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PLANAR DYNAMICS OF A UNIFORM BEAM WITH RIGID BODIES AFFIXED TO THE ENDS

by

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16 Abstract

Analyses of the planar dynamics of a uniform elastic beam subject to a variety of geometric and natural boundary conditions and external excitations are presented. The beams are inextensible and capable of small transverse bending deformations only. Classical beam vibration eigenvalue problems for a cantilever with tip mass, a cantilever with tip body and an unconstrained beam with rigid bodies at each end are examined. The characteristic equations, eigenfunctions and orthogonality relations for each are derived. The forced vibration of a cantilever with tip body subject to base acceleration is analyzed. The exact solution of the governing nonhomogeneous partial differential equation with time dependent boundary conditions is presented and compared with a Rayleigh-Ritz approximate solution. The arbitrary planar motion of an elastic beam with rigid bodies at the ends is addressed. Equations of motion are derived for two modal expansions of the beam deflection. The motion equations are cast in a first order form suitable for numerical integration. Selected FORTRAN programs are provided.

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TABLE OF CONTENTS

TABLE OF CONTENTS (Cont.)

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

An important class of Shuttle deployed payloads consists of cantilevered beam-like structures with massive tip bodies. This report presents analytical dynamics models for the planar motion of such Orbiter-payload systems (Figure 1.1). The models are specifically intended for use in stability studies of the Orbiter flight control system. Well established engineering approximations are invoked in the interests of simplicity and tractability. The format is a stepwise progression of mechanics problems each providing useful results and insight and forms a basis from which to address more complicated situations.

The payload beam-like structures are taken throughout to be long slender uniform beams capable of transverse bending only. A sequence of classical beam vibration eigenvalue problems are examined, namely: a cantilever with tip mass, a cantilever with tip body and an unconstrained beam with rigid bodies at each end. In each case the frequency equations, eigenfunctions and orthogonality relations are derived. The analytical treatment permits the free vibration characterization in terms of a minimum number of dimensionless parameters. As a precursor of the ultimate problem, the forced vibration of a cantilevered beam with tip body subject to base acceleration is studied. The exact solution to the nonhomogeneous partial differential equation with time dependent boundary conditions is presented. Natural "modal parameters" are defined and important identities in terms of these quantities derived. An approximate solution using the assumed modes method proves revealing and serves

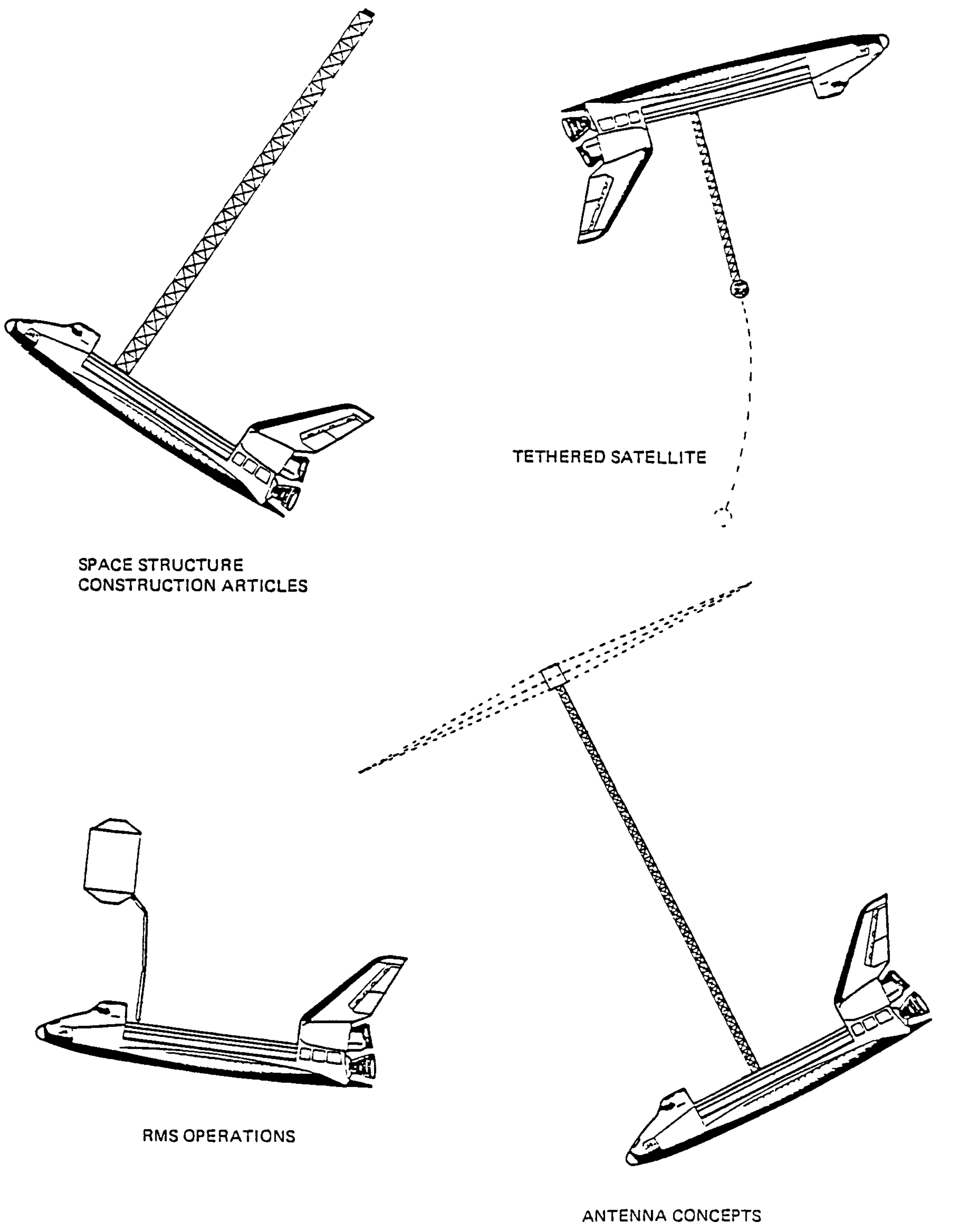


Figure 1.1. Representative cantilevered beam-like structures with tip bodies.

to check the exact solution. Finally, the planar dynamics of an elastic beam with rigid bodies affixed to the ends is addressed. This model of the Orbiter-payload system is capable of arbitrary motion in the plane accompanied by small elastic deformations. External forces and torques acting on the rigid bodies are accommodated. The vehicle motion equations are derived for two disparate modal expansions of the beam deflections. The equations of motion are cast in a first order form suitable for numerical integration. FORTRAN computer programs implementing the motion equations are given.

CHAPTER 2

EIGENVALUE PROBLEMS

2.1 Natural Frequencies and Mode Shapes of a Cantilevered Beam with Tip Mass

The partial differential equation governing the transverse vibration of a beam is given by

$$EI \frac{\partial^4 u}{\partial x^4} + \rho \frac{\partial^2 u}{\partial t^2} = 0$$

This equation assumes a uniform distribution of stiffness EI and mass per unit length ρ. The beam is clamped at x = 0

$$u(0,t) = \frac{\partial u}{\partial x}(0,t) = 0 \quad \text{for all } t \geq 0$$

at the other end of the beam: $x = \ell$, there is a point mass m_t (Figure 2.1).

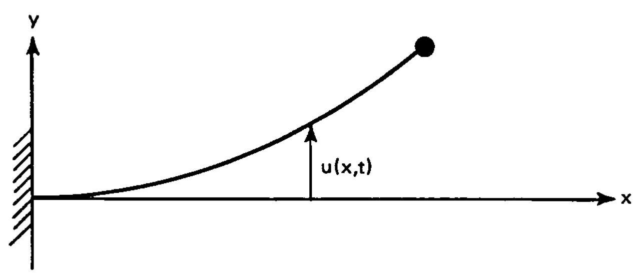


Figure 2.1. Clamped beam with tip mass.

$$\frac{\partial^2 u}{\partial x^2}(\ell,t) = 0$$

and

$$EI\,\frac{\partial^3 u}{\partial x^3}(\ell,t) = m_t\,\frac{\partial^2 u}{\partial t^2}(\ell,t)$$

Assume a solution of the form $u = \phi(x)e^{i\omega t}$. We then obtain the following boundary value problem for the mode shapes ϕ

$$\frac{d^4\phi}{dx^4} - \lambda\phi = 0 \qquad \phi(0) = \phi'(0) = 0$$

$$(\lambda \equiv \rho\omega^2/EI) \qquad \phi''(\ell) = 0, \quad \phi'''(\ell) = -\frac{m_t}{\rho}\lambda\phi(\ell) \tag{2-1}$$

The orthogonality condition in this case can be arrived at by a consideration of the kinetic energy expression for the system

$$T = \frac{1}{2}\rho\int_0^\ell \dot{u}(x,t)^2\,dx + \frac{1}{2}m_t\dot{u}(\ell,t)^2$$

Expanding u(x,t) in a series of modes

$$u = \sum_k \phi_k(x)q_k(t)$$

the kinetic energy can be written as a quadratic form in $\{\dot{q}_k\}$

$$T = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ \rho \int_0^{\ell} \phi_n(x)\phi_m(x)\, dx + m_t \phi_n(\ell)\phi_m(\ell) \right\} \dot{q}_n \dot{q}_m$$

We suspect that the natural modes will diagonalize this form, i.e.

$$\rho \int_0^{\ell} \phi_n(x)\phi_m(x)\, dx + m_t \phi_n(\ell)\phi_m(\ell) = 0 \tag{2-2}$$

where $\phi_n(x)$ and $\phi_m(x)$ are characteristic functions corresponding to distinct characteristic numbers. Indeed this orthogonality condition can be verified directly by appealing to the differential equation and boundary conditions (2.1).*

If we set $\lambda = \alpha^4$ the general solution of the differential equation is

$$\phi(x) = c_1 \sin \alpha x + c_2 \cos \alpha x + c_3 \sinh \alpha x + c_4 \cosh \alpha x$$

($\lambda = 0$ is not an eigenvalue)

The boundary conditions at $x = 0$ require

$$c_2 + c_4 = 0$$

and

$$c_1 + c_3 = 0$$

Applying the boundary conditions at $x = \ell$ we are lead to the frequency equation

---

* This alternate method of arriving at the orthogonality condition was first suggested by P.C. Hughes of UTIAS.

$$\begin{vmatrix} \cos \alpha\ell + \cosh \alpha\ell + \frac{m_t}{\rho} \alpha(\sinh \alpha\ell - \sin \alpha\ell) & \sinh \alpha\ell - \sin \alpha\ell + \frac{m_t}{\rho} \alpha(\cosh \alpha\ell - \cos \alpha\ell) \\ \sin \alpha\ell + \sinh \alpha\ell & \cos \alpha\ell + \cosh \alpha\ell \end{vmatrix} = 0$$

Expansion and simplification yields

$$\frac{m_t}{\rho\ell} \beta(\sin \beta \cosh \beta - \cos \beta \sinh \beta) = 1 + \cos \beta \cosh \beta \qquad (2\text{-}3)$$

where we have introduced the dimensionless parameter $\beta \equiv \alpha\ell$. Note that the roots of this equation only depend upon the ratio of the tip mass to the mass of the beam.

The natural frequencies ω_k are given by

$$\omega_k = \sqrt{\frac{EI}{\rho\ell^4}}\, \beta_k^2 \qquad (2\text{-}4)$$

and the corresponding mode shapes $\phi_k(x)$ are

$$\phi_k(x) = A_k\Big[(\cos \beta_k + \cosh \beta_k)\Big(\sin \beta_k \frac{x}{\ell} - \sinh \beta_k \frac{x}{\ell}\Big) - (\sin \beta_k + \sinh \beta_k)\Big(\cos \beta_k \frac{x}{\ell} - \cosh \beta_k \frac{x}{\ell}\Big)\Big] \qquad (2\text{-}5)$$

If we set $m_t = 0$, the orthogonality condition (2-2) and the frequency equation, Eq. (2-3), reduce to those for an ordinary clamped-free beam. In the limit as $m_t \to \infty$ the boundary conditions at $x = \ell$ approach those of a pinned connection: zero displacement and zero moment. And indeed the frequency equation, Eq. (2-3), takes the form appropriate to a cantilevered-pinned beam ($\tan \beta = \tanh \beta$).

2.2 Natural Frequencies and Mode Shapes of a Cantilevered Beam with Tip Body

Figure 2.2 depicts the situation in the undeformed position. A uniform beam of mass density ρ, bending stiffness EI, and length ℓ coincides with the x axis. At the tip (P) a rigid body is attached of mass m and inertia J (about P). The distance between P and the rigid body mass center is c and this directed line segment makes an angle γ with the positive x axis.

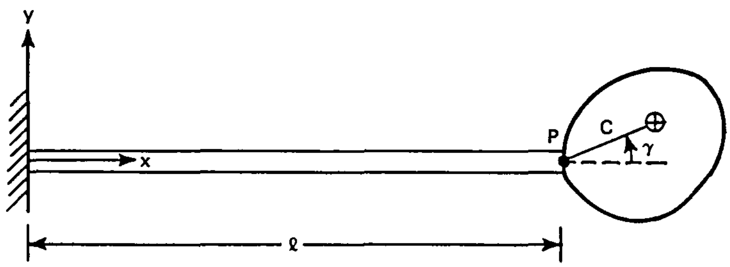


Figure 2.2. Clamped beam with tip body - undeformed.

Figure 2.3 shows the system in a deformed position. Note that the tip body is 'rigidly' attached at P. Denote the inertial velocity of P by $\bar{v}_p$ and the angular velocity of a body frame (fixed in the tip body) by $\bar{\omega}_p$

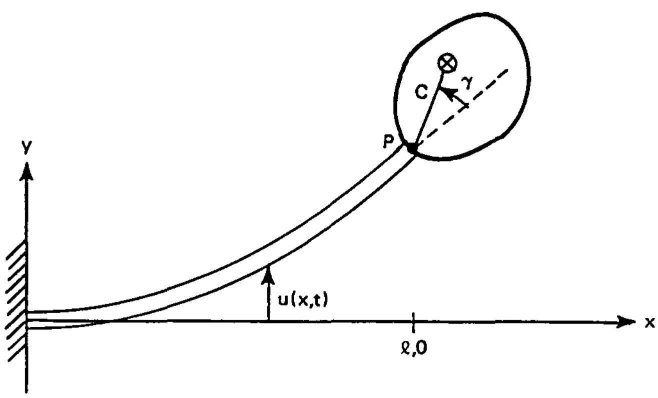


Figure 2.3. Clamped beam with tip body - deformed.

$$\bar{v}_p = \dot{u}(\ell,t)\bar{j}, \quad \bar{\omega}_p = \frac{\partial^2 u}{\partial x\,\partial t}(\ell,t)\bar{k}$$

where u(x,t) is the elastic deflection of the beam and $\bar{i}$, $\bar{j}$, $\bar{k}$ are the unit vectors along axes x, y, z. Denoting the velocity of the mass center of the tip body by $\bar{v}_{\oplus}$ and the vector from p to the mass center by $\bar{c}$ we have

$$\bar{v}_{\oplus} = \bar{v}_p + \bar{\omega}_p \times \bar{c}$$

observing that $\bar{c} = c\cos(\gamma + \theta_p)\bar{i} + c\sin(\gamma + \theta_p)\bar{j}$, where θ_p is the angle between the positive x-axis and the beam tip tangent. Noting that $|\theta_p| << 1$ so that $\sin\theta_p \cong \theta_p$, $\cos\theta_p \cong 1$ we arrive at

$$\bar{v}_{\oplus} = \left[\dot{u}(\ell,t) + c\frac{\partial^2 u}{\partial x\,\partial t}(\ell,t)(\cos\gamma - \theta_p\sin\gamma)\right]\bar{j}$$

$$- c\frac{\partial^2 u}{\partial x\,\partial t}(\ell,t)(\sin\gamma + \theta_p\cos\gamma)\bar{i}$$

The nonlinear term $\theta_p \frac{\partial^2 u}{\partial x\,\partial t}(\ell,t) = \theta_p\dot{\theta}_p$ will be dropped

$$\bar{v}_{\oplus} = \left[\frac{\partial u}{\partial t}(\ell,t) + c\cos\gamma\frac{\partial^2 u}{\partial x\,\partial t}(\ell,t)\right]\bar{j}$$

$$- c\sin\gamma\frac{\partial^2 u}{\partial x\,\partial t}(\ell,t)\bar{i} \tag{2-6}$$

The linear and angular accelerations of the tip body follow directly by differentiation of the above expressions. It will be observed that in general the mass center of the tip body will have a component of acceleration along the x axis

$$-c \sin \gamma \frac{\partial^3 u}{\partial t^2 \partial x} (\ell, t)$$

This implies a force acting on the beam along the x direction.

A problem arises at this point if we wish to accommodate the affect of axial loading on the transverse bending of the beam since nonlinear equations would result in the context of the present investigation. For simplicity we therefore assume that $\gamma = 0$ throughout the remainder of this section. The mass center offset of the tip body is therefore restricted to be directed axially from p and this results in no axial force being applied to the beam.

$$\bar{a}_{\oplus} = \left[\frac{\partial^2 u}{\partial t^2} (\ell, t) + c \frac{\partial^3 u}{\partial x \, \partial t^2} (\ell, t) \right] \bar{j}$$

$$\bar{\dot{\omega}}_p = \frac{\partial^3 u}{\partial x \, \partial t^2} (\ell, t) \bar{k}$$

(2-7)

$\bar{a}_{\oplus}$ is the acceleration of the mass center of the tip body and $\bar{\dot{\omega}}_p$ is the angular acceleration of the tip body.

In order to write the boundary conditions for $u(x,t)$ at the endpoint $x = \ell$ we consider a free body diagram of the tip body. As indicated in Figure 2.4 the beam exerts a force S directed along the y axis at p and a moment M directed along z. The equation of motion of the tip body along the y axis is

$$S = m \left[\frac{\partial^2 u}{\partial t^2} (\ell, t) + c \frac{\partial^3 u}{\partial x \, \partial t^2} (\ell, t) \right]$$

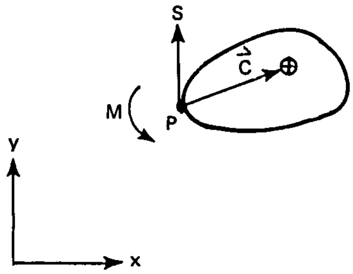


Figure 2.4. Free body diagram of tip body.

The shearing force in the beam at $x = \ell$ is given by $S = EI \frac{\partial^3 u}{\partial x^3}(\ell,t)$. This gives one of the required boundary conditions

$$EI \frac{\partial^3 u}{\partial x^3}(\ell,t) - m\left[\frac{\partial^2 u}{\partial t^2}(\ell,t) + c\,\frac{\partial^3 u}{\partial x\,\partial t^2}(\ell,t)\right] = 0 \qquad (2\text{-}8)$$

The second boundary condition is obtained by writing rotational motion equations for the tip body. Let [I] denote the inertia matrix of the tip body about its mass center and $\underline{h}$ its angular momentum both referred to body axes at the mass center.

$$\frac{d}{dt}\underline{h} = [I]\dot{\underline{\omega}}_p + \underline{\omega}_p \times [I]\underline{\omega}_p$$

Dropping the nonlinear term in ω_p it follows that

$$\hat{k} \cdot \frac{d}{dt}\overline{h} = (J - mc^2)\,\frac{\partial^3 u}{\partial x\,\partial t^2}(\ell,t)$$

where J is the moment of inertia of the tip body about an axis parallel to z and passing through P.

From Figure 2.4, the net moment about the mass center is

$$\overline{M} - \overline{c} \times \overline{S}$$

$\overline{M}$ can be calculated from beam theory in terms of stiffness and curvature. Specifically

$$\overline{M} = -EI \frac{\partial^2 u}{\partial x^2} (\ell,t)\hat{k}$$

From the work above we readily obtain

$$\overline{c} \times \overline{S} = cm\left[\frac{\partial^2 u}{\partial t^2} (\ell,t) + c \frac{\partial^3 u}{\partial x\, \partial t^2} (\ell,t)\right]\hat{k}$$

Taking the z component of the rotational motion equation

$$\frac{d}{dt} \overline{h} = \overline{M} - \overline{c} \times \overline{S}$$

results in the 2nd boundary condition at $x = \ell$

$$EI \frac{\partial^2 u}{\partial x^2} (\ell,t) + mc \frac{\partial^2 u}{\partial t^2} (\ell,t) + J \frac{\partial^3 u}{\partial x\, \partial t^2} (\ell,t) = 0 \qquad (2\text{-}9)$$

Since the beam is clamped at $x = 0$ we have the two geometric boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x} (0,t) = 0 \qquad (2\text{-}10)$$

The partial differential equation for free vibration is of course

$$EI \frac{\partial^4 u}{\partial x^4} + \rho \frac{\partial^2 u}{\partial t^2} = 0 \qquad (2\text{-}11)$$

We now proceed to solve the partial differential equation (2-11) subject to the geometric boundary conditions (2-10) and the natural boundary conditions (2-8) and (2-9). Seeking solutions of the form $e^{i\omega t}\phi(x)$ we are led to the eigenvalue problem

$$\frac{d^4\phi}{dx^4} - \lambda\phi = 0 \tag{2-12}$$

$$\phi'''(\ell) + m\frac{\lambda}{\rho}[\phi(\ell) + c\phi'(\ell)] = 0 \tag{2-13}$$

$$\phi''(\ell) - \frac{\lambda}{\rho}[mc\phi(\ell) + J\phi'(\ell)] = 0 \tag{2-14}$$

$$\phi(0) = \phi'(0) = 0 \tag{2-15}$$

Proceeding as in previous sections, the orthogonality condition can be arrived at by considering the kinetic energy, T, of the system.

$$T = \frac{1}{2}\int_0^\ell \dot{u}(x,t)^2\rho\, dx + \frac{1}{2}m\left[\dot{u}(\ell,t) + c\frac{\partial^2 u}{\partial x\,\partial t}(\ell,t)\right]^2 + \frac{1}{2}(J - mc^2)\left[\frac{\partial^2 u}{\partial x\,\partial t}(\ell,t)\right]^2$$

Expanding $u(x,t) = \sum_k \phi_k(x)q_k(t)$ the orthogonality condition is

$$\rho\int_0^\ell \phi_i(x)\phi_j(x)\,dx + m\phi_i(\ell)\phi_j(\ell) + J\phi_i'(\ell)\phi_j'(\ell) + mc[\phi_i(\ell)\phi_j'(\ell) + \phi_i'(\ell)\phi_j(\ell)] = 0 \tag{2-16}$$

where $\phi_i(x)$, $\phi_j(x)$ are eigenfunctions corresponding to distinct eigenvalues.

Returning to the eigenvalue problem, it can be readily shown that $\lambda = 0$ is not an eigenvalue. The general solution of (2-12) can be written as

$$\phi(x) = c_1 \sin \alpha x + c_2 \cos \alpha x + c_3 \sinh \alpha x + c_4 \cosh \alpha x$$

$$(\lambda \equiv \alpha^4 \neq 0)$$

The boundary conditions (2-15) require $c_4 = -c_2$ and $c_3 = -c_1$. Applying the boundary conditions (2-13), (2-14) and eliminating c_3 and c_4 we arrive at the following simultaneous equations in c_1 and c_2

$$\left[\frac{m}{\rho}(\sin\beta - \sinh\beta) + \frac{m}{\rho\ell}c\beta(\cos\beta - \cosh\beta) - \frac{\ell}{\beta}(\cos\beta + \cosh\beta)\right]c_1$$
$$+ \left[\frac{m}{\rho}(\cos\beta - \cosh\beta) - \frac{m}{\rho\ell}c\beta(\sin\beta + \sinh\beta) + \frac{\ell}{\beta}(\sin\beta - \sinh\beta)\right]c_2 = 0 \tag{2-17}$$

$$\left[-\frac{mc}{\rho}(\sin\beta - \sinh\beta) - \frac{J}{\rho\ell}\beta(\cos\beta - \cosh\beta) - \frac{\ell^2}{\beta^2}(\sin\beta + \sinh\beta)\right]c_1$$
$$+ \left[-\frac{mc}{\rho}(\cos\beta - \cosh\beta) + \frac{J}{\rho\ell}\beta(\sin\beta + \sinh\beta) - \frac{\ell^2}{\beta^2}(\cos\beta + \cosh\beta)\right]c_2 = 0 \tag{2-18}$$

where we have introduced the symbol $\beta \equiv \alpha\ell$. Setting the determinant of the system (2-17), (2-18) to zero gives the transcendental frequency equation

$$m^*(J^* - m^*c^{*2})\beta^4(1 - \cos\beta\cosh\beta) + m^*\beta(\cos\beta\sinh\beta - \sin\beta\cosh\beta)$$
$$- 2m^*c^*\beta^2\sin\beta\sinh\beta - J^*\beta^3(\sin\beta\cosh\beta + \sinh\beta\cos\beta)$$
$$+ 1 + \cos\beta\cosh\beta = 0 \tag{2-19}$$

where the dimensionless tip body parameters are defined by

$$m^* = \frac{m}{\rho\ell}; \quad c^* = \frac{c}{\ell}; \quad J^* = \frac{J}{\rho\ell^3}$$

The natural frequencies are given by

$$\omega_k = \sqrt{\frac{EI}{\rho\ell^4}}\,\beta_k^2 \tag{2-20}$$

and the corresponding eigenfunctions by

$$\begin{aligned}\phi_k(x) = {} & \Big[m^*\beta_k(\cos\beta_k - \cosh\beta_k) - m^*c^*\beta_k^2(\sin\beta_k + \sinh\beta_k) \\ & + \sin\beta_k - \sinh\beta_k\Big] \cdot \left(\sin\beta_k\frac{x}{\ell} - \sinh\beta_k\frac{x}{\ell}\right) \\ & + \Big[m^*\beta_k(\sinh\beta_k - \sin\beta_k) + m^*c^*\beta_k^2(\cosh\beta_k - \cos\beta_k) \\ & + \cos\beta_k + \cosh\beta_k\Big] \cdot \left(\cos\beta_k\frac{x}{\ell} - \cosh\beta_k\frac{x}{\ell}\right)\end{aligned} \tag{2-21}$$

Note: the eigenfunctions given in (2-21) are not normalized.

2.3. Natural Frequencies and Mode Shapes of an Unconstrained Beam with Two Tip Bodies

Figure 2.5 depicts the situation in the undeformed state. A uniform beam of mass per unit length ρ, bending stiffness EI, and length ℓ

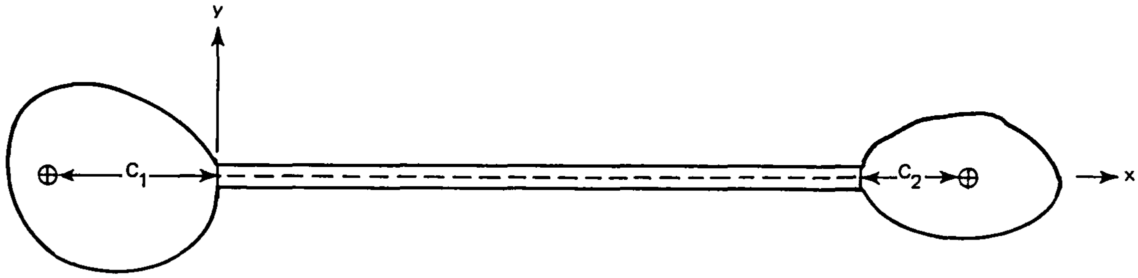


Figure 2.5. Unconstrained beam with rigid bodies attached to each end - undeformed.

lies on the x axis ($0 \leq x \leq \ell$). A rigid body of mass m_1 and inertia J_1 about its mass center, is attached at $x = 0$. The mass center offset is c_1 directed axially. A second rigid body is attached at $x = \ell$ with associated parameters m_2, J_2 and c_2 defined similarly. The motion is planar and the beam is capable of bending only.

The partial differential equation for free vibration is

$$EI \frac{\partial^4 u}{\partial x^4} + \rho \frac{\partial^2 u}{\partial t^2} = 0$$

where u(x,t) is the transverse displacement of the beam from the neutral axis. The boundary conditions at $x = \ell$ can be obtained directly from the analysis in section 2.2. Due to subtle sign changes the kinematics of body 1 will be derived here so as to obtain the correct boundary conditions at $x = 0$. Denote the inertial velocity of the attachment point at $x = 0$ by $\bar{v}_1$ and the angular velocity of a body frame fixed in body 1 by $\bar{\omega}_1$

$$\bar{v}_1 = \dot{u}(0,t)\hat{j}$$

$$\bar{\omega}_1 = \frac{\partial^2 u}{\partial x\, \partial t}(0,t)\hat{k}$$

$$\theta_1 \equiv \frac{\partial u}{\partial x}(0,t)$$

Referring to Figure 2.6, the angle of inclination of the beam with respect to the x axis is $\theta_1(t)$ and the vector from the attachment point to the mass center of body 1 is $\bar{c}_1$. If $\bar{v}^1_\oplus$ is the velocity of the mass center of body 1 we can write

$$\bar{v}^1_\oplus = \bar{v}_1 + \bar{\omega}_1 \times \bar{c}_1$$

where

$$\bar{c}_1 = -c_1(\cos\theta_1 \bar{i} + \sin\theta_1 \bar{j})$$

for

$$|\theta_1| << 1; \quad \bar{c}_1 \cong -c_1(\hat{i} + \theta_1 \hat{j})$$

Hence

$$\bar{v}^1_\oplus = \left[\frac{\partial u}{\partial t}(0,t) - c_1 \frac{\partial^2 u}{\partial x\, \partial t}(0,t)\right]\hat{j}$$

where we have dropped the nonlinear term $\theta_1\dot{\theta}_1$. By straightforward differentiation the acceleration of the mass center of body 1 ($\bar{a}^1_\oplus$) and the angular acceleration of body 1 ($\dot{\bar{\omega}}_1$) are given by

$$\bar{a}^1_\oplus = \left[\frac{\partial^2 u}{\partial t^2}(0,t) - c_1 \frac{\partial^3 u}{\partial x\, \partial t^2}(0,t)\right]\hat{j} \tag{2-22}$$

$$\dot{\bar{\omega}}_1 = \frac{\partial^3 u}{\partial x\, \partial t^2}(0,t)\hat{k} \tag{2-23}$$

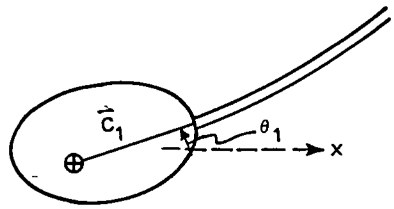


Figure 2.6. Beam and tip body 1 - deformed.

We now proceed to write dynamic equilibrium equations for body 1. For this purpose let S be the force exerted by the beam on the tip body and M the moment exerted by the beam on the tip body.

$$S = m_1 \left[\frac{\partial^2 u}{\partial t^2} (0,t) - c_1 \frac{\partial^3 u}{\partial x \, \partial t^2} (0,t) \right] \tag{2-24}$$

By the sign conventions of shearing forces in beams $S = -EI \frac{\partial^3 u}{\partial x^3}$ so we obtain the following boundary condition at x = 0

$$EI \frac{\partial^3 u}{\partial x^3} (0,t) + m_1 \left[\frac{\partial^2 u}{\partial t^2} (0,t) - c_1 \frac{\partial^3 u}{\partial x \, \partial t^2} (0,t) \right] = 0 \tag{2-25}$$

The z component of the time rate of change of the angular momentum of body 1 about its mass center is

$$J_1 \frac{\partial^3 u}{\partial x \, \partial t^2} (0,t)$$

$$\text{Net moment about mass center} = \overline{M} - \overline{c}_1 \times \overline{S}$$

From beam theory $\overline{M} = EI \frac{\partial^2 u}{\partial x^2}(0,t)\hat{k}$ and from (2-24)

$$-\overline{c}_1 \times \overline{S} = m_1 c_1 \left[\frac{\partial^2 u}{\partial t^2}(0,t) - c_1 \frac{\partial^3 u}{\partial x\, \partial t^2}(0,t)\right]\hat{k}$$

Equating the time rate of change of the angular momentum to the net moment we arrive at the second boundary condition at x = 0

$$EI \frac{\partial^2 u}{\partial x^2}(0,t) + m_1 c_1 \frac{\partial^2 u}{\partial t^2}(0,t) - \left(m_1 c_1^2 + J_1\right) \frac{\partial^3 u}{\partial x\, \partial t^2}(0,t) = 0 \tag{2-26}$$

The boundary conditions at $x = \ell$ can be obtained from Eqs. (2-8) and (2-9) (recall that in these equations J was the moment of inertia of the tip body about the attachment point)

$$EI \frac{\partial^3 u}{\partial x^3}(\ell,t) - m_2\left[\frac{\partial^2 u}{\partial t^2}(\ell,t) + c_2 \frac{\partial^3 u}{\partial x\, \partial t^2}(\ell,t)\right] = 0 \tag{2-27}$$

$$EI \frac{\partial^2 u}{\partial x^2}(\ell,t) + m_2 c_2 \frac{\partial^2 u}{\partial t^2}(\ell,t) + \left(J_2 + m_2 c_2^2\right) \frac{\partial^3 u}{\partial x\, \partial t^2}(\ell,t) = 0 \tag{2-28}$$

We now proceed to the eigenvalue problem associated with the beam bending equation and boundary conditions (2-25) - (2-28). Assuming a solution $u(x,t) = e^{i\omega t}\phi(x)$ we arrive at the eigenvalue problem

$$\frac{d^4\phi}{dx^4} - \alpha^4\phi = 0 \quad (\alpha^4 \equiv \rho\omega^2/EI)$$

Boundary conditions at $x = 0$

$$\phi'''(0) + \frac{m_1\alpha^4}{\rho}\left[-\phi(0) + c_1\phi'(0)\right] = 0 \qquad (2\text{-}29)$$

$$\phi''(0) - \frac{m_1c_1}{\rho}\alpha^4\phi(0) + \frac{\alpha^4}{\rho}\left(m_1c_1^2 + J_1\right)\phi'(0) = 0 \qquad (2\text{-}30)$$

Boundary conditions at $x = \ell$

$$\phi'''(\ell) + m_2\frac{\alpha^4}{\rho}\left[\phi(\ell) + c_2\phi'(\ell)\right] = 0 \qquad (2\text{-}31)$$

$$\phi''(\ell) - \frac{m_2c_2}{\rho}\alpha^4\phi(\ell) - \frac{\alpha^4}{\rho}\left(J_2 + m_2c_2^2\right)\phi'(\ell) = 0 \qquad (2\text{-}32)$$

<u>Orthogonality Relation</u>

The kinetic energy, T, of the system is given by

$$T = \frac{1}{2}\int_0^\ell \dot{u}^2(x,t)\rho\,dx + \frac{1}{2}m_1\left[\dot{u}(0,t) - c_1\frac{\partial^2 u}{\partial x\,\partial t}(0,t)\right]^2$$
$$+ \frac{1}{2}J_1\left[\frac{\partial^2 u}{\partial x\,\partial t}(0,t)\right]^2 + \frac{1}{2}m_2\left[\dot{u}(\ell,t) + c_2\frac{\partial^2 u}{\partial x\,\partial t}(\ell,t)\right]^2$$
$$+ \frac{1}{2}J_2\left[\frac{\partial^2 u}{\partial x\,\partial t}(\ell,t)\right]^2$$

Expanding $u(x,t)$ in a series of eigenfunctions and assuming the quadratic form in $\dot{q}_i(t)\dot{q}_j(t)$ is diagonal we arrive at

$$\rho \int_0^{\ell} \phi_i(x)\phi_j(x)\,dx + m_1[\phi_i(0) - c_1\phi_i'(0)][\phi_j(0) - c_1\phi_j'(0)]$$
$$+ J_1\phi_i'(0)\phi_j'(0) + m_2[\phi_i(\ell) + c_2\phi_i'(\ell)][\phi_j(\ell) + c_2\phi_j'(\ell)]$$
$$+ J_2\phi_i'(\ell)\phi_j'(\ell) = 0 \tag{2-33}$$

where $\phi_i(x)$ and $\phi_j(x)$ are eigenfunctions corresponding to distinct eigenvalues.

It can be readily shown that $\omega = 0$ is an eigenvalue of the problem corresponding to two linearly independent eigenfunctions: rigid body translation, and rigid body rotation.

Nonzero Bending Modes ($\alpha \neq 0$)

The eigenfunction has the form

$$\phi(x) = A_1 \sin \alpha x + A_2 \cos \alpha x + A_3 \sinh \alpha x + A_4 \cosh \alpha x \tag{2-34}$$

Applying the boundary conditions at x = 0 (2-29), (2-30) yields

$$\left(m_1^* c_1^* \beta^2 - 1\right)A_1 - m_1^*\beta A_2 + \left(1 + m_1^* c_1^* \beta^2\right)A_3 - m_1^*\beta A_4 = 0 \tag{2-35}$$

$$\left(m_1^* c_1^{*2} + J_1^*\right)\beta^3 A_1 - \left(1 + m_1^* c_1^* \beta^2\right)A_2 + \left(m_1^* c_1^{*2} + J_1^*\right)\beta^3 A_3 + \left(1 - m_1^* c_1^* \beta^2\right)A_4 = 0 \tag{2-36}$$

where we have introduced the dimensionless tip body parameters

$$m_1^* = \frac{m_1}{\rho\ell} \qquad c_1^* = \frac{c_1}{\ell} \qquad J_1^* = \frac{J_1}{\rho\ell^3} \qquad \text{and} \qquad \beta \equiv \alpha\ell$$

Applying the boundary conditions at $x = \ell$ (2-31), (2-32) yields

$$\left[m_2^*\beta \sin\beta - \left(1 - m_2^*c_2^*\beta^2\right)\cos\beta\right]A_1 + \left[m_2^*\beta\cos\beta + \left(1 - m_2^*c_2^*\beta^2\right)\sin\beta\right]A_2$$
$$+ \left[m_2^*\beta\sinh\beta + \left(1 + m_2^*c_2^*\beta^2\right)\cosh\beta\right]A_3$$
$$+ \left[m_2^*\beta\cosh\beta + \left(1 + m_2^*c_2^*\beta^2\right)\sinh\beta\right]A_4 = 0 \qquad (2\text{-}37)$$

$$-\left[\left(1 + m_2^*c_2^*\beta^2\right)\sin\beta + \left(J_2^* + m_2^*c_2^{*2}\right)\beta^3\cos\beta\right]A_1$$
$$+ \left[\left(J_2^* + m_2^*c_2^{*2}\right)\beta^3\sin\beta - \left(1 + m_2^*c_2^*\beta^2\right)\cos\beta\right]A_2$$
$$+ \left[\left(1 - m_2^*c_2^*\beta^2\right)\sinh\beta - \left(J_2^* + m_2^*c_2^{*2}\right)\beta^3\cosh\beta\right]A_3$$
$$+ \left[\left(1 - m_2^*c_2^*\beta^2\right)\cosh\beta - \left(J_2^* + m_2^*c_2^{*2}\right)\beta^3\sinh\beta\right]A_4 = 0 \qquad (2\text{-}38)$$

The dimensionless tip body parameters m_2^*, c_2^*, J_2^* are defined exactly as those for body 1.

Equations (2-35) - (2-38) constitute a system of 4 homogeneous linear equations in A_1, A_2, A_3 and A_4. A nontrivial solution exists if and only if the coefficient matrix is singular.

Writing the equations in matrix-vector form with $\bar{A} = (A_1 \quad A_2 \quad A_3 \quad A_4)^T$

$$[M]\bar{A} = \bar{0}$$

Note that

$$M = M(\beta;\, m_1^*,\, c_1^*,\, J_1^*,\, m_2^*,\, c_2^*,\, J_2^*)$$

The permissible values of β are determined from det[M] = 0 and are functions of only the six dimensionless tip body parameters even though there are nine system parameters. The natural frequencies ω_k are obtained from

$$\omega_k = \sqrt{\frac{EI}{\rho \ell^4}}\, \beta_k^2 \qquad (k = 1, 2, \ldots)$$

Corresponding to β_k, there will be a vector $\bar{A}^{(k)} \neq \bar{0}$ which in conjunction with Eq. (2-34) gives the corresponding eigenfunction.

CHAPTER 3

FORCED VIBRATION OF A CANTILEVERED BEAM WITH TIP BODY SUBJECT TO BASE ACCELERATION

3.1 Exact Solution

In Section 2.2 the natural frequencies and mode shapes for a cantilevered beam with tip body were investigated. We now wish to examine the more general situation in which the root of the beam (x = 0) is not inertially fixed and external forces are present. As noted in Section 2.2 we must be careful that no axial loads (impressed or inertial) are acting on the beam if we want to use the simple bending theory. Referring to Figure 3.1 let the x-y frame be a body fixed frame attached to the left end of the beam (x = 0) and denote by u(x,t) the elastic deflection of the beam along the y axis. The beam has a translational acceleration $a_0(t)$ at x = 0 directed along the y axis and the body frame

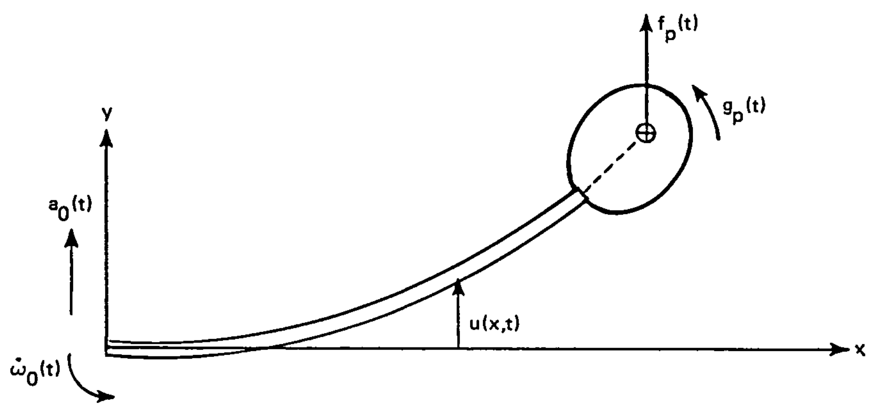


Figure 3.1. Cantilevered beam with tip body subject to base acceleration and external force and torque.

has an inertial angular acceleration $\dot{\omega}_0(t)$ perpendicular to the plane of motion (along z axis). Let $f_p(t)$ denote the external force acting upon the tip body through its center of mass directed along the y axis, and $g_p(t)$ the external moment on the tip body directed along the z axis.

The partial differential equation for the elastic displacement u(x,t), is essentially (2-11) modified by D'Alembert's principle.

$$EI \frac{\partial^4 u}{\partial x^4} + \rho \frac{\partial^2 u}{\partial t^2} = -\rho a_0(t) - \rho x \dot{\omega}_0(t) \tag{3-1}$$

By definition of the body frame we have two geometric boundary conditions at x = 0

$$u(0,t) = \frac{\partial u}{\partial x}(0,t) = 0 \quad \text{for all } t \geq 0 \tag{3-2}$$

The natural boundary conditions at $x = \ell$ are obtained exactly as in Section 2.2 taking into account the effects of $a_0(t)$ and $\dot{\omega}_0(t)$

$$\begin{aligned} EI \frac{\partial^2 u}{\partial x^2}(\ell,t) &+ mc \frac{\partial^2 u}{\partial t^2}(\ell,t) \\ &+ J \frac{\partial^3 u}{\partial x\, \partial t^2}(\ell,t) = g_p(t) + cf_p(t) - mca_0(t) \\ &\qquad - (mc\ell + J)\dot{\omega}_0(t) \end{aligned} \tag{3-3}$$

$$\begin{aligned} EI \frac{\partial^3 u}{\partial x^3}(\ell,t) &- m \frac{\partial^2 u}{\partial t^2}(\ell,t) \\ &- mc \frac{\partial^3 u}{\partial x\, \partial t^2}(\ell,t) = -f_p(t) + ma_0(t) + m(\ell + c)\dot{\omega}_0(t) \end{aligned} \tag{3-4}$$

The boundary value problem (3-1) - (3-4) consists of a nonhomogeneous partial differential equation and time dependent boundary conditions. A key in solving this problem consists of finding solutions of the associated homogeneous problem for which the base acceleration and external excitation is zero, i.e., $a_0(t) = \dot{\omega}_0(t) = g_p(t) = f_p(t) \equiv 0$. This problem has been solved in Section 2.2 where the eigenfunctions $\phi_k(x)$ and eigenvalues β_k were derived (see Eqs. (2-21) and (2-19)). It will prove more convenient to work with the dimensionless eigenfunctions $S_k(\eta)$ where $\eta \equiv x/\ell$. These are essentially given by (2-21) with the replacement $\frac{x}{\ell} \to \eta$. The eigenvalue problem for $S_k(\eta)$ can be obtained from (2-12) - (2-15)

$$\frac{d^4 S_k(\eta)}{d\eta^4} - \lambda_k S_k(\eta) = 0$$

$$S_k(0) = 0; \quad \frac{d}{d\eta} S_k(0) = 0$$

$$-\frac{d^3}{d\eta^3} S_k(\eta)\Big|_{\eta=1} = m^*\lambda_k\left[S_k(1) + c^* \frac{d}{d\eta} S_k(\eta)\Big|_{\eta=1}\right]$$

$$\frac{d^2}{d\eta^2} S_k(\eta)\Big|_{\eta=1} = \lambda_k\left[m^*c^*S_k(1) + J^* \frac{d}{d\eta} S_k(\eta)\Big|_{\eta=1}\right] \qquad (3\text{-}5)$$

The eigenvalue problem for $S_k(\eta)$ is in terms of the dimensionless tip body parameters m^*, J^* and c^*. The natural frequency of vibration Ω_k is related to the eigenvalue λ_k by

$$\lambda_k = \frac{\rho\ell^4}{EI} \Omega_k^2$$

The orthogonality condition (2-16) gives the natural inner product for the eigenfunctions $S_i(\eta)$. We assume henceforth that the eigenfunctions are orthonormal with respect to this inner product.*

$$\int_0^1 S_i(\eta) S_j(\eta)\, d\eta + m^* S_i(1) S_j(1)$$

$$+ J^* S_i'(1) S_j'(1) + m^* c^* [S_i(1) S_j'(1) + S_i'(1) S_j(1)] = \delta_{ij} \quad \text{(3-6)}$$

The boundary value problem (3-1) - (3-4) can be written in operator notation as

$$D[u(x,t)] = -\rho a_0(t) - \rho x \dot{\omega}_0(t)$$

$$D_1[u(x,t)]_{x=0} = 0; \qquad D_2[u(x,t)]_{x=0} = 0$$

$$D_3[u(x,t)]_{x=\ell} = f_1(t); \qquad D_4[u(x,t)]_{x=\ell} = f_2(t)$$

where the partial differential operators D, D_1, D_2, D_3, D_4 are

$$D \equiv EI \frac{\partial^4}{\partial x^4} + \rho \frac{\partial^2}{\partial t^2}; \qquad D_1 \equiv 1; \qquad D_2 \equiv \frac{\partial}{\partial x}$$

$$D_3 \equiv EI \frac{\partial^2}{\partial x^2} + mc \frac{\partial^2}{\partial t^2} + J \frac{\partial^3}{\partial x\, \partial t^2}; \qquad D_4 \equiv EI \frac{\partial^3}{\partial x^3} - m \frac{\partial^2}{\partial t^2} - mc \frac{\partial^3}{\partial x\, \partial t^2}$$

and

* The details of this normalization are given in Appendix A.

$$f_1(t) = g_p(t) + cf_p(t) - mca_0(t) - (mc\ell + J)\dot{\omega}_0(t)$$

$$f_2(t) = -f_p(t) + ma_0(t) + m(\ell + c)\dot{\omega}_0(t)$$

We write $u(x,t)$ as the sum

$$u(x,t) = v(x,t) + h_1(x) f_1(t) + h_2(x) f_2(t)$$

The approach is to choose the functions h_1, h_2 such that the boundary conditions on $v(x,t)$ are rendered homogeneous. In order that $D_1[v(x,t)]_{x=0} = 0$ we must have $h_1(0)f_1(t) + h_2(0)f_2(t) = 0$ for all $t > 0$. This will be satisfied if $h_1(0) = 0$ and $h_2(0) = 0$. Similarly

$$D_2[v(x,t)]_{x=0} = 0 \quad \text{if} \quad h_1'(0) = 0 \quad \text{and} \quad h_2'(0) = 0$$

$$D_3[v(x,t)]_{x=\ell} = f_1(t) - D_3[h_1(x) f_1(t)]_{x=\ell} - D_3[h_2(x) f_2(t)]_{x=\ell}$$

$D_3[v(x,t)]_{x=\ell}$ will be zero if

$$[EIh_1''(\ell) - 1] f_1(t) + EIh_2''(\ell) f_2(t)$$
$$+ [mch_1(\ell) + Jh_1'(\ell)]\ddot{f}_1(t)$$
$$+ [mch_2(\ell) + Jh_2'(\ell)]\ddot{f}_2(t) = 0 \quad \text{(for all } t > 0)$$

The above condition will be satisfied if

$$EIh_1''(\ell) - 1 = 0; \qquad h_2''(\ell) = 0;$$

$$mch_1(\ell) + Jh_1'(\ell) = 0; \qquad mch_2(\ell) + Jh_2'(\ell) = 0$$

Similarly $D_4[v(x,t)]_{x=\ell} = 0$ if the following conditions are met

$$h_1'''(\ell) = 0; \qquad EIh_2'''(\ell) - 1 = 0;$$

$$h_1(\ell) + ch_1'(\ell) = 0; \qquad h_2(\ell) + ch_2'(\ell) = 0$$

In summary, we must find functions $h_1(x)$ and $h_2(x)$ which satisfy the respective equations

$$\begin{aligned} h_1'''(\ell) &= 0 & h_2(0) &= 0 \\ h_1(0) &= 0 & h_2'(0) &= 0 \\ h_1'(0) &= 0 & h_2''(\ell) &= 0 \\ EIh_1''(\ell) &= 1 & h_2(\ell) &= 0 \\ h_1(\ell) &= 0 & EIh_2'''(\ell) &= 1 \\ h_1'(\ell) &= 0 & h_2'(\ell) &= 0 \end{aligned}$$

Clearly, these conditions do not determine h_1 and h_2 uniquely. For convenience we use fifth degree polynomials. Applying the above conditions to a general polynomial of fifth degree we obtain

$$h_1(x) = \frac{1}{EI}\left(\frac{3}{2}x^2 - \frac{4}{\ell}x^3 + \frac{7}{2\ell^2}x^4 - \frac{1}{\ell^3}x^5\right) \tag{3-7}$$

$$h_2(x) = \frac{1}{EI}\left(-\frac{\ell}{6}x^2 + \frac{1}{2}x^3 - \frac{1}{2\ell}x^4 + \frac{1}{6\ell^2}x^5\right) \tag{3-8}$$

We must still construct the function $v(x,t)$ which in addition to satisfying a particular differential equation (considered below) has to satisfy the homogeneous boundary conditions

$$D_1[v(x,t)]_{x=0} = 0; \qquad D_2[v(x,t)]_{x=0} = 0;$$

$$D_3[v(x,t)]_{x=\ell} = 0; \qquad D_4[v(x,t)]_{x=\ell} = 0$$

If we formally write

$$v(x,t) = \ell \sum_{k=1}^{\infty} q_k(t) S_k(\eta)$$

the first two boundary conditions are satisfied for arbitrary $q_k(t)$. However, since the differential operators D_3 and D_4 contain time derivatives as well as spatial derivatives the last two boundary conditions can not be met for arbitrary q_k, indeed constraints amongst these coordinates are obtained by demanding that these two boundary conditions are satisfied. Inserting this expansion for $v(x,t)$ into the last two boundary conditions (assuming term by term differentiation is valid) and using the results (3-5) we obtain the following

$$\sum_{k=1}^{\infty} \left[\ddot{q}_k(t) + \Omega_k^2 q_k(t)\right] \frac{S_k''(1)}{\lambda_k} = 0 \qquad (3\text{-}9)$$

$$\sum_{k=1}^{\infty} \left[\ddot{q}_k(t) + \Omega_k^2 q_k(t)\right] \frac{S_k'''(1)}{\lambda_k} = 0 \qquad (3\text{-}10)$$

Recall that

$$v(x,t) \equiv u(x,t) - h_1(x) f_1(t) - h_2(x) f_2(t)$$

and

$$D[u(x,t)] = -\rho a_0(t) - \rho x \dot{\omega}_0(t)$$

Hence the partial differential equation for $v(x,t)$ is

$$EI \frac{\partial^4 v}{\partial x^4} + \rho \frac{\partial^2 v}{\partial t^2} = R(x,t) \qquad 0 < x < \ell \qquad \text{(3-11)}$$

where

$$R(x,t) = -\rho a_0(t) - \rho x \dot{\omega}_0(t) - EIh_1^{iv}(x) f_1(t) - EIh_2^{iv}(x) f_2(t)$$
$$- \rho h_1(x) \ddot{f}_1(t) - \rho h_2(x) \ddot{f}_2(t) \qquad \text{(3-12)}$$

(The superscript "iv" indicates a fourth derivative with respect to the argument of the function).

Inserting the series expansion for $v(x,t)$ and using the differential equation for the eigenfunction $S_k(\eta)$; (3-11) becomes

$$\rho \ell \sum_k \left(\ddot{q}_k + \Omega_k^2 q_k\right) S_k(\eta) = R(x,t)$$

We observe that $R(0,t) \neq 0$ and $\frac{\partial}{\partial x} R(0,t) \neq 0$ in general. Therefore the above expansion for $R(x,t)$ cannot hold pointwise on $0 \leq x \leq \ell$. We will interpret the above expansion in the sense of convergence in the mean. This will suffice for our purposes since we will immediately form an integral of both sides with $S_1(\eta)$.

$$\rho \ell \sum_{k=1}^{\infty} \left(q_k + \Omega_k^2 q_k\right) \int_0^1 S_1(\eta) S_k(\eta) \, d\eta = \int_0^1 R(x,t) S_1(\eta) \, d\eta \qquad \text{(3-13)}$$

The orthogonality condition (3-6) can be rewritten with the aid of (3-5) as

$$\int_0^1 S_i(\eta)S_k(\eta)\ d\eta \;=\; \delta_{ik} - S_i'(1)\,\frac{S_k''(1)}{\lambda_k} + S_i(1)\,\frac{S_k'''(1)}{\lambda_k}$$

This result in conjunction with (3-13) and observance of the identities (3-9), (3-10) uncouples the modal coordinate equations for v(x,t).

$$\rho\ell\left(\ddot{q}_i + \Omega_i^2 q_i\right) \;=\; \int_0^1 R(x,t)S_i(\eta)\ d\eta \tag{3-14}$$

This modal coordinate equation can be expanded as

$$\ddot{q}_i + \Omega_i^2 q_i \;=\; F_i(t) + t_{1i}\ddot{f}_1(t) + t_{2i}\ddot{f}_2(t) \qquad (i = 1,\ 2,\ \ldots)$$

where

$$-F_i(t) \;=\; \frac{1}{\ell}\int_0^1 S_i(\eta)\ d\eta \cdot a_0(t) + \int_0^1 \eta S_i(\eta)\ d\eta\ \dot{\omega}_0(t)$$

$$+ \frac{EI}{\rho\ell}\int_0^1 h_1^{iv}(x)S_i(\eta)\ d\eta\ f_1(t)$$

$$+ \frac{EI}{\rho\ell}\int_0^1 h_2^{iv}(x)S_i(\eta)\ d\eta\ f_2(t)$$

$$t_{1i} \;=\; -\frac{1}{\ell}\int_0^1 h_1(x)S_i(\eta)\ d\eta$$

$$t_{2i} \;=\; -\frac{1}{\ell}\int_0^1 h_2(x)S_i(\eta)\ d\eta$$

It is possible to make a transformation to a new set of coordinates $p_k(t)$ such that the differential equations on $p_k(t)$ do not contain $\ddot{f}_1(t)$ and $\ddot{f}_2(t)$. Indeed if we set

$$q_i(t) = p_i(t) + t_{1i}f_1(t) + t_{2i}f_2(t) \tag{3-15}$$

The modal coordinate equation (3-14) transforms to

$$\ddot{p}_i + \Omega_i^2 p_i = F_i(t) - t_{1i}\Omega_i^2 f_1(t) - t_{2i}\Omega_i^2 f_2(t) \tag{3-16}$$

In terms of the new modal coordinates $p_k(t)$ the expression for $v(x,t)$ is

$$v(x,t) = \ell \sum_{k=1}^{\infty} p_k(t)S_k(\eta) + \ell \sum_{k=1}^{\infty} t_{1k}S_k(\eta)f_1(t) + \ell \sum_{k=1}^{\infty} t_{2k}S_k(\eta)f_2(t)$$

The beam deflection $u(x,t)$ assumes the form

$$u(x,t) = \ell \sum_{k=1}^{\infty} p_k(t)S_k(\eta) + \left[h_1(x) + \ell \sum_{k=1}^{\infty} t_{1k}S_k(\eta)\right]f_1(t)$$
$$+ \left[h_2(x) + \ell \sum_{k=1}^{\infty} t_{2k}S_k(\eta)\right]f_2(t)$$

with the substitution $x = \eta\ell$ Eqs. (3-7), (3-8) become

$$h_1(x) = \frac{\ell^2}{EI}\left(\frac{3}{2}\eta^2 - 4\eta^3 + \frac{7}{2}\eta^4 - \eta^5\right) \equiv g_1(\eta)$$

$$h_2(x) = \frac{\ell^3}{EI}\left(-\frac{1}{6}\eta^2 + \frac{1}{2}\eta^3 - \frac{1}{2}\eta^4 + \frac{1}{6}\eta^5\right) \equiv g_2(\eta) \tag{3-17}$$

For future reference we also have the expressions

$$h_1^{iv}(x) = \frac{1}{EI\ell^2}(84 - 120\eta)$$

$$h_2^{iv}(x) = \frac{1}{EI\ell}(-12 + 20\eta) \qquad (3\text{-}18)$$

Consider the expansion of $h_1(x)\,(g_1(\eta))$ in terms of the eigenfunctions $S_k(\eta)$.

$$g_1(\eta) = c_1S_1(\eta) + c_2S_2(\eta) + \ldots$$

taking inner products with $S_k(\eta)$ yields

$$c_k = \left\langle S_k(\eta),\ g_1(\eta)\right\rangle = \int_0^1 S_k(\eta)g_1(\eta)\,d\eta$$

since

$$g_1(1) = g_1'(1) = 0$$

hence

$$h_1(x) = -\sum_{k=1}^{\infty} \ell t_{1k}S_k(\eta)$$

Similarly it can be shown that

$$h_2(x) = -\sum_{k=1}^{\infty} \ell t_{2k}S_k(\eta)$$

Hence the beam deflection is expressible as a series in the mode shapes $S_k(\eta)$

$$u(x,t) = \ell \sum_{k=1}^{\infty} p_k(t) S_k(\eta) \qquad (3\text{-}19)$$

Returning to the modal coordinate Eq. (3-16) we have

$$\ddot{p}_1 + \Omega_1^2 p_1 = -\frac{1}{\ell}\int_0^1 S_1(\eta)\, d\eta \cdot a_0(t) - \int_0^1 \eta S_1(\eta)\, d\eta\, \dot{\omega}_0(t)$$

$$-\frac{1}{\ell}\int_0^1 \left[\frac{EI}{\rho} h_1^{iv}(x) - \Omega_1^2 h_1(x)\right] S_1(\eta)\, d\eta\, f_1(t) \qquad (3\text{-}20)$$

$$-\frac{1}{\ell}\int_0^1 \left[\frac{EI}{\rho} h_2^{iv}(x) - \Omega_1^2 h_2(x)\right] S_1(\eta)\, d\eta\, f_2(t)$$

Now

$$\int_0^1 \left[\frac{EI}{\rho} h_1^{iv}(x) - \Omega_1^2 h_1(x)\right] S_1(\eta)\, d\eta = \frac{4}{\rho\ell^2}\int_0^1 (21 - 30\eta) S_1(\eta)\, d\eta$$

$$- \frac{\lambda_1}{\rho\ell^2}\int_0^1 \left(\frac{3}{2}\eta^2 - 4\eta^3 + \frac{7}{2}\eta^4 - \eta^5\right) S_1(\eta)\, d\eta \qquad (3\text{-}21)$$

and

$$\int_0^1 \left[\frac{EI}{\rho} h_2^{iv}(x) - \Omega_1^2 h_2(x)\right] S_1(\eta)\, d\eta = \frac{1}{\rho\ell}\int_0^1 (-12 + 20\eta) S_1(\eta)\, d\eta$$

$$- \frac{\lambda_1}{\rho\ell}\int_0^1 \left(-\frac{1}{6}\eta^2 + \frac{1}{2}\eta^3 - \frac{1}{2}\eta^4 + \frac{1}{6}\eta^5\right) S_1(\eta)\, d\eta \qquad (3\text{-}22)$$

We must evaluate four weighted integrals of the eigenfunction. This can be readily accomplished through integration by parts and use of the geometric boundary conditions.

$$\int_0^1 \eta^2 S_1(\eta)\, d\eta = \frac{1}{\lambda_1} S_1'''(1) - \frac{2}{\lambda_1} S_1''(1) + \frac{2}{\lambda_1} S_1'(1)$$

$$\int_0^1 \eta^3 S_1(\eta)\, d\eta = \frac{1}{\lambda_1}\left[S_1'''(1) - 3S_1''(1) + 6S_1'(1) - 6S_1(1)\right]$$

$$(3\text{-}23)$$

$$\int_0^1 \eta^4 S_1(\eta)\, d\eta = \frac{1}{\lambda_1}\left[S_1'''(1) - 4S_1''(1) + 12S_1'(1) - 24S_1(1) + 24\int_0^1 S_1(\eta)\, d\eta\right]$$

$$\int_0^1 \eta^5 S_1(\eta)\, d\eta = \frac{1}{\lambda_1}\left[S_1'''(1) - 5S_1''(1) + 20S_1'(1) - 60S_1(1) + 120\int_0^1 \eta S_1(\eta)\, d\eta\right]$$

Using these results in (3-21) and (3-22) we obtain

$$\int_0^1 \left[\frac{EI}{\rho} h_1^{iv}(x) - \Omega_1^2 h_1(x)\right] S_1(\eta)\, d\eta = -\frac{1}{\rho\ell^2} S_1'(1)$$

$$\int_0^1 \left[\frac{EI}{\rho} h_2^{iv}(x) - \Omega_1^2 h_2(x)\right] S_1(\eta)\, d\eta = \frac{1}{\rho\ell} S_1(1)$$

Inserting the expressions for $f_1(t)$ and $f_2(t)$ and using the above results, the Eq. (3-20) can be written in its final form

$$\ddot{p}_i + \Omega_i^2 p_i = -u_{3i} \frac{a_0(t)}{\ell} - u_{4i}\dot{\omega}_0(t) + u_{1i} \frac{g_p(t)}{\rho\ell^3} + u_{2i} \frac{f_p(t)}{\rho\ell^2} \tag{3-24}$$

where the dimensionless modal parameters are defined by

$$\begin{aligned}
u_{1k} &= S_k'(1) \\
u_{2k} &= S_k(1) + c^*S_k'(1) \\
u_{3k} &= \int_0^1 S_k(\eta)\, d\eta + m^*S_k(1) + m^*c^*S_k'(1) \\
u_{4k} &= \int_0^1 \eta S_k(\eta)\, d\eta + m^*(1 + c^*)S_k(1) + (m^*c^* + J^*)S_k'(1)
\end{aligned} \tag{3-25}$$

3.2 Modal Parameter Identities

The modal parameter identities derived in this section are obtained by many formal operations and all series expansions are to be interpreted in the sense of convergence in the mean. Expanding $1 \cong c_1S_1(\eta) + c_2S_2(\eta) + \ldots$ and forming the inner product of both sides with $S_k(\eta)$ we find

$$c_k = \langle S_k, 1\rangle = \int_0^1 S_k(\eta)\, d\eta + m^*S_k(1) + m^*c^*S_k'(1) = u_{3k}$$

Hence

$$1 \cong \sum_{k=1}^{\infty} u_{3k} S_k(\eta)$$

Proceeding in the same fashion we have

$$\eta \cong \sum_{k=1}^{\infty} u_{4k} S_k(\eta)$$

Keeping the orthogonality condition (3-6) in mind we can write

$$1 = \sum_i \sum_j u_{3i} u_{3j} \int_0^1 S_i(\eta) S_j(\eta)\, d\eta$$

also

$$m^* = \sum_i \sum_j u_{3i} u_{3j} m^* S_i(1) S_j(1)$$

Differentiating the series expansion of 1: $0 \cong \sum_k u_{3k} S_k'(1)$, so

$$0 = \sum_i \sum_j u_{3i} u_{3j} J^* S_i'(1) S_j'(1)$$

Multiplying the series expansion of 1 with the above for 0

$$1 \cdot 0 = \sum_i \sum_j u_{3i} u_{3j} S_i(1) S_j'(1)$$

therefore

$$0 = \sum_i \sum_j u_{3i} u_{3j} m^* c^* \Big(S_i(1) S_j'(1) + S_i'(1) S_j(1) \Big)$$

Adding the above results we obtain

$$1 + m^* = \sum_{k=1}^{\infty} u_{3k}^2 \tag{3-26}$$

To get an identity on u_{4k} we proceed as above. Firstly

$$\eta^2 = \sum_i \sum_j u_{4i} u_{4j} S_i(\eta) S_j(\eta)$$

therefore

$$\frac{1}{3} = \sum_i \sum_j u_{4i} u_{4j} \int_0^1 S_i(\eta) S_j(\eta) \, d\eta$$

also

$$m^* = \sum_i \sum_j u_{4j} u_{4i} m^* S_i(1) S_j(1)$$

Differentiating the expansion for η

$$1 = \sum_{k=1}^{\infty} u_{4k} S_k'(1)$$

hence

$$J^* = \sum_i \sum_j u_{4i} u_{4j} J^* S_i'(1) S_j'(1)$$

We also have

$$1 = \sum_k u_{4k} S_k(1)$$

Multiplying by the expansion above

$$1 = \sum_i \sum_j u_{4i} u_{4j} S_i(1) S'_j(1)$$

Noting the symmetry of this expression in i, j

$$2m^*c^* = \sum_i \sum_j u_{4i} u_{4j} m^*c^* \Big(S_i(1) S'_j(1) + S_j(1) S'_i(1) \Big)$$

It follows by addition of the above results that

$$\sum_{k=1}^{\infty} u_{4k}^2 = \frac{1}{3} + m^* + J^* + 2m^*c^* \qquad (3\text{-}27)$$

We now proceed to get an identity on $u_{3k} \cdot u_{4k}$. From the first expansions

$$\eta = \sum_i \sum_j u_{3i} u_{4j} S_i(\eta) S_j(\eta)$$

Integrating over (0,1)

$$\frac{1}{2} = \sum_i \sum_j u_{3i} u_{4j} \int_0^1 S_i(\eta) S_j(\eta)\, d\eta$$

$$m^* = \sum_i \sum_j u_{3i} u_{4j} m^* S_i(1) S_j(1)$$

$$0 = \sum_i u_{3i} S'_i(1)$$

and

$$1 = \sum_j u_{4j} S'_j(1)$$

Multiplying these series

$$0 = \sum_i \sum_j u_{3i} u_{4j} S'_i(1) S'_j(1)$$

$$1 = \sum_i \sum_j u_{3i} u_{4j} S_i(1) S'_j(1)$$

and

$$0 = \sum_i \sum_j u_{3i} u_{4j} S'_i(1) S_j(1)$$

Hence

$$m^*c^* = \sum_i \sum_j u_{3i} u_{4j} m^*c^* \left(S_i(1) S'_j(1) + S'_i(1) S_j(1) \right)$$

Adding the above results we obtain

$$\sum_{k=1}^{\infty} u_{3k} u_{4k} = \frac{1}{2} + m^* + m^*c^* \tag{3-28}$$

We can obtain a simple identity amongst the parameters $\{u_{1k}\}$ by considering the expansion of η^2 over (0,1) in terms of the eigenfunctions, writing

$$\eta^2 = c_1 S_1(\eta) + c_2 S_2(\eta) + \ldots$$

we have

$$c_k = \left\langle \eta^2, S_k(\eta) \right\rangle$$

$$c_k = \int_0^1 \eta^2 S_k(\eta)\, d\eta + m^* S_k(1) + 2J^* S_k'(1) + m^* c^* \left(S_k'(1) + 2S_k(1) \right)$$

Evaluating the integral by the first of relations (3-23) and using the boundary conditions (3-5) we obtain

$$\eta^2 = \sum_{k=1}^{\infty} \frac{2}{\lambda_k} u_{1k} S_k(\eta)$$

Differentiating this result and evaluating at $\eta = 1$ we obtain

$$\sum_{k=1}^{\infty} \frac{u_{1k}^2}{\lambda_k} = 1 \qquad (3\text{-}29)$$

Now consider the expansion $\eta^3 = c_1 S_1(\eta) + c_2 S_2(\eta) + \ldots$

$$c_k = \int_0^1 \eta^3 S_k(\eta)\, d\eta + m^* S_k(1) + 3J^* S_k'(1) + m^* c^* [S_k'(1) + 3S_k(1)]$$

Evaluating the integral by the second relation in (3-23) and invoking (3-5) we obtain

$$\eta^3 = \sum_{k=1}^{\infty} \frac{6}{\lambda_k} [S_k'(1) - S_k(1)] S_k(\eta)$$

or equivalently

$$\eta^3 = 6 \sum_{k=1}^{\infty} \frac{1}{\lambda_k} [(1 + c^*) u_{1k} - u_{2k}] S_k(\eta)$$

Differentiating this result and evaluating at $\eta = 1$ we obtain

$$\frac{1}{2} = (1 + c^*) \sum_k \frac{u_{1k}^2}{\lambda_k} - \sum_k \frac{u_{2k} u_{1k}}{\lambda_k}$$

The first series is given by (3-29). Hence

$$\sum_{k=1}^{\infty} \frac{u_{1k} u_{2k}}{\lambda_k} = \frac{1}{2} + c^* \tag{3-30}$$

Evaluating the expansion for η^3 and its derivative at $\eta=1$ we can write

$$\frac{1}{6} + \frac{1}{2} c^* = \sum_k \frac{1}{\lambda_k} (1 + c^*) u_{1k} S_k(1) + \sum_k \frac{1}{\lambda_k} (c^* + c^{*2}) u_{1k} S_k'(1)$$
$$- \sum_k \frac{1}{\lambda_k} [u_{2k} (S_k(1) + c^* S_k'(1))]$$

$$\frac{1}{6} + \frac{1}{2} c^* = (1 + c^*) \sum_k \frac{u_{1k} u_{2k}}{\lambda_k} - \sum_k \frac{u_{2k}^2}{\lambda_k}$$

Using (3-30) we obtain

$$\sum_{k=1}^{\infty} \frac{u_{2k}^2}{\lambda_k} = \frac{1}{3} + c^* + c^{*2} \tag{3-31}$$

The model parameter identities derived above will prove extremely useful for numerical validation of digital simulations.

3.3 Approximate Solution

In the previous section the dynamics of the excited beam with tip body was solved in an exact fashion. The partial differential equation of motion was derived along with the time dependent boundary conditions. A great deal of labor was expended in constructing a solution which satisfied the governing equation and all boundary conditions. Indeed, it will be recalled, that satisfaction of the two natural boundary conditions at $x = \ell$ proved most difficult. The end result, (3-19), was that the structural deformation could be expressed as a series in the mode shapes $S_k(\eta)$ - mode shapes for a clamped beam with tip body. Although we were able to solve the boundary value problem (3-5), (3-6) for $S_k(\eta)$; these functions depend upon the parameters m*, c* and J* in a complicated fashion. The question arises whether we can expand the deformation in a series of simpler functions. Specifically, the mode shapes for a clamped beam without tip body suggest themselves, since they will serve the purpose for all beams with any tip body. Since these later eigenfunctions satisfy simple homogeneous boundary conditions at $x = \ell$, the boundary conditions (3-3), (3-4) will not be satisfied when the structural deformation is expanded in terms of these modes. This will lead to poor convergence of the series solution compared with the series solution based upon the former eigenfunctions.

We use the symbol $()^0$ to indicate a function or parameter based upon the simple clamped-free eigenfunctions as opposed to the clamped-tip body eigenfunctions. Hence

$$\frac{d^4}{d\eta^4} S_k^0 - \lambda_k^0 S_k^0 = 0$$

$$S_k^0(0) = S_k^{0\prime}(0) = 0$$

$$\frac{d^2}{d\eta^2} S_k^0 = \frac{d^3}{d\eta^3} S_k^0 = 0 \quad \text{at } \eta = 1 \tag{3-32}$$

These functions enjoy the simple orthonormality condition

$$\int_0^1 S_i^0(\eta) S_j^0(\eta)\, d\eta = \delta_{ij}$$

Analogous to (3-25) we have the tip bodyless modal parameters

$$u_{1k}^0 = S_k^{0\prime}(1)$$

$$u_{2k}^0 = S_k^0(1)$$

$$u_{3k}^0 = \int_0^1 S_k^0(\eta)\, d\eta$$

$$u_{4k}^0 = \int_0^1 \eta S_k^0(\eta)\, d\eta \tag{3-33}$$

Expanding the structural deformation in terms of these modes

$$u(x,t) = \ell \sum_k S_k^0(\eta) p_k(t) \tag{3-34}$$

It should be understood that this is only an approximation to $u(x,t)$ and that we sum over a finite number of terms. The generalized coordinates $p_k(t)$ will be determined via Lagrange's Equations for which expressions are required for the kinetic and potential energies in terms of the p_k and $\dot{p}_k$.

The potential energy in the beam is given by

$$V = \frac{1}{2} EI \int_0^{\ell} \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx$$

Using the expansion (3-34), integrating by parts, and invoking (3-32) we arrive at

$$V(t) = \frac{EI}{2\ell} \sum_k \lambda_k^0 p_k^2(t) \tag{3-35}$$

If we neglect terms of the order structural deformation × angular rate then the kinetic energy of the beam T_r can be written

$$T_r = \frac{1}{2} \rho \int_0^{\ell} [V_0 + x\omega_0 + \dot{u}(x,t)]^2 \, dx$$

where $V_0(t)$ is the inertial velocity at x = 0, directed along the y-axis. Inserting the expansion (3-34) we arrive at

$$T_r(t) = \frac{1}{6} \rho\ell^3\omega_0^2(t) + \frac{1}{2} \rho\ell^3 \sum_k \dot{p}_k^2(t) + \rho\ell^2 \sum_k (V_0(t)u_{3k}^0 + \ell\omega_0(t)u_{4k}^0)\dot{p}_k(t)$$

$$+ \frac{1}{2} \rho\ell V_0^2 + \frac{1}{2} \rho\ell^2 V_0\omega_0 \tag{3-36}$$

Similarly the kinetic energy of the tip body, T_p, can be expressed as

$$T_p = \frac{1}{2} m[V_0 + \omega_0\ell + \dot{u}(\ell,t)]^2 + \frac{1}{2} J\left[\omega_0 + \frac{\partial^2 u}{\partial t \, \partial x} (\ell,t)\right]^2$$

$$+ mc[V_0 + \omega_0\ell + \dot{u}(\ell,t)]\left[\omega_0 + \frac{\partial^2 u}{\partial t \, \partial x} (\ell,t)\right]$$

or

$$T_p = \frac{1}{2} m(V_0 + \omega_0 \ell)^2 + \frac{1}{2} J\omega_0^2 + mc\omega_0(V_0 + \omega_0 \ell)$$

$$+ \ell m(V_0 + \omega_0 \ell + c\omega_0) \sum_k u^0_{2k} \dot{p}_k(t) + [J\omega_0 + mc(V_0 + \omega_0 \ell)] \sum_k u^0_{1k} \dot{p}_k(t)$$

$$+ \frac{1}{2} m\ell^2 \sum_i \sum_j u^0_{2i} u^0_{2j} \dot{p}_i \dot{p}_j + \frac{1}{2} J \sum_i \sum_j u^0_{1i} u^0_{1j} \dot{p}_i \dot{p}_j + mc\ell \sum_i \sum_j u^0_{2i} u^0_{1j} \dot{p}_i \dot{p}_j \quad (3\text{-}37)$$

The last quadratic form in (3-37) can be rewritten in symmetric form as

$$\frac{1}{2} mc\ell \sum_i \sum_j (u^0_{2i} u^0_{1j} + u^0_{2j} u^0_{1i}) \dot{p}_i \dot{p}_j$$

Adding the expressions (3-36) and (3-37) we obtain the total kinetic energy. The virtual work δw performed by f_p and g_p is given by

$$\delta w = f_p \delta u(\ell,t) + (g_p + cf_p)\delta(u_x(\ell,t))$$

Using (3-34) we can write

$$\delta w = \sum_i P_i \delta p_i$$

where

$$P_i = (\ell u^0_{2i} + cu^0_{1i}) f_p + u^0_{1i} g_p \quad (3\text{-}38)$$

Lagranges equations are given by

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{p}_k} - \frac{\partial T}{\partial p_k} = - \frac{\partial V}{\partial p_k} + P_k \quad (k = 1, 2, 3, \ldots)$$

Inserting the appropriate expressions for T, V and P_k we arrive at the system of differential equations

$$[M]\ddot{\underline{p}} + \frac{EI}{\rho \ell^4} [K]\underline{p} = \underline{V}_1 \left(\frac{g_p}{\rho \ell^3}\right) + \underline{V}_2 \left(\frac{f_p}{\rho \ell^2}\right) - \underline{V}_3 \left(\frac{a_0}{\ell}\right) - \underline{V}_4 \dot{\omega}_0 \qquad (3\text{-}39)$$

where the symmetric positive definite matrices [M] and [K] are given by

$$M_{ij} = m^* u_{2i}^0 u_{2j}^0 + J^* u_{1i}^0 u_{1j}^0 + m^* c^* (u_{2i}^0 u_{1j}^0 + u_{2j}^0 u_{1i}^0) + \delta_{ij}$$

$$(i, j = 1, 2, \ldots)$$

$$(3\text{-}40)$$

$$[K] = \mathrm{diag}(\lambda_1^0, \lambda_2^0, \lambda_3^0, \ldots)$$

and the column vectors $\underline{V}_1$, $\underline{V}_2$, $\underline{V}_3$ and $\underline{V}_4$ are defined as

$$V_{1k} = u_{1k}^0$$

$$V_{2k} = u_{2k}^0 + c^* u_{1k}^0$$

$$V_{3k} = u_{3k}^0 + m^* c^* u_{1k}^0 + m^* u_{2k}^0$$

$$V_{4k} = u_{4k}^0 + m^*(1 + c^*) u_{2k}^0 + (m^* c^* + J^*) u_{1k}^0$$

$$(3\text{-}41)$$

Associated with the system (3-39) we have the eigenvalue problem

$$[K]\overline{e} = \mu [M]\overline{e} \qquad (3\text{-}42)$$

Here the eigenvalues $\mu_1, \mu_2, \ldots$ are all positive. Denote by [E] the matrix whose columns are the eigenvectors $\bar{e}_1, \bar{e}_2, \ldots$ and normalize the eigenvectors according to $[E^T][M][E] = [I]$. Hence

$$[E^T][K][E] = \mathrm{diag}(\mu_1, \mu_2, \ldots)$$

The equations (3-39) can now be uncoupled by applying the linear transformation

$$\underline{p} = [E]\underline{q}$$

$$\dot{\underline{q}} + \frac{EI}{\rho \ell^4} \mathrm{diag}(\mu_1, \mu_2, \ldots)\underline{q} = [E^T]\underline{V}_1\left(\frac{g_p}{\rho \ell^3}\right) + [E^T]\underline{V}_2\left(\frac{f_p}{\rho \ell^2}\right) - [E^T]\underline{V}_3\left(\frac{a_0}{\ell}\right) - [E^T]\underline{V}_4\dot{\omega}_0 \qquad (3\text{-}43)$$

In Section 3.1 we expanded the structural deformation in terms of the eigenfunctions $S_k(\eta)$ and arrived directly at the uncoupled system (3-24). Here we have expanded the structural deformation in terms of the eigenfunctions $S_k^0(\eta)$ and, as would be expected, the modal coordinate equations (3-39) do not uncouple. A transformation was required to arrive at the uncoupled set (3-43). It is reasonable to expect that as the number of retained modes $S_k^0(\eta)$ is increased the finite system (3-43) should approach the infinite system (3-24). More specifically

$$\mu_k \cong \lambda_k$$

$$\underline{u}_1 \cong [E^T]\underline{V}_1$$

$$\underline{u}_2 \cong [E^T]\underline{V}_2$$

$$\underline{u}_3 \cong [E^T]\underline{V}_3$$

$$\underline{u}_4 \cong [E^T]\underline{V}_4$$

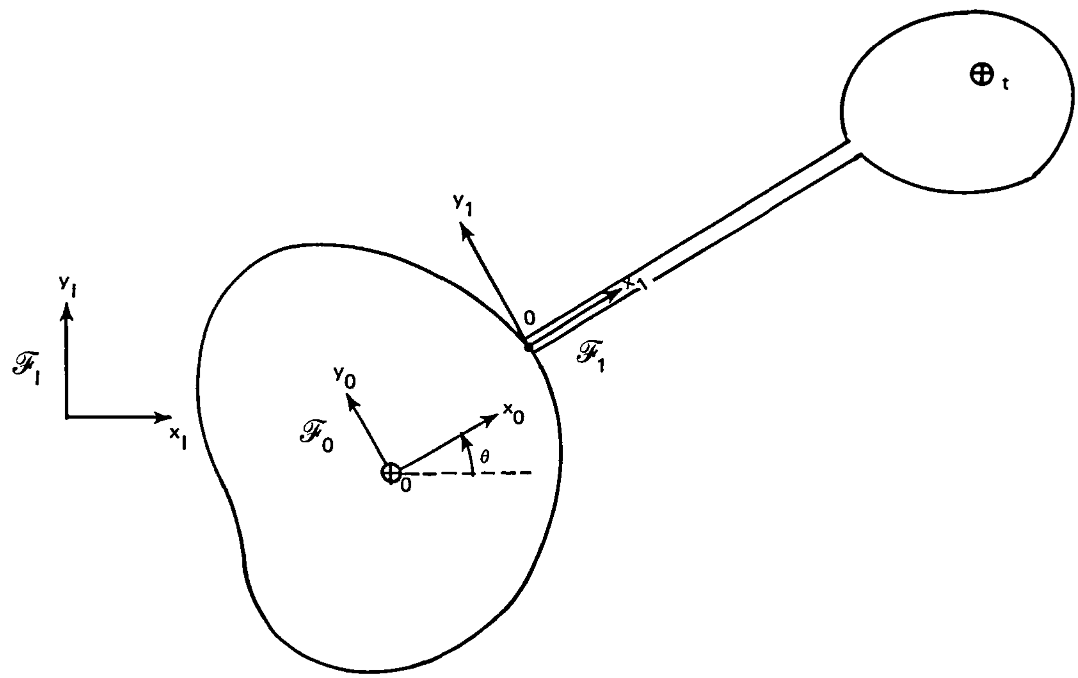
CHAPTER 4

A MODEL OF THE SHUTTLE ORBITER WITH DEPLOYED PAYLOAD

4.1 Motion Equations in Terms of Clamped-Tip Body Eigenfunctions

In this section we derive equations of motion for the vehicle presented in Figure 4.1. A large class of shuttle deployed payloads can be approximated in this fashion, specifically, as long slender beams with attached tip bodies. The orbiter and tip body are assumed rigid while the beam is allowed to undergo small, elastic transverse bending. The attachment point of the beam to the orbiter is located arbitrarily with respect to the mass center of the orbiter, but in keeping with other sections of this report, the mass center of the tip body is located along the tip tangent of the beam. (It should be noted that the difficulty discussed in Section 2.2 is still present due to axial loads exerted by the orbiter on the beam. This complicating affect will be ignored in the present analysis.) In order to fully understand the interaction between the flight control system and the flexible body dynamics, all rigid body motion and bending will be restricted to the orbiter pitch plane. Hence there are essentially three rigid body degrees of freedom—two translational and one rotational, and an infinite number of elastic degrees of freedom. Since the flight control system is primarily concerned with orbiter attitude control we analytically eliminate the translational coordinates so that the final set of differential equations of motion only involve orbiter attitude and elastic degrees of freedom.

The attitude control system exerts a net force $\underline{F}_0$ and moment $\underline{G}_0$ on the orbiter. There is an external torque $\underline{g}_p$ acting on the tip body and external force $\underline{f}_p$ applied at its mass center perpendicular to the neutral axis of the beam.



$\mathcal{F}_I$ INERTIAL FRAME

$\mathcal{F}_0$ ORBITER BODY FRAME FIXED AT ORBITER MASS CENTER ($\oplus_0$)

$\mathcal{F}_1$ BODY FRAME AT 0 CANTILEVERED AT BEAM TIP, PARALLEL TO $\mathcal{F}_0$
x_1 AXIS PASSES THROUGH TIP BODY MASS CENTER ($\oplus_t$)

θ PITCH ANGLE OF ORBITER MEASURED FROM x_I TO x_0

Figure 4.1. Planar dynamics model of Shuttle Orbiter with deployed payload.

Basic Mass and Geometric Parameters

m_0 - orbiter mass

I_0 - orbiter moment of inertia about its mass center perpendicular to plane of motion

$\underline{a}$ - vector from orbiter mass center to beam attachment point

ρ - mass per unit length of beam

ℓ - beam length

c - tip body mass center offset

m_t - tip body mass

I_t - tip body moment of inertia about its mass center perpendicular to plane of motion

It will prove convenient to define the following quantities.

$$m_1 = \rho\ell + m_t \qquad \text{mass of beam plus tip body}$$

$$m = m_0 + m_1 \qquad \text{total vehicle mass} \tag{4-1}$$

$$\mu_0 = m_0/m$$

$$\mu_1 = m_1/m$$

dimensionless mass ratios

$$b_1 = \frac{\rho\ell^2/2 + m_t(\ell + c)}{m_1}$$

location of mass center of beam + tip body relative to 0 under no deformation (4-2)

Primordial Motion Equations

Figure 4.2 is the free body diagram of two sub-bodies associated with the vehicle being analyzed: the orbiter, and the composite body of beam + tip body. $\underline{F}$ and $\underline{G}$ are respectively the interbody force and moment exerted by the beam on the orbiter at the attachment point 0.

We have the translational equations

$$\begin{aligned} m_0 \underline{a}^0_\oplus &= \underline{F}_0 + \underline{F} \\ m_1 \underline{a}^1_\oplus &= \underline{f}_p - \underline{F} \end{aligned} \tag{4-3}$$

where $\underline{a}^0_\oplus$ and $\underline{a}^1_\oplus$ are the inertial accelerations of the orbiter mass center and beam + tip body mass center respectively. The corresponding rotational equations are

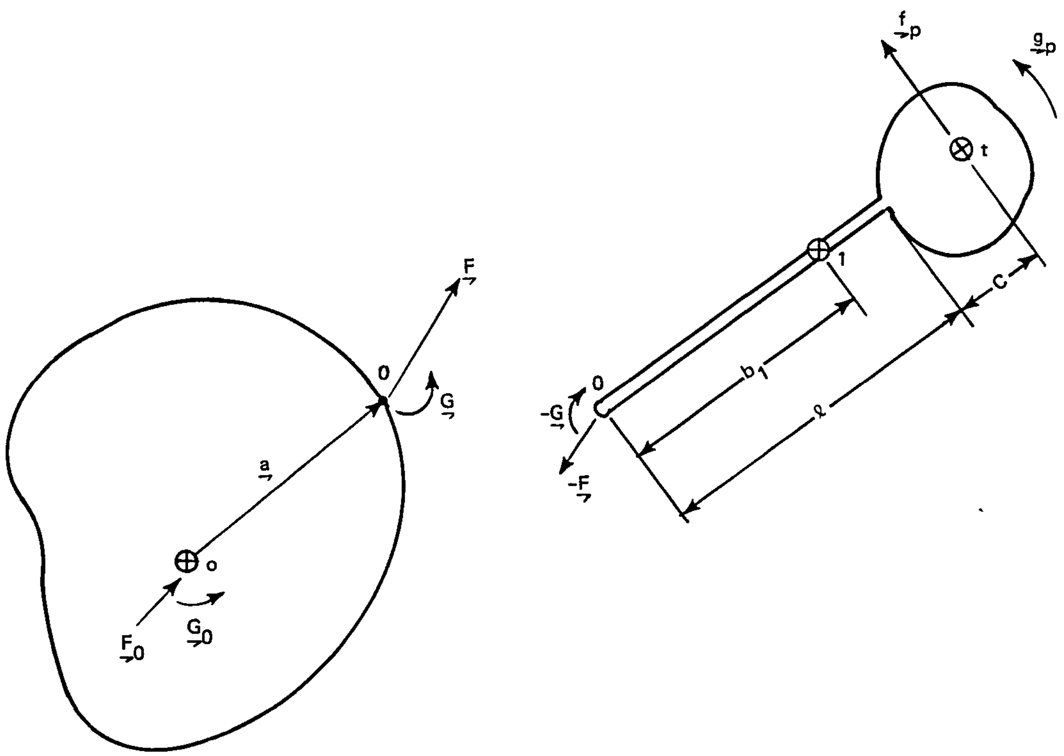


Figure 4.2. Free body diagrams.

$$\frac{d}{dt}\underline{h}^0_{\oplus} = \underline{G}_0 + \underline{G} + \underline{a} \times \underline{F} \tag{4-4}$$

$$\frac{d}{dt}\underline{h}^1_{\oplus} = -\underline{G} + \underline{g}_p + \underline{b} \times \underline{F} + (\underline{\ell} + \underline{c} - \underline{b} + \underline{\delta}_p) \times \underline{f}_p \tag{4-5}$$

where $\underline{h}^0_{\oplus}$, $\underline{h}^1_{\oplus}$ are the angular momenta of the orbiter and beam + tip body about their respective mass centers. The vector $\underline{b}$ locates the mass center of the deformed beam + tip body relative to 0 and is illustrated along with $\underline{\ell}$, $\underline{\delta}_p$ and $\underline{c}$ in Figure 4-3.

If $\underline{a}_0$ denotes the inertial acceleration of the attachment point 0, then we have the simple conditions

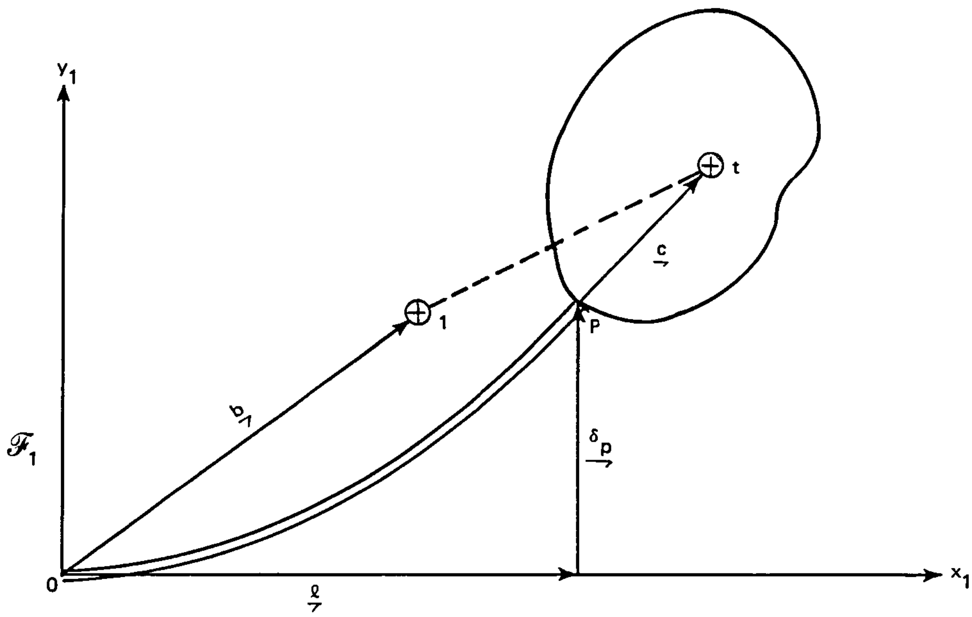


Figure 4-3. Deformed beam + tip body - vector geometry.

$$\underline{a}_0 = \underline{a}^0_\oplus + \frac{d^2}{dt^2}\underline{a}$$

$$\underline{a}^1_\oplus = \underline{a}_0 + \frac{d^2}{dt^2}\underline{b} \qquad (4\text{-}6)$$

Equation (4-5) can be rewritten in terms of $\underline{h}^1_0$ - the angular momentum of the beam + tip body about 0. Using the second of results (4-3) and (4-6) and the translation theorem for angular momentum we arrive at the more useful form

$$\frac{d}{dt}\underline{h}^1_0 = -\underline{G} - m_1\underline{b} \times \underline{a}_0 + \underline{g}_p + (\underline{\ell} + \underline{c} + \underline{\delta}_p) \times \underline{f}_p \qquad (4\text{-}7)$$

Ignoring completely the affect of axial acceleration on the beam + tip body, the elastic deformation of the system in Figure 4.3 is identical to the problem considered in Section 3.1. Indeed, if u(x,t) denotes the elastic deflection of the beam along the y axis of $\mathscr{F}_1$ the governing partial differential equation is given by (3-1) where $a_0(t)$ is now the component of $\underline{a}_0$ along the y_1 axis and $\dot{\omega}_0(t)$ is the inertial angular acceleration of $\mathscr{F}_1$ perpendicular to the plane of motion. By definition of the body frame $\mathscr{F}_1$, the geometric boundary conditions (3-2) apply. The natural boundary conditions are given by (3-3) and (3-4) with the following replacements

$$m \rightarrow m_t$$

$$J \rightarrow I_t + m_t c^2$$

It was shown that $u(x,t) = \ell \sum_{k=1}^{\infty} p_k(t) S_k(\eta)$ with the modal coordinate equations given by (3-24).

Elimination of Interbody Force $\underline{F}$ from Rotational Equations

Adding the translational equations (4-3) and invoking (4-6) we can show

$$\underline{a}_0 = \mu_0 \frac{d^2}{dt^2} \underline{a} - \mu_1 \frac{d^2}{dt^2} \underline{b} + m^{-1}(\underline{F}_0 + \underline{f}_p) \tag{4-8}$$

Inserting this expression into the first of (4-3) we obtain

$$\underline{F} = -m\mu_0\mu_1 \left(\frac{d^2}{dt^2} \underline{a} + \frac{d^2}{dt^2} \underline{b} \right) - \mu_1 \underline{F}_0 + \mu_0 \underline{f}_p \tag{4-9}$$

which expresses the interbody force in terms of kinematic quantities and the external forces on the vehicle.

The orbiter rotational motion equation (4-4) can now be written

$$\frac{d}{dt}\underline{h}^0_{\oplus} = \underline{G}_0 + \underline{G} - m\mu_0\mu_1\underline{a} \times \frac{d^2}{dt^2}(\underline{a} + \underline{b}) - \mu_1\underline{a} \times \underline{F}_0 + \mu_0\underline{a} \times \underline{f}_p \tag{4-10}$$

and the rotational motion equation (4-7) becomes

$$\frac{d}{dt}\underline{h}^1_0 = -\underline{G} - m\mu_1\underline{b} \times \frac{d^2}{dt^2}(\mu_0\underline{a} - \mu_1\underline{b}) - \mu_1\underline{b} \times \underline{F}_0 + \underline{g}_p + (\underline{\ell} + \underline{c} + \underline{\delta}_p - \mu_1\underline{b}) \times \underline{f}_p \tag{4-11}$$

Angular Momentum Calculations

Let $\bar{\imath}$, $\bar{\jmath}$, $\bar{k}$ be unit vectors along the x, y and z axes of $\mathscr{F}_0$ ($\mathscr{F}_1$) and let the plane of motion be the xy plane. If $\dot{\theta}$ is the orbiter pitch rate then

$$\underline{h}^0_{\oplus} = I_0\dot{\theta}\bar{k} + (\quad)\bar{\imath} + (\quad)\bar{\jmath} \tag{4-12}$$

If u(x,t) is the elastic deflection of the beam and we neglect the term $\dot{\theta}u$ then the angular momentum of the beam about 0 has a nonzero component along z given by

$$\int_0^{\ell} \rho x \frac{\partial u}{\partial t}(x,t)\,dx + \frac{1}{3}\rho\ell^3\dot{\theta}$$

The z component of the angular momentum of the tip body about 0 is

$$I_t\left[\dot{\theta} + \frac{\partial^2 u}{\partial t\,\partial x}(\ell,t)\right] + m_t\left[(\underline{\ell} + \underline{\delta}_p + \underline{c}) \times \frac{d}{dt}(\underline{\ell} + \underline{\delta}_p + \underline{c})\right]\cdot \bar{k}$$

We have

$$\underline{\ell} + \underline{\delta}_p + \underline{c} = (c + \ell)\,\overline{i} + \left[u(\ell,t) + c\,\frac{\partial u}{\partial x}(\ell,t)\right]\overline{j}$$

From this point onward terms of the order structural deflection × angular rate will be dropped as well as any derivatives of such quantities. It then can be shown that the z component of the angular momentum of the tip body about 0 is given by

$$I_t\left[\dot{\theta} + \frac{\partial^2 u}{\partial t\,\partial x}(\ell,t)\right] + m_t(c + \ell)\left[(c + \ell)\,\dot{\theta} + \frac{\partial u}{\partial t}(\ell,t) + c\,\frac{\partial^2 u}{\partial t\,\partial x}(\ell,t)\right]$$

Adding these two contributions it follows that

$$\underline{h}_0^1 \cdot \hat{k} = J_0\dot{\theta} + \int_0^{\ell} \rho x\,\frac{\partial u}{\partial t}(x,t)\,dx + I_t\,\frac{\partial^2 u}{\partial t\,\partial x}(\ell,t) + m_t(c + \ell)\left[\frac{\partial u}{\partial t}(\ell,t) + c\,\frac{\partial^2 u}{\partial t\,\partial x}(\ell,t)\right] \tag{4-13}$$

where J_0 is the inertia of the beam + tip body about 0 in the undeformed state

$$J_0 = \frac{1}{3}\rho\ell^3 + I_t + m_t(c + \ell)^2 \tag{4-14}$$

Differentiating (4-12), (4-13) it follows that

$$\left(\frac{d}{dt}\,\underline{h}_{\oplus}^0\right) \cdot \hat{k} = I_0\dot{\theta} \tag{4-15}$$

$$\left(\frac{d}{dt} \underline{h}_0^1\right) \cdot \hat{k} = J_0 \ddot{\theta} + \int_0^\ell \rho x \frac{\partial^2 u}{\partial t^2} (x,t)\, dx + I_t \frac{\partial^3 u}{\partial t^2 \partial x} (\ell,t)$$

$$+ m_t (\ell + c) \left[\frac{\partial^2 u}{\partial t^2} (\ell,t) + c \frac{\partial^3 u}{\partial t^2 \partial x} (\ell,t) \right] \tag{4-16}$$

Vector Geometry

The motion equations (4-10), (4-11) call for various geometric vectors and their time derivatives. Recall that $\underline{b}$ is the position vector of the mass center of beam + tip body relative to 0.

$$\underline{b} = b_1 \bar{\imath} + \frac{1}{m_1} \left\{ \int_0^\ell u(x,t) \rho\, dx + m_t \left[u(\ell,t) + c \frac{\partial u}{\partial x} (\ell,t) \right] \right\} \bar{\jmath}$$

Using the series expansion for u(x,t) and definition (3-25) for the modal parameter u_{3k} we can show

$$\underline{b} = b_1 \bar{\imath} + u_c(t) \bar{\jmath} \tag{4-17}$$

where $u_c(t)$, the shift in the mass center of the beam + tip body due to elastic deformation, is given by

$$u_c(t) = \frac{\rho \ell^2}{m_1} \sum_{k=1}^{\infty} u_{3k}\, p_k(t) \tag{4-18}$$

It follows that

$$\frac{d}{dt} \underline{b} = -u_c(t) \dot{\theta} \bar{\imath} + (b_1 \dot{\theta} + \dot{u}_c) \bar{\jmath}$$

$$\cong (b_1 \dot{\theta} + \dot{u}_c) \bar{\jmath}$$

(since u_c is of order structural deflection)

$$\frac{d^2}{dt^2}\underline{b} = -b_1\dot{\theta}^2\bar{\imath} + (b_1\ddot{\theta} + \dot{u}_c)\bar{\jmath} \tag{4-19}$$

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} \text{ in } \mathscr{F}_0(\mathscr{F}_1) \tag{4-20}$$

It follows that

$$\frac{d}{dt}\underline{a} = (-a_2\dot{\theta},\ a_1\dot{\theta},\ 0)^T$$

$$\frac{d^2}{dt^2}\underline{a} = \begin{pmatrix} -a_2\ddot{\theta} - a_1\dot{\theta}^2 \\ a_1\ddot{\theta} - a_2\dot{\theta}^2 \\ 0 \end{pmatrix} \tag{4-21}$$

$$\underline{a} \times \frac{d^2}{dt^2}\underline{a} = \left(a_1^2 + a_2^2\right)\ddot{\theta}\,\bar{k}$$

$$\underline{a} \times \frac{d^2}{dt^2}\underline{b} = \left(a_1\ddot{u}_c + a_1b_1\ddot{\theta} + a_2b_1\dot{\theta}^2\right)\bar{k}$$

$$\underline{b} \times \frac{d^2}{dt^2}\underline{a} = \left(b_1a_1\ddot{\theta} - b_1a_2\dot{\theta}^2\right)\bar{k}$$

$$\underline{b} \times \frac{d^2}{dt^2}\underline{b} = \left(b_1\ddot{u}_c + b_1^2\ddot{\theta}\right)\bar{k}$$

(4-22)

Writing $\underline{F}_0 = \begin{pmatrix} F_{01} \\ F_{02} \\ 0 \end{pmatrix}, \quad \underline{f}_p = \begin{pmatrix} 0 \\ f_p \\ 0 \end{pmatrix}$ in $\mathscr{F}_0(\mathscr{F}_1)$

$$\underline{a} \times \underline{F}_0 = (a_1 F_{02}(t) - a_2 F_{01}(t))\overline{k}$$

$$\underline{a} \times \underline{f}_p = a_1 f_p(t)\overline{k}$$

$$\underline{b} \times \underline{F}_0 = (b_1 F_{02}(t) - u_c(t) F_{01}(t))\overline{k}$$

$$(\underline{\ell} + \underline{c} + \underline{\delta}_p - \mu_1 \underline{b}) \times \underline{f}_p = (\ell + c - \mu_1 b_1) f_p(t)\overline{k} \tag{4-23}$$

Rotational Motion Equations - Expanded Form

Using the above results with (4-15) allows us to write (4-10) in the form

$$\left[I_0 + m\mu_0\mu_1\left(a_1^2 + a_2^2 + a_1 b_1\right)\right]\ddot{\theta} + m\mu_0\mu_1 a_1 \ddot{u}_c = G_0(t) + G_z(t) - \mu_1 a_1 F_{02}(t) + \mu_1 a_2 F_{01}(t) + \mu_0 a_1 f_p(t) - m\mu_0\mu_1 a_2 b_1 \dot{\theta}^2 \tag{4-24}$$

Here $G_0(t)$ is the component of $\underline{G}_0$ perpendicular to the plane of motion (only nonzero component by assumption) and $G_z(t)$ is the corresponding component of $\underline{G}$.

Similarly with the aid of (4-16) we can write (4-11) in the form

$$\left(J_0 + m\mu_0\mu_1 b_1 a_1 - m\mu_1^2 b_1^2\right)\ddot{\theta}$$

$$+ \int_0^\ell \rho x \frac{\partial^2 u}{\partial t^2}(x,t)\, dx + I_t \frac{\partial^3 u}{\partial t^2\, \partial x}(\ell,t)$$

$$+ m_t(\ell + c)\left[\frac{\partial^2 u}{\partial t^2}(\ell,t) + c\frac{\partial^3 u}{\partial t^2\, \partial x}(\ell,t)\right]$$

$$- m\mu_1^2 b_1 \ddot{u}_c = -G_z(t) - \mu_1 b_1 F_{02}(t) + \mu_1 u_c(t) F_{01}(t)$$

$$+ g_p(t) + (\ell + c - \mu_1 b_1) f_p(t)$$

$$+ m\mu_0\mu_1 b_1 a_2 \dot{\theta}^2 \qquad (4\text{-}25)$$

Using the series expansion for u(x,t) and definition (3-25) for the modal parameter u_{4k} we can show that

$$\int_0^\ell \rho x \frac{\partial^2 u}{\partial t^2}(x,t)\, dx + I_t \frac{\partial^3 u}{\partial t^2\, \partial x}(\ell,t)$$

$$+ m_t(\ell + c)\left[\frac{\partial^2 u}{\partial t^2}(\ell,t) + c\frac{\partial^3 u}{\partial t^2\, \partial x}(\ell,t)\right] = \rho\ell^3 \sum_{k=1}^{\infty} u_{4k}\ddot{p}_k(t)$$

Introducing this result into (4-25) and adding with (4-24) eliminates the interbody torque $G_z(t)$

$$\left[I_0 + J_0 + m\mu_0\mu_1\left(a_1^2 + a_2^2 + 2a_1b_1\right) - m\mu_1^2b_1^2\right]\ddot{\theta}$$

$$+ \rho\ell^2 \sum_{k=1}^{\infty} (\mu_0 a_1 u_{3k} + \ell u_{4k} - \mu_1 b_1 u_{3k}) p_k = G_0(t) - \mu_1(a_1 + b_1)F_{02}(t)$$

$$+ \mu_1(a_2 + u_c(t))F_{01}(t) + g_p(t)$$

$$+ (\mu_0 a_1 + \ell + c - \mu_1 b_1) f_p(t)$$

(4-26)

As discussed above, the modal coordinates $p_k(t)$ are governed by (3-24) with $\dot{\omega}_0(t) = \ddot{\theta}$ and $a_0(t)$ being the component of $\underline{a}_0$ along the y_1 axis. From (4-8) with (4-19) and (4-21)

$$a_0(t) = (\mu_0 a_1 - \mu_1 b_1)\ddot{\theta} - \frac{\rho\ell^2}{m} \sum_k u_{3k}\ddot{p}_k + \frac{1}{m}(F_{02} + f_p) - \mu_0 a_2 \dot{\theta}^2$$

The modal coordinate differential equation assumes the form

$$\left(\mu_0 \frac{a_1}{\ell} u_{3i} - \mu_1 \frac{b_1}{\ell} u_{3i} + u_{4i}\right)\ddot{\theta}$$

$$+ \sum_{k=1}^{\infty} \left(\delta_{ik} - \frac{\rho\ell}{m} u_{3i}u_{3k}\right)\ddot{p}_k = u_{1i} \frac{g_p(t)}{\rho\ell^3} + u_{2i} \frac{f_p(t)}{\rho\ell^2}$$

$$- \frac{u_{3i}}{m\ell}(F_{02} + f_p) + u_{3i}\mu_0 \frac{a_2}{\ell}\dot{\theta}^2 - \Omega_i^2 p_i(t)$$

$$(i = 1, 2, 3, \ldots)$$

(4-27)

Note the coupling of the modal coordinate equation with all the elastic coordinates as well as orbiter pitch. Introduce the following notation

$$a_{00} = \frac{I_0 + J_0}{\rho \ell^3} + (m/\rho\ell)\mu_0\mu_1 \left(\frac{a_1^2}{\ell^2} + \frac{a_2^2}{\ell^2} + 2\frac{a_1}{\ell} \cdot \frac{b_1}{\ell} \right) - (m/\rho\ell)\mu_1^2 b_1^2/\ell^2$$

$$a_{1k} = \mu_0 \frac{a_1}{\ell} u_{3k} + u_{4k} - \mu_1 \frac{b_1}{\ell} u_{3k} \quad (k = 1,2,3, \ldots) \tag{4-28}$$

Equations (4-26), (4-27) can now be written

$$a_{00}\ddot{\theta} + \sum_{k=1}^{\infty} a_{1k}\ddot{p}_k = \frac{G_0(t)}{\rho\ell^3} - \mu_1 \left(\frac{a_1 + b_1}{\ell} \right) \frac{F_{02}(t)}{\rho\ell^2} + \mu_1 \frac{(a_2 + u_c)}{\ell} \cdot \frac{F_{01}(t)}{\rho\ell^2} + \frac{g_p(t)}{\rho\ell^3} + \left(\mu_0 \frac{a_1}{\ell} + 1 + c^* - \mu_1 \frac{b_1}{\ell} \right) \frac{f_p(t)}{\rho\ell^2} \tag{4-29}$$

$$a_{1i}\ddot{\theta} + \sum_{k=1}^{\infty} \left(\delta_{ik} - \frac{\rho\ell}{m} u_{3i}u_{3k} \right) \ddot{p}_k = u_{1i} \frac{g_p(t)}{\rho\ell^3} + u_{2i} \frac{f_p(t)}{\rho\ell^2} - \frac{u_{3i}}{m\ell}(F_{02}(t) + f_p(t)) + u_{3i}\mu_0 \frac{a_2}{\ell} \dot{\theta}^2 - \Omega_i^2 p_i(t)$$

$$(i = 1, 2, 3, \ldots) \tag{4-30}$$

Equations (4-29) and (4-30) are the final set of differential equations of motion involving orbiter pitch and elastic coordinates of the beam. The coefficients of the accelerations are constant but (4-29) contains a time dependent coefficient through the term $F_{01}(t)u_c(t)$ while (4-30) has a nonlinear term in $\dot{\theta}$.

Matrix-Vector Form of Motion Equations

Define

$$\mathcal{F}_0(t) \equiv \frac{G_0(t)}{\rho \ell^3} - \mu_1 \left(\frac{a_1 + b_1}{\ell}\right) \frac{F_{02}(t)}{\rho \ell^2} + \mu_1 \frac{a_2}{\ell} \frac{F_{01}(t)}{\rho \ell^2} + \frac{g_p(t)}{\rho \ell^3}$$

$$+ \left(1 + c^* + \mu_0 \frac{a_1}{\ell} - \mu_1 \frac{b_1}{\ell}\right) \frac{f_p(t)}{\rho \ell^2}$$

$$\mathcal{F}_i(t) \equiv u_{1i} \frac{g_p(t)}{\rho \ell^3} + u_{2i} \frac{f_p(t)}{\rho \ell^2} - \frac{u_{3i}}{m\ell} \left(F_{02}(t) + f_p(t)\right)$$

$$(i = 1, 2, 3, \ldots) \tag{4-31}$$

The generalized forces $\mathcal{F}_0(t)$ and $\mathcal{F}_i(t)$ are functions only of the external forces and moments acting upon the vehicle. Let

$$\bar{x} = (\theta, p_1, p_2, p_3, \ldots)^T, \quad \bar{\mathcal{F}} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots)^T$$

$$\bar{N} = \begin{bmatrix} \mu_1 \dfrac{u_c(t)}{\ell} \dfrac{F_{01}(t)}{\rho \ell^2} \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} + \mu_0 \frac{a_2}{\ell} \begin{bmatrix} 0 \\ u_{31} \\ u_{32} \\ u_{33} \\ \vdots \end{bmatrix} \dot{\theta}^2(t) \tag{4-32}$$

We have the symmetric generalized mass matrix A

$$A = \begin{bmatrix} a_{00} & a_{11} & a_{12} & a_{13} & \cdots \\ a_{11} & \left(1 - \frac{\rho\ell}{m} u_{31}^2\right) & -\frac{\rho\ell}{m} u_{31}u_{32} & -\frac{\rho\ell}{m} u_{31}u_{33} & \cdots \\ a_{12} & -\frac{\rho\ell}{m} u_{32}u_{31} & \left(1 - \frac{\rho\ell}{m} u_{32}^2\right) & -\frac{\rho\ell}{m} u_{32}u_{33} & \cdots \\ a_{13} & -\frac{\rho\ell}{m} u_{33}u_{31} & -\frac{\rho\ell}{m} u_{33}u_{32} & \left(1 - \frac{\rho\ell}{m} u_{33}^2\right) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix} \tag{4-33}$$

and stiffness matrix

$$[K] = \operatorname{diag}\left(0,\ \Omega_1^2,\ \Omega_2^2,\ \Omega_3^2,\ \ldots\right) \tag{4-34}$$

In terms of the above notation Eqs. (4-29), (4-30) are

$$[A] \frac{d^2\overline{x}}{dt^2} + [K]\overline{x} = \overline{\mathscr{F}}(t) + \overline{N}(t;\ \overline{x}, \dot{\overline{x}})$$

4.2 Motion Equations in Terms of Clamped-Free Eigenfunctions

We treat here the same vehicle as in Section 4.1; the only difference being the set of eigenfunctions used to expand the elastic displacement in the beam. Here we use the tip-bodyless eigenfunctions $S_k^0(\eta)$ described in Section 3.3. For the same reasons as given there we expect that, for the same degree of truncation, the results will be inferior to those obtained by use of the motion equations of the previous section. The results of the present section in conjunction with our previous analysis can serve as a basis for numerical confirmation of this statement.

Throughout this section it will be understood that the elastic deflection in the beam is expanded in terms of the set $\{S_k^0(\eta)\}$

$$u(x,t) \cong \ell \sum_k S_k^0(\eta) p_k(t)$$

No confusion should arise between the modal coordinates $p_k(t)$ and those appearing in Section 4.1. Virtually all results previously obtained remain intact except in those instances where explicit use was made of the series expansion for $u(x,t)$. We list sequentially those equations in Section 4.1 which require modification due to a switch of the modal basis from $\{S_k(\eta)\}$ to $\{S_k^0(\eta)\}$.

(1) Equation (4-18) is replaced by

$$u_c(t) = \frac{\rho \ell^2}{m_1} \sum_k V_{3k} p_k(t) \qquad (V_{3k} \text{ is given by Eq. (3-41)})$$

(2) Equation (4-26) - second term on left hand side is replaced by

$$\rho \ell^3 \sum_k \left(\mu_0 \frac{a_1}{\ell} V_{3k} + V_{4k} - \mu_1 \frac{b_1}{\ell} V_{3k}\right) \ddot{p}_k(t)$$

(3) The modal coordinates are now governed by Eq. (3-39) with

$$a_0(t) = (\mu_0 a_1 - \mu_1 b_1)\ddot{\theta} - \frac{\rho \ell^2}{m} \sum_k V_{3k} \ddot{p}_k + \frac{1}{m}(F_{02} + f_p) - \mu_0 a_2 \dot{\theta}^2$$

and $\dot{\omega}_0 = \ddot{\theta}$. Thus Equation (4-27) is replaced by

$$\left(V_{4i} + \mu_0 \frac{a_1}{\ell} V_{3i} - \mu_1 \frac{b_1}{\ell} V_{3i}\right)\dot{\theta}$$

$$+ \sum_k \left(M_{ik} - \frac{\rho\ell}{m} V_{3i}V_{3k}\right)\dot{p}_k = V_{1i} \frac{g_p}{\rho\ell^3} + V_{2i} \frac{f_p}{\rho\ell^2} - \frac{V_{3i}}{m\ell} (F_{02} + f_p)$$

$$+ V_{3i}\mu_0 \frac{a_2}{\ell} \dot{\theta}^2 - \frac{EI}{\rho\ell^4} \lambda_i^0 p_i(t)$$

where M_{ik} is given by Eq. (3-40).

(4) Equation (4-28) is replaced by

$$a_{1k} = \mu_0 \frac{a_1}{\ell} V_{3k} + V_{4k} - \mu_1 \frac{b_1}{\ell} V_{3k} \quad (k = 1, 2, 3, \ldots)$$

(5) Equation (4-30) is replaced by

$$a_{1i}\ddot{\theta} + \sum_k \left(M_{ik} - \frac{\rho\ell}{m} V_{3i}V_{3k}\right)\ddot{p}_k = V_{1i} \frac{g_p}{\rho\ell^3} + V_{2i} \frac{f_p}{\rho\ell^2} - \frac{V_{3i}}{m\ell} (F_{02} + f_p)$$

$$+ V_{3i}\mu_0 \frac{a_2}{\ell} \dot{\theta}^2 - \frac{EI}{\rho\ell^4} \lambda_i^0 p_i(t)$$

(with a_{1i} given by (4) above).

(6) The second definition (4-31) is replaced by

$$\mathscr{F}_i(t) \equiv V_{1i} \frac{g_p}{\rho\ell^3} + V_{2i} \frac{f_p}{\rho\ell^2} - \frac{V_{3i}}{m\ell} (F_{02} + f_p)$$

(7) In Eq. (4-32) use the expression for $u_c(t)$ given by (1) above and replace second vector with

$$\mu_0 \frac{a_2}{\ell} \dot{\theta}^2(t) (0 \quad V_{31} \quad V_{32} \quad V_{33} \quad V_{34} \quad \ldots)^T$$

(8) Definition (4-33) is replaced with

$$A = \begin{bmatrix} a_{00} & a_{11} & a_{12} & a_{13} & \cdots \\ a_{11} & \left(M_{11} - \frac{\rho\ell}{m} V_{31}V_{31}\right) & \left(M_{12} - \frac{\rho\ell}{m} V_{31}V_{32}\right) & \left(M_{13} - \frac{\rho\ell}{m} V_{31}V_{33}\right) & \cdots \\ a_{12} & \left(M_{21} - \frac{\rho\ell}{m} V_{32}V_{31}\right) & \left(M_{22} - \frac{\rho\ell}{m} V_{32}V_{32}\right) & \left(M_{23} - \frac{\rho\ell}{m} V_{32}V_{33}\right) & \cdots \\ a_{13} & \left(M_{31} - \frac{\rho\ell}{m} V_{33}V_{31}\right) & \left(M_{32} - \frac{\rho\ell}{m} V_{33}V_{32}\right) & \left(M_{33} - \frac{\rho\ell}{m} V_{33}V_{33}\right) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

(9) Definition (4-34) is replaced by

$$[K] = \frac{EI}{\rho\ell^4} \operatorname{diag}(0, \lambda_1^0, \lambda_2^0, \lambda_3^0, \ldots)$$

Note: these are the squares of the natural frequencies of a uniform clamped beam without tip body.

4.3 Differential Equations of Motion - First Order Form

In the last two sections equations of motion were derived for the vehicle depicted in Figure 4.1. These equations were of the form of a system of second order differential equations and are not directly suitable for digital implementation. Since these equations are linear in $\frac{d^2\bar{x}}{dt^2}$ they can be transformed into an equivalent set of first order differential equations. In effecting this transformation the "system natural frequencies" reveal themselves and the option of further modal truncation (beyond the truncation of elastic degrees of freedom in the beam) can be adopted.

Our motion equations can be written

$$[A]\frac{d^2}{dt^2}\underline{x} + \frac{EI}{\rho\ell^4}[B]\underline{x} = \underline{\mathscr{F}}(t) + \underline{N}(t;\ \underline{x},\ \dot{\underline{x}}) \tag{4-35}$$

where

$$[B] = \mathrm{diag}(0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots\ \lambda_n)$$

If we follow the procedure in Section 4.1 the λ_i are the eigenvalues of the problem for a clamped beam with tip body (Equation 3-5) while if we follow the procedure in Section 4.2 they are the eigenvalues appropriate to a clamped beam without tip body (Equation 3-32).

The free vibration of the system associated with (4-35) leads to the eigenvalue problem

$$[B]\underline{V} = \mu[A]\underline{V} \tag{4-36}$$

and the "system natural frequencies" ω^s are given by

$$\omega^s = \left(\frac{EI}{\rho\ell^4}\right)^{1/2}\sqrt{\mu_s}$$

[B] is symmetric, positive-semidefinite and [A] is symmetric, positive definite. All eigenvalues are therefore nonnegative. Specifically $\mu = 0$ is a simple eigenvalue with eigenvector $(1,\ 0,\ 0,\ \ldots,\ 0)^T$ corresponding to rigid body rotation. This eigenvalue problem has a full set of linearly independent eigenvectors $\underline{V}^{(1)}$, $\underline{V}^{(2)}$, ..., $\underline{V}^{(n+1)}$ and we can always arrange that they be orthonormal relative to [A], i.e.

$$\underline{V}^{(i)T}[A]\underline{V}^{(j)} = \delta_{ij}$$

Define

$$[Z] = [\underline{V}^{(1)} \quad \underline{V}^{(2)} \quad \underline{V}^{(3)} \quad \cdots \quad \underline{V}^{(n+1)}] \tag{4-37}$$

If we make the linear transformation $\underline{x} = [Z]\underline{q}$ in (4-35) we arrive at

$$\frac{d^2}{dt^2}\underline{q} + \frac{EI}{\rho\ell^4}\,\mathrm{diag}(\mu_1\mu_2 \cdots \mu_{n+1})\underline{q} = [Z^T](\mathscr{F}(t) + \underline{N}(t;\ \underline{x},\ \dot{\underline{x}})) \tag{4-38}$$

Let

$$\underline{y}(t) = \left[\begin{array}{c}\underline{q}(t)\\ \hline \dot{\underline{q}}(t)\end{array}\right]$$

and label the eigenvalues so that $\mu_1 = 0$.

The system (4-38) can now be written as

$$\frac{d}{dt}\underline{y} = \left[\begin{array}{c} y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ \vdots \\ y_{2n+2} \\ \hline [Z^T](\mathscr{F} + \underline{N}) - \frac{EI}{\rho\ell^4}(0,\ \mu_2 y_2,\ \mu_3 y_3,\ \ldots,\ \mu_{n+1}y_{n+1})^T \end{array}\right] \tag{4-39}$$

CHAPTER 5

FORTRAN PROGRAMS

Two separate FORTRAN computer programs have been created for the purpose of numerically implementing the motion equations of Sections 4.1 and 4.3. Complete listings of these programs accompanied by annotated sample input and output data are provided in Appendices B and C. Each is extensively commented throughout and in most instances the FORTRAN variable names are mnemonically similar to corresponding analytical quantities. Where relevant, reference is made to specific equations of the report.

The program of Appendix B computes the eigenvalues, modal parameters and modal parameter identities of a cantilevered beam with tip body. Note that these computations are performed in (IBM) quadruple precision due to the numerical sensitivity of the transcendental expressions involved. The roots of the characteristic equation (Eq. 2-19) are estimated by incrementing the parameter β and searching for sign changes in the left-hand-side. Root estimates are then improved by Newton-Raphson iteration. The modal parameters of Eq. 3-25 are computed and together with the roots of Eq. 2-19 (raised to the fourth power) written to a disc data set to be used by the motion equation program (described below). The partial sums and asymptotic values of the modal parameter identities (Eqs. 3-26 thru 3-31) are evaluated and output. This information serves to check the calculations of the modal parameters and can indicate "missed" roots of the characteristic equation by showing poor convergence.

The program of Appendix C numerically synthesizes and integrates the motion equations for the planar dynamics of an elastic beam with rigid bodies at each end. The formulation implemented is that for which the beam deflection is expanded as a series of eigenfunctions appropriate to a clamped beam with tip body (as per Section 4.1). The eigenvalues and modal parameters generated by the above program are read and checked for consistency with the NAMELIST input data. Note that the initial conditions on the attitude angle and its rate may be specified arbitrarily as input, while those of the modal coordinates and their time derivatives are internally set to zero. The algebraic eigenvalue problem of Eq. 4-36 is solved via the double precision IMSL subroutine EIGZS. The "system natural frequencies" are output and are helpful for selection of an integration time step. External forces and torques on the "Orbiter" and tip body are specified through subroutines ORBFOR and TPBFOR respectively. The motion equations in first order form, Eq. 4-39 are integrated using the Adams Method with third order differences.

APPENDIX A

EIGENFUNCTION NORMALIZATION FOR CANTILEVERED BEAM WITH TIP BODY

In Section 3.1 extensive use was made of the eigenfunctions $S_k(\eta)$ defined by the boundary value problem (3-5). In this appendix we formulate a procedure to obtain these normalized eigenfunctions from a set of non-normalized eigenfunctions. Certain definite integrals involving the eigenfunctions are evaluated to facilitate computation of the modal parameters given in Eqs. (3-25).

Recall that for the case of a simple beam with no tip body the normalized clamped-free eigenfunctions are given by

$$Y_k(\eta) = \cosh \beta_k \eta - \cos \beta_k \eta + \frac{\sin \beta_k - \sinh \beta_k}{\cos \beta_k + \cosh \beta_k} (\sinh \beta_k \eta - \sin \beta_k \eta)$$

where β_k is a root of

$$\cosh \beta \cos \beta + 1 = 0$$

and the normalization condition is

$$\int_0^1 Y_k^2(\eta)\, d\eta = 1$$

To obtain a set of eigenfunctions for the case of a beam with tip body we can, in the discussion preceding Eqs. (2-17), (2-18), choose the constant $c_4 = 1$. (So $c_2 = -1$). Invoking the boundary condition (2-18) the resulting unnormalized eigenfunction $\psi_k(\eta)$ is given by

$$\psi_k(\eta) = \cosh \beta_k\eta - \cos \beta_k\eta + \gamma_k(\sinh \beta_k\eta - \sin \beta_k\eta) \qquad (A.1)$$

where now β_k are the roots of Eq. (2-19) and the parameter γ_k is given by

$$\gamma = \frac{m^*c^*\beta^2(\cosh \beta - \cos \beta) + J^*\beta^3(\sin \beta + \sinh \beta) - \cos \beta - \cosh \beta}{m^*c^*\beta^2(\sin \beta - \sinh \beta) + J^*\beta^3(\cos \beta - \cosh \beta) + \sin \beta + \sinh \beta}$$

where subscript k is deleted for clarity. (Note that for $m^* = J^* = 0$ this eigenfunction is already normalized).

Using the eigenfunction (A.1) we readily evaluate the following integrals

$$\begin{aligned}\int_0^1 \psi^2(\eta)\, d\eta = {} & 1 + \frac{1}{2\beta}(\sinh \beta \cosh \beta + \sin \beta \cos \beta) \\ & - \frac{1}{\beta}(\sinh \beta \cos \beta + \cosh \beta \sin \beta) \\ & + \frac{\gamma^2}{2\beta}[\sinh \beta \cosh \beta - \sin \beta \cos \beta \\ & - 2(\sin \beta \cosh \beta - \cos \beta \sinh \beta)] \\ & + \frac{\gamma}{\beta}(\sin \beta - \sinh \beta)^2\end{aligned} \qquad (A.2)$$

$$\int_0^1 \psi(\eta)\, d\eta = \frac{1}{\beta}[\sinh \beta - \sin \beta + \gamma(\cosh \beta + \cos \beta - 2)] \qquad (A.3)$$

$$\int_0^1 \eta\psi(\eta)\, d\eta = \frac{1}{\beta}\,[\sinh\beta - \sin\beta + \gamma(\cosh\beta + \cos\beta)]$$

$$+ \frac{1}{\beta^2}\,[2 - \cosh\beta - \cos\beta - \gamma(\sin\beta + \sinh\beta)] \qquad \text{(A.4)}$$

We can now evaluate the square of the norm of the eigenfunction $\psi_k(n)$ with respect to the inner product in (3-6)

$$P_k \equiv \int_0^1 \psi_k^2(\eta)\, d\eta + m^*\psi_k^2(1) + J^*\left[\psi_k'(1)\right]^2 + 2m^*c^*\psi_k(1)\psi_k'(1) \qquad \text{(A.5)}$$

The normalized eigenfunctions are given by

$$S_k(\eta) = \frac{1}{\sqrt{P_k}}\,\psi_k(\eta)$$

APPENDIX B

"EIGENVALUE"-MODAL PARAMETER
FORTRAN PROGRAM LISTING

```
              C *********************************************************************00000100
              C *  THIS PROGRAM COMPUTES THE EIGENVALUES, MODAL PARAMETERS AND       *00000110
              C *  MODAL PARAMETER IDENTITIES OF A CLAMPED BEAM WITH TIP BODY        *00000120
              C *  THE EIGENVALUES AND MODAL PARAMETERS WHICH ARE NECESSARY INPUT    *00000130
              C *  TO THE PLANAR DYNAMICS PROGRAM ARE WRITTEN TO A DISC FILE         *00000140
              C *  ( WRITTEN BY JOEL STORCH & STEPHEN GATES  C S D L   BASED UPON    *00000200
              C *     C S D L  REPORT # R-1629  MAY 1983 )                           *00000300
              C *********************************************************************00000400
              C                                                                       00000500
              C    NOTE  ARRAYS DIMENSIONED FOR A MAXIMUM OF 50 EIGENVALUES           00000510
              C                                                                       00000520
ISN 0002            IMPLICIT REAL*16(A-H,O-Z)                                         00000600
ISN 0003            REAL*16 LAM,MSTAR,JSTAR                                           00000700
ISN 0004            DIMENSION LAM(50),U1(50),U2(50),U3(50),U4(50)                     00000900
ISN 0005            NAMELIST /INPUT/ MSTAR,JSTAR,CSTAR,NF                             00000910
              C                                                                       00000918
              C  ********************* INPUT - OUTPUT FILES ***********************00000926
              C  FILE #5  NAMELIST INPUT FILE                                         00000934
              C  FILE #6  PRINTED OUTPUT FILE                                         00000942
              C  FILE #8  DISC OUTPUT FILE FOR EIGENVALUES & MODAL PARAMETERS         00000950
              C                                                                       00000958
              C  *************** DESCRIPTION OF /INPUT/ LIST ITEMS *****************00000966
              C                                                                       00000374
              C  "MSTAR" MASS RATIO (OF EQ  2-19)                                     00000982
              C  "JSTAR" INERTIA RATIO (OF EQ  2-19)                                  00000983
              C  "CSTAR" OFFSET RATIO (OF EQ  2-19)                                   00000984
              C  "NF"    NUMBER OF EIGENVALUES & MODAL PARAMETERS TO BE COMPUTED      00000985
              C  *******************************************************************00000986
              C                                                                       00000990
              C  PROMPT FOR NAMELIST INPUT DATA                                       00000993
              C                                                                       00000996
ISN 0006            WRITE(5,1)                                                        00001000
ISN 0007      1     FORMAT(1H ,5X,'INPUT DATA',/)                                     00001100
ISN 0008            READ(5,INPUT)                                                     00001200
ISN 0009            IF(NF  LE  50) GO TO 3                                            00001300
ISN 0011            WRITE(6,2) NF                                                     00001400
ISN 0012            STOP                                                              00001500
ISN 0013      2     FORMAT(1HO,5X,I3,'  MODES REQUESTED  MAXIMUM ALLOWABLE IS 50')    00001600
ISN 0014      3     WRITE(6,6)                                                        00001610
ISN 0015            WRITE(6,4) MSTAR,JSTAR,CSTAR                                      00001700
ISN 0016      4     FORMAT(1HO,5X,'MSTAR=',F8 4,4X,'JSTAR=',F8 4,4X,'CSTAR=',F8 4)    00001800
ISN 0017            WRITE(6,5) NF                                                     00001810
ISN 0018      5     FORMAT(1H ,5X,'NF=',I3)                                           00001820
ISN 0019      6     FORMAT(1H1,15X,'DATA FROM NAMELIST INPUT')                        00001830
ISN 0020            CALL EIGEN(NF,MSTAR,JSTAR,CSTAR,LAM,U1,U2,U3,U4)                  00001900
              C                                                                       00001910
              C  OUTPUT EIGENVALUES AND MODAL PARAMETERS (NOTE  ARRAY LAM CONTAINS THE00001920
              C   ROOTS OF EQ  2-19 RAISED TO THE FOURTH POWER)                       00001930
              C                                                                       00001940
ISN 0021            WRITE(6,13)                                                       00001950
ISN 0022            WRITE(6,11)                                                       00002000
ISN 0023            WRITE(8) NF,MSTAR,JSTAR,CSTAR                                     00002100
ISN 0024            DO 10 N=1,NF                                                      00002200
```

```
ISN 0025          WRITE(8) LAM(N),U1(N),U2(N),U3(N),U4(N)
ISN 0026    10    WRITE(6,12) N,LAM(N),U1(N),U2(N),U3(N),U4(N)
ISN 0027    11    FORMAT(1HO,T4,'N',T12,'LAM',T27,'U1',T40,'U2',T49,'U3',T66,'U4')
ISN 0028    12    FORMAT(1HO,T3,I2,T8,G12 5,T24,F9 4,T36,F9.4,T47,F9 5,T60,G13 5)
ISN 0029    13    FORMAT(1HO,/,17X,'"EIGENVALUES" & MODAL PARAMETERS')
            C
            C  MODAL PARAMETER IDENTITIES
            C
ISN 0030          WRITE(6,20)
ISN 0031    20    FORMAT(1H1,35X,'MODAL PARAMETER IDENTITIES')
ISN 0032          WRITE(6,23)
ISN 0033          SUM1=0 000
ISN 0034          SUM2=0 000
ISN 0035          SUM3=0.000
ISN 0036          SUM4=0 000
ISN 0037          SUM5=0 000
ISN 0038          SUM6=0 000
ISN 0039          DO 22 N=1,NF
ISN 0040          SUM1=SUM1+U3(N)**2
ISN 0041          SUM2=SUM2+U4(N)**2
ISN 0042          SUM3=SUM3+U3(N)*U4(N)
ISN 0043          SUM4=SUM4+U1(N)**2/LAM(N)
ISN 0044          SUM5=SUM5+U1(N)*U2(N)/LAM(N)
ISN 0045          SUM6=SUM6+U2(N)**2/LAM(N)
ISN 0046          WRITE(6,21)N,SUM1,SUM2,SUM3,SUM4,SUM5,SUM6
ISN 0047    21    FORMAT(1HO,5X,I2,6(G13 5,2X))
ISN 0048    22    CONTINUE
ISN 0049    23    FORMAT(1HO,42X,'PARTIAL SUMS',//////)
            C
            C  EXACT SUMS
            C
ISN 0050          SUM1=1 +MSTAR
ISN 0051          SUM2=1 /3 +MSTAR+JSTAR+2 *MSTAR*CSTAR
ISN 0052          SUM3=.5+MSTAR*(1 +CSTAR)
ISN 0053          SUM4=1
ISN 0054          SUM5= 5+CSTAR
ISN 0055          SUM6=1 /3.+CSTAR*(1.+CSTAR)
ISN 0056          WRITE(6,30) SUM1,SUM2,SUM3,SUM4,SUM5,SUM6
ISN 0057    30    FORMAT(1HO,//////,8X,6(G13 5,2X))
ISN 0058          STOP
ISN 0059          END
```

```
ISN 0002          FUNCTION CHARDT(ALF,MSTAR,JSTAR,CSTAR)
            C
            C  THIS FUNCTION COMPUTES THE EQUIVALENT OF EQ  2-19
            C
ISN 0003          IMPLICIT REAL*16(A-H,O-Z)
ISN 0004          REAL*16 MSTAR,JSTAR
ISN 0005          S=QSIN(ALF)
ISN 0006          C=QCOS(ALF)
ISN 0007          SH=QSINH(ALF)
ISN 0008          CH=QCOSH(ALF)
ISN 0009          C1=MSTAR*ALF
ISN 0010          C2=C1*CSTAR*ALF
ISN 0011          C3=JSTAR*ALF**3
ISN 0012          A11=C2*(C-CH)-C3*(S+SH)+C+CH
ISN 0013          A12=C2*(S-SH)+C3*(C-CH)+S+SH
ISN 0014          A21=C1*(CH-C)+C2*(S+SH)-S+SH
ISN 0015          A22=C1*(SH-S)+C2*(CH-C)+C+CH
ISN 0016          CHARDT=A11*A22-A12*A21
ISN 0017          RETURN
ISN 0018          END
```

```
ISN 0002          SUBROUTINE EIGEN(NF,MSTAR,JSTAR,CSTAR,LAM,U1,U2,U3,U4)
            C
            C    THIS SUBROUTINE COMPUTES "NF" EIGENVALUES (ROOTS OF EQ  2-19) AND
            C    MODAL PARAMETERS (EQ'S 3-25) OF A CLAMPED BEAM WITH TIP BODY
            C
ISN 0003          IMPLICIT REAL*16(A-H,O-Z)
ISN 0004          REAL*16 MSTAR,JSTAR,LAM(NF),LEFT
ISN 0005          DIMENSION U1(NF),U2(NF),U3(NF),U4(NF)
            C
            C    ESTIMATE ROOT OF CHARACTERISTIC EQUATION  (EQ  2-19)
            C
            C    N IS THE NUMBER OF ROOTS FOUND
            C
ISN 0006          N=0
ISN 0007          STEP=1 0Q-02
ISN 0008          LEFT=0 000
ISN 0009    1     RIGHT=LEFT+STEP
            C
            C    COMPARE THE SIGNS OF THE FUNCTION "CHARDT" AT THE TWO POINTS
            C
ISN 0010          IF(CHARDT(LEFT,MSTAR,JSTAR,CSTAR)*CHARDT(RIGHT,MSTAR,JSTAR,CSTAR)
                 1  LE  0 000) GO TO 2
ISN 0012          LEFT=RIGHT
ISN 0013          GO TO 1
ISN 0014    2     N=N+1
ISN 0015          ALF=RIGHT
            C
            C    IMPROVE ROOT ESTIMATE WITH NEWTON-RAPHSON ITERATION
            C
ISN 0016          NIT=0
ISN 0017    3     S=QSIN(ALF)
ISN 0018          C=QCOS(ALF)
ISN 0019          SH=QSINH(ALF)
ISN 0020          CH=QCOSH(ALF)
ISN 0021          C1=MSTAR*ALF
ISN 0022          C2=C1*CSTAR*ALF
ISN 0023          C3=JSTAR*ALF**3
ISN 0024          A11=C2*(C-CH)-C3*(S+SH)+C+CH
ISN 0025          A12=C2*(S-SH)+C3*(C-CH)+S+SH
ISN 0026          A21=C1*(CH-C)+C2*(S+SH)-S+SH
ISN 0027          A22=C1*(SH-S)+C2*(CH-C)+C+CH
ISN 0028          A11P=MSTAR*CSTAR*(2 000*ALF*(C-CH)-ALF**2*(S+SH))-C3*(C+CH)
                 1 -3 000*JSTAR*ALF**2*(S+SH)-S+SH
ISN 0029          A12P=MSTAR*CSTAR*(2 000*ALF*(S-SH)+ALF**2*(C-CH))-C3*(S+SH)
                 1 +3 000*JSTAR*ALF**2*(C-CH)+C+CH
ISN 0030          A21P=C1*(S+SH)+C2*(C+CH)+2 000*MSTAR
                 1 *CSTAR*ALF*(S+SH)-C+CH+MSTAR*(CH-C)
ISN 0031          A22P=MSTAR*(SH-S)+C1*(CH-C)+C2*(SH+S)
                 1 +2 000*MSTAR*CSTAR*ALF*(CH-C)-S+SH
ISN 0032          F=A11*A22-A12*A21
ISN 0033          FP=A11P*A22+A11*A22P-A12P*A21-A12*A21P
ISN 0034          DA=F/FP
ISN 0035          ALF=ALF-DA
```

```
ISN 0036          IF(QABS(DA/ALF)  LE  1.0Q-09) GO TO 6                          00012600
ISN 0038          NIT=NIT+1                                                        00012700
ISN 0039          IF(NIT  GT. 10) GO TO 4                                          00012800
ISN 0041          GO TO 3                                                          00012900
ISN 0042     4    WRITE(6,5) NIT,N,DA,ALF                                          00013000
ISN 0043     5    FORMAT(1HO,5X,'NEWTON RAPHSON ITERATION FAILED TO CONVERGE IN',1X,00013100
                 1 I2,' ITERATIONS. ROOT # ',I2,2X,'DA=',E13 5,2X,'ALF=',E13 5)    00013200
ISN 0044     6    S=QSIN(ALF)                                                      00013300
ISN 0045          C=QCOS(ALF)                                                      00013400
ISN 0046          SH=QSINH(ALF)                                                    00013500
ISN 0047          CH=QCOSH(ALF)                                                    00013600
ISN 0048          C1=MSTAR*ALF                                                     00013700
ISN 0049          C2=C1*CSTAR*ALF                                                  00013800
ISN 0050          C3=JSTAR*ALF**3                                                  00013900
ISN 0051          A11=C2*(C-CH)-C3*(S+SH)+C+CH                                     00014000
ISN 0052          A12=C2*(S-SH)+C3*(C-CH)+S+SH                                     00014100
ISN 0053          LAM(N)=ALF**4                                                    00014200
ISN 0054          BETA=-A11/A12                                                    00014300
ISN 0055          T1=ALF*(SH+S+BETA*(CH-C))                                        00014400
ISN 0056          T2=CH-C+BETA*(SH-S)                                              00014500
ISN 0057          T3=(SH-S+BETA*(CH+C-2.0Q0))/ALF                                  00014600
ISN 0058          T4=(SH-S+BETA*(CH+C))/ALF-(CH+C-2 0Q0+BETA*(S+SH))/ALF**2        00014700
             C                                                                     00014800
             C  CALCULATE MODAL PARAMETERS WITH NON NORMALIZED EIGENFUNCTIONS      00014900
             C                                                                     00015000
ISN 0059          U1(N)=T1                                                         00015100
ISN 0060          U2(N)=T2+CSTAR*T1                                                00015200
ISN 0061          U3(N)=T3+MSTAR*T2+MSTAR*CSTAR*T1                                 00015300
ISN 0062          U4(N)=T4+MSTAR*(1 0Q0+CSTAR)*T2+(MSTAR*CSTAR+JSTAR)*T1           00015400
             C                                                                     00015500
             C  CALCULATE INTEGRAL OF SQUARE OF EIGENFUNCTION                      00015600
             C                                                                     00015700
ISN 0063          C1=1 0Q0+BETA**2                                                 00015800
ISN 0064          C2=BETA**2-1 0Q0                                                 00015900
ISN 0065          T5=(C1*CH+2 0Q0*BETA*SH)*SH/(2 0Q0*ALF)                          00016000
ISN 0066          T6=C2*SH*C/ALF-C1*CH*S/ALF-2.0Q0*BETA*SH*S/ALF                   00016100
ISN 0067          T7=S*(2 0Q0*BETA*S-C2*C)/(2.0Q0*ALF)+1 0Q0                       00016200
ISN 0068          VALINT=T5+T6+T7                                                  00016300
             C                                                                     00016400
             C  CALCULATE NORM OF EIGENFUNCTION & MODAL PARAMETERS FOR NORMALIZED  00016500
             C   EIGENFUNCTIONS                                                    00016600
             C                                                                     00016610
ISN 0069          P2=VALINT+MSTAR*T2**2+JSTAR*T1**2+2 0Q0*MSTAR*CSTAR*T1*T2        00016700
ISN 0070          P=QSQRT(P2)                                                      00016800
ISN 0071          U1(N)=U1(N)/P                                                    00016900
ISN 0072          U2(N)=U2(N)/P                                                    00017000
ISN 0073          U3(N)=U3(N)/P                                                    00017100
ISN 0074          U4(N)=U4(N)/P                                                    00017200
ISN 0075          IF(N .EQ  NF) RETURN                                             00017300
ISN 0077          LEFT=RIGHT                                                       00017400
ISN 0078          GO TO 1                                                          00017500
ISN 0079          END                                                              00017600
```

OUTPUT FROM "EIGENVALUE"-MODAL PARAMETER PROGRAM

DATA FROM NAMELIST INPUT

MSTAR= 2 0000 JSTAR= 0 0280 CSTAR= 0 1000
NF= 10

"EIGENVALUES" & MODAL PARAMETERS

N	LAM	U1	U2	U3	U4
1	1 0310	0 9087	0 6760	1 56911	1 6540
2	143 31	-4 8354	-0 1266	0 52240	14854
3	1220 0	6 0703	-0 0027	0 29800	505870-01
4	5231 5	-4 9666	0 0552	0 22042	259090-01
5	16775	3 5599	-0 0608	0 17072	150010-01
6	42936	-2 6385	0 0551	0 13693	951230-02
7	93095	2 0584	-0 0485	0 11354	650020-02
8	178940+06	-1 6739	0 0427	0 09673	470290-02
9	314510+06	1 4044	-0 0380	0 08415	355330-02
10	516170+06	-1 2066	0 0341	0 07442	277650-02

MODAL PARAMETER IDENTITIES

PARTIAL SUMS

N	$\sum_{k=1}^{N} u_{3k}^2$	$\sum_{k=1}^{N} u_{4k}^2$	$\sum_{k=1}^{N} u_{3k}u_{4k}$	$\sum_{k=1}^{N} \frac{u_{1k}^2}{\lambda_k}$	$\sum_{k=1}^{N} \frac{u_{1k}u_{2k}}{\lambda_k}$	$\sum_{k=1}^{N} \frac{u_{2k}^2}{\lambda_k}$
1	2 4621	2.7356	2 5953	80094	59581	44322
2	2 7350	2 7577	2 6729	96409	60008	44333
3	2 8238	2 7602	2 6879	99429	60007	44333
4	2 8724	2 7609	2 6936	99901	60002	44333
5	2 9015	2 7611	2 6962	99976	60001	44333
6	2 9203	2.7612	2 6975	99993	60000	44333
7	2 9332	2 7613	2 6983	99997	60000	44333
8	2 9425	2 7613	2 6987	99999	60000	44333
9	2 9496	2.7613	2 6990	99999	60000	44333
10	2 9552	2 7613	2 6992	1 0000	60000	44333

ASYMPTOTIC VALUES

$1 + m^*$	$1/3 + m^* + J^* + 2m^*c^*$	$1/2 + m^* + m^*c^*$	1 0	$1/2 + c^*$	$1/3 + c^* + c^{*2}$
3 0000	2 7613	2 7000	1 0000	60000	44333

APPENDIX C

PLANAR DYNAMICS FORTRAN PROGRAM LISTING

```
                C
                C************************************************************************
                C* THIS PROGRAM SOLVES THE EQUATIONS OF MOTION FOR THE PLANAR DYNAMICS *
                C* OF A FLEXIBLE BEAM WITH  RIGID BODIES ATTACHED TO EACH END  THE      *
                C* BEAM DEFORMATION IS EXPANDED IN TERMS OF THE MODES APPROPRIATE TO    *
                C* A CLAMPED BEAM WITH TIP BODY  ONLY THE ATTITUDE AND BENDING COOR-    *
                C* DINATES ARE CALCULATED  ( WRITTEN BY JOEL STORCH & STEPHEN GATES     *
                C* BASED UPON C S D L  REPORT # R-1629  MAY 1983 )                      *
                C************************************************************************
                C
                C  NOTE  ARRAYS DIMENSIONED FOR A MAXIMUM OF 20 CLAMPED-TIP BODY MODES
                C
ISN 0002              IMPLICIT REAL*8(A-H,O-Z)
ISN 0003              REAL*16 LAM,U1,U2,U3,U4,MSTAR,JSTAR,CST
ISN 0004              REAL*8 M,L,MO,IO,MT,IT,M1,MUO,MU1,JO,NA
ISN 0005              DIMENSION LAM(20),U1(20),U2(20),U3(20),U4(20),A1V(20),AS(231),
                     1 BS(231),Z(21,21),EV(21),WK(483),FV(21),CN5(20),CN6(20),CN7(20),
                     2 CN8(20),FO(2),FA(21),NA(21),W1(21),W2(21),Y(42),YDOT(42)
ISN 0006              COMMON /DIM/ NF,NFP1,NSZ
ISN 0007              COMMON /EXTFC/ CN1,CN2,CN3,CN4,CN5,CN6,CN7
ISN 0008              COMMON /NLKTC/ CN9,U3,CN8
ISN 0009              COMMON /STATE/ X(21),XDOT(21)
ISN 0010              NAMELIST /INPUT/ MO,IO,A1,A2,L,PHO,C,MT,IT,NF,EI,THETA,THETAD,
                     1 DT,TSTOP,TPRT
                C
                C   ************** INPUT - OUTPUT FILES *****************************
                C   FILE #5  NAMELIST INPUT FILE
                C   FILE #6  PRINTED OUTPUT FILE
                C   FILE #8  INPUT FILE OF EIGENVALUES & MODAL PARAMETERS FOR BEAM WITH
                C                 TIP BODY
                C
                C   *****************DESCRIPTION OF /INPUT/ LIST ITEMS*****************
                C
                C  "MO"  ORBITER MASS
                C  "IO"  ORBITER MOMENT OF INERTIA ABOUT ITS MASS CENTER
                C        PERPENDICULAR TO PLANE OF MOTION
                C  "A1","A2" VECTOR FROM ORBITER MASS CENTER TO BEAM ATTACHMENT POINT
                C             (EXPRESSED IN ORBITER FRAME)
                C  "L"   BEAM LENGTH
                C  "RHO" MASS PER UNIT LENGTH OF BEAM
                C  "EI"  BENDING STIFFNESS IN BEAM
                C  "C"   TIP BODY MASS CENTER OFFSET
                C  "MT"  TIP BODY MASS
                C  "IT"  TIP BODY MOMENT OF INERTIA ABOUT ITS MASS CENTER FOR AXIS
                C        PERPENDICULAR TO PLANE OF MOTION
                C  "NF"  NUMBER OF CANTILEVERED-TIP BODY MODES TO BE RETAINED
                C        FOR EXPANSION OF BEAM ELASTIC DISPLACEMENT
                C  "THETA"  ORBITER INITIAL ATTITUDE (DEG )
                C  "THETAD" ORBITER INITIAL ATTITUDE RATE (DEG /SEC)
                C  "DT"     INTEGRATION TIME STEP
                C  "TSTOP"  FINAL TIME
                C  "TPRT"   TIME INTERVAL BETWEEN CONSECUTIVE PRINTOUTS
                C   *************************************************************
```

```
             C
ISN 0011              READ(5,INPUT)
ISN 0012              WRITE(6,7)
ISN 0013              WRITE(6,1) MO,IO
ISN 0014      1       FORMAT(1HO,'MO = ',1PE17.8,5X,'IO = ',1PE17.8)
ISN 0015              WRITE(6,2) MT,IT
ISN 0016      2       FORMAT(1H ,'MT = ',1PE17 8,5X,'IT = ',1PE17.8)
ISN 0017              WRITE(6,3) A1,A2
ISN 0018      3       FORMAT(1H ,'A1 = ',1PE17 8,5X,'A2 = ',1PE17 8)
ISN 0019              WRITE(6,4) L,C
ISN 0020      4       FORMAT(1H ,'L =  ',1PE17 8,5X,'C  = ',1PE17 8)
ISN 0021              WRITE(6,5) RHO,EI
ISN 0022      5       FORMAT(1H ,'RHO =',1PE17 8,5X,'EI = ',1PE17 8)
ISN 0023              WRITE(6,6) DT,TPRT
ISN 0024      6       FORMAT(1H ,'DT = ',1PE17 8,5X,'TPRT =',1PE16 8)
ISN 0025              WRITE(6,8) THETA,THETAD
ISN 0026      7       FORMAT(1H1,13X,'DATA FROM NAMELIST INPUT')
ISN 0027      8       FORMAT(1H ,'THETA =',1PE15 8,3X,'THETADOT ='1PE15 8)
              C
              C   TEST FOR DATA CONSISTENCY WITH VALUES ON DISC DATA SET
              C
ISN 0028              READ(8) NFT,MSTAR,JSTAR,CST
ISN 0029              D1=DABS(DBLEQ(MSTAR)-MT/(RHO*L))
ISN 0030              D2=DABS(DBLEQ(JSTAR)-(IT+MT*C**2)/(RHO*L**3))
ISN 0031              D3=DABS(DBLEQ(CST)-C/L)
ISN 0032              D1=D1/DBLEQ(MSTAR)
ISN 0033              D2=D2/DBLEQ(JSTAR)
ISN 0034              D3=D3/DBLEQ(CST)
ISN 0035              IER=0
ISN 0036              IF(NFT  GE  NF) GO TO 11
ISN 0038              IER=1
ISN 0039              WRITE(6,10) NF,NFT
ISN 0040      10      FORMAT(1HO,5X,'* * * FATAL ERROR * * *',2X,I3,' CANTILEVERED',
                    1 ' TIP-BODY MODES REQUESTED  DATA ON DISC ONLY FOR',I3,
                    2 '  MODES')
ISN 0041      11      IF(D1  LE   001DO) GO TO 13
ISN 0043              IER=1
ISN 0044              WRITE(6,12) MSTAR
ISN 0045      12      FORMAT(1HO,5X,'* * * FATAL ERROR * * *',2X,'MSTAR(DISC)=',G13 5)
ISN 0046      13      IF(D2  LE   001DO) GO TO 15
ISN 0048              IER=1
ISN 0049              WRITE(6,14) JSTAR
ISN 0050      14      FORMAT(1HO,5X,'* * * FATAL ERROR* * * ',2X,'JSTAR(DISC)=',G13 5)
ISN 0051      15      IF(D3 .LE   001DO) GO TO 17
ISN 0053              IER=1
ISN 0054              WRITE(6,16) CST
ISN 0055      16      FORMAT(1HO,5X,'* * * FATAL ERROR* * * ',2X,'CSTAR(DISC)=',G13 5)
ISN 0056      17      IF(IER  NE  0) STOP
              C
              C  COMPUTE CONSTANT QUANTITIES
              C
ISN 0058              R5=EI/(RHO*L**4)
ISN 0059              FC= 1591549*DSQRT(R5)
ISN 0060              M1=RHO*L+MT
ISN 0061              M=MO+M1
ISN 0062              MUO=MO/M
ISN 0063              MU1=M1/M
ISN 0064              B1=( 5*RHO*L**2+MT*(L+C))/M1
```

```
ISN 0065          JO=RHO*L**3/3 ODO+IT+MT*(C+L)**2
ISN 0066          R1=RHO*L/M
ISN 0067          R2=A1/L
ISN 0068          R3=A2/L
ISN 0069          R4=B1/L
ISN 0070          CSTAR=C/L
ISN 0071          AOO=(IO+JO)/(RHO*L**3)+MUO*MU1*(R2**2+R3**2+2 ODO*R2*R4)/R1
                 1 -MU1**2*R4**2/R1
             C
             C  READ & PRINT EIGENVALUES AND MODAL PARAMETERS FOR BEAM+TIP BODY
             C
ISN 0072          WRITE(6,23)
ISN 0073          WRITE(6,21)
ISN 0074          DO 20 N=1,NF
ISN 0075          READ(8) LAM(N),U1(N),U2(N),U3(N),U4(N)
ISN 0076     20   WRITE(6,22) N,LAM(N),U1(N),U2(N),U3(N),U4(N)
ISN 0077     21   FORMAT(1HO,T4,'N',T12,'LAM',T27,'U1',T40,'U2',T49,'U3',T66,'U4')
ISN 0078     22   FORMAT(1H ,T3,I2,T8,G12 5,T24,F9 4,T36,F9 4,T47,F9 5,T60,G13 5)
ISN 0079     23   FORMAT(1HO,/,11X,'EIGENVALUES & MODAL PARAMETERS FROM DISC FILE')
ISN 0080          WRITE(6,24)
ISN 0081     24   FORMAT(1HO,/,7X,'SYSTEM NATURAL FREQUENCIES')
             C
             C  COMPUTE TERMS A1I
             C
ISN 0082          DO 30 K=1,NF
ISN 0083     30   A1V(K)=MUO*R2*U3(K)+U4(K)-MU1*R4*U3(K)
             C
             C   STORE "A" IN SYMMETRIC STORAGE MODE - "AS"
             C   STORF "K" IN SYMMETRIC STORAGE MODE - "BS"
             C
ISN 0084          AS(1)=AOO
ISN 0085          N=1
ISN 0086          DO 41 I=1,NF
ISN 0087          N=N+1
ISN 0088          AS(N)=A1V(I)
ISN 0089          DO 40 J=1,I
ISN 0090          N=N+1
ISN 0091          AS(N)=-R1*U3(J)*U3(I)
ISN 0092          IF(I EQ J) AS(N)=AS(N)+1 ODO
ISN 0094     40   CONTINUE
ISN 0095     41   CONTINUE
ISN 0096          BS(1)=0 ODO
ISN 0097          N=1
ISN 0098          DO 51 I=1,NF
ISN 0099          N=N+1
ISN 0100          BS(N)=0 ODO
ISN 0101          DO 50 J=1,I
ISN 0102          N=N+1
ISN 0103          BS(N)=0 ODO
ISN 0104          IF(I EQ J) BS(N)=LAM(I)
ISN 0106     50   CONTINUE
ISN 0107     51   CONTINUE
             C
             C   GET EIGENVALUES AND EIGENVECTORS FOR 2,3,   ,NFP1 DEGREES OF FREEDOM
             C
ISN 0108          NFP1=NF+1
ISN 0109          NSZ=2*NFP1
ISN 0110          DO 66 N=2,NFP1
```

```
ISN 0111          CALL EIGZS(BS,AS,N,1,EV,Z,21,WK,IER)                                   00016600
ISN 0112          IF(IER  EQ. 0) GO TO 62                                                00016700
ISN 0114          WRITE(6,61) N,IER                                                      00016800
ISN 0115     61   FORMAT(1HO,5X,'ERROR IN EIGENVALUE EXTRACTION  MATRIX ORDER=',I2,  00016900
                1 3X,'IER=',I3)                                                          00017000
ISN 0116          STOP                                                                   00017100
ISN 0117     62   WRITE(6,63) N                                                          00017200
ISN 0118     63   FORMAT(1HO,I2,' DEGREES OF FREEDOM IN EIGENVALUE PROBLEM')             00017300
ISN 0119          DO 64 I=1,N                                                            00017500
ISN 0120          WS=FC*DSQRT(EV(I))                                                     00017600
ISN 0121     64   WRITE(6,65) I,WS                                                       00017700
ISN 0122     65   FORMAT(1H ,2X,'MODE',I2,2X,'SYSTEM FREQUENCY(HZ )=',G13 5)             00017800
ISN 0123     66   CONTINUE                                                               00017900
             C                                                                           00018000
             C   NORMALIZE EIGENVECTORS                                                  00018100
             C                                                                           00018200
ISN 0124          DO 70 N=1,NFP1                                                         00018300
ISN 0125     70   CALL NORM(AS,NFP1,Z(1,N))                                              00018400
             C                                                                           00018500
             C   CALCULATE CONSTANTS IN MOTION EQUATIONS                                 00018600
             C                                                                           00018700
ISN 0126          DO 80 I=1,NFP1                                                         00018800
ISN 0127     80   FV(I)=R5*EV(I)                                                         00018900
ISN 0128          CN1=RHO*L**3                                                           00019000
ISN 0129          CN2=MU1*(A1+B1)/CN1                                                    00019100
ISN 0130          CN3=MU1*A2/CN1                                                         00019200
ISN 0131          CN4=(1 ODO+CSTAR+MUO*R2-MU1*R4)/(RHO*L**2)                             00019300
ISN 0132          CN9=1 ODO/(M*L)                                                        00019400
ISN 0133          DO 90 I=1,NF                                                           00019500
ISN 0134          CN5(I)=U1(I)/CN1                                                       00019600
ISN 0135          CN6(I)=U2(I)/(RHO*L**2)                                                00019700
ISN 0136          CN7(I)=U3(I)*CN9                                                       00019800
ISN 0137     90   CN8(I)=MUO*R3*U3(I)                                                    00019900
             C                                                                           00020000
             C   SET INITIAL CONDITIONS                                                  00020100
             C                                                                           00020200
ISN 0138          T=0 ODO                                                                00020300
ISN 0139          CPRT=0 ODO                                                             00020400
ISN 0140          X(1)=THETA* 0174532                                                    00020500
ISN 0141          XDOT(1)=THETAD* 0174532                                                00020600
             C                                                                           00020700
             C   INITIAL DEFORMATION AND RATE ARE SET TO ZERO                            00020800
             C                                                                           00020900
ISN 0142          DO 100 I=1,NF                                                          00021000
ISN 0143          X(I+1)=0.ODO                                                           00021100
ISN 0144     100  XDOT(I+1)=0.ODO                                                        00021200
             C                                                                           00021300
             C   CALCULATE Y AT T=0                                                      00021400
             C                                                                           00021500
ISN 0145          DO 112 I=1,NFP1                                                        00021600
ISN 0146          II=I                                                                   00021700
ISN 0147          W1(I)=0 ODO                                                            00021800
ISN 0148          W2(I)=0 ODO                                                            00021900
ISN 0149          DO 111 J=1,NFP1                                                        00022000
ISN 0150          JJ=J                                                                   00022100
ISN 0151          IF( I  GE. J) GO TO 110                                                00022200
ISN 0153          II=J                                                                   00022300
ISN 0154          JJ=I                                                                   00022400
```

```
ISN 0155    110   LL=II*(II-1)/2+JJ
ISN 0156          W1(I)=W1(I)+AS(LL)*X(J)
ISN 0157    111   W2(I)=W2(I)+AS(LL)*XDOT(J)
ISN 0158    112   CONTINUE
ISN 0159          DO 114 I=1,NFP1
ISN 0160          Y(I)=0 ODO
ISN 0161          Y(NFP1+I)=0 ODO
ISN 0162          DO 113 J=1,NFP1
ISN 0163          Y(I)=Y(I)+Z(J,I)*W1(J)
ISN 0164    113   Y(NFP1+I)=Y(NFP1+I)+Z(J,I)*W2(J)
ISN 0165    114   CONTINUE
ISN 0166          WRITE(6,115)
ISN 0167    115   FORMAT(1HO,/,20X,'TIME RESPONSE')
            C
            C   CALCULATE EXTERNAL FORCES ON ORBITER AND TIP BODY
            C
ISN 0168    120   CALL ORBFOR(T,FO,GO)
ISN 0169          CALL TPBFOR(T,FP,GP)
ISN 0170          CALL EXTF(FO,GO,FP,GP,FA)
            C
            C   CALCULATE NON-LINEAR TERMS
            C
ISN 0171          CALL NLKT(FO,NA)
ISN 0172          DO 131 I=1,NFP1
ISN 0173          W1(I)=0 ODO
ISN 0174          DO 130 J=1,NFP1
ISN 0175    130   W1(I)=W1(I)+Z(J,I)*(FA(J)+NA(J))
ISN 0176    131   CONTINUE
            C
            C   CALCULATE "YDOT"
            C
ISN 0177          DO 140 I=1,NFP1
ISN 0178          YDOT(I)=Y(NFP1+I)
ISN 0179          YDOT(NFP1+I)=W1(I)-FV(I)*Y(I)
ISN 0180    140   CONTINUE
            C
            C   INTEGRATE DIFFERENTIAL EQUATIONS IN FIRST ORDER FORM
            C
ISN 0181          CALL ODESLV(NSZ,Y,YDOT,DT)
ISN 0182          T=T+DT
ISN 0183          CPRT=CPRT+DT
ISN 0184          DO 151 I=1,NFP1
ISN 0185          X(I)=0 ODO
ISN 0186          XDOT(I)=0 ODO
ISN 0187          DO 150 J=1,NFP1
ISN 0188          X(I)=X(I)+Z(I,J)*Y(J)
ISN 0189    150   XDOT(I)=XDOT(I)+Z(I,J)*Y(NFP1+J)
ISN 0190    151   CONTINUE
ISN 0191          THETA=X(1)*57 29578
ISN 0192          THETAD=XDOT(1)*57 29578
            C
            C  PRINT OUTPUT
            C
ISN 0193          IF(CPRT  LT  TPRT) GO TO 170
ISN 0195          CPRT=0 ODO
ISN 0196          WRITE(6,160) T
ISN 0197    160   FORMAT(1HO,'TIME =',F7 3)
ISN 0198          WRITE(6,161) THETA
```

```
ISN 0199          WRITE(6,165) THETAD                                                       00028010
ISN 0200    161   FORMAT(1H ,'THETA =',1PE17 8,' DEG')                                      00028100
ISN 0201          WRITE(6,162)                                                              00028200
ISN 0202    162   FORMAT(1HO,'    MODAL COORDINATES',10X,'DERIV MODAL COORDINATES')         00028300
ISN 0203          DO 163 I=2,NFP1                                                           00028400
ISN 0204          J=I-1                                                                     00028500
ISN 0205    163   WRITE(6,164) J,X(I),XDOT(I)                                               00028600
ISN 0206    164   FORMAT(1H ,I2,2X,1PE17 8,10X,1PE17 8)                                     00028700
ISN 0207    165   FORMAT(1H ,'THETADOT =',1PE17 8,' DEG/SEC')                               00028710
            C                                                                               00028800
ISN 0208    170   IF(T  GE  TSTOP) STOP                                                     00028900
ISN 0210          GO TO 120                                                                 00029000
ISN 0211          END                                                                       00029100
```

```
ISN 0002          SUBROUTINE ORBFOR(T,FO,GO)                                                00029200
            C                                                                               00029300
            C   THIS SUBROUTINE CALCULATES THE EXTERNAL FORCE "FO" ON THE ORBITER           00029400
            C   (IN ORBITER FRAME) AND NET MOMENT  "GO" PERPENDICULAR TO PLANE OF           00029500
            C   MOTION AT TIME "T"                                                          00029600
            C                                                                               00029700
ISN 0003          IMPLICIT REAL*8(A-H,O-Z)                                                  00029800
ISN 0004          DIMENSION FO(2)                                                           00029900
ISN 0005          FO(1)=0 ODO                                                               00030000
ISN 0006          FO(2)=0 ODO                                                               00030100
ISN 0007          GO=0 ODO                                                                  00030200
ISN 0008     10   FORMAT(1H ,3X,'FO=',2E13 5,3X,'GO=',E13 5)                                00030300
ISN 0009          RETURN                                                                    00030400
ISN 0010          END                                                                       00030500
```

```
ISN 0002          SUBROUTINE TPBFOR(T,F,G)                                                  00030600
            C                                                                               00030700
            C   THIS SUBROUTINE CALCULATES THE EXTERNAL FORCE "F" ON THE TIP BODY           00030800
            C   (ACTING TRANSVERSE TO BEAM NEUTRAL AXIS) AND MOMENT "G"                     00030900
            C   PERPENDICULAR TO PLANE OF MOTION AT TIME "T"                                00031000
            C                                                                               00031100
ISN 0003          F=0 ODO                                                                   00031200
ISN 0004          G=0 ODO                                                                   00031300
ISN 0005          RETURN                                                                    00031400
ISN 0006          END                                                                       00031500
```

```
ISN 0002          SUBROUTINE EXTF(FO,GO,FP,GP,F)                                            00031600
            C                                                                               00031700
            C   THIS SUBROUTINE ASSEMBLES THE VECTOR OF EXTERNAL FORCES "F" GIVEN           00031800
            C   BY EQ (4-31) THE ORBITER FORCE AND MOMENT "FO","GO" AS WELL AS THE          00031900
            C   FORCE AND MOMENT ON THE TIP BODY "FP","GP" ARE INPUT                        00032000
            C                                                                               00032100
ISN 0003          IMPLICIT REAL*8(A-H,O-Z)                                                  00032200
ISN 0004          DIMENSION FO(2),F(NFP1)                                                   00032300
ISN 0005          COMMON /DIM/ NF,NFP1,NSZ                                                  00032400
ISN 0006          COMMON /EXTFC/ CN1,CN2,CN3,CN4,CN5(20),CN6(20),CN7(20)                    00032500
ISN 0007          F(1)=GO/CN1-CN2*FO(2)+CN3*FO(1)+GP/CN1+CN4*FP                             00032600
ISN 0008          DO 10 I=1,NF                                                              00032700
ISN 0009     10   F(I+1)=CN5(I)*GP+CN6(I)*FP-CN7(I)*(FO(2)+FP)                              00032800
ISN 0010          RETURN                                                                    00032900
ISN 0011          END                                                                       00033000
```

```
ISN 0002          SUBROUTINE NLKT(FO,N)
            C
            C   THIS SUBROUTINE ASSEMBLES THE VECTOR OF NON-LINEAR TERMS "N"
            C   GIVEN BY EQ (4-32)  THE EXTERNAL FORCE ON THE ORBITER "FO" IS INPUT
            C
ISN 0003          IMPLICIT REAL*8(A-H,O-Z)
ISN 0004          REAL*8 N
ISN 0005          REAL*16 U3
ISN 0006          COMMON /NLKTC/ CN9,U3(20),CN8(20)
ISN 0007          COMMON /STATE/ X(21),XDOT(21)
ISN 0008          COMMON /DIM/ NF,NFP1,NSZ
ISN 0009          DIMENSION FO(2),N(NFP1)
ISN 0010          N(1)=0 ODO
ISN 0011          DO 10 K=1,NF
ISN 0012    10    N(1)=N(1)+DBLEQ(U3(K))*X(K+1)
ISN 0013          N(1)=N(1)*CN9*FO(1)
ISN 0014          THD2=XDOT(1)**2
ISN 0015          DO 20 K=1,NF
ISN 0016    20    N(K+1)=CN8(K)*THD2
ISN 0017          RETURN
ISN 0018          END
```

```
ISN 0002          SUBROUTINE NORM(AS,N,X)
            C
            C   THIS SUBROUTINE NORMALIZES THE EIGENVECTOR "X" WITH RESPECT TO THE
            C   SYMMETPIC POSITIVE DEFINITE MATRIX 'A' OF ORDER "N" STORED IN
            C   SYMMETRIC STORAGE MODE AS THE VECTOR "AS"
            C
ISN 0003          IMPLICIT REAL*8(A-H,O-Z)
ISN 0004          DIMENSION AS(1),X(1)
ISN 0005          S=AS(1)*X(1)**2
ISN 0006          NC=1
ISN 0007          DO 10 I=2,N
ISN 0008          IM1=I-1
ISN 0009          DO 20 J=1,IM1
ISN 0010          NC=NC+1
ISN 0011    20    S=S+2 ODO*AS(NC)*X(I)*X(J)
ISN 0012          NC=NC+1
ISN 0013          S=S+AS(NC)*X(I)**2
ISN 0014    10    CONTINUE
ISN 0015          S=DSQRT(S)
ISN 0016          DO 30 I=1,N
ISN 0017    30    X(I)=X(I)/S
ISN 0018          RETURN
ISN 0019          END
```

```
ISN 0002          SUBROUTINE ODESLV(N,Y,DERIV,H)
ISN 0003          IMPLICIT REAL*8(A-H,O-Z)
ISN 0004          DIMENSION DERIV(N),Y(N),DERIVO(42),BD1(42,2),BD2(42,2),BD3(42)
ISN 0005          DATA INTF/1/,C1/0 0/,C2/0 0/,C3/0 /
            C
            C   THIS SUBROUTINE INTEGRATES THE FIRST ORDER SYSTEM OF ORDINARY
            C   DIFFERENTIAL EQUATIONS  "DY/DT=DERIV" BY THE ADAMS METHOD
            C   USING THIRD ORDER DIFFERENCES
            C     N- SIZE OF SYSTEM
            C     Y- VECTOR OF INITIAL VALUES ON INPUT  "Y" IS OVERWRITTEN
            C        WITH THE NEW SOLUTION
            C     H- STEP SIZE
```

```
                C
ISN 0006              IF(N  LE  42) GO TO 10
ISN 0008              WRITE(6,12) N
ISN 0009              STOP
ISN 0010        12    FORMAT(1HO,5X,'ERROR IN SUBROUTINE **ODESLV** CALLED WITH STATE
                     1SIZE =',I3,'  EXCEEDS DIMENSION SIZE OF ARRAYS')
ISN 0011        10    GO TO(1000,2000,3000,4000),INTF
                C
                C   FIRST CALL TO ROUTINE  - EULER INTEGRATION
                C
ISN 0012        1000  DO 20 I=1,N
ISN 0013        20    DERIVO(I)=DERIV(I)
ISN 0014              INTF=2
ISN 0015              GO TO 5000
                C
                C   SECOND CALL TO ROUTINE - FIRST ORDER DIFFERENCES
                C
ISN 0016        2000  DO 30 I=1,N
ISN 0017              BD1(I,1)=DERIV(I)-DERIVO(I)
ISN 0018              BD1(I,2)=BD1(I,1)
ISN 0019        30    DERIVO(I)=DERIV(I)
ISN 0020              C1= 5
ISN 0021              INTF=3
ISN 0022              GO TO 5000
                C
                C   THIRD CALL TO ROUTINE - SECOND ORDER FIFFERENCES
                C
ISN 0023        3000  DO 40 I=1,N
ISN 0024              BD1(I,2)=DERIV(I)-DERIVO(I)
ISN 0025              BD2(I,1)=BD1(I,2)-BD1(I,1)
ISN 0026              BD2(I,2)=BD2(I,1)
ISN 0027              DERIVO(I)=DERIV(I)
ISN 0028        40    BD1(I,1)=BD1(I,2)
ISN 0029              INTF=4
ISN 0030              C2=5 0/12 0
ISN 0031              GO TO 5000
                C
                C   ADAMS METHOD WITH 3RD ORDER DIFFERENCES
                C
ISN 0032        4000  DO 50 I=1,N
ISN 0033              BD1(I,2)=DERIV(I)-DERIVO(I)
ISN 0034              BD2(I,2)=BD1(I,2)-BD1(I,1)
ISN 0035              BD3(I)=BD2(I,2)-BD2(I,1)
ISN 0036              DERIVO(I)=DERIV(I)
ISN 0037              BD1(I,1)=BD1(I,2)
ISN 0038        50    BD2(I,1)=BD2(I,2)
ISN 0039              C3=3 0/8 0
ISN 0040              GO TO 5000
                C
                C   UPDATE VECTOR 'Y'
                C
ISN 0041        5000  DO 60 I=1,N
ISN 0042        60    Y(I)=Y(I)+H*(DERIV(I)+C1*BD1(I,2)+C2*BD2(I,2)+C3*BD3(I))
ISN 0043              RETURN
ISN 0044              END
```

EXAMPLE PROBLEM PARAMETERS

m_0 = 98739.5 kg

I_0 = 9769869.5 kg-m^2

m_t = 875.32 kg

I_t = 1400.512 kg-m^2

a_1 = 2.0 m

a_2 = 0.0 m

ℓ = 20.0 m

c = 2.0 m

ρ = 21.883 kg/m

EI = 353520.0 N-m^2

m^* = 2.0

J^* = 0.028

c^* = 0.1

NAMELIST INPUT DATA

```
&INPUT MO=98739 5,IO=9769869 5,A1=2 ,A2=0 ,L=20 ,RHO=21 883,EI=353520 ,C=2 ,
MT=875 32,IT=1400 512,NF=3,THETA=0 ,THETAD=0 ,DT= 01,TSTOP=1 ,TPRT= 02,&END
```

OUTPUT DATA FROM PLANAR DYNAMICS PROGRAM

Excitation: $G_0 = 4 \times 10^4$ N for all $t \geq 0$

All units are metric (MKS)

```
            DATA FROM NAMELIST INPUT

MO =     9 87395000D+04      IO =     9 76986950D+06
MT =     8 75320000D+02      IT =     1 40051200D+03
A1 =     2 00000000D+00      A2 =     0 0
L =      2 00000000D+01      C  =     2 00000000D+00
RHO =    2 18830000D+01      EI =     3 53520000D+05
DT =     1 00000000D-02      TPRT =   2 00000000D-02
THETA = 0 0                THETADOT = 0 0

         EIGENVALUES & MODAL PARAMETERS FROM DISC FILE

 N       LAM           U1          U2         U3              U4
 1     1 0310        0 9087      0 6760     1 56911        1 6540
 2     143 31       -4 8354     -0 1266     0 52240         14854
 3     1220 0        6 0703     -0 0027     0 29800         505870-01

     SYSTEM NATURAL FREQUENCIES

 2 DEGREES OF FREEDOM IN EIGENVALUE PROBLEM
  MODE 1   SYSTEM FREQUENCY(HZ )=     0
  MODE 2   SYSTEM FREQUENCY(HZ )=     53106D-01

 3 DEGREES OF FREEDOM IN EIGENVALUE PROBLEM
  MODE 1   SYSTEM FREQUENCY(HZ )=     0
  MODE 2   SYSTEM FREQUENCY(HZ )=     53106D-01
  MODE 3   SYSTEM FREQUENCY(HZ )=     60600

 4 DEGREES OF FREEDOM IN EIGENVALUE PROBLEM
  MODE 1   SYSTEM FREQUENCY(HZ )=     0
  MODE 2   SYSTEM FREQUENCY(HZ )=     53106D-01
  MODE 3   SYSTEM FREQUENCY(HZ )=     60600
  MODE 4   SYSTEM FREQUENCY(HZ )=   1 7669

                TIME RESPONSE

TIME =   0 020
THETA =    3 51839150D-05 DEG
THETADOT =    4 69118867D-03 DEG/SEC

    MODAL COORDINATES            DERIV MODAL COORDINATES
 1     -1 11189128D-06             -1 48252171D-04
 2     -1 23251214D-07             -1 64334952D-05
 3     -4 93395608D-08             -6 57860810D-06

TIME =   0 040
THETA =    1 75919099D-04 DEG
THETADOT =    9 38231697D-03 DEG/SEC

    MODAL COORDINATES            DERIV MODAL COORDINATES
 1     -5 55938160D-06             -2 96494856D-04
 2     -6 15468068D-07             -3 27670730D-05
 3     -2 44025899D-07             -1 28184228D-05
```

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