_\xi s_\nu - c_\xi c_\eta c_\nu s_\zeta)\right]$$

$$Z_{19} = \frac{1}{Z_4}\left[\frac{2aa'}{\ell^2} c_\nu c_\zeta + \frac{2a'}{\ell}(-c_\nu c_\zeta c_\xi s_\eta + c_\xi c_\eta c_\nu s_\zeta)\right] - \frac{Z_7}{Z_8}\left[\frac{2aa'}{\ell^2} s_\nu s_\zeta + \frac{2a'}{\ell}(-s_\nu s_\zeta c_\xi s_\eta\right.$$

$$\left. - c_\xi c_\eta s_\nu c_\zeta)\right]$$

$$Z_{20} = Z_3$$

$$Z_{21} = c_\nu c_\zeta$$

$$Z_{22} = -s_\nu c_\zeta Z_1$$

$$Z_{23} = -s_\nu c_\zeta Z_2$$

$$Z_{24} = -s_\nu c_\zeta Z_3 - c_\nu s_\zeta$$

$$Z_{25} = s_\nu$$

$$Z_{26} = c_\nu Z_1$$

$$Z_{27} = c_\nu Z_2$$

$$Z_{28} = c_\nu Z_3$$

$$Z_{29} = -c_\nu s_\zeta$$

$$Z_{30} = s_\nu s_\zeta Z_1$$

$$Z_{31} = s_\nu s_\zeta Z_2$$

$$Z_{32} = s_\nu s_\zeta Z_3 - c_\nu c_\zeta$$

$$Z_{33} = s_\nu$$

$$Z_{34} = c_\nu Z_1$$

$$Z_{35} = c_\nu Z_2$$

$$Z_{36} = c_\nu Z_3$$

$$Z_{37} = 1$$

$$Z_{38} = s_\delta s_\zeta - c_\delta c_\zeta s_\nu$$

$$Z_{39} = -c_\delta c_\zeta c_\nu Z_1$$

$$Z_{40} = -c_\delta c_\zeta c_\nu Z_2$$

$$Z_{41} = s_\delta c_\zeta + c_\delta s_\zeta s_\nu - c_\delta c_\zeta c_\nu Z_3$$

$$Z_{42} = c_\delta s_\zeta + s_\delta c_\zeta s_\nu$$

$$Z_{43} = c_\delta c_\nu$$

$$Z_{44} = -c_\delta s_\nu Z_1$$

$$Z_{45} = -c_\delta s_\nu Z_2$$

$$Z_{46} = -c_\delta s_\nu Z_3$$

$$Z_{47} = -s_\delta c_\nu$$

$$Z_{48} = s_\zeta c_\delta s_\nu + s_\delta c_\zeta$$

$$Z_{49} = s_\zeta c_\delta c_\nu Z_1$$

$$Z_{50} = s_\zeta c_\delta c_\nu Z_2$$

$$Z_{51} = c_\zeta c_\delta s_\nu + s_\zeta c_\delta c_\nu Z_3 - s_\delta s_\zeta$$

$$Z_{52} = -s_\zeta s_\delta s_\nu + c_\delta c_\zeta$$

$$Z_{53} = s_\delta(Z_9 + Z_2 Z_{11} + Z_1 Z_{16})$$

$$Z_{54} = s_\delta(Z_{10} + Z_1 Z_{11})$$

$$Z_{55} = s_\delta(Z_3 Z_{11} + Z_{12} + Z_1 Z_{18}) - c_\delta s_\nu Z_1$$

$$Z_{56} = s_\delta Z_{14}$$

$$Z_{57} = s_\delta(Z_{15} + Z_2 Z_{16})$$

$$Z_{58} = s_\delta(Z_{13} + Z_3 Z_{16} + Z_2 Z_{18}) - c_\delta s_\nu Z_2$$

$$Z_{59} = s_\delta Z_{17}$$

$$Z_{60} = s_\delta(Z_{19} + Z_3 Z_{18}) - c_\delta s_\nu Z_3$$

$$Z_{61} = s_\delta Z_{20} + c_\delta c_\nu$$

$$Z_{62} = c_\delta Z_1$$

$$Z_{63} = c_\delta Z_2$$

$$Z_{64} = c_\delta Z_3 - s_\delta c_\nu$$

$$Z_{65} = c_\delta s_\zeta + c_\zeta s_\delta s_\nu$$

$$Z_{66} = c_\zeta s_\delta c_\nu Z_1$$

$$Z_{67} = c_\zeta s_\delta c_\nu Z_2$$

$$Z_{68} = c_\delta c_\zeta - s_\zeta s_\delta s_\nu + c_\zeta s_\delta c_\nu Z_3$$

$$Z_{69} = -s_\delta s_\zeta + c_\zeta c_\delta s_\nu$$

$$Z_{70} = -s_\delta c_\nu$$

$$Z_{71} = s_\delta s_\nu Z_1$$

$$Z_{72} = s_\delta s_\nu Z_2$$

$$Z_{73} = s_\delta s_\nu Z_3$$

$$Z_{74} = -c_\delta c_\nu$$

$$Z_{75} = -s_\zeta s_\delta s_\nu + c_\delta c_\zeta$$

$$Z_{76} = -s_\zeta s_\delta c_\nu Z_1$$

$$Z_{77} = -s_\zeta s_\delta c_\nu Z_2$$

$$Z_{78} = -s_\zeta s_\delta c_\nu Z_3 - c_\zeta s_\delta s_\nu - c_\delta s_\zeta$$

$$Z_{79} = -s_\zeta c_\delta s_\nu - s_\delta c_\zeta$$

$$Z_{80} = c_\delta(Z_9 + Z_2 Z_{11} + Z_1 Z_{16})$$

$$Z_{81} = c_\delta(Z_{10} + Z_1 Z_{11})$$

$$Z_{82} = c_\delta(Z_3 Z_{11} + Z_{12} + Z_1 Z_{18}) + s_\delta s_\nu Z_1$$

$$Z_{83} = c_\delta Z_{14}$$

$$Z_{84} = c_\delta(Z_{15} + Z_2 Z_{16})$$

$$Z_{85} = c_\delta(Z_{13} + Z_3 Z_{16} + Z_2 Z_{18}) + s_\delta s_\nu Z_2$$

$$Z_{86} = c_\delta Z_{17}$$

$$Z_{87} = c_\delta(Z_{19} + Z_3 Z_{18}) + s_\delta s_\nu Z_3$$

$$Z_{88} = c_\delta Z_{20} - s_\delta c_\nu$$

$$Z_{89} = -s_\delta Z_1$$

$$Z_{90} = -s_\delta Z_2$$

$$Z_{91} = -s_\delta Z_3 - c_\delta c_\nu$$

$$Z_{92} = R + \ell c_\xi c_\eta - \frac{a'}{2} c_\nu s_\zeta - hs_\zeta s_\delta s_\nu + hc_\delta c_\zeta$$

$$Z_{93} = \ell s_\xi - \frac{a'}{2} s_\nu + hs_\delta c_\nu$$

$$Z_{94} = -Z_{93}$$

$$Z_{95} = Z_{92}$$

$$Z_{96} = \frac{a}{2} - \ell c_\xi s_\eta - \frac{a'}{2} c_\nu c_\zeta - hc_\delta s_\zeta - hc_\zeta s_\delta s_\nu$$

$$Z_{97} = Z_{96}$$

$$Z_{98} = -2\ell c_\xi s_\eta + (a' s_\nu s_\zeta - 2hs_\zeta s_\delta c_\nu) Z_1$$

$$Z_{99} = -2\ell s_\xi c_\eta + (a' s_\nu s_\zeta - 2hs_\zeta s_\delta c_\nu) Z_2$$

$$Z_{100} = -a' c_\nu c_\zeta - 2hc_\zeta s_\delta s_\nu - 2hc_\delta s_\zeta + (a' s_\nu s_\zeta - 2hs_\zeta s_\delta c_\nu) Z_3$$

$$Z_{101} = -2hs_\zeta c_\delta s_\nu - 2hs_\delta c_\zeta$$

$$Z_{102} = (-a' c_\nu - 2hs_\delta s_\nu) Z_1$$

$$Z_{103} = 2\ell c_\xi + (-a' c_\nu - 2hs_\delta s_\nu) Z_2$$

$$Z_{104} = (-a' c_\nu - 2hs_\delta s_\nu) Z_3$$

$$Z_{105} = 2hc_\delta c_\nu$$

$$Z_{106} = \ell c_\xi c_\eta + \left(- \frac{a'}{2} s_\nu c_\zeta + hs_\delta c_\nu c_\zeta\right) Z_{14}$$

$$Z_{107} = -\ell c_\xi s_\eta + \left(- \frac{a'}{2} s_\nu c_\zeta + hs_\delta c_\nu c_\zeta\right)(Z_{10} + Z_1 Z_{11}) + \left(- \frac{a'}{2} c_\nu c_\zeta - hs_\delta s_\nu c_\zeta\right) Z_1^2$$

$$Z_{108} = -2\ell s_\xi c_\eta + \left(- \frac{a'}{2} s_\nu c_\zeta + hs_\delta c_\nu c_\zeta\right)(Z_9 + Z_2 Z_{11} + Z_1 Z_{16}) + 2\left(- \frac{a'}{2} c_\nu c_\zeta - hs_\delta s_\nu c_\zeta\right) Z_1 Z_2$$

$$Z_{109} = -\ell s_\xi s_\eta + \left(- \frac{a'}{2} s_\nu c_\zeta + hs_\delta c_\nu c_\zeta\right) Z_{17}$$

$$Z_{110} = -\ell c_\xi s_\eta + \left(- \frac{a'}{2} s_\nu c_\zeta + hs_\delta c_\nu c_\zeta\right)(Z_{15} + Z_2 Z_{16}) + \left(- \frac{a'}{2} c_\nu c_\zeta - hs_\delta s_\nu c_\zeta\right) Z_2^2$$

$$Z_{111} = (a's_\nu s_\zeta - 2hs_\zeta s_\delta c_\nu)Z_1 + 2\left(-\frac{a'}{2} c_\nu c_\zeta - hs_\delta s_\nu c_\zeta\right)Z_1 Z_3 + \left(-\frac{a'}{2} s_\nu c_\zeta + hs_\delta c_\nu c_\zeta\right)(Z_3 Z_{11} + Z_{12} + Z_1 Z_{18})$$

$$Z_{112} = (a's_\nu s_\zeta - 2hs_\zeta s_\delta c_\nu)Z_2 + 2\left(-\frac{a'}{2} c_\nu c_\zeta - hs_\delta s_\nu c_\zeta\right)Z_2 Z_3 + \left(-\frac{a'}{2} s_\nu c_\zeta + hs_\delta c_\nu c_\zeta\right)(Z_{13} + Z_3 Z_{16} + Z_2 Z_{18})$$

$$Z_{113} = -\frac{a'}{2} c_\nu s_\zeta - hs_\zeta s_\delta s_\nu + hc_\delta c_\zeta + \left(-\frac{a'}{2} s_\nu c_\zeta + hs_\delta c_\nu c_\zeta\right)Z_{20}$$

$$Z_{114} = -\frac{a'}{2} c_\nu c_\zeta - hc_\zeta s_\delta s_\nu - hc_\delta s_\zeta + (a's_\nu s_\zeta - 2hs_\zeta s_\delta c_\nu)Z_3 + \left(-\frac{a'}{2} c_\nu c_\zeta - hs_\delta s_\nu c_\zeta\right)Z_3^2 + \left(-\frac{a'}{2} s_\nu s_\zeta - hc_\zeta s_\delta c_\nu\right)(Z_{19} + Z_3 Z_{18})$$

$$Z_{115} = 2hc_\nu c_\delta c_\zeta Z_1$$

$$Z_{116} = 2hc_\nu c_\zeta c_\delta Z_2$$

$$Z_{117} = -2hs_\zeta c_\delta s_\nu - 2hs_\delta c_\zeta + 2hc_\delta c_\nu c_\zeta Z_3$$

$$Z_{118} = -hs_\zeta c_\delta - hs_\nu s_\delta c_\zeta$$

$$Z_{119} = -hs_\zeta s_\delta + hs_\nu c_\delta c_\zeta$$

$$Z_{120} = -Z_{92}$$

$$Z_{121} = -Z_{96}$$

$$Z_{122} = -Z_{96}$$

$$Z_{123} = Z_{92}$$

$$Z_{124} = Z_{93}$$

$$Z_{125} = Z_{93}$$

$$Z_{126} = -Z_{98}$$

$$Z_{127} = -Z_{99}$$

$$Z_{128} = -Z_{100}$$

$$Z_{129} = -Z_{101}$$

$$Z_{130} = 2\ell c_\xi c_\eta + (-a's_\nu c_\zeta + 2hc_\zeta s_\delta c_\nu)Z_1$$

$$Z_{131} = -2\ell s_\xi s_\eta + (-a's_\nu c_\zeta + 2hs_\delta c_\nu c_\zeta)Z_2$$

$$Z_{132} = -a'c_\nu s_\zeta + 2hc_\delta c_\zeta - 2hs_\zeta s_\delta s_\nu + (-a's_\nu c_\zeta + 2hs_\delta c_\nu c_\zeta)Z_3$$

$$Z_{133} = 2Z_{119}$$

$$Z_{134} = \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)Z_{14}$$

$$Z_{135} = \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)(Z_{10} + Z_1Z_{11}) + \left(-\frac{a'}{2} s_\nu + hs_\delta c_\nu\right)Z_1^2$$

$$Z_{136} = \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)(Z_9 + Z_2Z_{11} + Z_1Z_{16}) + \left(-\frac{a'}{2} s_\nu + hs_\delta c_\nu\right)(2Z_1Z_2)$$

$$Z_{137} = -\ell c_\xi + \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)Z_{17}$$

$$Z_{138} = \ell s_\xi + \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)(Z_{15} + Z_2Z_{16}) + \left(-\frac{a'}{2} s_\nu + hs_\delta c_\nu\right)Z_2^2$$

$$Z_{139} = \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)(Z_3Z_{11} + Z_{12} + Z_1Z_{18}) + \left(-\frac{a'}{2} s_\nu + hs_\delta c_\nu\right)(2Z_1Z_3)$$

$$Z_{140} = \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)(Z_{13} + Z_3Z_{16} + Z_2Z_{18}) + \left(-\frac{a'}{2} s_\nu + hs_\delta c_\nu\right)(2Z_2Z_3)$$

$$Z_{141} = \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)Z_{20}$$

$$Z_{142} = \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)(Z_{19} + Z_3Z_{18}) + \left(-\frac{a'}{2} s_\nu + hs_\delta c_\nu\right)Z_3^2$$

$$Z_{143} = 2hc_\delta s_\nu Z_1$$

$$Z_{144} = 2hc_\delta s_\nu Z_2$$

$$Z_{145} = 2hc_\delta s_\nu Z_3$$

$$Z_{146} = hs_\delta c_\nu$$

$$Z_{147} = -hc_\delta c_\nu$$

$$Z_{148} = -Z_{93}$$

$$Z_{149} = Z_{96}$$

$$Z_{150} = -Z_{96}$$

$$Z_{151} = -Z_{93}$$

$$Z_{152} = -Z_{92}$$

$$Z_{153} = -Z_{92}$$

$$Z_{154} = -Z_{102}$$

$$Z_{155} = -Z_{103}$$

$$Z_{156} = -Z_{104}$$

$$Z_{157} = -Z_{105}$$

$$Z_{158} = -2\ell c_\xi c_\eta + (a's_\nu c_\zeta - 2hc_\zeta s_\delta c_\nu)Z_1$$

$$Z_{159} = 2\ell s_\xi s_\eta + (a's_\nu c_\zeta - 2hc_\zeta s_\delta c_\nu)Z_2$$

$$Z_{160} = a'c_\nu s_\zeta - 2hc_\delta c_\zeta + 2hs_\zeta s_\delta s_\nu + (a's_\nu c_\zeta - 2hc_\zeta s_\delta c_\nu)Z_3$$

$$Z_{161} = -2Z_{119}$$

$$Z_{162} = -\ell c_\xi s_\eta + \left(\frac{a'}{2} s_\nu s_\zeta - hs_\delta c_\nu s_\zeta\right)Z_{14}$$

$$Z_{163} = -\ell c_\xi c_\eta + \left(\frac{a'}{2} s_\nu s_\zeta - hs_\delta c_\nu s_\zeta\right)(Z_{10} + Z_1 Z_{11}) + \left(\frac{a'}{2} c_\nu s_\zeta + hs_\delta s_\nu s_\zeta\right)Z_1^2$$

$$Z_{164} = 2\ell s_\xi s_\eta + \left(\frac{a'}{2} s_\nu s_\zeta - hs_\delta c_\nu s_\zeta\right)(Z_9 + Z_2 Z_{11} + Z_1 Z_{16}) + \left(\frac{a'}{2} c_\nu s_\zeta + hs_\delta s_\nu s_\zeta\right)(2Z_1 Z_2)$$

$$Z_{165} = -\ell s_\xi c_\eta + \left(\frac{a'}{2} s_\nu s_\zeta - hs_\delta c_\nu s_\zeta\right)Z_{17}$$

$$Z_{166} = -\ell c_\xi c_\eta + \left(\frac{a'}{2} s_\nu s_\zeta - hs_\delta c_\nu s_\zeta\right)(Z_{15} + Z_2 Z_{16}) + \left(\frac{a'}{2} c_\nu s_\zeta + hs_\delta s_\nu s_\zeta\right)Z_2^2$$

$$Z_{167} = (a's_\nu c_\zeta - 2hs_\delta c_\nu c_\zeta)Z_1 + (a'c_\nu s_\zeta + 2hs_\delta s_\nu s_\zeta)Z_1 Z_3 + \left(\frac{a'}{2} s_\nu s_\zeta - hs_\delta c_\nu s_\zeta\right)(Z_{12} + Z_3 Z_{11} + Z_1 Z_{18})$$

$$Z_{168} = (a's_\nu c_\zeta - 2hs_\delta c_\nu c_\zeta)Z_2 + (a'c_\nu s_\zeta + 2hs_\delta s_\nu s_\zeta)Z_2 Z_3 + \left(\frac{a'}{2} s_\nu s_\zeta - hs_\delta c_\nu s_\zeta\right)(Z_{13} + Z_2 Z_{18} + Z_3 Z_{16})$$

$$Z_{169} = \frac{-a'}{2} c_\nu c_\zeta - hc_\delta s_\zeta - hc_\zeta s_\delta s_\nu + \left(\frac{a'}{2} s_\nu s_\zeta - hs_\delta c_\nu s_\zeta\right)Z_{20}$$

$$Z_{170} = \frac{a'}{2} c_\nu s_\zeta - hc_\delta c_\zeta + hs_\zeta s_\delta s_\nu + (a's_\nu c_\zeta - 2hc_\zeta s_\delta c_\nu)Z_3 + \left(\frac{a'}{2} s_\nu s_\zeta - hs_\delta c_\nu s_\zeta\right)(Z_{19} + Z_3 Z_{18}) + \left(\frac{a'}{2} c_\nu s_\zeta + hs_\delta s_\nu s_\zeta\right)Z_3^2$$

$$Z_{171} = -2hc_\delta c_\nu s_\zeta Z_1$$

$$Z_{172} = -2hc_\delta c_\nu s_\zeta Z_2$$

$$Z_{173} = -2hc_\delta c_\nu s_\zeta Z_3 - 2hc_\zeta c_\delta s_\nu + 2hs_\delta s_\zeta$$

$$Z_{174} = -hc_\delta c_\zeta + hs_\delta s_\zeta s_\nu$$

$$Z_{175} = -hs_\delta c_\zeta - hc_\delta s_\zeta s_\nu$$

$$Z_{176} = c_\nu c_\zeta E_1 + (-c_\delta s_\zeta - c_\zeta s_\delta s_\nu)E_{13}$$

$$Z_{177} = s_\nu E_1 + s_\delta c_\nu E_{13}$$

$$Z_{178} = -c_\nu s_\zeta E_1 + (s_\zeta s_\delta s_\nu - c_\delta c_\zeta)E_{13}$$

$$Z_{179} = s_\nu E_1 + (s_\delta c_\nu - c_\delta Z_3)E_{13}$$

$$Z_{180} = -c_\delta Z_1 E_{13}$$

$$Z_{181} = -c_\delta Z_2 E_{13}$$

$$Z_{182} = E_1$$

$$Z_{183} = (s_\delta s_\zeta - c_\delta c_\zeta s_\nu)E_2$$

$$Z_{184} = c_\delta c_\nu E_2$$

$Z_{185} = (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)E_2$

$Z_{186} = (c_\delta c_\nu + s_\delta Z_3)E_2$

$Z_{187} = s_\delta Z_1 E_2$

$Z_{188} = s_\delta Z_2 E_2$

$Z_{189} = (c_\delta s_\zeta + c_\zeta s_\delta s_\nu)E_3 - c_\nu c_\zeta E_{13}$

$Z_{190} = -s_\delta c_\nu E_3 - s_\nu E_{13}$

$Z_{191} = (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta)E_3 + c_\nu s_\zeta E_{13}$

$Z_{192} = (-s_\delta c_\nu + c_\delta Z_3)E_3 - s_\nu E_{13}$

$Z_{193} = c_\delta Z_1 E_3$

$Z_{194} = c_\delta Z_2 E_3$

$Z_{195} = -E_{13}$

$Z_{196} = (c_\delta s_\zeta + c_\zeta s_\delta s_\nu)Z_{183} + (-s_\delta s_\zeta + c_\delta c_\zeta s_\nu)Z_{189}$

$Z_{197} = -s_\delta c_\nu Z_{183} + (c_\delta s_\zeta + c_\zeta s_\delta s_\nu)Z_{184} - c_\delta c_\nu Z_{189} + (-s_\delta s_\zeta + c_\delta c_\zeta s_\nu)Z_{190}$

$Z_{198} = (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta)Z_{183} + (c_\delta s_\zeta + c_\zeta s_\delta s_\nu)Z_{185} + (-s_\zeta c_\delta s_\nu - s_\delta c_\zeta)Z_{189} + (-s_\delta s_\zeta$
$+ c_\delta c_\zeta s_\nu)Z_{191}$

$Z_{199} = c_\delta Z_1 Z_{183} + (c_\delta s_\zeta + c_\zeta s_\delta s_\nu)Z_{187} - S_\delta Z_1 Z_{189} + (-s_\delta s_\zeta + c_\delta c_\zeta s_\nu)Z_{193}$

$Z_{200} = c_\delta Z_2 Z_{183} + (c_\delta s_\zeta + c_\zeta s_\delta s_\nu)Z_{188} - s_\delta Z_2 Z_{189} + (-s_\delta s_\zeta + c_\delta c_\zeta s_\nu)Z_{194}$

$Z_{201} = (-s_\delta c_\nu + c_\delta Z_3)Z_{183} + (c_\delta s_\zeta + c_\zeta s_\delta s_\nu)Z_{186} + (-s_\delta Z_3 - c_\delta c_\nu)Z_{189} + (-s_\delta s_\zeta$
$+ c_\delta c_\zeta s_\nu)Z_{192}$

$Z_{202} = (-s_\delta s_\zeta + c_\delta c_\zeta s_\nu)Z_{195}$

$Z_{203} = -s_\delta c_\nu Z_{184} - c_\delta c_\nu Z_{190}$

$Z_{204} = (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta)Z_{184} - s_\delta c_\nu Z_{185} + (-s_\zeta c_\delta s_\nu - s_\delta c_\zeta)Z_{190} - c_\delta c_\nu Z_{191}$

$Z_{205} = c_\delta Z_1 Z_{184} - s_\delta c_\nu Z_{187} - s_\delta Z_1 Z_{190} - c_\delta c_\nu Z_{193}$

$Z_{206} = c_\delta Z_2 Z_{184} - s_\delta c_\nu Z_{188} - s_\delta Z_2 Z_{190} - c_\delta c_\nu Z_{194}$

$Z_{207} = (c_\delta Z_3 - s_\delta c_\nu)Z_{184} - s_\delta c_\nu Z_{186} + (-s_\delta Z_3 - c_\delta c_\nu)Z_{190} - c_\delta c_\nu Z_{192}$

$Z_{208} = -c_\delta c_\nu Z_{195}$

$Z_{209} = (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta)Z_{185} + (-s_\zeta c_\delta s_\nu - s_\delta c_\zeta)Z_{191}$

$$Z_{210} = c_\delta Z_1 Z_{185} + (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta)Z_{187} - s_\delta Z_1 Z_{191} + (-s_\zeta c_\delta s_\nu - s_\delta c_\zeta)Z_{193}$$

$$Z_{211} = c_\delta Z_2 Z_{185} + (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta)Z_{188} - s_\delta Z_2 Z_{191} + (-s_\zeta c_\delta s_\nu - s_\delta c_\zeta)Z_{194}$$

$$Z_{212} = (-s_\delta c_\nu + c_\delta Z_3)Z_{185} + (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta)Z_{186} + (-c_\delta c_\nu - s_\delta Z_3)Z_{191} + (-s_\zeta c_\delta s_\nu - s_\delta c_\zeta)Z_{192}$$

$$Z_{213} = (-s_\zeta c_\delta s_\nu - s_\delta c_\zeta)Z_{195}$$

$$Z_{214} = (-s_\delta c_\nu + c_\delta Z_3)Z_{186} + (-c_\delta c_\nu - s_\delta Z_3)Z_{192}$$

$$Z_{215} = c_\delta Z_1 Z_{186} + (c_\delta Z_3 - c_\nu s_\delta)Z_{187} - s_\delta Z_1 Z_{192} + (-c_\delta c_\nu - s_\delta Z_3)Z_{193}$$

$$Z_{216} = c_\delta Z_2 Z_{186} + (c_\delta Z_3 - s_\delta c_\nu)Z_{188} - s_\delta Z_2 Z_{192} + (-c_\delta c_\nu - s_\delta Z_3)Z_{194}$$

$$Z_{217} = (-c_\delta c_\nu - s_\delta Z_3)Z_{195}$$

$$Z_{218} = c_\delta Z_1 Z_{187} - s_\delta Z_1 Z_{193}$$

$$Z_{219} = c_\delta Z_2 Z_{187} + c_\delta Z_1 Z_{188} - s_\delta Z_2 Z_{193} - s_\delta Z_1 Z_{194}$$

$$Z_{220} = -s_\delta Z_1 Z_{195}$$

$$Z_{221} = -s_\delta Z_2 Z_{195}$$

$$Z_{222} = c_\delta Z_2 Z_{188} - s_\delta Z_2 Z_{194}$$

$$Z_{223} = (-c_\delta s_\zeta - c_\zeta s_\delta s_\nu)Z_{176} + c_\nu c_\zeta Z_{189}$$

$$Z_{224} = s_\delta c_\nu Z_{176} + (-c_\delta s_\zeta - c_\zeta s_\delta s_\nu)Z_{177} + s_\nu Z_{189} + c_\nu c_\zeta Z_{190}$$

$$Z_{225} = (s_\zeta s_\delta s_\nu - c_\delta c_\zeta)Z_{176} + (-c_\delta s_\zeta - c_\zeta s_\delta s_\nu)Z_{178} - c_\nu s_\zeta Z_{189} + c_\nu c_\zeta Z_{191}$$

$$Z_{226} = -c_\delta Z_1 Z_{176} + (-c_\delta s_\zeta - c_\zeta s_\delta s_\nu)Z_{180} + c_\nu c_\zeta Z_{193}$$

$$Z_{227} = -c_\delta Z_2 Z_{176} + (-c_\delta s_\zeta - c_\zeta s_\delta s_\nu)Z_{181} + c_\nu c_\zeta Z_{194}$$

$$Z_{228} = (s_\delta c_\nu - c_\delta Z_3)Z_{176} + (-c_\delta s_\zeta - c_\zeta s_\delta s_\nu)Z_{179} + s_\nu Z_{189} + c_\nu c_\zeta Z_{192}$$

$$Z_{229} = (-c_\delta s_\zeta - c_\zeta s_\delta s_\nu)Z_{182} + Z_{189} + c_\nu c_\zeta Z_{195}$$

$$Z_{230} = s_\delta c_\nu Z_{177} + s_\nu Z_{190}$$

$$Z_{231} = (s_\zeta s_\delta s_\nu - c_\delta c_\zeta)Z_{177} + s_\delta c_\nu Z_{178} - c_\nu s_\zeta Z_{190} - s_\nu Z_{191}$$

$$Z_{232} = -c_\delta Z_1 Z_{177} + s_\delta c_\nu Z_{180} + s_\nu Z_{193}$$

$$Z_{233} = -c_\delta Z_2 Z_{177} + s_\delta c_\nu Z_{181} + s_\nu Z_{194}$$

$$Z_{234} = (-c_\delta Z_3 + s_\delta c_\nu)Z_{177} + s_\delta c_\nu Z_{179} + s_\nu Z_{190} + s_\nu Z_{192}$$

$$Z_{235} = s_\delta c_\nu Z_{182} + Z_{190} + s_\nu Z_{195}$$

$$Z_{236} = (s_\zeta s_\delta s_\nu - c_\delta c_\zeta)Z_{178} - c_\nu s_\zeta Z_{191}$$

$$Z_{237} = -c_\delta Z_1 Z_{178} + (s_\zeta s_\delta s_\nu - c_\delta c_\zeta)Z_{180} - c_\nu s_\zeta Z_{193}$$

$$Z_{238} = -c_\delta Z_2 Z_{178} + (s_\zeta s_\delta s_\nu - c_\delta c_\zeta)Z_{181} - c_\nu s_\zeta Z_{194}$$

$$Z_{239} = (s_\delta c_\nu - c_\delta Z_3)Z_{178} + (s_\zeta s_\delta s_\nu - c_\delta c_\zeta)Z_{179} + s_\nu Z_{191} - c_\nu s_\zeta Z_{192}$$

$$Z_{240} = (s_\zeta s_\delta s_\nu - c_\delta c_\zeta)Z_{182} + Z_{191} - c_\nu s_\zeta Z_{195}$$

$$Z_{241} = (s_\delta c_\nu - c_\delta Z_3)Z_{179} + s_\nu Z_{192}$$

$$Z_{242} = -c_\delta Z_1 Z_{179} + (s_\delta c_\nu - c_\delta Z_3)Z_{180} + s_\nu Z_{193}$$

$$Z_{243} = -c_\delta Z_2 Z_{179} + (s_\delta c_\nu - c_\delta Z_3)Z_{181} + s_\nu Z_{194}$$

$$Z_{244} = (s_\delta c_\nu - c_\delta Z_3)Z_{182} + Z_{192} + s_\nu Z_{195}$$

$$Z_{245} = -c_\delta Z_1 Z_{180}$$

$$Z_{246} = -c_\delta Z_2 Z_{180} - c_\delta Z_1 Z_{181}$$

$$Z_{247} = -c_\delta Z_1 Z_{182} + Z_{193}$$

$$Z_{248} = -c_\delta Z_2 Z_{182} + Z_{194}$$

$$Z_{249} = -c_\delta Z_2 Z_{181}$$

$$Z_{250} = Z_{195}$$

$$Z_{251} = (s_\delta s_\zeta - c_\delta c_\zeta s_\nu)Z_{176} - c_\nu c_\zeta Z_{183}$$

$$Z_{252} = c_\delta c_\nu Z_{176} + (s_\delta s_\zeta - c_\delta c_\zeta s_\nu)Z_{177} - s_\nu Z_{183} - c_\nu c_\zeta Z_{184}$$

$$Z_{253} = (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)Z_{176} + (s_\delta s_\zeta - c_\delta c_\zeta s_\nu)Z_{178} + c_\nu s_\zeta Z_{183} - c_\nu c_\zeta Z_{185}$$

$$Z_{254} = s_\delta Z_1 Z_{176} + (s_\delta s_\zeta - c_\delta c_\zeta s_\nu)Z_{180} - c_\nu c_\zeta Z_{187}$$

$$Z_{255} = s_\delta Z_2 Z_{176} + (s_\delta s_\zeta - c_\delta c_\zeta s_\nu)Z_{181} - c_\nu c_\zeta Z_{188}$$

$$Z_{256} = (c_\delta c_\nu + s_\delta Z_3)Z_{176} + (s_\delta s_\zeta - c_\delta c_\zeta s_\nu)Z_{179} - s_\nu Z_{183} - c_\nu c_\zeta Z_{186}$$

$$Z_{257} = (s_\delta s_\zeta - c_\delta c_\zeta s_\nu)Z_{182} - Z_{183}$$

$$Z_{258} = c_\delta c_\nu Z_{177} - s_\nu Z_{184}$$

$$Z_{259} = (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)Z_{177} + c_\delta c_\nu Z_{178} + c_\nu s_\zeta Z_{184} - s_\nu Z_{185}$$

$$Z_{260} = s_\delta Z_1 Z_{177} + c_\delta c_\nu Z_{180} - s_\nu Z_{187}$$

$$Z_{261} = s_\delta Z_2 Z_{177} + c_\delta c_\nu Z_{181} - s_\nu Z_{188}$$

$$Z_{262} = (c_\delta c_\nu + s_\delta Z_3)Z_{177} + c_\delta c_\nu Z_{179} - s_\nu Z_{184} - s_\nu Z_{186}$$

$$Z_{263} = c_\delta c_\nu Z_{182} - Z_{184}$$

$$Z_{264} = (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)Z_{178} + c_\nu s_\zeta Z_{185}$$

$$Z_{265} = s_\delta Z_1 Z_{178} + (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)Z_{180} + c_\nu s_\zeta Z_{187}$$

$$Z_{266} = s_\delta Z_2 Z_{178} + (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)Z_{181} + c_\nu s_\zeta Z_{188}$$

$$Z_{267} = (s_\delta Z_3 + c_\delta c_\nu)Z_{178} + (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)Z_{179} - s_\nu Z_{185} + c_\nu s_\zeta Z_{186}$$

$$Z_{268} = (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)Z_{182} - Z_{185}$$

$$Z_{269} = (c_\delta c_\nu + s_\delta Z_3)Z_{179} - s_\nu Z_{186}$$

$$Z_{270} = s_\delta Z_1 Z_{179} + (c_\delta c_\nu + s_\delta Z_3)Z_{180} - s_\nu Z_{187}$$

$$Z_{271} = s_\delta Z_2 Z_{179} + (c_\delta c_\nu + s_\delta Z_3)Z_{181} - s_\nu Z_{188}$$

$$Z_{272} = (c_\delta c_\nu + s_\delta Z_3)Z_{182} - Z_{186}$$

$$Z_{273} = s_\delta Z_1 Z_{180}$$

$$Z_{274} = s_\delta Z_2 Z_{180} + s_\delta Z_1 Z_{181}$$

$$Z_{275} = s_\delta Z_1 Z_{182} - Z_{187}$$

$$Z_{276} = s_\delta Z_2 Z_{182} - Z_{188}$$

$$Z_{277} = s_\delta Z_2 Z_{181}$$

$$Z_{278} = -E_1 Z_{21} + E_{13} Z_{65}$$

$$Z_{279} = Z_{196}$$

$$Z_{280} = Z_{197}$$

$$Z_{281} = Z_{198}$$

$$Z_{282} = -E_1 Z_{22} + E_{13} Z_{66} + Z_{199}$$

$$Z_{283} = -E_1 Z_{23} + E_{13} Z_{67} + Z_{200}$$

$$Z_{284} = -E_1 Z_{24} + E_{13} Z_{68} + Z_{201}$$

$$Z_{285} = E_{13} Z_{69} + Z_{202}$$

$$Z_{286} = -E_1 Z_{25} + E_{13} Z_{70}$$

$$Z_{287} = Z_{203}$$

$$Z_{288} = Z_{204}$$

$$Z_{289} = -E_1 Z_{26} + E_{13} Z_{71} + Z_{205}$$

$$Z_{290} = -E_1Z_{27} + E_{13}Z_{72} + Z_{206}$$

$$Z_{291} = -E_1Z_{28} + E_{13}Z_{73} + Z_{207}$$

$$Z_{292} = E_{13}Z_{74} + Z_{208}$$

$$Z_{293} = -E_1Z_{29} + E_{13}Z_{75}$$

$$Z_{294} = Z_{209}$$

$$Z_{295} = -E_1Z_{30} + E_{13}Z_{76} + Z_{210}$$

$$Z_{296} = -E_1Z_{31} + E_{13}Z_{77} + Z_{211}$$

$$Z_{297} = -E_1Z_{32} + E_{13}Z_{78} + Z_{212}$$

$$Z_{298} = E_{13}Z_{79} + Z_{213}$$

$$Z_{299} = E_{13}Z_{80} + Z_{219}$$

$$Z_{300} = -E_1Z_{34} + E_{13}Z_{82} + Z_{215}$$

$$Z_{301} = E_{13}Z_{81} + Z_{218}$$

$$Z_{302} = E_{13}Z_{83}$$

$$Z_{303} = E_{13}Z_{84}Z_{222}$$

$$Z_{304} = -E_1Z_{35} + E_{13}Z_{85} + Z_{216}$$

$$Z_{305} = E_{13}Z_{86}$$

$$Z_{306} = -E_1Z_{36} + E_{13}Z_{87} + Z_{214}$$

$$Z_{307} = -E_1Z_{33} + E_{13}Z_{88}$$

$$Z_{308} = E_{13}Z_{89} + Z_{220}$$

$$Z_{309} = E_{13}Z_{90} + Z_{221}$$

$$Z_{310} = E_{13}Z_{91} + Z_{217}$$

$$Z_{311} = -E_1Z_{37}$$

$$Z_{312} = -E_2Z_{38}$$

$$Z_{313} = Z_{223}$$

$$Z_{314} = Z_{224}$$

$$Z_{315} = Z_{225}$$

$$Z_{316} = -E_2Z_{39} + Z_{226}$$

$Z_{317} = -E_2 Z_{40} + Z_{227}$

$Z_{318} = -E_2 Z_{41} + Z_{228}$

$Z_{319} = -E_2 Z_{42} + Z_{229}$

$Z_{320} = -E_2 Z_{43}$

$Z_{321} = Z_{230}$

$Z_{322} = Z_{231}$

$Z_{323} = -E_2 Z_{44} + Z_{232}$

$Z_{324} = -E_2 Z_{45} + Z_{233}$

$Z_{325} = -E_2 Z_{46} + Z_{234}$

$Z_{326} = -E_2 Z_{47} + Z_{235}$

$Z_{327} = -E_2 Z_{48}$

$Z_{328} = Z_{236}$

$Z_{329} = -E_2 Z_{49} + Z_{237}$

$Z_{330} = -E_2 Z_{50} + Z_{238}$

$Z_{331} = -E_2 Z_{51} + Z_{239}$

$Z_{332} = -E_2 Z_{52} + Z_{240}$

$Z_{333} = -E_2 Z_{53} + Z_{246}$

$Z_{334} = -E_2 Z_{55} + Z_{242}$

$Z_{335} = -E_2 Z_{54} + Z_{245}$

$Z_{336} = -E_2 Z_{56}$

$Z_{337} = -E_2 Z_{57} + Z_{249}$

$Z_{338} = -E_2 Z_{58} + Z_{243}$

$Z_{339} = -E_2 Z_{59}$

$Z_{340} = -E_2 Z_{60} + Z_{241}$

$Z_{341} = -E_2 Z_{61}$

$Z_{342} = -E_2 Z_{62} + Z_{247}$

$Z_{343} = -E_2 Z_{63} + Z_{248}$

$Z_{344} = -E_2Z_{64} + Z_{244}$

$Z_{345} = Z_{250}$

$Z_{346} = -E_3Z_{65} + E_{13}Z_{21}$

$Z_{347} = Z_{251}$

$Z_{348} = Z_{252}$

$Z_{349} = Z_{253}$

$Z_{350} = -E_3Z_{66} + E_{13}Z_{22} + Z_{254}$

$Z_{351} = -E_3Z_{67} + E_{13}Z_{23} + Z_{255}$

$Z_{352} = -E_3Z_{68} + E_{13}Z_{24} + Z_{256}$

$Z_{353} = -E_3Z_{69} + Z_{257}$

$Z_{354} = -E_3Z_{70} + E_{13}Z_{25}$

$Z_{355} = Z_{258}$

$Z_{356} = Z_{259}$

$Z_{357} = -E_3Z_{71} + E_{13}Z_{26} + Z_{260}$

$Z_{358} = -E_3Z_{72} + E_{13}Z_{27} + Z_{261}$

$Z_{359} = -E_3Z_{73} + E_{13}Z_{28} + Z_{262}$

$Z_{360} = -E_3Z_{74} + Z_{263}$

$Z_{361} = -E_3Z_{75} + E_{13}Z_{29}$

$Z_{362} = Z_{264}$

$Z_{363} = -E_3Z_{76} + E_{13}Z_{30} + Z_{265}$

$Z_{364} = -E_3Z_{77} + E_{13}Z_{31} + Z_{266}$

$Z_{365} = -E_3Z_{78} + E_{13}Z_{32} + Z_{267}$

$Z_{366} = -E_3Z_{79} + Z_{268}$

$Z_{367} = -E_3Z_{80} + Z_{274}$

$Z_{368} = -E_3Z_{82} + E_{13}Z_{34} + Z_{270}$

$Z_{369} = -E_3Z_{81} + Z_{273}$

$Z_{370} = -E_3Z_{83}$

$$Z_{371} = -E_3Z_{84} + Z_{277}$$

$$Z_{372} = -E_3Z_{85} + E_{13}Z_{35} + Z_{271}$$

$$Z_{373} = -E_3Z_{86}$$

$$Z_{374} = -E_3Z_{87} + E_{13}Z_{36} + Z_{269}$$

$$Z_{375} = -E_3Z_{88} + E_{13}Z_{33}$$

$$Z_{376} = -E_3Z_{89} + Z_{275}$$

$$Z_{377} = -E_3Z_{90} + Z_{276}$$

$$Z_{378} = -E_3Z_{91} + Z_{272}$$

$$Z_{379} = E_{13}Z_{37}$$

$$Z_{380} = -R - \ell c_\xi c_\eta + \frac{a'}{2} c_\nu s_\zeta + hs_\zeta s_\delta s_\nu - hc_\delta c_\zeta$$

$$Z_{381} = -\ell S_\xi + \frac{a'}{2} s_\nu - hs_\delta c_\nu$$

$$Z_{382} = s_\nu Z_{380} - c_\nu s_\zeta Z_{381}$$

$$Z_{383} = c_\delta c_\nu Z_{380} + (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)Z_{381}$$

$$Z_{384} = -s_\delta c_\nu Z_{380} + (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta)Z_{381}$$

$$Z_{385} = c_\theta s_\phi Z_{380} + c_\theta c_\phi Z_{381}$$

$$Z_{386} = c_\nu c_\zeta$$

$$Z_{387} = s_\delta s_\zeta - c_\delta c_\zeta s_\nu$$

$$Z_{388} = c_\delta s_\zeta + c_\zeta s_\delta s_\nu$$

$$Z_{389} = 0$$

$$Z_{390} = -mZ_{380}$$

$$Z_{391} = -mZ_{381}$$

$$Z_{392} = 0$$

$$Z_{393} = 0$$

$$Z_{394} = -mZ_{380}$$

$$Z_{395} = mZ_{380}$$

$$Z_{396} = mZ_{381}$$

$$Z_{397} = -mZ_{381}$$

$$Z_{398} = -mZ_{380}Z_{120} - mZ_{381}Z_{148} + Z_{386}Z_{278} + Z_{387}Z_{312} + Z_{388}Z_{346}$$

$$Z_{399} = -mZ_{381}Z_{149} + Z_{386}Z_{286} + Z_{387}Z_{320} + Z_{388}Z_{354}$$

$$Z_{400} = -mZ_{380}Z_{121} + Z_{386}Z_{291} + Z_{387}Z_{327} + Z_{388}Z_{361}$$

$$Z_{401} = -mZ_{380}Z_{122} + Z_{386}Z_{280} + Z_{387}Z_{314} + Z_{388}Z_{348}$$

$$Z_{402} = -mZ_{381}Z_{150} + Z_{386}Z_{281} + Z_{387}Z_{315} + Z_{388}Z_{349}$$

$$Z_{403} = -mZ_{380}Z_{123} - mZ_{381}Z_{151} + Z_{386}Z_{288} + Z_{387}Z_{322} + Z_{388}Z_{356}$$

$$Z_{404} = -mZ_{380}Z_{124} - mZ_{381}Z_{152} + Z_{386}Z_{279} + Z_{387}Z_{313} + Z_{388}Z_{347}$$

$$Z_{405} = -mZ_{381}Z_{153} + Z_{386}Z_{287} + Z_{387}Z_{321} + Z_{388}Z_{355}$$

$$Z_{406} = -mZ_{380}Z_{125} + Z_{386}Z_{293} + Z_{387}Z_{328} + Z_{388}Z_{362}$$

$$Z_{407} = -mZ_{380}Z_{126} - mZ_{381}Z_{154} + Z_{386}Z_{282} + Z_{387}Z_{316} + Z_{388}Z_{350}$$

$$Z_{408} = -mZ_{380}Z_{127} - mZ_{381}Z_{155} + Z_{386}Z_{283} + Z_{387}Z_{317} + Z_{388}Z_{351}$$

$$Z_{409} = -mZ_{380}Z_{128} - mZ_{381}Z_{156} + Z_{386}Z_{284} + Z_{387}Z_{318} + Z_{388}Z_{352}$$

$$Z_{410} = -mZ_{380}Z_{129} - mZ_{381}Z_{157} + Z_{386}Z_{285} + Z_{387}Z_{319} + Z_{388}Z_{353}$$

$$Z_{411} = -mZ_{381}Z_{158} + Z_{386}Z_{289} + Z_{387}Z_{323} + Z_{388}Z_{357}$$

$$Z_{412} = -mZ_{381}Z_{159} + Z_{386}Z_{311} + Z_{387}Z_{324} + Z_{388}Z_{358}$$

$$Z_{413} = -mZ_{381}Z_{160} + Z_{386}Z_{290} + Z_{387}Z_{325} + Z_{388}Z_{359}$$

$$Z_{414} = -mZ_{381}Z_{161} + Z_{386}Z_{292} + Z_{387}Z_{326} + Z_{388}Z_{360}$$

$$Z_{415} = -mZ_{380}Z_{130} + Z_{386}Z_{294} + Z_{387}Z_{329} + Z_{388}Z_{363}$$

$$Z_{416} = -mZ_{380}Z_{131} + Z_{386}Z_{295} + Z_{387}Z_{330} + Z_{388}Z_{364}$$

$$Z_{417} = -mZ_{380}Z_{132} + Z_{386}Z_{296} + Z_{387}Z_{331} + Z_{388}Z_{365}$$

$$Z_{418} = -mZ_{380}Z_{133} + Z_{386}Z_{298} + Z_{387}Z_{332} + Z_{388}Z_{366}$$

$$Z_{419} = -mZ_{380}Z_{134} - mZ_{381}Z_{162} + Z_{386}Z_{302} + Z_{387}Z_{336} + Z_{388}Z_{370}$$

$$Z_{420} = -mZ_{380}Z_{135} - mZ_{381}Z_{163} + Z_{386}Z_{304} + Z_{387}Z_{335} + Z_{388}Z_{369}$$

$$Z_{421} = -mZ_{380}Z_{136} - mZ_{381}Z_{164} + Z_{386}Z_{303} + Z_{387}Z_{333} + Z_{388}Z_{367}$$

$$Z_{422} = -mZ_{380}Z_{139} - mZ_{381}Z_{167} + Z_{386}Z_{306} + Z_{387}Z_{334} + Z_{388}Z_{368}$$

$$Z_{423} = -mZ_{380}Z_{143} - mZ_{381}Z_{171} + Z_{386}Z_{308} + Z_{387}Z_{342} + Z_{388}Z_{376}$$

$$Z_{424} = -mZ_{380}Z_{137} - mZ_{381}Z_{165} + Z_{386}Z_{305} + Z_{387}Z_{339} + Z_{388}Z_{373}$$

$$Z_{425} = -mZ_{380}Z_{138} - mZ_{381}Z_{166} + Z_{386}Z_{301} + Z_{387}Z_{337} + Z_{388}Z_{371}$$

$$Z_{426} = -mZ_{380}Z_{140} - mZ_{381}Z_{168} + Z_{386}Z_{300} + Z_{387}Z_{338} + Z_{388}Z_{372}$$

$$Z_{427} = -mZ_{380}Z_{144} - mZ_{381}Z_{172} + Z_{386}Z_{309} + Z_{387}Z_{343} + Z_{388}Z_{377}$$

$$Z_{428} = -mZ_{380}Z_{141} - mZ_{381}Z_{169} + Z_{386}Z_{297} + Z_{387}Z_{341} + Z_{388}Z_{375}$$

$$Z_{429} = -mZ_{380}Z_{142} - mZ_{381}Z_{170} + Z_{386}Z_{307} + Z_{387}Z_{340} + Z_{388}Z_{374}$$

$$Z_{430} = -mZ_{380}Z_{145} - mZ_{381}Z_{173} + Z_{386}Z_{310} + Z_{387}Z_{344} + Z_{388}Z_{378}$$

$$Z_{431} = -mZ_{380}Z_{147} - mZ_{381}Z_{175} + Z_{386}Z_{299} + Z_{388}Z_{379}$$

$$Z_{432} = -mZ_{380}Z_{146} - mZ_{381}Z_{174} + Z_{387}Z_{345}$$

$$Z_{433} = -Z_{380}$$

$$Z_{434} = \frac{a}{2} - \ell c_\xi s_\eta - \frac{a'}{2} c_\nu c_\zeta - hc_\delta s_\zeta - hc_\zeta s_\delta s_\nu$$

$$Z_{435} = -Z_{380}$$

$$Z_{436} = s_\nu$$

$$Z_{437} = c_\delta c_\nu$$

$$Z_{438} = -s_\delta c_\nu$$

$$Z_{439} = c_\nu c_\zeta Z_{433} - c_\nu s_\zeta Z_{434}$$

$$Z_{440} = (s_\delta s_\zeta - c_\delta c_\zeta s_\nu)Z_{433} + (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)Z_{434}$$

$$Z_{441} = (c_\delta s_\zeta + c_\zeta s_\delta s_\nu)Z_{433} + (-s_\zeta s_\delta s_\nu + c_\delta c_\zeta)Z_{434}$$

$$Z_{442} = -s_\theta Z_{433} + c_\theta c_\phi Z_{434}$$

$$Z_{443} = s_\nu$$

$$Z_{444} = c_\delta c_\nu$$

$$Z_{445} = s_\delta c_\nu$$

$$Z_{446} = -mZ_{435}$$

$$Z_{447} = 0$$

$$Z_{448} = -mZ_{434}$$

$$Z_{449} = mZ_{435}$$

$$Z_{450} = -mZ_{435}$$

$$Z_{451} = 0$$

$Z_{452} = 0$

$Z_{453} = mZ_{434}$

$Z_{454} = -mZ_{434}$

$Z_{455} = -mZ_{434}Z_{148} + Z_{436}Z_{278} + Z_{312}Z_{437} + Z_{438}Z_{346}$

$Z_{456} = -mZ_{435}Z_{92} - mZ_{434}Z_{149} + Z_{436}Z_{286} + Z_{320}Z_{437} + Z_{438}Z_{354}$

$Z_{457} = -mZ_{435}Z_{93} + Z_{436}Z_{291} + Z_{437}Z_{327} + Z_{438}Z_{361}$

$Z_{458} = -mZ_{435}Z_{94} + Z_{436}Z_{280} + Z_{437}Z_{314} + Z_{438}Z_{348}$

$Z_{459} = -mZ_{435}Z_{95} - mZ_{434}Z_{150} + Z_{436}Z_{281} + Z_{437}Z_{315} + Z_{349}Z_{438}$

$Z_{460} = -mZ_{434}Z_{151} + Z_{436}Z_{288} + Z_{437}Z_{322} + Z_{438}Z_{356}$

$Z_{461} = -mZ_{434}Z_{152} + Z_{436}Z_{279} + Z_{437}Z_{313} + Z_{438}Z_{347}$

$Z_{462} = -mZ_{435}Z_{96} - mZ_{434}Z_{153} + Z_{436}Z_{287} + Z_{437}Z_{321} + Z_{438}Z_{355}$

$Z_{463} = -mZ_{435}Z_{97} + Z_{436}Z_{293} + Z_{437}Z_{328} + Z_{438}Z_{362}$

$Z_{464} = -mZ_{434}Z_{154} + Z_{436}Z_{282} + Z_{437}Z_{316} + Z_{438}Z_{350}$

$Z_{465} = -mZ_{434}Z_{155} + Z_{436}Z_{283} + Z_{437}Z_{317} + Z_{438}Z_{351}$

$Z_{466} = -mZ_{434}Z_{156} + Z_{436}Z_{284} + Z_{437}Z_{318} + Z_{438}Z_{352}$

$Z_{467} = -mZ_{434}Z_{157} + Z_{437}Z_{319} + Z_{438}Z_{353} + Z_{436}Z_{285}$

$Z_{468} = -mZ_{435}Z_{98} - mZ_{434}Z_{158} + Z_{436}Z_{289} + Z_{437}Z_{323} + Z_{438}Z_{357}$

$Z_{469} = -mZ_{435}Z_{99} - mZ_{434}Z_{159} + Z_{436}Z_{311} + Z_{437}Z_{324} + Z_{438}Z_{358}$

$Z_{470} = -mZ_{435}Z_{100} - mZ_{434}Z_{160} + Z_{436}Z_{290} + Z_{437}Z_{325} + Z_{438}Z_{359}$

$Z_{471} = -mZ_{435}Z_{101} - mZ_{434}Z_{161} + Z_{437}Z_{326} + Z_{438}Z_{360} + Z_{436}Z_{292}$

$Z_{472} = -mZ_{435}Z_{102} + Z_{436}Z_{294} + Z_{437}Z_{329} + Z_{433}Z_{363}$

$Z_{473} = -mZ_{435}Z_{103} + Z_{436}Z_{295} + Z_{437}Z_{330} + Z_{438}Z_{364}$

$Z_{474} = -mZ_{435}Z_{104} + Z_{436}Z_{296} + Z_{437}Z_{331} + Z_{438}Z_{365}$

$Z_{475} = -mZ_{435}Z_{105} + Z_{437}Z_{332} + Z_{438}Z_{336} + Z_{436}Z_{298}$

$Z_{476} = -mZ_{435}Z_{106} - mZ_{434}Z_{162} + Z_{437}Z_{336} + Z_{438}Z_{370} + Z_{436}Z_{302}$

$Z_{477} = -mZ_{435}Z_{107} - mZ_{434}Z_{163} + Z_{436}Z_{304} + Z_{437}Z_{335} + Z_{438}Z_{369}$

$Z_{478} = -mZ_{435}Z_{108} - mZ_{434}Z_{164} + Z_{436}Z_{303} + Z_{437}Z_{333} + Z_{438}Z_{367}$

$Z_{479} = -mZ_{435}Z_{111} - mZ_{434}Z_{167} + Z_{436}Z_{306} + Z_{437}Z_{334} + Z_{438}Z_{368}$

$Z_{480} = -mZ_{435}Z_{115} - mZ_{434}Z_{171} + Z_{437}Z_{342} + Z_{438}Z_{376} + Z_{436}Z_{308}$

$Z_{481} = -mZ_{435}Z_{109} - mZ_{434}Z_{165} + Z_{437}Z_{339} + Z_{438}Z_{373} + Z_{436}Z_{305}$

$Z_{482} = -mZ_{435}Z_{110} - mZ_{434}Z_{166} + Z_{436}Z_{301} + Z_{437}Z_{337} + Z_{438}Z_{371}$

$Z_{483} = -mZ_{435}Z_{112} - mZ_{434}Z_{168} + Z_{436}Z_{300} + Z_{437}Z_{338} + Z_{438}Z_{372}$

$Z_{484} = -mZ_{435}Z_{116} - mZ_{434}Z_{172} + Z_{437}Z_{343} + Z_{438}Z_{377}$

$Z_{485} = -mZ_{435}Z_{113} - mZ_{434}Z_{169} + Z_{436}Z_{297} + Z_{437}Z_{341} + Z_{438}Z_{375}$

$Z_{486} = -mZ_{435}Z_{114} - mZ_{434}Z_{170} + Z_{436}Z_{307} + Z_{437}Z_{340} + Z_{438}Z_{374}$

$Z_{487} = -mZ_{435}Z_{117} - mZ_{434}Z_{173} + Z_{437}Z_{344} + Z_{438}Z_{378}$

$Z_{488} = -mZ_{435}Z_{119} - mZ_{434}Z_{175} + Z_{436}Z_{299} + Z_{438}Z_{379}$

$Z_{489} = -mZ_{435}Z_{118} - mZ_{434}Z_{174} + Z_{437}Z_{345}$

$Z_{490} = -c_\nu c_\zeta Z_{381} - s_\nu Z_{434}$

$Z_{491} = (-s_\delta s_\zeta + c_\delta c_\zeta s_\nu)Z_{381} - c_\delta c_\nu Z_{434}$

$Z_{492} = (-c_\delta s_\zeta - c_\zeta s_\delta s_\nu)Z_{411} + s_\delta c_\nu Z_{464}$

$Z_{493} = s_\theta Z_{381} - c_\theta s_\phi Z_{434}$

$Z_{494} = -c_\nu s_\zeta$

$Z_{495} = s_\zeta c_\delta s_\nu + s_\delta c_\zeta$

$Z_{496} = -s_\zeta s_\delta s_\nu + c_\delta c_\zeta$

$Z_{497} = mZ_{381}$

$Z_{498} = mZ_{434}$

$Z_{499} = 0$

$Z_{500} = -mZ_{381}$

$Z_{501} = mZ_{381}$

$Z_{502} = mZ_{434}$

$Z_{503} = -mZ_{434}$

$Z_{504} = 0$

$Z_{505} = 0$

$$Z_{506} = +mZ_{434}Z_{120} + Z_{493}Z_{278} + Z_{494}Z_{312} + Z_{495}Z_{346}$$

$$Z_{507} = +mZ_{381}Z_{92} + Z_{493}Z_{286} + Z_{320}Z_{494} + Z_{495}Z_{354}$$

$$Z_{508} = +mZ_{381}Z_{93} + mZ_{434}Z_{121} + Z_{493}Z_{291} + Z_{494}Z_{327} + Z_{495}Z_{361}$$

$$Z_{509} = +mZ_{381}Z_{94} + mZ_{434}Z_{122} + Z_{493}Z_{280} + Z_{494}Z_{314} + Z_{495}Z_{348}$$

$$Z_{510} = +mZ_{381}Z_{95} + Z_{493}Z_{281} + Z_{494}Z_{315} + Z_{495}Z_{349}$$

$$Z_{511} = +mZ_{434}Z_{123} + Z_{493}Z_{288} + Z_{494}Z_{322} + Z_{495}Z_{356}$$

$$Z_{512} = +mZ_{434}Z_{124} + Z_{493}Z_{279} + Z_{494}Z_{313} + Z_{495}Z_{347}$$

$$Z_{513} = +mZ_{381}Z_{96} + Z_{493}Z_{287} + Z_{494}Z_{321} + Z_{355}Z_{495}$$

$$Z_{514} = +mZ_{381}Z_{97} + mZ_{434}Z_{95} + Z_{493}Z_{293} + Z_{494}Z_{328} + Z_{495}Z_{362}$$

$$Z_{515} = +mZ_{434}Z_{126} + Z_{493}Z_{282} + Z_{494}Z_{316} + Z_{495}Z_{350}$$

$$Z_{516} = +mZ_{434}Z_{127} + Z_{493}Z_{283} + Z_{494}Z_{317} + Z_{495}Z_{351}$$

$$Z_{517} = +mZ_{434}Z_{128} + Z_{493}Z_{284} + Z_{494}Z_{318} + Z_{495}Z_{352}$$

$$Z_{518} = +mZ_{434}Z_{129} + Z_{494}Z_{319} + Z_{495}Z_{353} + Z_{493}Z_{285}$$

$$Z_{519} = +mZ_{381}Z_{98} + Z_{493}Z_{289} + Z_{494}Z_{323} + Z_{495}Z_{357}$$

$$Z_{520} = +mZ_{381}Z_{99} + Z_{493}Z_{311} + Z_{494}Z_{324} + Z_{495}Z_{358}$$

$$Z_{521} = +mZ_{381}Z_{100} + Z_{493}Z_{290} + Z_{494}Z_{325} + Z_{495}Z_{359}$$

$$Z_{522} = +mZ_{381}Z_{101} + Z_{494}Z_{326} + Z_{495}Z_{360} + Z_{493}Z_{292}$$

$$Z_{523} = +mZ_{381}Z_{102} + mZ_{434}Z_{130} + Z_{493}Z_{294} + Z_{494}Z_{329} + Z_{495}Z_{363}$$

$$Z_{524} = +mZ_{381}Z_{103} + mZ_{434}Z_{131} + Z_{493}Z_{295} + Z_{494}Z_{330} + Z_{495}Z_{364}$$

$$Z_{525} = +mZ_{381}Z_{104} + mZ_{434}Z_{132} + Z_{493}Z_{296} + Z_{494}Z_{331} + Z_{495}Z_{365}$$

$$Z_{526} = +mZ_{381}Z_{105} + mZ_{434}Z_{133} + Z_{494}Z_{332} + Z_{495}Z_{366} + Z_{493}Z_{298}$$

$$Z_{527} = +mZ_{381}Z_{106} + mZ_{434}Z_{134} + Z_{494}Z_{336} + Z_{495}Z_{370} + Z_{493}Z_{302}$$

$$Z_{528} = +mZ_{381}Z_{107} + mZ_{434}Z_{135} + Z_{493}Z_{304} + Z_{494}Z_{335} + Z_{495}Z_{369}$$

$$Z_{529} = +mZ_{381}Z_{108} + mZ_{434}Z_{136} + Z_{493}Z_{303} + Z_{494}Z_{333} + Z_{495}Z_{367}$$

$$Z_{530} = +mZ_{381}Z_{111} + mZ_{434}Z_{139} + Z_{493}Z_{306} + Z_{494}Z_{334} + Z_{495}Z_{368}$$

$$Z_{531} = mZ_{381}Z_{115} + mZ_{434}Z_{143} + Z_{494}Z_{342} + Z_{495}Z_{376} + Z_{493}Z_{308}$$

$$Z_{532} = mZ_{381}Z_{109} + mZ_{434}Z_{137} + Z_{494}Z_{339} + Z_{495}Z_{373} + Z_{493}Z_{305}$$

$Z_{533} = mZ_{381}Z_{110} + mZ_{434}Z_{138} + Z_{493}Z_{301} + Z_{494}Z_{337} + Z_{495}Z_{371}$

$Z_{534} = mZ_{381}Z_{112} + mZ_{434}Z_{140} + Z_{493}Z_{300} + Z_{494}Z_{338} + Z_{495}Z_{372}$

$Z_{535} = mZ_{381}Z_{116} + mZ_{434}Z_{144} + Z_{493}\ Z_{309} + Z_{494}Z_{343} + Z_{495}Z_{377}$

$Z_{536} = mZ_{381}Z_{113} + mZ_{434}Z_{141} + Z_{493}Z_{297} + Z_{494}Z_{341} + Z_{495}Z_{375}$

$Z_{537} = +mZ_{381}Z_{114} + mZ_{434}Z_{142} + Z_{493}Z_{307} + Z_{494}Z_{340} + Z_{495}Z_{374}$

$Z_{538} = +mZ_{381}Z_{117} + mZ_{434}Z_{145} + Z_{493}Z_{310} + Z_{494}Z_{344} + Z_{495}Z_{378}$

$Z_{539} = +mZ_{381}Z_{119} + mZ_{434}Z_{147} + Z_{493}Z_{299} + Z_{495}Z_{379}$

$Z_{540} = +mZ_{118}Z_{381} + mZ_{434}Z_{146} + Z_{494}Z_{345}$

$Z_{541} = \ell c_\xi c_\eta - \left(\frac{a'}{2} s_\nu c_\zeta - hs_\delta c_\nu c_\zeta\right)Z_1$

$Z_{542} = \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right)Z_1$

$Z_{543} = -\ell c_\xi s_\eta + \left(\frac{a'}{2} s_\nu s_\zeta - hs_\delta c_\nu s_\zeta\right)Z_1$

$Z_{544} = Z_{541}$

$Z_{545} = Z_{542}$

$Z_{546} = Z_{543}$

$Z_{547} = mZ_{544}Z_{106} + mZ_{545}Z_{134} + mZ_{546}Z_{162} - s_\delta Z_1 Z_{336} - c_\delta Z_1 Z_{370}$

$Z_{548} = c_\nu c_\zeta Z_{541} + s_\nu Z_{542} - c_\nu s_\zeta Z_{543}$

$Z_{549} = (s_\delta s_\zeta - c_\delta c_\zeta s_\nu)Z_{541} + c_\delta c_\nu Z_{542} + (s_\zeta c_\delta s_\nu + s_\delta c_\zeta)Z_{543}$

$Z_{550} = (c_\delta s_\zeta + c_\zeta s_\delta s_\nu)Z_{541} - s_\delta c_\nu Z_{542} + (s_\zeta s_\delta s_\nu + c_\delta c_\zeta)Z_{543}$

$Z_{551} = -s_\theta Z_{541} + c_\theta s_\phi Z_{542} + c_\theta c_\phi Z_{543}$

$Z_{552} = s_\delta Z_1$

$Z_{553} = c_\delta Z_1$

$Z_{554} = -mZ_{544}$

$Z_{555} = -mZ_{545}$

$Z_{556} = -mZ_{546}$

$Z_{557} = mZ_{544}$

$Z_{558} = -mZ_{544}$

$Z_{559} = -mZ_{545}$

$$Z_{560} = mZ_{545}$$

$$Z_{561} = mZ_{546}$$

$$Z_{562} = -mZ_{546}$$

$$Z_{563} = -mZ_{545}Z_{120} - mZ_{546}Z_{148} + s_{\delta}Z_{1}Z_{312} + c_{\delta}Z_{1}Z_{346}$$

$$Z_{564} = -mZ_{544}Z_{92} - mZ_{546}Z_{149} + s_{\delta}Z_{1}Z_{320} + c_{\delta}Z_{1}Z_{354}$$

$$Z_{565} = -mZ_{544}Z_{93} - mZ_{545}Z_{121} + s_{\delta}Z_{1}Z_{327} + c_{\delta}Z_{1}Z_{361}$$

$$Z_{566} = -mZ_{544}Z_{94} - mZ_{545}Z_{122} + s_{\delta}Z_{1}Z_{314} + c_{\delta}Z_{1}Z_{348}$$

$$Z_{567} = -mZ_{544}Z_{95} - mZ_{546}Z_{150} + s_{\delta}Z_{1}Z_{315} + c_{\delta}Z_{1}Z_{349}$$

$$Z_{568} = -mZ_{545}Z_{123} - mZ_{546}Z_{151} + s_{\delta}Z_{1}Z_{322} + c_{\delta}Z_{1}Z_{356}$$

$$Z_{569} = -mZ_{545}Z_{124} - mZ_{546}Z_{152} + s_{\delta}Z_{1}Z_{313} + c_{\delta}Z_{1}Z_{347}$$

$$Z_{570} = -mZ_{544}Z_{96} - mZ_{546}Z_{153} + s_{\delta}Z_{1}Z_{321} + c_{\delta}Z_{1}Z_{355}$$

$$Z_{571} = -mZ_{544}Z_{97} - mZ_{545}Z_{125} + s_{\delta}Z_{1}Z_{328} + c_{\delta}Z_{1}Z_{362}$$

$$Z_{572} = -mZ_{545}Z_{126} - mZ_{546}Z_{154} + s_{\delta}Z_{1}Z_{316} + c_{\delta}Z_{1}Z_{350}$$

$$Z_{573} = -mZ_{545}Z_{127} - mZ_{546}Z_{155} + s_{\delta}Z_{1}Z_{317} + c_{\delta}Z_{1}Z_{351}$$

$$Z_{574} = -mZ_{545}Z_{128} - mZ_{546}Z_{156} + s_{\delta}Z_{1}Z_{318} + c_{\delta}Z_{1}Z_{352}$$

$$Z_{575} = -mZ_{545}Z_{129} - mZ_{546}Z_{157} + s_{\delta}Z_{1}Z_{319} + c_{\delta}Z_{1}Z_{353}$$

$$Z_{576} = -mZ_{544}Z_{98} - mZ_{546}Z_{158} + s_{\delta}Z_{1}Z_{323} + c_{\delta}Z_{1}Z_{357}$$

$$Z_{577} = -mZ_{544}Z_{99} - mZ_{546}Z_{159} + s_{\delta}Z_{1}Z_{324} + c_{\delta}Z_{1}Z_{358}$$

$$Z_{578} = -mZ_{544}Z_{100} - mZ_{546}Z_{160} + s_{\delta}Z_{1}Z_{325} + c_{\delta}Z_{1}Z_{359}$$

$$Z_{579} = -mZ_{544}Z_{101} - mZ_{546}Z_{161} + s_{\delta}Z_{1}Z_{326} + c_{\delta}Z_{1}Z_{360}$$

$$Z_{580} = -mZ_{544}Z_{102} - mZ_{545}Z_{130} + s_{\delta}Z_{1}Z_{329} + c_{\delta}Z_{1}Z_{363}$$

$$Z_{581} = -mZ_{544}Z_{103} - mZ_{545}Z_{131} + s_{\delta}Z_{1}Z_{330} + c_{\delta}Z_{1}Z_{364}$$

$$Z_{582} = -mZ_{544}Z_{104} - mZ_{545}Z_{132} + s_{\delta}Z_{1}Z_{331} + c_{\delta}Z_{1}Z_{365}$$

$$Z_{583} = -mZ_{544}Z_{105} - mZ_{545}Z_{133} + s_{\delta}Z_{1}Z_{332} + c_{\delta}Z_{1}Z_{366}$$

$$Z_{584} = -mZ_{544}Z_{107} - mZ_{545}Z_{135} - mZ_{546}Z_{163} + s_{\delta}Z_{1}Z_{335} + c_{\delta}Z_{1}Z_{369}$$

$$Z_{585} = -mZ_{544}Z_{108} - mZ_{545}Z_{136} - mZ_{546}Z_{164} + s_{\delta}Z_{1}Z_{333} + c_{\delta}Z_{1}Z_{367}$$

$$Z_{586} = -mZ_{544}Z_{111} - mZ_{545}Z_{139} - mZ_{546}Z_{167} + s_{\delta}Z_{1}Z_{334} + c_{\delta}Z_{1}Z_{368}$$

$$Z_{587} = -mZ_{544}Z_{115} - mZ_{545}Z_{143} - mZ_{546}Z_{171} + s_\delta Z_1 Z_{342} + c_\delta Z_1 Z_{376}$$

$$Z_{588} = -mZ_{544}Z_{109} - mZ_{545}Z_{137} - mZ_{546}Z_{165} + s_\delta Z_1 Z_{339} + c_\delta Z_1 Z_{373}$$

$$Z_{589} = -mZ_{544}Z_{110} - mZ_{545}Z_{138} - mZ_{546}Z_{166} + s_\delta Z_1 Z_{337} + c_\delta Z_1 Z_{371}$$

$$Z_{590} = -mZ_{544}Z_{112} - mZ_{545}Z_{140} - mZ_{546}Z_{168} + s_\delta Z_1 Z_{338} + c_\delta Z_1 Z_{372}$$

$$Z_{591} = -mZ_{544}Z_{116} - mZ_{545}Z_{144} - mZ_{546}Z_{172} + s_\delta Z_1 Z_{343} + c_\delta Z_1 Z_{377}$$

$$Z_{592} = -mZ_{544}Z_{113} - mZ_{545}Z_{141} - mZ_{546}Z_{169} + s_\delta Z_1 Z_{341} + c_\delta Z_1 Z_{375}$$

$$Z_{593} = -mZ_{544}Z_{114} - mZ_{545}Z_{142} - mZ_{546}Z_{170} + s_\delta Z_1 Z_{340} + c_\delta Z_1 Z_{374}$$

$$Z_{594} = -mZ_{544}Z_{117} - mZ_{545}Z_{145} - mZ_{546}Z_{173} + s_\delta Z_1 Z_{344} + c_\delta Z_1 Z_{378}$$

$$Z_{595} = -mZ_{544}Z_{119} - mZ_{545}Z_{147} - mZ_{545}Z_{175} + c_\delta Z_1 Z_{379}$$

$$Z_{596} = -mZ_{544}Z_{118} - mZ_{545}Z_{146} - mZ_{546}Z_{174} + s_\delta Z_1 Z_{345}$$

$$Z_{597} = -\ell s_\xi s_\eta - \left(\frac{a'}{2} s_\nu c_\zeta - h s_\delta c_\nu c_\zeta\right) Z_2$$

$$Z_{598} = -\ell c_\xi + \left(\frac{a'}{2} c_\nu + h s_\delta s_\nu\right) Z_2$$

$$Z_{599} = -\ell s_\xi c_\eta + \left(\frac{a'}{2} s_\nu s_\zeta - h s_\delta c_\nu s_\zeta\right) Z_2$$

$$Z_{600} = mZ_{597}Z_{109} + mZ_{598}Z_{137} + mZ_{599}Z_{165} - s_\delta Z_2 Z_{339} - c_\delta Z_2 Z_{373}$$

$$Z_{601} = c_\nu c_\zeta Z_{597} + s_\nu Z_{598} - c_\nu s_\zeta Z_{599}$$

$$Z_{602} = (s_\delta s_\zeta - c_\delta c_\zeta s_\nu) Z_{597} + c_\delta c_\nu Z_{598} + (s_\zeta c_\delta s_\nu + s_\delta c_\zeta) Z_{599}$$

$$Z_{603} = (c_\delta s_\zeta + c_\zeta s_\delta s_\nu) Z_{597} - s_\delta c_\nu Z_{598} + (s_\zeta s_\delta s_\nu + c_\delta c_\zeta) Z_{599}$$

$$Z_{604} = -s_\theta Z_{597} + c_\theta s_\phi Z_{598} + c_\theta c_\phi Z_{599}$$

$$Z_{605} = s_\delta Z_2$$

$$Z_{606} = c_\delta Z_2$$

$$Z_{607} = -mZ_{597}$$

$$Z_{608} = -mZ_{538}$$

$$Z_{609} = -mZ_{599}$$

$$Z_{610} = mZ_{597}$$

$$Z_{611} = -mZ_{597}$$

$$Z_{612} = -mZ_{598}$$

$$Z_{613} = +mZ_{598}$$

$$Z_{614} = mZ_{599}$$

$$Z_{615} = -mZ_{599}$$

$$Z_{616} = -mZ_{598}Z_{120} - mZ_{599}Z_{148} + s_\delta Z_2 Z_{312} + c_\delta Z_2 Z_{346}$$

$$Z_{617} = -mZ_{597}Z_{92} - mZ_{599}Z_{149} + s_\delta Z_2 Z_{320} + c_\delta Z_2 Z_{354}$$

$$Z_{618} = -mZ_{597}Z_{93} - mZ_{598}Z_{124} + s_\delta Z_2 Z_{327} + c_\delta Z_2 Z_{361}$$

$$Z_{619} = -mZ_{597}Z_{94} - mZ_{598}Z_{122} + s_\delta Z_2 Z_{314} + c_\delta Z_2 Z_{348}$$

$$Z_{620} = -mZ_{597}Z_{95} - mZ_{599}Z_{150} + s_\delta Z_2 Z_{315} + c_\delta Z_2 Z_{349}$$

$$Z_{621} = -mZ_{598}Z_{123} - mZ_{599}Z_{151} + s_\delta Z_2 Z_{322} + c_\delta Z_2 Z_{356}$$

$$Z_{622} = -mZ_{598}Z_{124} - mZ_{599}Z_{152} + s_\delta Z_2 Z_{313} + c_\delta Z_2 Z_{347}$$

$$Z_{623} = -mZ_{597}Z_{96} - mZ_{599}Z_{153} + s_\delta Z_2 Z_{321} + c_\delta Z_2 Z_{355}$$

$$Z_{624} = -mZ_{597}Z_{97} - mZ_{598}Z_{125} + s_\delta Z_2 Z_{328} + c_\delta Z_2 Z_{362}$$

$$Z_{625} = -mZ_{598}Z_{126} - mZ_{599}Z_{154} + s_\delta Z_2 Z_{316} + c_\delta Z_2 Z_{350}$$

$$Z_{626} = -mZ_{598}Z_{127} - mZ_{599}Z_{155} + s_\delta Z_2 Z_{317} + c_\delta Z_2 Z_{351}$$

$$Z_{627} = -mZ_{598}Z_{128} - mZ_{599}Z_{156} + s_\delta Z_2 Z_{318} + c_\delta Z_2 Z_{352}$$

$$Z_{628} = -mZ_{598}Z_{129} - mZ_{599}Z_{157} + s_\delta Z_2 Z_{319} + c_\delta Z_2 Z_{353}$$

$$Z_{629} = -mZ_{597}Z_{98} - mZ_{599}Z_{158} + s_\delta Z_2 Z_{323} + c_\delta Z_2 Z_{357}$$

$$Z_{630} = -mZ_{597}Z_{99} - mZ_{599}Z_{159} + s_\delta Z_2 Z_{324} + c_\delta Z_2 Z_{358}$$

$$Z_{631} = -mZ_{597}Z_{100} - mZ_{599}Z_{160} + s_\delta Z_2 Z_{325} + c_\delta Z_2 Z_{359}$$

$$Z_{632} = -mZ_{597}Z_{101} - mZ_{599}Z_{161} + s_\delta Z_2 Z_{326} + c_\delta Z_2 Z_{360}$$

$$Z_{633} = -mZ_{597}Z_{102} - mZ_{598}Z_{130} + s_\delta Z_2 Z_{329} + c_\delta Z_3 Z_{363}$$

$$Z_{634} = -mZ_{597}Z_{103} - mZ_{598}Z_{131} + s_\delta Z_2 Z_{330} + c_\delta Z_2 Z_{364}$$

$$Z_{635} = -mZ_{597}Z_{104} - mZ_{598}Z_{132} + s_\delta Z_2 Z_{331} + c_\delta Z_2 Z_{365}$$

$$Z_{636} = -mZ_{597}Z_{105} - mZ_{598}Z_{133} + s_\delta Z_2 Z_{332} + c_\delta Z_2 Z_{366}$$

$$Z_{637} = -mZ_{597}Z_{106} - mZ_{598}Z_{134} - mZ_{599}Z_{162} + s_\delta Z_2 Z_{335} + c_\delta Z_2 Z_{370}$$

$$Z_{638} = -mZ_{597}Z_{107} - mZ_{598}Z_{135} - mZ_{599}Z_{163} + s_\delta Z_2 Z_{335} + c_\delta Z_2 Z_{369}$$

$$Z_{639} = -mZ_{597}Z_{108} - mZ_{598}Z_{136} - mZ_{599}Z_{164} + s_\delta Z_2 Z_{333} + c_\delta Z_2 Z_{367}$$

$$Z_{640} = -mZ_{597}Z_{111} - mZ_{598}Z_{139} - mZ_{599}Z_{167} + s_\delta Z_2 Z_{334} + c_\delta Z_2 Z_{368}$$

$$Z_{641} = -mZ_{597}Z_{115} - mZ_{598}Z_{143} - mZ_{599}Z_{171} + s_\delta Z_2 Z_{342} + c_\delta Z_2 Z_{376}$$

$$Z_{642} = -mZ_{597}Z_{110} - mZ_{598}Z_{138} - mZ_{599}Z_{166} + s_\delta Z_2 Z_{337} + c_\delta Z_2 Z_{371}$$

$$Z_{643} = -mZ_{597}Z_{112} - mZ_{598}Z_{140} - mZ_{599}Z_{168} + s_\delta Z_2 Z_{338} + c_\delta Z_2 Z_{372}$$

$$Z_{644} = -mZ_{597}Z_{116} - mZ_{598}Z_{144} - mZ_{599}Z_{172} + s_\delta Z_2 Z_{343} + c_\delta Z_2 Z_{377}$$

$$Z_{645} = -mZ_{597}Z_{113} - mZ_{598}Z_{141} - mZ_{599}Z_{169} + s_\delta Z_2 Z_{341} + c_\delta Z_2 Z_{375}$$

$$Z_{646} = -mZ_{597}Z_{114} - mZ_{598}Z_{142} - mZ_{599}Z_{170} + s_\delta Z_2 Z_{340} + c_\delta Z_2 Z_{374}$$

$$Z_{647} = -mZ_{597}Z_{117} - mZ_{598}Z_{145} - mZ_{599}Z_{173} + s_\delta Z_2 Z_{344} + c_\delta Z_2 Z_{378}$$

$$Z_{648} = -mZ_{597}Z_{119} - mZ_{598}Z_{147} - mZ_{599}Z_{175} + c_\delta Z_2 Z_{379}$$

$$Z_{649} = -mZ_{597}Z_{118} - mZ_{598}Z_{146} - mZ_{599}Z_{174} + s_\delta Z_2 Z_{345}$$

$$Z_{650} = -hZ_{387}$$

$$Z_{651} = -hZ_{437}$$

$$Z_{652} = -hZ_{494}$$

$$Z_{653} = mZ_{650}Z_{119} + mZ_{651}Z_{147} + mZ_{652}Z_{175} - Z_{299}$$

$$Z_{654} = Z_{386}Z_{650} + Z_{436}Z_{651} + Z_{438}Z_{652}$$

$$Z_{655} = Z_{387}Z_{650} + Z_{437}Z_{651} + Z_{494}Z_{652}$$

$$Z_{656} = Z_{65}Z_{650} + Z_{70}Z_{651} + (-s_\zeta s_\delta s_\nu + c_\delta s_\zeta)Z_{652}$$

$$Z_{657} = -s_\theta Z_{650} + c_\theta s_\phi Z_{651} + c_\theta c_\phi Z_{652}$$

$$Z_{658} = 1$$

$$Z_{659} = -mZ_{650}$$

$$Z_{660} = -mZ_{651}$$

$$Z_{661} = -mZ_{652}$$

$$Z_{662} = mZ_{650}$$

$$Z_{663} = -mZ_{650}$$

$$Z_{664} = -mZ_{651}$$

$$Z_{665} = +mZ_{651}$$

$$Z_{666} = mZ_{652}$$

$$Z_{667} = -mZ_{652}$$

$$Z_{668} = -mZ_{651}Z_{120} - mZ_{652}Z_{148} + Z_{278}$$

$$Z_{669} = -mZ_{650}Z_{92} - mZ_{652}Z_{149} + Z_{286}$$

$$Z_{670} = -mZ_{650}Z_{93} - mZ_{651}Z_{121} + Z_{291}$$

$$Z_{671} = -mZ_{650}Z_{94} - mZ_{651}Z_{122} + Z_{280}$$

$$Z_{672} = -mZ_{650}Z_{95} - mZ_{652}Z_{150} + Z_{281}$$

$$Z_{673} = -mZ_{651}Z_{123} - mZ_{652}Z_{151} + Z_{288}$$

$$Z_{674} = -mZ_{651}Z_{124} - mZ_{652}Z_{152} + Z_{279}$$

$$Z_{675} = -mZ_{650}Z_{96} - mZ_{652}Z_{153} + Z_{287}$$

$$Z_{676} = -mZ_{650}Z_{97} - mZ_{651}Z_{125} + Z_{293}$$

$$Z_{677} = -mZ_{651}Z_{126} - mZ_{652}Z_{154} + Z_{282}$$

$$Z_{678} = -mZ_{651}Z_{127} - mZ_{652}Z_{155} + Z_{283}$$

$$Z_{679} = -mZ_{651}Z_{128} - mZ_{652}Z_{156} + Z_{284}$$

$$Z_{680} = -mZ_{651}Z_{129} - mZ_{652}Z_{157} + Z_{285}$$

$$Z_{681} = -mZ_{650}Z_{98} + Z_{289} - mZ_{652}Z_{158}$$

$$Z_{682} = -mZ_{650}Z_{99} + Z_{311} - mZ_{652}Z_{159}$$

$$Z_{683} = -mZ_{650}Z_{100} + Z_{290} - mZ_{652}Z_{160}$$

$$Z_{684} = -mZ_{650}Z_{101} - mZ_{652}Z_{161} + Z_{292}$$

$$Z_{685} = -mZ_{650}Z_{102} - mZ_{651}Z_{130} + Z_{294}$$

$$Z_{686} = -mZ_{650}Z_{103} - mZ_{651}Z_{131} + Z_{295}$$

$$Z_{687} = -mZ_{650}Z_{104} - mZ_{651}Z_{132} + Z_{296}$$

$$Z_{688} = -mZ_{650}Z_{105} - mZ_{651}Z_{133} + Z_{298}$$

$$Z_{689} = -mZ_{650}Z_{106} - mZ_{651}Z_{134} - mZ_{652}Z_{162} + Z_{302}$$

$$Z_{690} = -mZ_{650}Z_{107} - mZ_{651}Z_{135} - mZ_{652}Z_{163} + Z_{304}$$

$$Z_{691} = -mZ_{650}Z_{108} - mZ_{651}Z_{136} - mZ_{652}Z_{164} + Z_{303}$$

$$Z_{692} = -mZ_{650}Z_{111} - mZ_{651}Z_{139} - mZ_{652}Z_{167} + Z_{306}$$

$$Z_{693} = -mZ_{650}Z_{115} - mZ_{651}Z_{143} - mZ_{652}Z_{171} + Z_{308}$$

$$Z_{694} = -mZ_{650}Z_{109} - mZ_{651}Z_{137} - mZ_{652}Z_{165} + Z_{305}$$

$Z_{695} = -mZ_{650}Z_{110} - mZ_{651}Z_{138} - mZ_{652}Z_{166} + Z_{301}$

$Z_{696} = -mZ_{650}Z_{112} - mZ_{651}Z_{140} - mZ_{652}Z_{168} + Z_{300}$

$Z_{697} = -mZ_{650}Z_{116} - mZ_{651}Z_{144} - mZ_{652}Z_{172} + Z_{309}$

$Z_{698} = -mZ_{650}Z_{113} - mZ_{651}Z_{141} - mZ_{652}Z_{169} + Z_{297}$

$Z_{699} = -mZ_{650}Z_{114} - mZ_{651}Z_{142} - mZ_{652}Z_{170} + Z_{307}$

$Z_{700} = -mZ_{650}Z_{117} - mZ_{651}Z_{145} - mZ_{652}Z_{173} + Z_{310}$

$Z_{701} = -mZ_{650}Z_{118} - mZ_{651}Z_{146} - mZ_{652}Z_{174}$

$Z_{702} = -\frac{a'}{2} c_\nu s_\zeta - hs_\zeta s_\delta s_\nu + hc_\delta c_\zeta - \left(\frac{a'}{2} s_\nu c_\zeta - hs_\delta c_\nu c_\zeta\right) Z_3$

$Z_{703} = \left(\frac{a'}{2} c_\nu + hs_\delta s_\nu\right) Z_3$

$Z_{704} = -\frac{a'}{2} c_\nu c_\zeta - hc_\delta s_\zeta - hc_\zeta s_\delta s_\nu + \left(\frac{a'}{2} s_\nu s_\zeta - hs_\delta c_\nu s_\zeta\right) Z_3$

$Z_{705} = Z_{436}$

$Z_{706} = c_\delta c_\nu + s_\delta Z_3$

$Z_{707} = -s_\delta c_\nu + c_\delta Z_3$

$Z_{708} = mZ_{702}Z_{113} + mZ_{703}Z_{141} + mZ_{704}Z_{169} - Z_{705}Z_{297} - Z_{706}Z_{341} - Z_{707}Z_{375}$

$Z_{709} = c_\nu c_\zeta Z_{702} + s_\nu Z_{703} - c_\nu s_\zeta Z_{704}$

$Z_{710} = (s_\delta s_\zeta - c_\delta c_\zeta s_\nu) Z_{702} + c_\delta c_\nu Z_{703} + (s_\zeta c_\delta s_\nu + s_\delta c_\zeta) Z_{704}$

$Z_{711} = Z_{42} Z_{702} - s_\delta s_\nu Z_{703} + Z_{75} Z_{704}$

$Z_{712} = -s_\theta Z_{702} + c_\theta s_\phi Z_{703} + c_\theta c_\phi Z_{704}$

$Z_{713} = s_\nu$

$Z_{714} = c_\delta c_\nu + s_\delta Z_3$

$Z_{715} = Z_{707}$

$Z_{716} = -mZ_{702}$

$Z_{717} = -mZ_{703}$

$Z_{718} = -mZ_{704}$

$Z_{719} = mZ_{702}$

$Z_{720} = -mZ_{702}$

$Z_{721} = -mZ_{703}$

$Z_{722} = mZ_{703}$

$Z_{723} = mZ_{704}$

$Z_{724} = -mZ_{704}$

$Z_{725} = -mZ_{703}Z_{120} - mZ_{704}Z_{148} + Z_{705}Z_{278} + Z_{706}Z_{312} + Z_{707}Z_{346}$

$Z_{726} = -mZ_{702}Z_{92} - mZ_{704}Z_{149} + Z_{705}Z_{286} + Z_{706}Z_{320} + Z_{707}Z_{354}$

$Z_{727} = -mZ_{702}Z_{93} - mZ_{703}Z_{121} + Z_{705}Z_{291} + Z_{706}Z_{327} + Z_{707}Z_{361}$

$Z_{728} = -mZ_{702}Z_{94} - mZ_{703}Z_{122} + Z_{705}Z_{280} + Z_{706}Z_{314} + Z_{707}Z_{348}$

$Z_{729} = -mZ_{702}Z_{95} - mZ_{704}Z_{150} + Z_{705}Z_{281} + Z_{706}Z_{315} + Z_{707}Z_{349}$

$Z_{730} = -mZ_{703}Z_{123} - mZ_{704}Z_{151} + Z_{705}Z_{288} + Z_{706}Z_{322} + Z_{707}Z_{356}$

$Z_{731} = -mZ_{703}Z_{124} - mZ_{704}Z_{152} + Z_{705}Z_{279} + Z_{706}Z_{313} + Z_{707}Z_{347}$

$Z_{732} = -mZ_{702}Z_{96} - mZ_{704}Z_{153} + Z_{705}Z_{287} + Z_{706}Z_{321} + Z_{707}Z_{355}$

$Z_{733} = -mZ_{702}Z_{97} - mZ_{703}Z_{125} + Z_{705}Z_{293} + Z_{706}Z_{328} + Z_{707}Z_{362}$

$Z_{734} = -mZ_{703}Z_{126} - mZ_{704}Z_{154} + Z_{705}Z_{282} + Z_{706}Z_{316} + Z_{707}Z_{350}$

$Z_{735} = -mZ_{703}Z_{127} - mZ_{704}Z_{155} + Z_{705}Z_{283} + Z_{706}Z_{317} + Z_{707}Z_{351}$

$Z_{736} = -mZ_{703}Z_{128} - mZ_{704}Z_{156} + Z_{705}Z_{284} + Z_{706}Z_{318} + Z_{707}Z_{352}$

$Z_{737} = -mZ_{703}Z_{129} - mZ_{704}Z_{157} + Z_{706}Z_{319} + Z_{707}Z_{353} + Z_{705}Z_{285}$

$Z_{738} = -mZ_{702}Z_{98} - mZ_{704}Z_{158} + Z_{705}Z_{289} + Z_{706}Z_{323} + Z_{707}Z_{357}$

$Z_{739} = -mZ_{702}Z_{99} - mZ_{704}Z_{159} + Z_{705}Z_{311} + Z_{706}Z_{324} + Z_{707}Z_{358}$

$Z_{740} = -mZ_{702}Z_{100} - mZ_{704}Z_{160} + Z_{705}Z_{290} + Z_{706}Z_{325} + Z_{707}Z_{359}$

$Z_{741} = -mZ_{702}Z_{101} - mZ_{704}Z_{161} + Z_{706}Z_{326} + Z_{707}Z_{360} + Z_{705}Z_{292}$

$Z_{742} = -mZ_{702}Z_{102} - mZ_{703}Z_{130} + Z_{705}Z_{294} + Z_{329}Z_{706} + Z_{707}Z_{363}$

$Z_{743} = -mZ_{702}Z_{103} - mZ_{703}Z_{131} + Z_{705}Z_{295} + Z_{706}Z_{330} + Z_{364}Z_{707}$

$Z_{744} = -mZ_{702}Z_{104} - mZ_{703}Z_{132} + Z_{705}Z_{296} + Z_{331}Z_{706} + Z_{707}Z_{365}$

$Z_{745} = -mZ_{702}Z_{105} - mZ_{703}Z_{133} + Z_{706}Z_{332} + Z_{707}Z_{366} + Z_{705}Z_{298}$

$Z_{746} = -mZ_{702}Z_{106} - mZ_{703}Z_{134} - mZ_{704}Z_{162} + Z_{706}Z_{336} + Z_{707}Z_{370} + Z_{705}Z_{302}$

$Z_{747} = -mZ_{702}Z_{107} - mZ_{703}Z_{135} - mZ_{704}Z_{163} + Z_{705}Z_{304} + Z_{706}Z_{335} + Z_{707}Z_{369}$

$Z_{748} = -mZ_{702}Z_{108} - mZ_{703}Z_{136} - mZ_{704}Z_{164} + Z_{705}Z_{303} + Z_{706}Z_{333} + Z_{707}Z_{367}$

$Z_{749} = -mZ_{702}Z_{111} - mZ_{703}Z_{139} - mZ_{704}Z_{167} + Z_{705}Z_{306} + Z_{706}Z_{334} + Z_{707}Z_{368}$

$Z_{750} = -mZ_{702}Z_{115} - mZ_{703}Z_{143} - mZ_{704}Z_{171} + Z_{706}Z_{342} + Z_{707}Z_{376} + Z_{705}Z_{308}$

$Z_{751} = -mZ_{702}Z_{109} - mZ_{703}Z_{137} - mZ_{704}Z_{165} + Z_{706}Z_{339} + Z_{707}Z_{373} + Z_{705}Z_{305}$

$Z_{752} = -mZ_{702}Z_{110} - mZ_{703}Z_{138} - mZ_{704}Z_{166} + Z_{705}Z_{301} + Z_{706}Z_{337} + Z_{707}Z_{371}$

$Z_{753} = -mZ_{702}Z_{112} - mZ_{703}Z_{140} - mZ_{704}Z_{168} + Z_{705}Z_{300} + Z_{706}Z_{338} + Z_{707}Z_{372}$

$Z_{754} = -mZ_{702}Z_{116} - mZ_{703}Z_{144} - mZ_{704}Z_{172} + Z_{705}Z_{309} + Z_{706}Z_{343} + Z_{707}Z_{377}$

$Z_{755} = -mZ_{702}Z_{114} - mZ_{703}Z_{142} - mZ_{704}Z_{170} + Z_{705}Z_{307} + Z_{706}Z_{340} + Z_{707}Z_{374}$

$Z_{756} = -mZ_{702}Z_{117} - mZ_{703}Z_{145} - mZ_{704}Z_{173} + Z_{705}Z_{310} + Z_{706}Z_{344} + Z_{707}Z_{378}$

$Z_{757} = -mZ_{702}Z_{118} - mZ_{703}Z_{146} - mZ_{704}Z_{174} + Z_{706}Z_{345}$

$Z_{758} = -mZ_{702}Z_{119} - mZ_{703}Z_{147} - mZ_{704}Z_{175} + Z_{707}Z_{379} + Z_{705}Z_{299}$

REFERENCES

1. Gupta, N. K.; and Bryson, A. E.: Near-Hover Control of a Helicopter with a Hanging Load. Journal of Aircraft, vol. 13, no. 3, 1976, pp. 217-222.

2. Prabhakar, A.: Stability of a Helicopter Carrying an Underslung Load. Vertica, vol. 2, 1978, pp. 121-143.

3. Abzug, M. J.: Dynamics and Control of Helicopters with Two-Cable Slung Loads. AIAA Paper 70-929, July 1970.

4. Garnett, T. S.; and Smith, J. H.: Active Arm (External Cargo) Stabilization System Flight Demonstration. USAAMRDL-TR-76-23, Sept. 1976.

5. Kane, Y. R.: Dynamics. Holt, Rinehart and Winston Inc., 1968.

6. Kane, Y. R.; and Tseng, Gan-tai: Dynamics of the Bifilar Pendulum. International Journal of Mechanical Sciences, vol. 9, 1967, pp. 83-96.

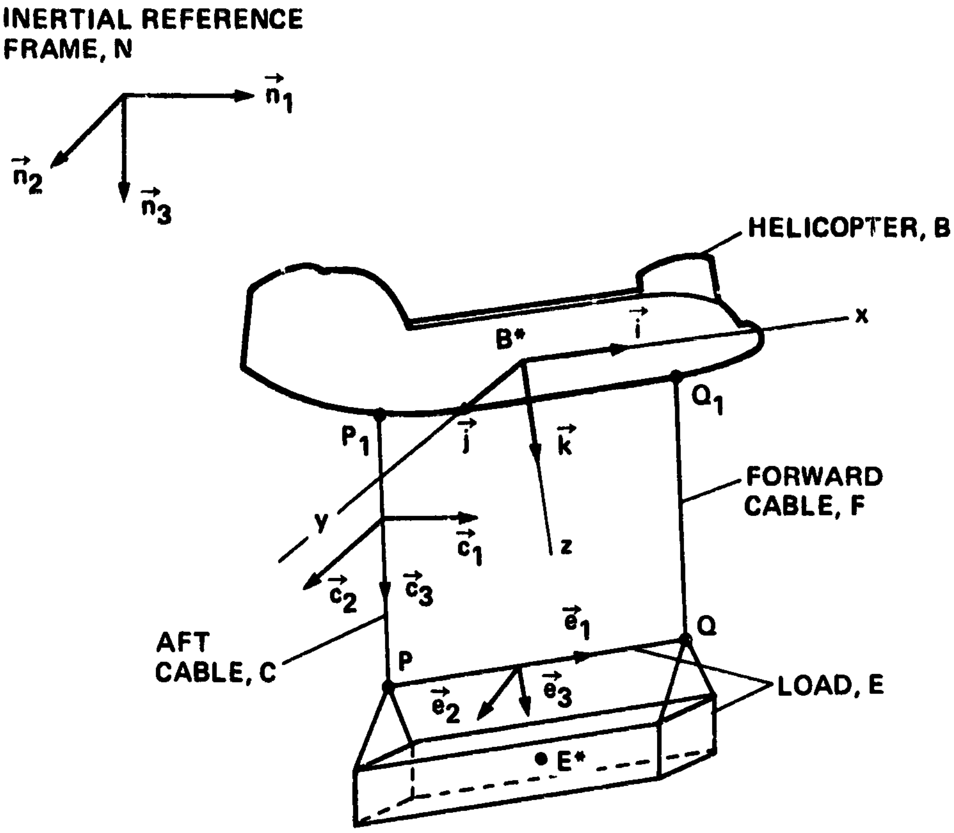


Figure 1.- Configuration of helicopter/slung load system.

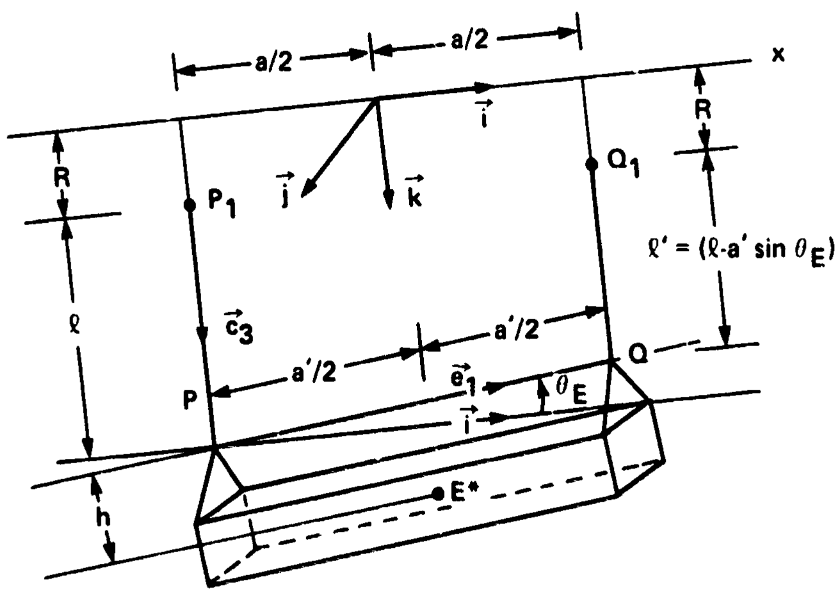


Figure 2.- Geometry of helicopter, cables and slung load at rest.

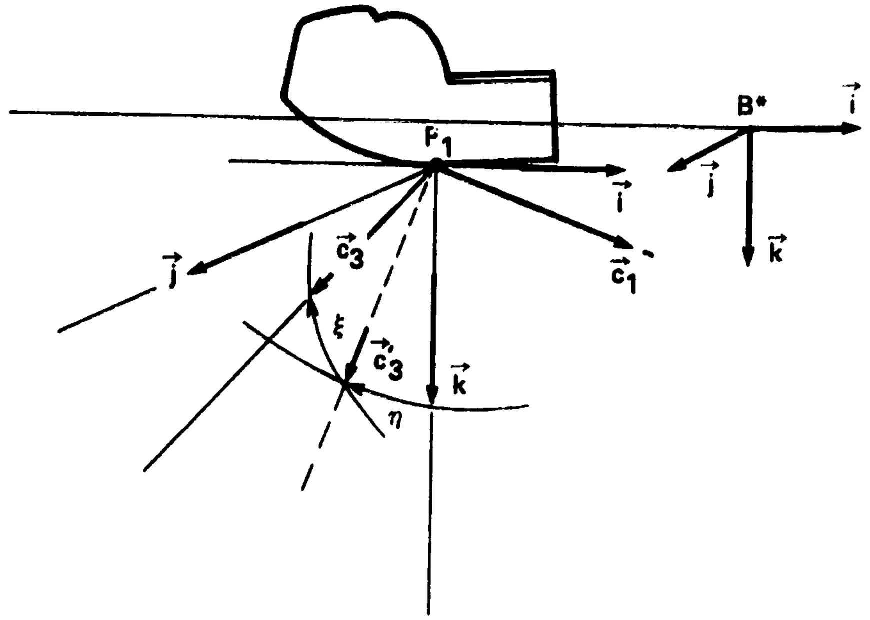


Figure 3.- Angle convention for describing aft cable motion.

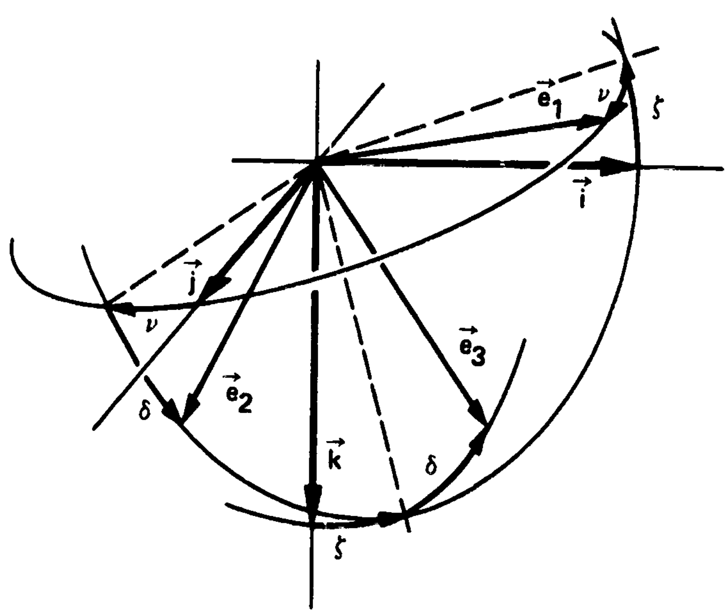


Figure 4.- Angle convention for describing suspended load motion.