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TRANSVERSE VIBRATION AND BUCKLING OF A CANTILEVERED BEAM WITH TIP BODY UNDER CONSTANT AXIAL BASE ACCELERATION

by

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16. Abstract

Analyses of the planar transverse bending behavior of a uniform cantilevered beam with rigid tip body subject to constant axial base acceleration are presented. The beam is inextensible and capable of small elastic transverse bending deformations only. Two classes of tip bodies are recognized, those with mass centers located along the beam tip tangent line (restricted mass center offsets) and those with mass centers having an arbitrary offset with respect to the beam attachment point. The steady-state response is studied for the beam end-condition cases: free, tip mass, tip body with restricted mass center offset, and tip body with arbitrary mass center offset. The first three cases constitute classical Euler buckling problems, and for each, the characteristic equation for the critical loads/accelerations are determined. A nonhomogeneous boundary value problem arises for the last case and it is shown that a unique steady-state solution exists, except at certain discrete values of base acceleration for which no such solution exists. The free vibration response is examined for the two classes of tip body. The characteristic equation, eigenfunctions and their orthogonality properties are obtained for the case of restricted mass center offset. The vibration problem is nonhomogeneous for the case of arbitrary mass center offset. The exact solution is obtained as a sum of the steady-state solution and a superposition of simple harmonic motions. Approximate solutions to the free vibration and buckling characteristics for the problem of restricted mass center offset are obtained using an assumed modes

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16. ABSTRACT (Cont.)

formulation. A method for recovering the strain energy from the governing partial differential equation and boundary conditions is set forth. Selected FORTRAN programs and numerical results are provided.

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CHAPTER 1

INTRODUCTION

Beam-like structures cantilevered to the Shuttle cargo bay will, in general, experience axial loads as a result of attitude maneuvers. In our previous work (Reference 1), we developed dynamics models for such a class of orbiter-payload systems with the effects of axial loads ignored for reasons of simplicity and tractability. In this report, we examine that approximation in a more simple setting which clearly exposes the phenomena of interest. Presented are analyses of the planar transverse bending behavior of a uniform cantilevered beam with rigid tip body under the action of constant axial base acceleration. Exact steady-state and free vibration solutions are obtained for various forms of this problem with additional approximate solutions for selected cases. While the results of this study are of limited direct applicability to the problem of Reference 1, they are of general interest, provide much useful insight, and serve as a basis from which to address more complicated situations.

The beam is taken to be long, slender, and inextensible with uniform mass, stiffness, and cross section. Only small elastic transverse bending deformations in a plane of symmetry of the cross section are considered. The tip body is rigidly attached to the beam, with its mass center having an arbitrary offset with respect to the attachment point. In all cases, the base acceleration is constant in magnitude and directed along the undeformed longitudinal axis of the beam. The governing partial differential equation is shown to be linear with variable coefficients, accompanied by nonhomogeneous boundary conditions for the general problem of arbitrary mass center offset. The steady-state response is examined for a number of beam end conditions. For the cases of free-end, tip mass, and tip body with mass center along the beam tip tangent line (henceforth referred to as restricted mass center offset), the boundary value problems are homogeneous. These classical Euler buckling problems are solved exactly for the critical buckling loads/accelerations. For the problem of arbitrary tip body mass center offset, it is shown that a unique steady-state solution exists, except for certain critical values of the base acceleration for which no such solution exists. The treatment of transverse vibration begins with the case of restricted mass center offset. The boundary value problem is homogeneous and admits the usual separation of variables solution with harmonic time dependence. The ensuing variable coefficient ordinary differential equation for the spatial functions is solved exactly in terms of a power series. Application of the boundary conditions yields the characteristic equation and eigenfunctions. For the case of arbitrary tip body mass center offset, the vibration problem is nonhomogeneous. It is shown that the exact solution may be written as the sum of the steady-state solution obtained previously, and a superposition of simple harmonic motions which correspond to the exact solution of the associated homogeneous problem. This approach, which takes advantage of the steady-state solution, yields a greatly simplified form of the final solution. An assumed modes formulation is detailed for the restricted mass center offset case. The approximate solutions to the free vibration and buckling characteristics serve to check the exact analyses. A particularly useful and general method for recovering the strain energy from the governing partial differential equation and boundary conditions is set forth. Selected FORTRAN programs and numerical results are provided.

CHAPTER 2

MOTION EQUATIONS

2.1 Derivation of the Partial Differential Equation and Boundary Conditions

In this chapter, we derive the partial differential equation and boundary conditions governing the planar transverse vibration of a beam with tip body under the action of an axial force. Figure 2-1 depicts the situation in the undeformed configuration. At one end of the beam, (x = 0), an axial force is applied imparting a velocity, $v_0(t)$, along the x axis. The beam is assumed inextensible and of length ℓ. At the other end of the beam, $(x = \ell)$, a tip body is rigidly attached at point P. The rigid tip body has mass m and moment of inertia about its mass center I_t. The distance from P to the mass center of the tip body is c, and this directed line segment makes an angle, γ, with the beam tip tangent at P. Figure 2-2 shows the system in the deformed state. The base O of the beam moves along the x direction while the deformation u(x, t), occurs along the y direction. Note that the angle γ is maintained constant.

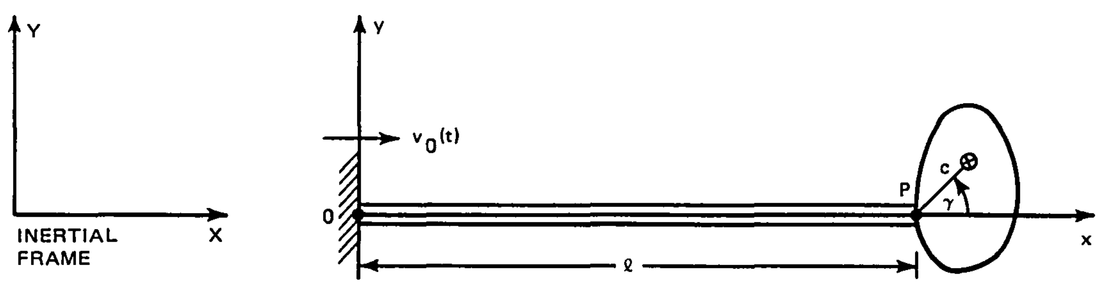


Figure 2-1.

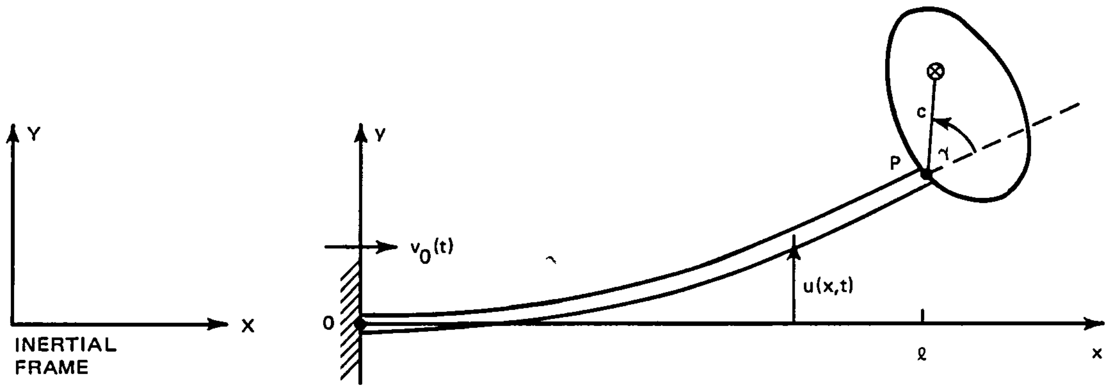


Figure 2-2.

Figure 2-3 is a free-body diagram of a beam element between x and x + Δx. T(x, t) is the internal tension, S(x, t) is the shear force, and M(x, t) is the bending moment. Since the beam is assumed to have no deformation along the x axis, the acceleration component along x is $a_0(t)$ for any point in the beam, where $a_0(t)$ is the prescribed base acceleration at 0.

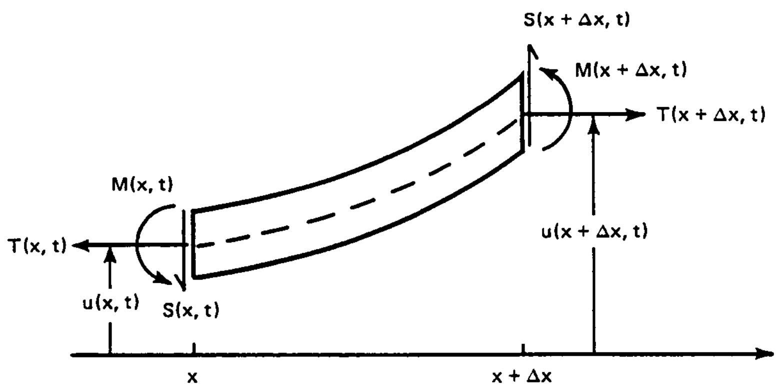


Figure 2-3.

Equilibrium of the beam element along the x direction gives

$$\frac{\partial T}{\partial x} = \rho a_o(t) \tag{2-1}$$

where ρ is the mass per unit length of the beam.

Similarly, equilibrium along the y-direction gives

$$\frac{\partial S}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \tag{2-2}$$

Applying rotational equilibrium to the beam element and neglecting rotary inertia we obtain

$$\frac{\partial M}{\partial x} + S(x,t) - T(x,t) \frac{\partial u}{\partial x} = 0 \tag{2-3}$$

Differentiating Eq. (2-3) and using Eq. (2-2), we obtain

$$\frac{\partial^2 M}{\partial x^2} + \rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left[T(x,t) \frac{\partial u}{\partial x}\right] = 0$$

Assuming an Euler-Bernoulli beam

$$M(x, t) = EI(x) \frac{\partial^2 u}{\partial x^2}$$

we arrive at the partial differential equation for the beam deflection

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 u}{\partial x^2}\right] - \frac{\partial}{\partial x} \left[T(x,t) \frac{\partial u}{\partial x}\right] + \rho \frac{\partial^2 u}{\partial t^2} = 0 \tag{2-4}$$

We also have the differential relations for the natural boundary conditions

$$M(x,t) = EI(x)\,\frac{\partial^2 u}{\partial x^2} \tag{2-5}$$

$$S(x,t) = T(x,t)\,\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left[EI(x)\,\frac{\partial^2 u}{\partial x^2}\right] \tag{2-6}$$

<u>Kinematics of the Tip Body</u>

Let $\vec{v}_p$ and $\vec{\omega}_p$ denote the inertial velocity of P and the angular velocity of the tip body, respectively

$$\vec{v}_p = v_o(t)\,\vec{i} + \frac{\partial u}{\partial t}(\ell,t)\,\vec{j} \tag{2-7}$$

$$\vec{\omega}_p = \frac{\partial^2 u}{\partial t\,\partial x}(\ell,t)\,\vec{k} \tag{2-8}$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ denote unit vectors along the directions of x, y, z, respectively.

If we let $\vec{c}$ be the vector from P to the mass center of the tip body and $\vec{v}_\oplus$ the velocity of the tip body mass center then

$$\vec{v}_\oplus = \vec{v}_p + \vec{\omega}_p \times \vec{c}$$

$$\vec{v}_\oplus = \left[v_o(t) - c\sin\gamma\,\frac{\partial^2 u}{\partial t\,\partial x}(\ell,t)\right]\vec{i} + \left[\frac{\partial u}{\partial t}(\ell,t) + c\cos\gamma\,\frac{\partial^2 u}{\partial t\,\partial x}(\ell,t)\right]\vec{j} \tag{2-9}$$

where we have dropped the nonlinear term $\frac{\partial u}{\partial x}(\ell,t)\ \frac{\partial^2 u}{\partial t \partial x}(\ell,t)$. Differentiating Eqs. (2-8) and (2-9), we have the acceleration expressions

$$\dot{\vec{\omega}}_p = \frac{\partial^3 u}{\partial t^2 \partial x}(\ell,t)\ \vec{k} \tag{2-10}$$

$$\vec{a}_\oplus = \left[a_o(t) - c \sin\gamma \frac{\partial^3 u}{\partial t^2 \partial x}(\ell,t)\right]\vec{i} + \left[\frac{\partial^2 u}{\partial t^2}(\ell,t) + c \cos\gamma \frac{\partial^3 u}{\partial t^2 \partial x}(\ell,t)\right]\vec{j} \tag{2-11}$$

<u>Natural Boundary Conditions at $x = \ell$</u>

Figure 2-4 is a free body diagram of the tip body indicating the force and moment exerted by the beam at P.

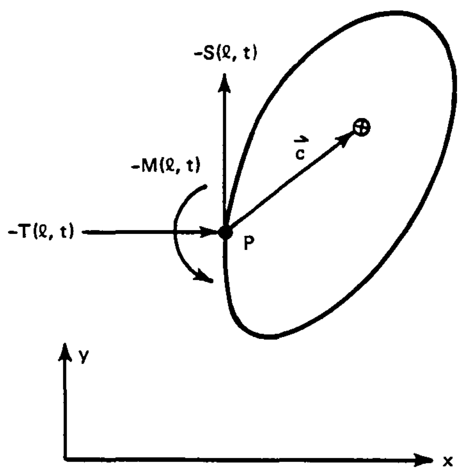


Figure 2-4.

The equation of motion of the tip body when resolved along the directions of x and y yields the two scalar equations

$$-T(\ell,t) = ma_o(t) - mc \sin\gamma \frac{\partial^3 u}{\partial x \partial t^2}(\ell,t) \qquad (2\text{-}12)$$

$$-S(\ell,t) = m \frac{\partial^2 u}{\partial t^2}(\ell,t) + mc \cos\gamma \frac{\partial^3 u}{\partial x \partial t^2}(\ell,t) \qquad (2\text{-}13)$$

Using Eqs. (2-6) and (2-12) allows us to write the first boundary condition at $x = \ell$ as

$$mc \cos\gamma \frac{\partial^3 u}{\partial x \partial t^2}(\ell,t) - EI \frac{\partial^3 u}{\partial x^3}(\ell,t) - ma_o(t) \frac{\partial u}{\partial x}(\ell,t) + m \frac{\partial^2 u}{\partial t^2}(\ell,t) = 0 \qquad (2\text{-}14)$$

where we have assumed a uniform EI and have dropped the nonlinear term $mc \sin\gamma \frac{\partial u}{\partial x}(\ell,t) \frac{\partial^3 u}{\partial x \partial t^2}(\ell,t)$.

It can be shown that the z component of the time rate of change of the angular momentum of the tip body about its mass center is given by

$$I_t \frac{\partial^3 u}{\partial x \partial t^2}(\ell,t)$$

If $\vec{M}_\oplus$ is the net moment about the mass center of the tip body

$$\vec{M}_\oplus = -M(\ell,t)\vec{k} + \vec{c} \times S(\ell,t)\vec{j} + \vec{c} \times T(\ell,t)\vec{i}$$

Using Eqs. (2-5) and (2-6) and dropping nonlinear terms we obtain

$$\vec{M}_{\oplus} = -\left[EI \frac{\partial^2 u}{\partial x^2} (\ell,t) + EI\, c \cos\gamma \frac{\partial^3 u}{\partial x^3} (\ell,t) + c \sin\gamma\, T(\ell,t)\right]\vec{k} \tag{2-15}$$

Using Eq. (2-12) for $T(\ell,t)$ in Eq. (2-15), the second boundary condition at $x = \ell$ is

$$(I_t + mc^2 \sin^2\gamma) \frac{\partial^3 u}{\partial x \partial t^2} (\ell,t) + EI\, c \cos\gamma \frac{\partial^3 u}{\partial x^3} (\ell,t) + EI \frac{\partial^2 u}{\partial x^2} (\ell,t) - mc \sin\gamma\, a_o(t) = 0 \tag{2-16}$$

The partial differential equation governing $u(x,t)$, Eq. (2-4), requires a specification of the internal tension $T(x,t)$ for $0 \le x \le \ell$. Integrating Eq. (2-1) and using Eq. (2-12) for a boundary condition we obtain

$$T(x,t) = -[\rho(\ell - x) + m]a_o(t) + mc \sin\gamma \frac{\partial^3 u}{\partial x \partial t^2} (\ell,t) \tag{2-17}$$

Assume we have a compressive load, $P_0(t) > 0$ applied at $x = 0$. Then

$$T(0, t) = -P_o(t)$$

Hence

$$P_o(t) = (\rho\ell + m)a_o(t) - mc \sin\gamma \frac{\partial^3 u}{\partial x \partial t^2} (\ell,t) \tag{2-18}$$

which relates the applied axial force to the base acceleration.

The governing equation of motion, Eq. (2-4), can be rewritten with the aid of Eqs. (2-17) and (2-18) as

$$EI \frac{\partial^4 u}{\partial x^4} + P_o(t) \frac{\partial}{\partial x} \left[\left(1 - \frac{\rho x}{\rho \ell + m}\right) \frac{\partial u}{\partial x}\right] + \rho \frac{\partial^2 u}{\partial t^2} = 0 \qquad (2-19)$$

We assume the beam to be cantilevered at $x = 0$ so that

$$u(0,t) = \frac{\partial u}{\partial x}(0, t) = 0 \quad \text{for all } t \qquad (2-20)$$

The partial differential Eq. (2-19) 's to be solved subject to the geometric boundary conditions of Eq. (2-20) and the natural boundary conditions of Eqs. (2-14) and (2-16).

In Reference 1, we studied the transverse vibration of a clamped beam with tip body with no base acceleration. On page 10 of that report it was stated that γ must be restricted to zero in order that no axial loads be introduced into the problem. With the present analysis, one can accommodate any axial loads that may be introduced by $\gamma \neq 0$, and will discover, surprisingly, that the results given in Reference 1 can be directly extended to the case of $\gamma \neq 0$.

CHAPTER 3

STEADY-STATE SOLUTIONS

3.1 Buckling of a Cantilevered-Free Beam Under Axial Thrust

In this section, the possibility of a cantilevered-free beam (no tip body) buckling under axial thrust is investigated. The governing equation is Eq. (2-19) with no time dependence and m = 0.

$$EI \frac{d^4y}{dx^4} + P_{cr} \frac{d}{dx} \left[(1 - x/\ell) \frac{dy}{dx} \right] = 0 \qquad (3\text{-}1)$$

where P_{cr} denotes those values of thrust which lead to buckling.

The two geometric boundary conditions at x = 0 are

$$y(0) = y'(0) = 0$$

The boundary conditions at $x = \ell$ are obtained from Eqs. (2-14) and (2-16) with m = c = 0

$$y''(\ell) = y'''(\ell) = 0$$

Integrating Eq. (3-1) and setting the constant of integration to zero we obtain

$$\frac{d^3y}{dx^3} + \frac{P_{cr}}{EI} (1 - x/\ell) \frac{dy}{dx} = 0 \qquad (3\text{-}2)$$

subject to the three boundary conditions

$$y(0) = y'(0) = y''(\ell) = 0$$

making the substitution

$$\frac{dy}{dx} = w, \quad z = \left(\frac{P_{cr}\ell^2}{EI}\right)^{1/3} (1 - x/\ell)$$

transforms Eq. (3-2) into

$$\frac{d^2w}{dz^2} + zw = 0$$

with boundary conditions

$$w = 0 \text{ at } z = \left(\frac{P_{cr}\ell^2}{EI}\right)^{1/3}$$

$$\frac{dw}{dz} = 0 \text{ at } z = 0$$

The general solution of the differential equation can be written

$$w = c_1\sqrt{z}\, J_{1/3}\left(\frac{2}{3} z^{3/2}\right) + c_2\sqrt{z}\, J_{-1/3}\left(\frac{2}{3} z^{3/2}\right)$$

where c_1 and c_2 are arbitrary constants and $J_{1/3}$ denotes the Bessel function of the first kind of order 1/3.

The boundary condition $\frac{dw}{dz} = 0$ at $z = 0$ implies that $c_1 = 0$. Let j_n denote the roots of the Bessel function of the first kind of order (-1/3), i.e.,

$$J_{-1/3}(j_n) = 0$$

Applying the remaining boundary condition we find that the critical buckling loads are given by

$$P_{cr,n} = \frac{9}{4} j_n^2 \frac{EI}{\ell^2} \qquad n = 1,2,3, \ldots \tag{3-3}$$

The first three buckling loads are

$$P_{cr1} = 7.8664 \frac{EI}{\ell^2} ; \quad P_{cr2} = 55.977 \frac{EI}{\ell^2} ; \quad P_{cr3} = 148.51 \frac{EI}{\ell^2}$$

3.2 Buckling of Cantilevered Beam with Tip Mass Under Axial Thrust

The governing equation is Eq. (2-19) with no time dependence and the boundary conditions are Eqs. (2-20), (2-14), and (2-16), with $I_t = 0$ and $c = 0$.

$$EI \frac{d^4y}{dx^4} + P_{cr} \frac{d}{dx} \left[\left(1 - \frac{\rho x}{\rho\ell + m}\right) \frac{dy}{dx}\right] = 0 \tag{3-4}$$

$$y(0) = y'(0) = 0 ; \; y''(\ell) = 0 ; \; EI\, y'''(\ell) + \frac{mP_{cr}}{\rho\ell + m} y'(\ell) = 0$$

where we have used Eq. (2-18) to express a_o in terms of P_{cr}.

Integrating Eq. (3-4) and setting the constant of integration to zero, we will satisfy the last boundary condition.

$$\frac{d^3y}{dx^3} + \frac{P_{cr}}{EI}\left(1 - \frac{\rho x}{\rho \ell + m}\right)\frac{dy}{dx} = 0$$

If we make the substitutions $\frac{dy}{dx} = w$ and

$$z = \left[\frac{P_{cr}}{EI}\left(\frac{\rho \ell + m}{\rho}\right)^2\right]^{1/3}\left(1 - \frac{\rho x}{\rho \ell + m}\right)$$

this equation transforms to

$$\frac{d^2w}{dz^2} + zw = 0 \qquad (3\text{-}5)$$

with boundary conditions

$$w = 0 \text{ at } z = z_1$$

and

$$\frac{dw}{dz} = 0 \text{ at } z = z_2$$

where

$$z_1 = \left(\frac{P_{cr}}{EI}\right)^{1/3}\left(\frac{\rho \ell + m}{\rho}\right)^{2/3}$$

and

$$z_2 = \frac{m}{\rho^{2/3}}\left[\frac{P_{cr}}{EI(\rho \ell + m)}\right]^{1/3}$$

As has already been shown

$$w = \sqrt{z}\left[c_1 J_{1/3}\left(\frac{2}{3} z^{3/2}\right) + c_2 J_{-1/3}\left(\frac{2}{3} z^{3/2}\right)\right]$$

$$\frac{dw}{dz} = z\left[c_1 J_{-2/3}\left(\frac{2}{3} z^{3/2}\right) - c_2 J_{2/3}\left(\frac{2}{3} z^{3/2}\right)\right]$$

Applying the boundary conditions, we obtain a homogeneous system in c_1 and c_2. Setting the determinant to zero (we want $y(x) \neq 0$), we obtain the following transcendental equation for the critical buckling loads.

$$J_{1/3}\left((1 + m^*)\lambda\right) \cdot J_{2/3}\left(\frac{m^*}{\sqrt{1 + 1/m^*}}\lambda\right) + J_{-1/3}\left((1 + m^*)\lambda\right) \cdot J_{-2/3}\left(\frac{m^*}{\sqrt{1 + 1/m^*}}\lambda\right) = 0 \quad (3\text{-}6)$$

where

$$m^* = m/\rho\ell$$

and

$$\lambda = \frac{2}{3}\sqrt{\frac{P_{cr}\ell^2}{EI}}$$

Note that the roots of Eq. (3-6) depend only upon the ratio of the tip mass to the beam mass.

In the limit as $m^* \to 0$, the critical buckling loads in Eq. (3-6) should approach those found previously for a cantilevered-free beam (Eq. 3-3). This is readily established if we recall the facts

$$\lim_{x \to 0} J_{2/3}(x) = 0$$

and

$$\lim_{x \to 0} J_{-2/3}(x) = \infty$$

3.3 Buckling of Cantilevered Beam with Tip Body Under Axial Thrust ($\gamma = 0$)

The governing equation is Eq. (2-19) with no time dependence and the boundary conditions are Eqs. (2-14), (2-16), and (2-20) with $\gamma = 0$.

$$EI \frac{d^4y}{dx^4} + P_{cr} \frac{d}{dx} \left[\left(1 - \frac{\rho x}{\rho \ell + m}\right) \frac{dy}{dx} \right] = 0$$

$$y(0) = y'(0) = 0 \; ; \quad EI\, c\, y'''(\ell) + EI\, y''(\ell) = 0 \qquad (3-6)$$

$$EI\, y'''(\ell) + m \frac{P_{cr}}{\rho \ell + m} y'(\ell) = 0 \qquad (3-7)$$

Integrating the differential equation and setting the constant of integration to zero, we will satisfy the boundary condition Eq. (3-7).

$$\frac{d^3y}{dx^3} + \frac{P_{cr}}{EI} \left(1 - \frac{\rho x}{\rho \ell + m}\right) \frac{dy}{dx} = 0$$

If we make the same substitutions as in the previous section we obtain

$$\frac{d^2w}{dz^2} + zw = 0 \; ; \quad w = 0 \text{ at } z = z_1$$

Substituting Eq. (3-7) into Eq. (3-6) allows us to write the second boundary condition as

$$(\rho \ell + m)\, EI\, y''(\ell) - mc\, P_{cr}\, y'(\ell) = 0 \qquad (3-8)$$

or in terms of $w(z)$

$$[(\rho \ell + m) EI]^{2/3} (P_{cr} \rho)^{1/3} \frac{dw}{dz}(z_2) + mc\, P_{cr} w(z_2) = 0 \qquad (3-9)$$

Using the results from the previous section, Eq. (3-9) can be written as

$$\left[\sqrt{1 + m^*}\ J_{-2/3}\left(\frac{m^*}{\sqrt{1 + 1/m^*}}\lambda\right) + \frac{3}{2}\sqrt{m^*}\ c^*\ \lambda\ J_{1/3}\left(\frac{m^*}{\sqrt{1 + 1/m^*}}\lambda\right)\right]c_1$$

$$+ \left[\frac{3}{2}\sqrt{m^*}\ c^*\ \lambda\ J_{-1/3}\left(\frac{m^*}{\sqrt{1 + 1/m^*}}\lambda\right) - \sqrt{1 + m^*}\ J_{2/3}\left(\frac{m^*}{\sqrt{1 + 1/m^*}}\lambda\right)\right]c_2 = 0 \qquad (3\text{-}10)$$

where $c^* = c/\ell$.

The first boundary condition at $z = z_1$ can be written as

$$J_{1/3}\big((1 + m^*)\lambda\big)c_1 + J_{-1/3}\big((1 + m^*)\lambda\big)c_2 = 0 \qquad (3\text{-}11)$$

The homogeneous system, Eqs. (3-10) and (3-11), will have a nontrivial solution if

$$J_{-1/3}(\alpha\lambda)\left[\sqrt{\alpha}\ J_{-2/3}(\beta\lambda) + \frac{3}{2}\sqrt{m^*}\ c^*\ \lambda\ J_{1/3}(\beta\lambda)\right]$$

$$+ J_{1/3}(\alpha\lambda)\left[\sqrt{\alpha}\ J_{2/3}(\beta\lambda) - \frac{3}{2}\sqrt{m^*}\ c^*\ \lambda\ J_{-1/3}(\beta\lambda)\right] = 0 \qquad (3\text{-}12)$$

where $\alpha = 1 + m^*$ and $\beta = m^*/\sqrt{1 + \frac{1}{m^*}}$. Note that the critical buckling loads are independent of the tip body inertia.

3.4 Steady-State Solutions of a Cantilevered Beam with Tip Body under Axial Thrust ($\gamma \neq 0$)

In the previous section, we investigated the buckling of a uniform cantilevered beam with tip body. It will be recalled that in that section, the mass center of the tip body was restricted to lie along the beam tip tangent line. Presently, we wish to investigate the possibility of buckling for the case of nonzero γ.

As before, we wish to determine solutions of Eq. (2-19) which are indeoendent of time, u = y(x). The governing equation is

$$EI \frac{d^4y}{dx^4} + P_{cr} \frac{d}{dx} \left[\left(1 - \frac{\rho x}{\rho \ell + m}\right) \frac{dy}{dx}\right] = 0 \qquad (3\text{-}13)$$

with geometric boundary conditions $y(0) = \frac{dy}{dx}(0) = 0$. The natural boundary conditions at $x = \ell$ may be written

$$EI\, c \cos\gamma\, y'''(\ell) + EI\, y''(\ell) - mc \sin\gamma \frac{P_{cr}}{\rho\ell + m} = 0 \qquad (3\text{-}14)$$

$$EI\, v'''(\ell) + m \frac{P_{cr}}{\rho\ell + m} y'(\ell) = 0 \qquad (3\text{-}15)$$

Inteqrating Eq. (3-13) and setting the constant of integration to zero, we will satisfy the boundary condition Eq. (3-15).

$$\frac{d^3y}{dx^3} + \frac{P_{cr}}{EI} \left(1 - \frac{\rho x}{\rho\ell + m}\right) \frac{dy}{dx} = 0 \qquad (3\text{-}16)$$

If we make the change of variables

$$\frac{dy}{dx} = w$$

and

$$z = \left[\frac{P_{cr}}{EI} \left(\frac{\rho\ell + m}{\rho}\right)^2\right]^{1/3} \left(1 - \frac{\rho x}{\rho\ell + m}\right)$$

the differential equation (3-16) transforms into

$$\frac{d^2w}{dz^2} + zw = 0 \tag{3-17}$$

$$x = 0 \text{ goes into } z = z_1 \equiv \left(\frac{P_{cr}}{EI}\right)^{1/3} \left(\frac{\rho\ell + m}{\rho}\right)^{2/3}$$

$$x = \ell \text{ goes into } z = z_2 \equiv \frac{m}{\rho^{2/3}} \left[\frac{P_{cr}}{EI(\rho\ell + m)}\right]^{1/3}$$

Substituting Eq. (3-15) into Eq. (3-14) and using the above transformation allows us to write the remaining natural boundary condition as

$$[(\rho\ell + m)EI]^{2/3}(P_{cr}\rho)^{1/3} \frac{dw}{dz}(z_2) + mc \cos\gamma\, P_{cr}w(z_2) = -mc \sin\gamma\, P_{cr} \tag{3-18}$$

we can write the general solution to Eq. (3-17) as

$$w = \sqrt{z}\left[c_1 J_{1/3}\left(\frac{2}{3} z^{3/2}\right) + c_2 J_{-1/3}\left(\frac{2}{3} z^{3/2}\right)\right]$$

where c_1 and c_2 are arbitrary constants.

Applying the boundary conditions $w(z_1) = 0$ and Eq. (3-18) we obtain the following *nonhomogeneous* system on c_1 and c_2

$$J_{1/3}(\alpha\lambda) \cdot c_1 + J_{-1/3}(\alpha\lambda) \cdot c_2 = 0$$

$$\left[\sqrt{\alpha}J_{-2/3}(\beta\lambda) + \frac{3}{2}\sqrt{m^*}\, c^* \cos\gamma \cdot \lambda J_{1/3}(\beta\lambda)\right]c_1$$

$$+ \left[\frac{3}{2}\sqrt{m^*}\, c^* \cos\gamma \cdot \lambda J_{-1/3}(\beta\lambda) - \sqrt{\alpha}\, J_{2/3}(\beta\lambda)\right]c_2 = -\sqrt{m^*}\, \frac{c^* \sin\gamma}{\beta^{1/3}} \left(\frac{3\lambda}{2}\right)^{2/3} \tag{3-19}$$

where the parameters m^*, c^*, λ, α, β are as defined previously.

In general these equations will have a unique solution. If λ (axial load) assumes a value such that the coefficient matrix in Eq. (3-19) is singular, it can be shown that the equations are inconsistent. Note that the transcendental equation obtained by setting the determinant of the system (3-19) to zero is exactly the same as Eq. (3-12) except for the replacement $c^* \to c^* \cos\gamma$. Hence the system (3-19) can become singular for an infinite number of real values of λ.

For those cases where a unique solution exists, we can integrate once more and obtain (recall $y = 0$ at $x = 0$)

$$y(x) = \frac{1}{s} \int_{\alpha\lambda}^{\zeta} \left(c_1 J_{1/3}(t) + c_2 J_{-1/3}(t) \right) dt \qquad (3\text{-}20)$$

where

$$s \equiv -\frac{1}{\ell} \left(\frac{9}{4} \frac{\lambda^2}{\alpha} \right)^{1/3}$$

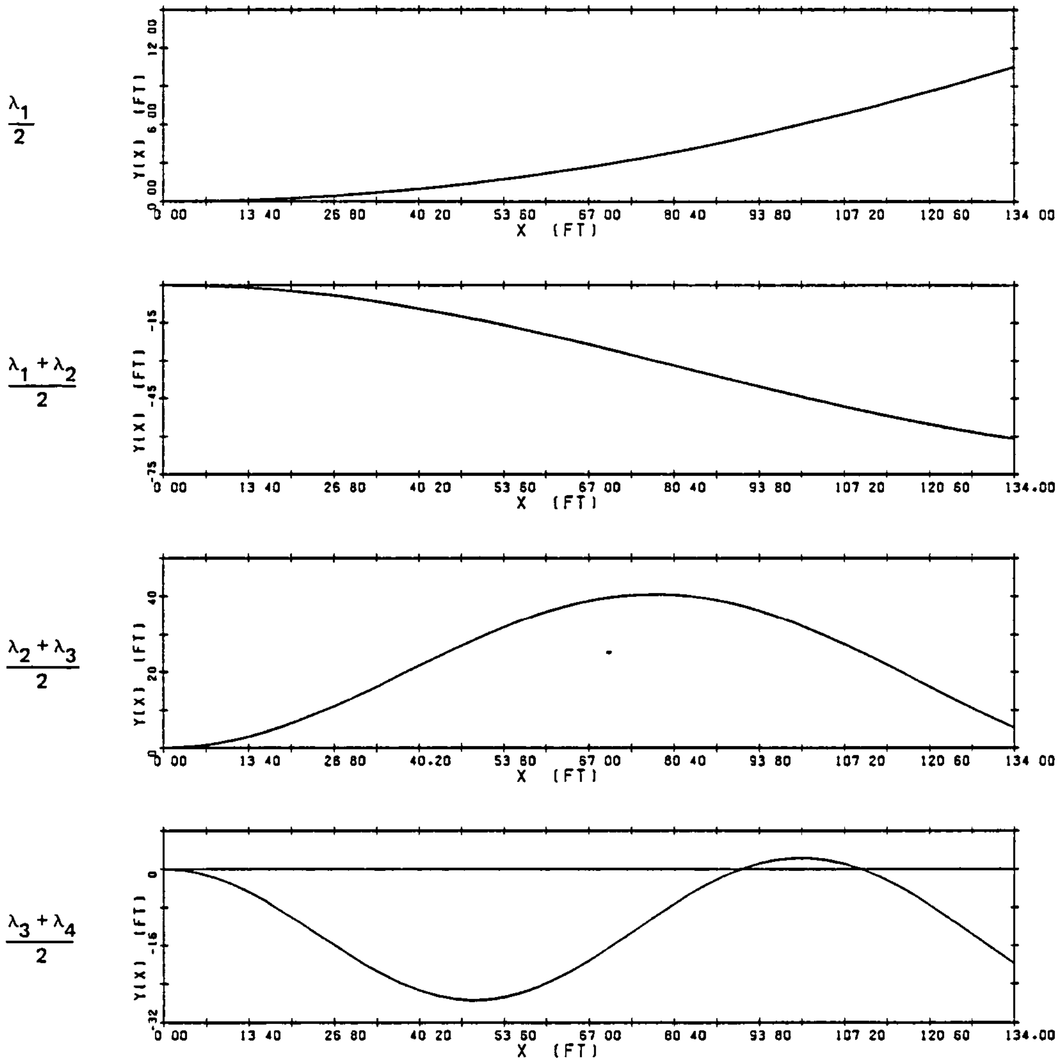
and

$$\zeta = \alpha\lambda \left(1 - \frac{x}{\alpha\ell} \right)^{3/2}$$

on the interval $0 < x < \ell$.

Figure 3-1 illustrates the steady-state solutions given by Eq. (3-20) at values of λ lying between the first four consecutive critical values. Numerical experiments revealed that the overall shapes of these curves are insensitive to m^*, c^*, ℓ, and γ within the respective ranges of λ.

We have shown that for the case $\gamma \neq 0$, a unique steady-state solution to the vibration equation exists, except when the axial thrust assumes certain critical values (such that the linear system (3-19) becomes singular) for which no solution exists.



PARAMETERS $m^* = 2\ 778$, $c^* = 0\ 5$, $\ell = 134\ 0$ ft, $\gamma = 45\ 0°$

FIRST FOUR CRITICAL VALUES OF λ

$\lambda_1 = 0\ 89$ $\lambda_2 = 2\ 74$ $\lambda_3 = 4\ 81$ $\lambda_4 = 6\ 97$

Figure 3-1. Representative steady-state solutions for $\gamma \neq 0$.

CHAPTER 4

VIBRATION SOLUTIONS

4.1 Natural Frequencies and Mode Shapes ($\gamma = 0$)--Power Series Solution

In this chapter, we will formulate the eigenvalue problem for the transverse vibration of a cantilevered beam with tip body under the action of axial thrust. Since an eigenvalue problem requires the solution of a homogeneous differential equation with homogeneous boundary conditions, we need to restrict γ to zero (see boundary condition (2-16). The axial thrust P_o is assumed constant.

Using Eqs. (2-18) through (2-20), (2-14) and (2-16), we have

$$EI \frac{\partial^4 u}{\partial x^4} + P_o \frac{\partial}{\partial x} \left[\left(1 - \frac{\rho x}{\rho \ell + m}\right) \frac{\partial u}{\partial x}\right] + \rho \frac{\partial^2 u}{\partial t^2} = 0 \qquad (4-1)$$

$$u(0,t) = \frac{\partial u}{\partial x}(0,t) = 0: \quad t \geq 0 \qquad (4-2)$$

$$I_t \frac{\partial^3 u}{\partial x \partial t^2}(\ell,t) + EI \cdot c \frac{\partial^3 u}{\partial x^3}(\ell,t) + EI \cdot \frac{\partial^2 u}{\partial x^2}(\ell,t) = 0 \qquad (4-3)$$

$$mc \frac{\partial^3 u}{\partial x \partial t^2}(\ell,t) - EI \cdot \frac{\partial^3 u}{\partial x^3}(\ell,t) - m \frac{P_o}{\rho \ell + m} \frac{\partial u}{\partial x}(\ell,t) + m \frac{\partial^2 u}{\partial t^2}(\ell,t) = 0 \qquad (4-4)$$

Assuming a solution $e^{i\omega t}\phi(x)$, we arrive at the eigenvalue problem

$$EI \frac{d^4\phi}{dx^4} + P_o \frac{d}{dx} \left[\left(1 - \frac{\rho x}{\rho \ell + m}\right) \frac{d\phi}{dx}\right] - \rho\omega^2 \phi(x) = 0 \tag{4-5}$$

$$\phi(0) = \frac{d\phi}{dx}(0) = 0 \tag{4-6}$$

$$EI\ \phi'''(\ell) + m\left(c\omega^2 + \frac{P_o}{\rho\ell + m}\right)\phi'(\ell) + m\omega^2\phi(\ell) = 0 \tag{4-7}$$

$$mc\omega^2\phi(\ell) + \left[(I_t + mc^2)\omega^2 + \frac{mcP_o}{\rho\ell + m}\right] \phi'(\ell) - EI\ \phi''(\ell) = 0 \tag{4-8}$$

We wish to find those values of ω for which the differential equation (4-5), subject to the boundary conditions (4-6) through (4-8) has nontrivial solutions.

Scaling of Eigenvalue Problem

Let $\xi = x/\ell$ and define $y(\xi) \equiv \phi(\ell\xi)$. Also define the dimensionless parameters

$$a^2 = \frac{P_o \ell^2}{EI} \qquad b^4 = \frac{\rho\ell^4}{EI}\,\omega^2 \tag{4-9}$$

$$m^* = m/\rho\ell, \quad c^* = c/\ell, \quad J^* = \frac{I_t + mc^2}{\rho\ell^3}$$

The eigenvalue problem will then transform into the convenient form

$$\frac{d^4y}{d\xi^4} + a^2 \frac{d}{d\xi}\left[\left(1 - \frac{\xi}{1 + m^*}\right)\frac{dy}{d\xi}\right] - b^4 y(\xi) = 0 \tag{4-10}$$

$$y(o) = y'(o) = 0 \tag{4-11}$$

$$y'''(1) + m^*\left(c^*b^4 + \frac{a^2}{1 + m^*}\right)y'(1) + m^*b^4 y(1) = 0 \tag{4-12}$$

$$m^*c^*b^4 y(1) + \left(J^*b^4 + \frac{m^*c^*}{1 + m^*}a^2\right)y'(1) - y''(1) = 0 \tag{4-13}$$

Here (') denotes differentiation with respect to ξ.

Let $y_i(\xi)$ and $y_j(\xi)$ be two eigenfunctions corresponding to distinct eigenvalues. It can be shown that the following orthogonality condition holds

$$\int_0^1 y_i(\xi)y_j(\xi)d\xi + J^*y_i'(1)y_j'(1) + m^*y_i(1)y_j(1)$$

$$+ m^*c^*\left[y_i(1)y_j'(1) + y_i'(1)y_j(1)\right] = 0$$

cf. Eq. (2-16) of Reference ´.

The general solution of Eq. (4-10) can be written as a linear combination of four linearly independent solutions.

$$y(\xi) = c_1y_1(\xi) + c_2y_2(\xi) + c_3y_3(\xi) + c_4y_4(\xi)$$

For convenience, we specify the initial values of these four functions and their first three derivatives as

$$y_i^{(j-1)}(0) = \delta_{ij} \qquad i = 1,2,3,4; \quad j = 1,2,3,4$$
$$(\delta_{ij} \text{ - Kronecker delta})$$

Applying the geometric boundary conditions (4-11) we find

$$c_1 = 0 \quad \text{and} \quad c_2 = 0$$

Application of the natural boundary conditions (4-12) and (4-13) yields the homogeneous system of equations on c_3 and c_4

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$\begin{aligned} a_{11} &= y_3'''(1) + m^* \left(c^*b^4 + \frac{a^2}{1+m^*}\right) y_3'(1) + m^*b^4 y_3(1) \\ a_{12} &= y_4'''(1) + m^* \left(c^*b^4 + \frac{a^2}{1+m^*}\right) y_4'(1) + m^*b^4 y_4(1) \\ a_{21} &= m^*c^*b^4 y_3(1) + \left(J^*b^4 + \frac{m^*c^*}{1+m^*} a^2\right) y_3'(1) - y_3''(1) \\ a_{22} &= m^*c^*b^4 y_4(1) + \left(J^*b^4 + \frac{m^*c^*}{1+m^*} a^2\right) y_4'(1) - y_4''(1) \end{aligned} \qquad (4\text{-}14)$$

In order for this system to have a nontrivial solution we require

$$\Delta(\omega^2) \equiv a_{11}a_{22} - a_{12}a_{21} = 0 \qquad (4\text{-}15)$$

Equation (4-15) determines the natural frequencies ω_i.

Power Series Expansion for Eigenfunctions

It still remains to find the functions y_3 and y_4. A purely numerical technique which is easy to apply consists of integrating two initial value problems and iterating on ω until values are found which satisfy Eq. (4-15). The well-known natural frequencies of a cantilevered-free beam serve to bracket the frequencies in the present case of a cantilevered beam with a tip body on an accelerating base. We choose to address the problem analytically by solving the differential Eq. (4-10) in the form of a power series, thus obtaining the eigenfunctions in functional form.

Note that for $P_0 = 0$, the differential equation has constant coefficients and can therefore be solved in closed form in terms of elementary functions (see Reference 1). For the more general case, $P_0 \neq 0$, the equation is linear with analytic coefficients and has a solution which is regular at $\xi = 0$. The series representing the solution will converge for all ξ (see Ince., E.L., "Ordinary Differential Equations").

If we assume a solution of Eq. (4-10) in the form

$$y = \sum_{k=0}^{\infty} A_k \xi^k$$

insert this expansion and its corresponding derivatives into Eq. (4-10), then we obtain the following recursion formulas for the coefficients

$$A_4 = \left[b^4 A_0 + a^2/(1 + m^*)\ A_1 - 2a^2 A_2\right]/24 \qquad (4\text{-}16)$$

$$A_{k+4} = \frac{b^4 A_k + a^2/(1 + m^*) \cdot (k + 1)^2 A_{k+1} - a^2 (k + 1)(k + 2) A_{k+2}}{(k + 1)(k + 2)(k + 3)(k + 4)} \qquad (4\text{-}17)$$

These conditions determine A_4, A_5, A_6, ... once we presribe values for A_0, A_1, A_2, and A_3.

To generate $y_3(\xi)$, we must have $A_0 = 0$, $A_1 = 0$, $A_2 = \frac{1}{2}$, $A_3 = 0$

To generate $y_4(\xi)$, we must have $A_0 = 0$, $A_1 = 0$, $A_2 = 0$, $A_3 = 1/6$

The eigenfunction $Y_i(\xi)$ corresponding to the eigenvalue ω_i is

$$Y_i(\xi) = a_{12}(\omega_i)\left[\frac{1}{2}\xi^2 - \frac{a^2}{24}\xi^4 + \frac{a^2}{60(1 + m^*)}\xi^5 + \frac{a^4 + b_i^4}{720}\xi^6 + \ldots\right]$$
$$- a_{11}(\omega_i)\left[\frac{1}{6}\xi^3 - \frac{a^2}{120}\xi^5 + \frac{a^2}{240(1 + m^*)}\xi^6 + \frac{a^4 + b_i^4}{5040}\xi^7 + \ldots\right] \tag{4-18}$$

Given the infinite series $\sum_{k=0}^{\infty} A_k \xi^k$, then by the ratio test the series converges (absolutely) for those values of ξ for which

$$\lim_{k\to\infty} \left|\frac{A_{k+1}}{A_k}\right| |\xi| < 1$$

We noted above that the differential equation (4-10) has solutions which are regular at $\xi = 0$ and that these series converge for all ξ. Using the recurrence formula Eq. (4-17) we can verify this fact. This more elementary procedure was used by Lamb ("Hydrodynamics", p. 335).

Write the recurrence formula in the form

$$\frac{A_{k+4}}{A_k} + \frac{a^2}{(k + 3)(k + 4)}\frac{A_{k+2}}{A_k} - \frac{[a^2/(1 + m^*)](k + 1)}{(k + 2)(k + 3)(k + 4)}\frac{A_{k+1}}{A_k}$$
$$- \frac{b^4}{(k+1)(k + 2)(k + 3)(k + 4)} = 0$$

If $\lim_{k\to\infty} \frac{A_{k+1}}{A_k} = L$ then $\lim_{k\to\infty} \frac{A_{k+2}}{A_k} = L^2$ and $\lim_{k\to\infty} \frac{A_{k+4}}{A_k} = L^4$. Taking the limit in the above formula as $k\to\infty$ we obtain the condition $L^4 = 0$ or $\lim_{k\to\infty} \frac{A_{k+1}}{A_k} = 0$. Hence the series converges for all ξ in agreement with the general theory.

4.2 Free Vibration of a Cantilevered Beam with Tip Body under Constant Base Acceleration ($\gamma \neq 0$)

In this section we investigate the vibrational response of the accelerating beam with $\gamma \neq 0$. It is important to note that in all cases considered thus far, $\gamma = 0$ or steady-state response, a constant axial force implied a constant base acceleration. In the present case this is no longer true (see Eq. (2-18). We will assume here that a base force $P_o(t)$ is applied such that the base acceleration a_o is constant.

Neglecting nonlinear terms in u(x,t), the equation of moton is

$$EI \frac{\partial^4 u}{\partial x^4} + a_o \frac{\partial}{\partial x} \left[(\rho \ell + m - \rho x) \frac{\partial u}{\partial x} \right] + \rho \frac{\partial^2 u}{\partial t^2} = 0 \qquad (4\text{-}19)$$

The boundary conditions are given by Eqs. (2-14), (2-16), and (2-20). The boundary condition (2-16) is inhomogeneous. In order to solve the differential equation (4-19) we write

$$u(x,t) = v(x,t) + f(x) \qquad (4\text{-}20)$$

and choose f(x) so that the boundary conditions on v are rendered homogeneous. We then have the following requirements on f(x)

$$f(0) = 0$$

$$f'(0) = 0 \qquad (4\text{-}21)$$

$$EI\, f'''(\ell) + ma_0 f'(\ell) = 0 \qquad (4\text{-}21)$$

$$c\, EI \cos\gamma\, f'''(\ell) + EI\, f''(\ell) = mc \sin\gamma\, a_0$$

The partial differential equation on v(x,t) can be written as

$$EI \frac{\partial^4 v}{\partial x^4} + a_o \frac{\partial}{\partial x} \left[(\rho\ell + m - \rho x) \frac{\partial v}{\partial x}\right] + \rho \frac{\partial^2 v}{\partial t^2} =$$

$$- \left\{EI \frac{d^4 f}{dx^4} + a_o \frac{d}{dx} \left[(\rho\ell + m - \rho x) \frac{df}{dx}\right]\right\} \qquad (4\text{-}22)$$

If, in addition to satisfying the conditions (4-21), we require that f(x) satisfies

$$EI \frac{d^4 f}{dx^4} + a_o \frac{d}{dx} \left[(\rho\ell + m - \rho x) \frac{df}{dx}\right] = 0 \qquad (4\text{-}23)$$

then the differential equation on v(x,t), as well as the boundary conditions, are rendered homogeneous.

The conditions on f(x), i.e., Eq. (4-23) with boundary conditions (4-21), are precisely those governing the steady-state solution of Eq. (4-19) with the boundary conditions (2-14), (2-16), and (2-20). It was shown in Section 3.4 that f(x) exists for all values of a_0, except when the system (3-19) becomes singular. f(x) is given by (3-20) with

$$\lambda = \frac{2}{3} \sqrt{\frac{\ell^2}{EI} (\rho\ell + m) a_0}$$

It is seen that v(x,t) is governed by the system of Equations (4-1) through (4-4) with the replacements

$$c \rightarrow c \cos \gamma, \qquad I_t \rightarrow I_t + mc^2 \sin^2 \gamma$$

$$p_o \rightarrow a_o(\rho\ell + m)$$

thus

$$v(x,t) = \sum_{k=1}^{\infty} (A_k \cos \omega_k t + B_k \sin \omega_k t) Y_k(\xi)$$

where the frequencies ω_k are solutions of Eq. (4-15) with $c^* \rightarrow c^* \cos \gamma$, $p_o \rightarrow a_o(\rho\ell + m)$, and the functions $Y_k(\xi)$ are given by Eq. (4-18).

All frequencies are positive for sufficiently small values of a_0. When a_0 assumes a critical value such that Eq. (3-12) is satisfied (with $c^* \rightarrow c^* \cos \gamma$), a frequency ω_k goes to zero. This corresponds to the situation where f(x) fails to exist.

CHAPTER 5

NATURAL FREQUENCIES AND MODE SHAPES ($\gamma = 0$) ASSUMED MODES SOLUTION

In this chapter, approximate solutions to the natural frequencies and mode shapes of a cantilevered beam with tip body subject to constant axial base acceleration are determined via an assumed modes formulation. As indicated in Section 4.1, γ must be restricted to zero. The mass per unit length, ρ, and bending stiffness, EI, are assumed to be constant.

5.1 Kinetic and Strain Energies

Let T_b, T_t, and T_s denote the kinetic energy of the beam, tip body and total system respectively. Using Eqs. (2-7), (2-8), and (2-9) we have in general

$$T_b = \frac{1}{2} \int_0^{\ell} [v_o^2 + (\frac{\partial u}{\partial t})^2]\rho \, dx$$

$$T_t = \frac{1}{2} I_t [\frac{\partial^2 u}{\partial x \partial t} (\ell,t)]^2 + \frac{1}{2} m \{[v_o - c \sin\gamma \frac{\partial^2 u}{\partial x \partial t} (\ell,t)]^2 + [\frac{\partial u}{\partial t} (\ell,t) + c \cos\gamma \frac{\partial^2 u}{\partial x \partial t} (\ell,t)]^2\}$$

$$T_s = \frac{1}{2} (\rho\ell + m) v_o^2(t) + \frac{1}{2} (I_t + mc^2)[\frac{\partial^2 u}{\partial x \partial t} (\ell,t)]^2 + \frac{1}{2} m[\frac{\partial u}{\partial t} (\ell,t)]^2 + mc \frac{\partial^2 u}{\partial x \partial t} (\ell,t) [\cos\gamma \frac{\partial u}{\partial t} (\ell,t) - \sin\gamma \, v_o(t)] + \frac{1}{2} \int_0^{\ell} [\frac{\partial u}{\partial t} (x,t)]^2 \rho \, dx \qquad (5\text{-}1)$$

For the case of $\gamma = 0$, this reduces to

$$T_s = \frac{1}{2}(\rho\ell + m)\, v_o^2(t) + \frac{1}{2}\int_o^{\ell} \left(\frac{\partial u}{\partial t}\right)^2 \rho\, dx + \frac{1}{2} I_t \left[\frac{\partial^2 u}{\partial t \partial x}(\ell,t)\right]^2$$

$$+ \frac{1}{2} m\left[\frac{\partial u}{\partial t}(\ell,t) + c\,\frac{\partial^2 u}{\partial x \partial t}(\ell,t)\right]^2 \qquad (5\text{-}2)$$

In Appendix A it is shown that the strain energy expression for the problem at hand is Eq. (A-11). Using Eq. (2-17) we obtain

$$V = \frac{1}{2} EI \int_o^{\ell} \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx + \frac{1}{2} a_o \rho\ell \int_o^{\ell} \left[\frac{x}{\ell} - (1 + \mu^*)\right]\left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$- \frac{1}{2} a_o\, mc\left[\frac{\partial u}{\partial x}(\ell,t)\right]^2 \qquad (5\text{-}3)$$

Discretization

Expressing the beam deformation as

$$u(x,t) = \sum_{i=1}^{n} \phi_i(x) q_i(t)$$

the energy expressions (5-2) and (5-3) become

$$T_s = \frac{1}{2}(\rho\ell + m)\, v_o^2(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}\dot{q}_i\dot{q}_j$$

where

$$M_{ij} = \rho \int_o^{\ell} \phi_i(x)\phi_j(x)\, dx + (I_t + mc^2)\phi_i'(\ell)\phi_j'(\ell) + m\phi_i(\ell)\phi_j(\ell)$$

$$+ mc\left[\phi_i(\ell)\phi_j'(\ell) + \phi_i'(\ell)\phi_j(\ell)\right] \qquad (5\text{-}4)$$

$$V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} K_{ij} q_i q_j$$

where

$$K_{ij} = EI \int_0^{\ell} \phi_i''(x)\phi_j''(x)\, dx + a_o \rho \ell \int_0^{\ell} \left[\frac{x}{\ell} - (1 + m^*)\right] \phi_i'(x)\phi_j'(x)\, dx$$

$$- a_o\, mc\; \phi_i'(\ell)\phi_j'(\ell) \qquad (5\text{-}5)$$

where

(') and (•) denote differentiation with respect to x and t, respectively

According to the assumed modes method, the spatial functions $\phi_i(x)$ may be any arbitrary functions which satisfy the geometric boundary conditions of the problem and are two times differentiable on the interval $0 < x < \ell$. Two sets of functions immediately suggest themselves as candidates: the fixed base eigenfunctions of a clamped-free beam and those of a clamped beam with tip body. Whereas the clamped-free eigenfunctions have the advantage that they can serve for beams with any tip body, superior convergence will be realized with use of the clamped-tip body eigenfunctions. Furthermore, the latter, which were derived in Reference 1, diagonalize the mass matrix, (5-4), as a result of their orthogonality property.

Either set of eigenfunctions satisfy differential equations of the form

$$\frac{d^4\phi_i}{dx^4} = \beta_i^4 \phi_i(x) \qquad (5\text{-}6)$$

For distinct eigenvalues β_i^4 and β_j^4, the following indefinite integral holds

$$\int \phi_i'(x)\phi_j'(x)\, dx = \frac{1}{(\beta_i^4 - \beta_j^4)} \left[\beta_i^4 \phi_i \phi_j' - \phi_i''' \phi_j'' + \phi_i'' \phi_j''' - \beta_j^4 \phi_i' \phi_j\right] \qquad (5\text{-}7)$$

This permits the analytical evaluation of the second integral in Eq. (5-5) via integration by parts

$$\int_0^\ell [\frac{x}{\ell} - (1 + m^*)]\ \phi_i'(x)\phi_j'(x)\ dx = \int_0^\ell [\frac{x}{\ell} - (1 + m^*)]\ \frac{d}{dx}\ [\int \phi_i'(x)\phi_j'(x)\ dx]\ dx$$

$$= \left\{ \frac{1}{(\beta_i^4 - \beta_j^4)}\ [\frac{x}{\ell} - (1 + m^*)][\beta_i^4 \phi_i \phi_j' - \phi_i''' \ \phi_j'' + \phi_i''\ \phi_j''' - \beta_j^4 \phi_i' \phi_j] \right\} \Bigg|_0^\ell$$

$$- \frac{1}{\ell}\ \frac{1}{(\beta_i^4 - \beta_j^4)} \left\{ \frac{2(\beta_i^4 + \beta_j^4)}{(\beta_i^4 - \beta_j^4)}\ [\phi_j'\ \phi_i''' - \phi_i''\ \phi_j'' + \phi_i'\ \phi_j''' - \beta_j^4 \phi_j \phi_i] \right.$$

$$\left. -2\ \beta_j^4 \phi_i \phi_j - \phi_i'''\ \phi_j' + \phi_i'\ \phi_j''' \right\} \Bigg|_0^\ell$$

for $i \neq j$.

Appendix B demonstrates a useful means by which to specialize the indefinite integral Eq. (5-7) to the case of $i = j$

$$\int [\phi_i'(x)]^2\ dx = \frac{x}{4}\ [(\phi_i')^2 - 2\phi_i \phi_i'' + \frac{1}{\beta_i^4}\ (\phi_i''')^2] - \frac{1}{4\beta_i^4}\ \phi_i''\phi_i''' + \frac{3}{4}\ \phi_i \phi_i' \qquad (5\text{-}8)$$

This permits the evaluation

$$\int_0^\ell [\frac{x}{\ell} - (1 + m^*)][\phi_i']^2\ dx = \left\{ [\frac{x}{\ell} - (1 + m^*)] \left(\frac{x}{4}[(\phi_i')^2 - 2\phi_i\phi_i'' + \frac{1}{\beta_i^4}(\phi_i''')^2] \right.\right.$$

$$\left.\left. - \frac{1}{4\beta_i^4}\ \phi_i''\ \phi_i''' + \frac{3}{4}\ \phi_i \phi_i' \right) \right\} \Bigg|_0^\ell$$

$$- \frac{1}{\ell} \left\{ \frac{x^2}{8}\ [(\phi_i')^2 - 2\phi_i\phi_i'' + \frac{1}{\beta_i^4}\ (\phi_i''')^2] \right.$$

$$\left. - \frac{1}{8\beta_i^4}\ (\phi_i'')^2 + \frac{3}{8}\ (\phi_i)^2 \right\} \Bigg|_0^\ell$$

5.2 Eigenvalue Problem

The application of Lagrange's equations yields the free vibration motion equations

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}$$

Seeking harmonic solutions, $\{q\} = \{U\}e^{i\Omega t}$ leads to the eigenvalue problem

$$([K] - \Omega^2[M])\{U\} = \{0\} \qquad (5\text{-}9)$$

The demand of nontrivial solutions of the assumed form yields the set of natural frequencies Ω_r and eigenfunctions

$$Y^{(r)}(x) = \sum_{i=1}^{n} \phi_i(x)\, U_i^{(r)} \qquad (r = 1,2, \ldots, n)$$

The natural frequencies, Ω_r and eigenfunctions, $Y^{(r)}$ are the sought approximations to those of Section 4.1. Note that the values of base acceleration for which

$$\det\,[K(a_o)] = 0$$

are approximations to the critical accelerations of Section 3.3.

CHAPTER 6

FORTRAN PROGRAMS

Two FORTRAN programs have been created to evaluate the free vibration and buckling characteristics of a uniform cantilevered beam with tip body under constant axial base acceleration. For each, the tip body mass center is restricted to lie along the beam tip tangent line. Complete listings of each program, accompanied by annotated sample input/output data, are provided in Appendices C and D.

The program of Appendix C computes the natural frequencies and mode shapes by implementing the assumed modes formulation of Chapter 5. The eigenfunctions of a clamped-free beam on a fixed base are used as the admissible spatial functions. The eigenvalues of the clamped-free beam, (β_i^4), are obtained from the roots of the corresponding characteristic equation

$$\cos \beta_i \ell \cosh \beta_i \ell + 1 = 0$$

The first fifty of these roots (valid for any beam), appear following the sample NAMELIST input data in the appendix. The program functions in one of two possible ways, depending upon the values of the parameters AI, AF, and AINC (which denote the initial, final, and incremental values of the base acceleration, respectively). If AI = AF (AINC arbitrary), the natural frequencies and mode shapes are computed for the single value of base acceleration. If AI < AF, then (only) the natural frequencies are computed for the values of base acceleration starting with AI and terminating with AF in increments of AINC. The general algebraic eigenvalue problem Eq. (5-9) is solved via IMSL subroutine EIGZS.

The program of Appendix D computes the first critical buckling load and acceleration by finding the first root of the buckling characteristic equation. Since available IMSL subroutines evaluate only Bessel functions of positive order, the actual characteristic equation implemented is not Eq. (3-12) but, rather, an equivalent form expressed in terms of positive order Bessel functions

$$\left[\frac{4}{3\alpha\lambda} J_{2/3}(\alpha\lambda) - J_{5/3}(\alpha\lambda)\right]\left[\left(\frac{2\sqrt{\alpha}}{3\beta\lambda} + \frac{3\sqrt{m^*}}{2} c^*\lambda\right) J_{1/3}(\beta\lambda) - \sqrt{\alpha}\, J_{4/3}(\beta\lambda)\right]$$

$$+ J_{1/3}(\alpha\lambda)\left[\left(\sqrt{\alpha} - 2\sqrt{m^*}\,\frac{c^*}{\beta}\right) J_{2/3}(\beta\lambda) + \frac{3}{2}\sqrt{m^*}\, c^*\lambda\, J_{5/3}(\beta\lambda)\right] = 0$$

The IMSL subroutines, MMBSJR and ZBRENT, are used to evaluate the Bessel functions and the first root of the above characteristic equation, respectively.

APPENDIX A

WORK-ENERGY BALANCE

Starting with the general partial differential equation governing transverse beam vibration under axial loading, we derive an energy-balance relationship. We then apply the boundary conditions and identify the potential energy expression. This derivation obviates the more common geometric arguments which lack a certain degree of rigor, and reveal the influence of the boundary conditions on the strain energy.

The starting point of our derivation is Eq. (2-4)

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\,\frac{\partial^2 u}{\partial x^2}\right] - \frac{\partial}{\partial x}\left[T(x,t)\,\frac{\partial u}{\partial x}\right] + \rho(x)\,\frac{\partial^2 u}{\partial t^2} = f(x,t) \qquad \text{(A-1)}$$

where we have inserted the term f(x,t) on the right-hand side to accommodate external loading/length on the beam perpendicular to the x axis. The reader can verify that Eq. (A-1) can be written in the alternate form

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\rho(x)\left(\frac{\partial u}{\partial t}\right)^2 + EI(x)\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + T(x,t)\cdot\left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$+ \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}\frac{\partial}{\partial x}\left(EI(x)\,\frac{\partial^2 u}{\partial x^2}\right)\right] - \frac{\partial}{\partial x}\left[EI(x)\,\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial x\partial t}\right]$$

$$- \frac{\partial}{\partial x}\left[T(x,t)\,\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right] - \frac{1}{2}\frac{\partial T}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 = f(x,t)\,\frac{\partial u}{\partial t} \qquad \text{(A-2)}$$

This equation holds for $0 < x < \ell$ and $t > 0$. We can then integrate over the region in the xt plane $\begin{cases} 0 < x < \ell \\ 0 < t < \bar{t} \end{cases}$ obtaining

$$\frac{1}{2} \int_0^{\ell} \left[\rho\left(\frac{\partial u}{\partial t}\right)^2 + EI\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + T\left(\frac{\partial u}{\partial x}\right)^2\right]_{t=0}^{\bar{t}} dx$$

$$+\int_0^{\bar{t}} \left[\frac{\partial u}{\partial t} \frac{\partial}{\partial x} \left(EI \frac{\partial^2 u}{\partial x^2}\right) - EI \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial x \partial t} - T \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial t}\right]_{x=0}^{\ell} dt$$

$$- \frac{1}{2} \int_0^{\bar{t}} \int_0^{\ell} \frac{\partial T}{\partial t} \left(\frac{\partial u}{\partial x}\right)^2 dx\, dt = \int_0^{\bar{t}} \int_0^{\ell} f(x,t) \frac{\partial u}{\partial t} dx\, dt \qquad \text{(A-3)}$$

The right-hand side of Eq. (A-3) represents the work performed by the external loading $f(x,t)$ on the beam between $t = 0$ and $t = \bar{t}$.

The work performed by the axial force $P_0(t)$ at the base of the beam between $t = 0$ and $t = \bar{t}$ is given by

$$\int_0^{\bar{t}} P_o(t) v_o(t)\, dt = \frac{1}{2} (\rho\ell + m)\, v_o^2(t)\Big|_{t=0}^{\bar{t}} - mc \sin\gamma \int_0^{\bar{t}} \frac{\partial^3 u}{\partial x \partial t^2} (\ell,t) v_o\, dt \qquad \text{(A-4)}$$

where we have used Eq. (2-18) (ρ assumed constant).

Now the total work done on the system by the external forces f and P_0 is equal to the change in kinetic plus potential energies. Using Eq. (5-1) for the kinetic energy and Eq. (A-3) and (A-4), this work-energy balance can be written as

$$\frac{1}{2}\int_0^{\ell}\left[EI\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + T\left(\frac{\partial u}{\partial x}\right)^2\right]_{t=0}^{\bar{t}} dx - mc \sin\gamma \int_0^{\bar{t}} \frac{\partial^3 u}{\partial x \partial t^2}(\ell,t) v_o \, dt$$

$$+ Q(\bar{t}) - \frac{1}{2}\int_0^{\bar{t}}\int_0^{\ell} \frac{\partial T}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dx\, dt = \frac{1}{2}\left(I_t + mc^2\right)\left[\frac{\partial^2 u}{\partial x \partial t}(\ell,t)\right]^2\Big|_{t=0}^{\bar{t}}$$

$$+ \frac{1}{2} m \left[\frac{\partial u}{\partial t}(\ell,t)\right]^2\Big|_{t=0}^{\bar{t}} + mc \frac{\partial^2 u}{\partial x \partial t}(\ell,t)$$

$$\cdot \left[\cos\gamma \frac{\partial u}{\partial t}(\ell,t) - v_o(t) \sin\gamma\right]\Big|_{t=0}^{\bar{t}}$$

$$+ V(t)\Big|_{t=0}^{\bar{t}} \tag{A-5}$$

$V(t)$ is the potential energy and $Q(\bar{t})$ is given by

$$Q(\bar{t}) = \int_0^{\bar{t}} \left[\frac{\partial u}{\partial t}\frac{\partial}{\partial x}\left(EI \frac{\partial^2 u}{\partial x^2}\right) - EI \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial x \partial t} - T \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right]_{x=0}^{\ell} dt \tag{A-6}$$

Assuming a uniform EI and using Eq. (2-12) along with the clamped boundary conditions at $x = 0$ we have

$$Q(\bar{t}) = \int_0^{\bar{t}} \Big\{\Big[EI \frac{\partial^3 u}{\partial x^3}(\ell,t) + m\, a_o(t) \frac{\partial u}{\partial x}(\ell,t)$$

$$- mc \sin\gamma \frac{\partial^3 u}{\partial x \partial t^2}(\ell,t) \frac{\partial u}{\partial x}(\ell,t)\Big] \frac{\partial u}{\partial t}(\ell,t)$$

$$- EI \frac{\partial^2 u}{\partial x^2}(\ell,t) \frac{\partial^2 u}{\partial x \partial t}(\ell,t)\Big\}\, dt \tag{A-7}$$

The natural boundary conditions (2-14) and (2-16) are equivalent to

$$mc \cos \gamma \frac{\partial^3 u}{\partial x \partial t^2} (\ell,t) - EI \frac{\partial^3 u}{\partial x^3} (\ell,t) - ma_o(t) \frac{\partial u}{\partial x} (\ell,t) + m \frac{\partial^2 u}{\partial t^2} (\ell,t) = 0 \qquad \text{(A-8)}$$

$$\left(I_t + mc^2\right) \frac{\partial^3 u}{\partial x \partial t^2} (\ell,t) - mc \cos \gamma \, a_o(t) \frac{\partial u}{\partial x} (\ell,t) + mc \cos \gamma \frac{\partial^2 u}{\partial t^2} (\ell,t) + EI \frac{\partial^2 u}{\partial x^2} (\ell,t) = mc \sin \gamma \cdot a_o(t) \qquad \text{(A-9)}$$

With the aid of (A-8), (A-9), Eq. (A-7) further simplifies to

$$\begin{aligned} Q(\bar{t}) = {} & mc \cos \gamma \frac{\partial u}{\partial t} (\ell,t) \frac{\partial^2 u}{\partial x \partial t} (\ell,t) \Big|_{t=0}^{\bar{t}} + \frac{1}{2} m \left[\frac{\partial u}{\partial t} (\ell,t)\right]^2 \Big|_{t=0}^{\bar{t}} \\ & + \frac{1}{2} (I_t + mc^2) \left[\frac{\partial^2 u}{\partial x \partial t} (\ell,t)\right]^2 \Big|_{t=0}^{\bar{t}} \\ & - mc \cos \gamma \int_0^{\bar{t}} a_o(t) \frac{\partial u}{\partial x} (\ell,t) \frac{\partial^2 u}{\partial x \partial t} (\ell,t) \, dt \\ & - mc \sin \gamma \int_0^{\bar{t}} a_o(t) \frac{\partial^2 u}{\partial x \partial t} (\ell,t) \, dt \\ & - mc \sin \gamma \int_0^{\bar{t}} \frac{\partial^3 u}{\partial x \partial t^2} (\ell,t) \frac{\partial u}{\partial x} (\ell,t) \frac{\partial u}{\partial t} (\ell,t) \, dt \end{aligned}$$

Now

$$\int_0^{\bar{t}} \frac{\partial^2 u}{\partial x \partial t}(\ell,t)\ a_o(t)\ dt = \frac{\partial^2 u}{\partial x \partial t}(\ell,t)\ v_o(t)\Big|_{t=0}^{\bar{t}} - \int_0^{\bar{t}} v_o\ \frac{\partial^3 u}{\partial x \partial t^2}(\ell,t)\ dt$$

Using the above results, Eq. (A-5) can be put in the form

$$\begin{aligned} V(\bar{t}) - V(0) = & \frac{1}{2} \int_0^{\ell} \left[EI \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + T(x,t)\left(\frac{\partial u}{\partial x}\right)^2\right]\Big|_{t=0}^{\bar{t}} dx \\ & - mc \cos\gamma \int_0^{\bar{t}} a_o(t)\ \frac{\partial u}{\partial x}(\ell,t)\ \frac{\partial^2 u}{\partial x \partial t}(\ell,t)\ dt \\ & - mc \sin\gamma \int_0^{\bar{t}} \frac{\partial^3 u}{\partial x \partial t^2}(\ell,t)\ \frac{\partial u}{\partial x}(\ell,t)\frac{\partial u}{\partial t}(\ell,t)\ dt \\ & - \frac{1}{2} \int_0^{\bar{t}} \int_0^{\ell} \frac{\partial T}{\partial t} \left(\frac{\partial u}{\partial x}\right)^2 dx\ dt \end{aligned} \qquad \text{(A-10)}$$

Equation (A-10) gives the potential energy of a clamped beam with tip body under axial acceleration $a_o(t)$; EI, and ρ are assumed constant. Recall that potential energy is defined only up to an arbitrary additive constant; hence we can set $V(0) = 0$.

For the special case $\gamma = 0$ and constant a_o

$$V(t) = \frac{1}{2} \int_0^{\ell} \left[EI \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + T(x) \left(\frac{\partial u}{\partial x}\right)^2\right] dx - \frac{1}{2}\, mc\ a_o \left[\frac{\partial u}{\partial x}(\ell,t)\right]^2 \qquad \text{(A-11)}$$

Admittedly, this derivation of the strain energy does not appear to be well motivated and indeed was developed with hindsight. A more direct but mathematically sophisticated derivation can be found in Reference 2.

APPENDIX B

INDEFINITE INTEGRALS OF EIGENFUNCTIONS

In this appendix we shall present a method for evaluating the indefinite integral

$$\int [\phi_i(x)]^2 \; dx$$

where $\phi_i(x)$ is the solution of the differential equation

$$\frac{d^4\phi}{dx^4} = \beta^4\phi(x) \qquad \text{(B-1)}$$

corresponding to $\beta = \beta_i$. This technique is extremely useful in the normalization of eigenfunctions and the evaluation of certain integrals arising in vibration problems in general. One of the earliest references to this method is S. Timoshenko ("Vibration Problems in Engineering," 2nd edition, p. 335).

The solution of the differential equation (B-1) depends upon the value of β as well as x. We indicate this by the notation $\phi = \phi(x;\, \beta)$. Formally expanding ϕ in a Taylor series we have

$$\phi(x;\, \beta + \delta\beta) = \phi(x;\, \beta) + \frac{\partial\phi}{\partial\beta}\,\delta\beta + \ldots$$

Denote by $\phi_i(x)$ the value of ϕ corresponding to $\beta = \beta_i$ and by $\phi_j(x)$ the value corresponding to $\beta_i + \delta\beta$. Thus

$$\phi_j(x) = \phi_i(x) + \left(\frac{\partial\phi}{\partial\beta}\right)_{\beta_i}\delta\beta + \ldots \qquad \text{(B-2)}$$

From Eq. (B-1) one can readily show

$$(\beta_i^4 - \beta_j^4) \int \phi_i \phi_j \, dx = \phi_j \phi_i''' - \phi_i \phi_j''' + \phi_i' \phi_j'' - \phi_j' \phi_i'' \qquad \text{(B-3)}$$

where β_i and β_j are distinct (not necessarily eigenvalues).
Substituting Eq. (B-2) into Eq. (B-3)

$$-4\beta_i^3 \delta\beta \int \phi_i(x) \left[\phi_i(x) + \left(\frac{\partial\phi}{\partial\beta}\right)_{\beta_i} \delta\beta + \ldots\right] dx = \left[\phi_i(x) + \left(\frac{\partial\phi}{\partial\beta}\right)_{\beta_i} \delta\beta + \ldots\right]\phi_i'''$$

$$- \phi_i\left[\phi_i''' + \frac{d^3}{dx^3}\left(\frac{\partial\phi}{\partial\beta}\right)_{\beta_i} \delta\beta + \ldots\right] + \phi_i'\left[\phi_i'' + \frac{d^2}{dx^2}\left(\frac{\partial\phi}{\partial\beta}\right)_{\beta_i} \delta\beta + \ldots\right]$$

$$- \phi_i''\left[\phi_i' + \frac{d}{dx}\left(\frac{\partial\phi}{\partial\beta}\right)_{\beta_i} \delta\beta + \ldots\right] \qquad \text{(B-4)}$$

where we have used the binomial expansion

$$(\beta_i + \delta\beta)^4 = \beta_i^4 + 4\beta_i^3 \delta\beta + \ldots$$

We observe from Eq. (B-1) that the functions ϕ depend upon x and β only through the argument βx. Therefore

$$\frac{\partial\phi}{\partial\beta} = \frac{\partial\phi}{\partial(\beta x)} \frac{\partial(\beta x)}{\partial\beta} = x \frac{\partial\phi}{\partial(\beta x)}$$

Similarly

$$\frac{\partial\phi}{\partial(\beta x)} = \frac{1}{\beta} \frac{\partial\phi}{\partial x}$$

Hence

$$\left(\frac{\partial\phi}{\partial\beta}\right)_{\beta_i} = \frac{x}{\beta_i}\,\phi_i'(x) \qquad \text{(B-5)}$$

Neglecting higher order terms in $\delta\beta$ and using Eq. (B-5), allows us to write Eq. (B-4) as

$$-4\beta_i^3\int\phi_i^2\,dx = \frac{x}{\beta_i}\,\phi_i'\,\phi_i''' - \phi_i\,\frac{1}{\beta_i}\,\frac{d^3}{dx^3}\,(x\phi_i')$$

$$+\ \phi_i'\,\frac{1}{\beta_i}\,\frac{d^2}{dx^2}\,(x\phi_i') - \phi_i''\,\frac{1}{\beta_i}\,\frac{d}{dx}\,(x\phi_i')$$

Expanding derivatives we obtain the final desired result

$$\int[\phi_i(x)]^2\,dx = \frac{x}{4\beta_i^4}\left[\beta_i^4\phi_i^2 - 2\phi_i'\,\phi_i''' + (\phi_i'')^2\right]$$

$$+\ \frac{3}{4\beta_i^4}\,\phi_i\,\phi_i''' - \frac{1}{4\beta_i^4}\,\phi_i'\,\phi_i'' \qquad \text{(B-6)}$$

Indeed, one can verify this result by direct differentiation.

Related integrals, e.g.

$$\int[\phi_i']^2\,dx$$

can be evaluated in the same fashion.

APPENDIX C

NATURAL FREQUENCIES AND MODE SHAPES
PROGRAM LISTING

```
          C***********************************************************************
          C  THIS PROGRAM COMPUTES THE NATURAL FREQUENCIES AND MODE SHAPES OF A
          C  UNIFORM CANTILEVERED BEAM WITH TIP BODY SUBJECT TO CONSTANT AXIAL
          C  BASE ACCELERATION. AN ASSUMED MODES FORMULATION IS IMPLEMENTED USING
          C  THE EIGENFUNCTIONS OF A CLAMPED-FREE BEAM ON A FIXED BASE AS THE
          C  BASIS  THE TIP BODY MASS CENTER IS RESTRICTED TO LIE ALONG THE BEAM
          C  TIP TANGENT LINE. ( WRITTEN BY JOEL STORCH & STEPHEN GATES BASED
          C  UPON CSDL R-1675  OCTOBER 1983 ).
          C***********************************************************************
          C
          C********************** INPUT/OUTPUT FILES ****************************
          C
          C  FILE#5. NAMELIST INPUT & "BETA * L" VALUES
          C  FILE#6  OUTPUT  ACCELERATIONS & NATURAL FREQUENCIES
          C  FILE#7: OUTPUT  EIGENFUNCTIONS EVALUATED AT DISCRETE LOCATIONS
          C
          C*************** DESCRIPTION OF NAMELIST INPUT ITEMS ******************
          C
          C  "EI"  BEAM BENDING STIFFNESS
          C  "RHO" MASS PER UNIT LENGTH OF BEAM
          C  "L"   BEAM LENGTH
          C  "MT"  TIP BODY MASS
          C  "IT"  TIP BODY INERTIA ABOUT IT'S MASS CENTER
          C  "CT"  TIP BODY MASS CENTER OFFSET FROM BEAM ATTACHMENT POINT
          C  "AI"  INITIAL VALUE OF BASE ACCELERATION
          C  "AF"  FINAL VALUE OF BASE ACCELERATION
          C "AINC" INCREMENT OF BASE ACCELERATION FOR SWEEP
          C  "N"   NUMBER OF CLAMPED-FREE EIGENFUNCTIONS USED
          C
          C***********************************************************************
          C
          C    NOTE ARRAYS DIMENSIONED FOR A MAXIMUM OF 50 DEGREES OF FREEDOM
          C
          C***********************************************************************
          C
ISN 0002        IMPLICIT REAL*8 (A-H,O-Z)
ISN 0003        REAL*8 L,M,K,MT,IT,MSTAR
ISN 0004        DIMENSION K(50,50),M(50,50),BETAL(50),B(50),B4(50),FOL(50),
               +F1L(50),F20(50),F30(50),GAMMA(50,50),AA(1275),BB(1275),D(50),
               +Z(50,50),WK(2600),PHI(51,50),Y(51,5)
ISN 0005        NAMELIST /INPUT/ EI,L,RHO,N,MT,IT,CT,AI,AF,AINC
ISN 0006        READ(5,INPUT)
ISN 0007        RHOL=RHO*L
ISN 0008        MSTAR=MT/RHOL
ISN 0009        SRHOL=DSQRT(RHOL)
          C
          C READ BETA*L VALUES
          C
ISN 0010        DO 1 I=1,N
ISN 0011        READ(5,2) BETAL(I)
ISN 0012        B(I)=BETAL(I)/L
ISN 0013      1 B4(I)=B(I)**4
ISN 0014      2 FORMAT(E17 8)
```

```
               C
               C ECHO PRINT INPUT DATA
               C
ISN 0015             WRITE(6,3)
ISN 0016           3 FORMAT(1H1,24X,'DATA FROM NAMELIST INPUT')
ISN 0017             WRITE(6,4) EI,RHO,L
ISN 0018           4 FORMAT(1HO,'  EI = ',1PE13.6,2X,' RHO = ',1PE13 6,2X,'   L = ',
                    +1PE13 6)
ISN 0019             WRITE(6,5) MT,IT,CT
ISN 0020           5 FORMAT(1HO,'  MT = ',1PE13 6,2X,'  IT = ',1PE13 6,2X,'  CT = ',
                    +1PE13.6)
ISN 0021             WRITE(6,6) AI,AF,AINC
ISN 0022           6 FORMAT(1HO,'  AI = ',1PE13.6,2X,'  AF = ',1PE13.6,2X,'AINC = ',
                    +1PE13.6)
ISN 0023             WRITE(6,7) N
ISN 0024           7 FORMAT(1HO,'NO. OF CLAMPED-FREE MODES USED = ',I2)
               C
               C  EVALUATE CLAMPED-FREE EIGENFN & DERIVATIVES @ O AND L
               C
ISN 0025             DO 10 I=1,N
ISN 0026             BL=BETAL(I)
ISN 0027             S=DSIN(BL)
ISN 0028             C=DCOS(BL)
ISN 0029             SH=DSINH(BL)
ISN 0030             CH=DCOSH(BL)
ISN 0031             FOL(I)= -2.ODO/SRHOL
ISN 0032             F1L(I)=FOL(I)*B(I)*(S*CH + C*SH)/(S*SH)
ISN 0033             F2O(I)=FOL(I)*(B(I)**2)*(C+CH)/(S*SH)
ISN 0034          10 F3O(I)=FOL(I)*(B(I)**3)*(S-SH)/(S*SH)
               C
               C ASSEMBLE MASS MATRIX
               C
ISN 0035             DO 20 I=1,N
ISN 0036             DO 20 J=I,N
ISN 0037             M(I,J)= (IT+(MT*CT*CT))*F1L(I)*F1L(J) + MT*FOL(I)*FOL(J)
                    ++ MT*CT*(FOL(I)*F1L(J) + F1L(I)*FOL(J))
ISN 0038          20 M(J,I)=M(I,J)
ISN 0039             DO 21 I=1,N
ISN 0040          21 M(I,I)=M(I,I)+1 ODO
               C
               C ASSEMBLE GAMMA MATRIX
               C
ISN 0041             DO 30 I=1,N
ISN 0042             IP1=I+1
ISN 0043             DO 30 J=IP1,N
ISN 0044             BMB=B4(I)-B4(J)
ISN 0045             T1=(1 ODO+MSTAR)*(F2O(I)*F3O(J) - F3O(I)*F2O(J))
ISN 0046             T2=MSTAR*(B4(I)*FOL(I)*F1L(J) - B4(J)*F1L(I)*FOL(J))
ISN 0047             T3=2 ODO*(B4(I)+B4(J))*F2O(I)*F2O(J)/(L*BMB)
ISN 0048             T4=4.ODO*B4(I)*B4(J)*FOL(I)*FOL(J)/(L*BMB)
ISN 0049             GAMMA(I,J)=(T1-T2-T3+T4)/BMB
ISN 0050          30 GAMMA(J,I)=GAMMA(I,J)
ISN 0051             DO 40 I=1,N
ISN 0052             T1=MSTAR*(L*(F1L(I)**2) + 3 ODO*FOL(I)*F1L(I))/4 ODO
ISN 0053             T2=(1 ODO+MSTAR)*F2O(I)*F3O(I)/(4 ODO*B4(I))
ISN 0054             T3=(L*F1L(I))**2 + 3 ODO*(FOL(I)**2) + (F2O(I)**2)/B4(I)
ISN 0055          40 GAMMA(I,I)=-T1-T2-T3/(8 ODO*L)
               C
```

```
              C ASSEMBLE STIFFNESS MATRIX                                                     00010100
              C                                                                               00010110
ISN 0056            AX=AI                                                                     00010200
ISN 0057         50 DO 60 I=1,N                                                               00010300
ISN 0058            IP1=I+1                                                                   00010400
ISN 0059            DO 60 J=IP1,N                                                             00010500
ISN 0060            K(I,J)=AX RHOL*GAMMA(I,J)-MT*CT*AX*F1L(I)*F1L(J)                          00010600
ISN 0061         60 K(J,I)=K( ,J)                                                             00010700
ISN 0062            DO 70 I=1 N                                                               00010800
ISN 0063         70 K(I,I)=EI*B4(I)/RHO + AX*RHOL*GAMMA(I,I)-MT*CT*AX*(F1L(I)**2)             00010900
              C                                                                               00010910
              C LOAD K INTO AA  &  M INTO BB   BOTH SYMMETRIC STORAGE                         00011000
              C                                                                               00011010
ISN 0064            II=1                                                                      00011100
ISN 0065            DO 80 I=1,N                                                               00011200
ISN 0066            DO 80 J=1,I                                                               00011300
ISN 0067            AA(II)=K(I,J)                                                             00011400
ISN 0068            BB(II)=M(I,J)                                                             00011500
ISN 0069            II=II+1                                                                   00011600
ISN 0070         80 CONTINUE                                                                  00011700
              C                                                                               00011710
              C COMPUTE EIGENVALUES & EIGENVECTORS                                            00011800
              C                                                                               00011810
ISN 0071            IJOB=2                                                                    00011900
ISN 0072            IZ=50                                                                     00012000
ISN 0073            CALL EIGZS(AA,BB,N,IJOB,D,Z,IZ,WK,IER)                                    00012100
              C                                                                               00012101
              C OUTPUT FIRST 5 EIGENVALUES & ACCELERATION                                     00012200
              C                                                                               00012210
ISN 0074            WRITE(6,90  AX                                                            00012300
ISN 0075         90 FORMAT(1HO,'BASE ACCELERATION = ',1PE13.6)                                00012400
ISN 0076            WRITE(6,91)                                                               00012500
ISN 0077         91 FORMAT(1HO 4X,'NATURAL FREQUENCIES')                                      00012600
ISN 0078            WRITE(6,92,                                                               00012610
ISN 0079         92 FORMAT(1H ,1X,'N',5X,'OMEGA(N)**2  (SEC**-2)')                            00012620
ISN 0080            DO 93 I=1,5                                                               00012630
ISN 0081         93 WRITE(6,94) I,D(I)                                                        00012640
ISN 0082         94 FORMAT(1H  I2,5X,1PE13 6)                                                 00012650
              C                                                                               00012700
              C TEST FOR EIGENVECTOR COMPUTATION OR ACCEL  SWEEP                              00012800
              C                                                                               00012810
ISN 0083            IF( AI .EQ  AF ) GO TO 100                                                00012900
ISN 0085            IF( AX .GE. AF ) GO TO 1000                                               00013000
ISN 0087            AX=AX+AINC                                                                00013100
ISN 0088            GO TO 50                                                                  00013200
              C                                                                               00013210
              C EVALUATE CLAMP?D-FREE EIGENFUNCTIONS AT 51 STATIONS                            00013300
              C                                                                               00013310
ISN 0089        100 DX=L/50 0D0                                                               00013400
ISN 0090            DO 110 I=1,N                                                              00013500
ISN 0091            BL=BETAL(I)                                                               00013600
ISN 0092            SS=DSIN(BL)*DSINH(BL)                                                     00013700
ISN 0093            SMS=DSIN(BL) - DSINH(BL)                                                  00013800
ISN 0094            CPC=DCOS(BL) + DCOSH(BL)                                                  00013900
ISN 0095            X=0.0D0                                                                   00014000
ISN 0096            DO 105 J=1,51                                                             00014100
ISN 0097            BX=B(I)*X                                                                 00014200
ISN 0098            PHI(J,I)=SMS*(DSIN(BX)-DSINH(BX)) + CPC*(DCOS(BX)-DCOSH(BX))              00014300
```

```
ISN 0099          PHI(J,I)=PHI(J,I)/(SRHOL*SS)                                   00014400
ISN 0100     105  X=X+DX                                                         00014500
ISN 0101     110  CONTINUE                                                       00014600
           C                                                                     00014610
           C COMPUTE FIRST 5 EIGENVECTORS                                        00014700
           C                                                                     00014710
ISN 0102          DO 125 IP=1,51                                                 00014800
ISN 0103          DO 120 IM=1,5                                                  00014900
ISN 0104          SUM=0 ODO                                                      00015000
ISN 0105          DO 115 J=1,N                                                   00015100
ISN 0106          PROD=PHI(IP,J)*Z(J,IM)                                         00015200
ISN 0107     115  SUM=SUM+PROD                                                   00015300
ISN 0108     120  Y(IP,IM)=SUM                                                   00015400
ISN 0109     125  CONTINUE                                                       00015500
           C                                                                     00015510
           C OUTPUT EIGENVECTORS FOR PLOTTING                                    00015600
           C                                                                     00015610
ISN 0110          X=0 ODO                                                        00015700
ISN 0111          DO 130 I=1,51                                                  00015800
ISN 0112          WRITE(7,131) X,(Y(I,J),J=1,5)                                  00015900
ISN 0113     130  X=X+DX                                                         00016000
ISN 0114     131  FORMAT(6(1X,E12 5))                                            00016100
ISN 0115    1000  STOP                                                           00016200
ISN 0116          END                                                            00016300
```

EXAMPLE PROBLEM PARAMETERS

BEAM PARAMETERS

$EI = 6\ 25 \cdot 10^7$ lb-ft^2

$L = 134 0$ ft

$\rho = 0 4172$ slug/ft

TIP BODY PARAMETERS

$m = 155\ 28$ slugs

$I_t = 0\ 0$ slug-ft^2

ι = 0 0 ft

NAMELIST INPUT AND CLAMPED-FREE BEAM "EIGENVALUES"

```
&INPUT EI=6 25E7,L=134 0,RHO=.4172,N=20
       MT=155.28,IT=0 0,CT=0.0
       AI=0 0,AF=60 0,AINC=20 0,&END
  0 18751041E+01
  0 46940911E+01
  0 78547574E+01
  0 10995541E+02
  0 14137168E+02
  0.17278760E+02
  0 20420352E+02
  0.23561945E+02
  0.26703538E+02
  0 29845130E+02
  0 32986723E+02
  0 36128316E+02
  0.39269908E+02
  0.42411501E+02
  0 45553093E+02
  0 48694686E+02
  0 51836279E+02
  0.54977871E+02
  0 58119464E+02
  0 61261057E+02
  0 64402649E+02
  0 67544242E+02
  0 70685835E+02
  0 73827427E+02
  0 76969020E+02
  0 80110613E+02
  0 83252205E+02
  0 86393798E+02
  0 89535391E+02
  0 92676983E+02
  0 95818576E+02
  0 98960169E+02
  0 10210176E+03
  0 10524335E+03
  0 10838495E+03
  0 11152654E+03
  0 11466813E+03
  0 11780972E+03
  0 12095132E+03
  0 12409291E+03
  0 12723450E+03
  0.13037610E+03
  0.13351769E+03
  0 13665928E+03
  0 13980087E+03
  0 14294247E+03
  0 14608406E+03
  0 14922565E+03
  0 15236724E+03
  0 15550884E+03
```

FIRST FIFTY ROOTS ($\beta_i \ell$) OF $\cos \beta_i \ell \cosh \beta_i \ell = -1$

PROGRAM OUTPUT
ACCELERATION SWEEP RUN

DATA FROM NAMELIST INPUT

```
EI =  6.250000E+07   RHO =  4 172000E-01     L =  1 340000E+02

MT =  1.552800E+02    IT =  0 000000E+00    CT =  0 000000E+00

AI =  0 000000E+00    AF =  6 000000E+01  AINC =  2 000000E+01

NO  OF CLAMPED-FREE MODES USED = 20

BASE ACCELERATION =  0 000000E+00

    NATURAL FREQUENCIES
 N     OMEGA(N)**2  (SEC**-2)
 1      4.625102E-01
 2       1 151594E+02
 3       1 176526E+03
 4       5 085561E+03
 5       1 483397E+04

BASE ACCELERATION =  2.000000E+01

    NATURAL FREQUENCIES
 N     OMEGA(N)**2  (SEC**-2)
 1       2 785320E-01
 2       1 094116E+02
 3       1 155686E+03
 4       5 039933E+03
 5       1 475390E+04

BASE ACCELERATION =  4 000000E+01

    NATURAL FREQUENCIES
 N     OMEGA(N)**2  (SEC**-2)
 1      9.289338E-02
 2       1 036581E+02
 3       1 134843E+03
 4       4 994303E+03
 5       1 467382E+04

BASE ACCELERATION =  6 000000E+01

    NATURAL FREQUENCIES
 N     OMEGA(N)**2  (SEC**-2)
 1      -9 466231E-02
 2      9.789877E+01
 3       1 113999E+03
 4       4 948672E+03
 5       1 459374E+04
```

PROGRAM OUTPUT
MODE SHAPES RUN

DATA FROM NAMELIST INPUT

```
EI =  6 250000E+07   RHO =  4 172000E-01      L =  1 340000E+02

MT =  1 552800E+02    IT =  0 000000E+00     CT =  0 000000E+00

AI =  3 220000E+01    AF =  3 220000E+01   AINC =  0 000000E+00

NO  OF CLAMPED-FREE MODES USED = 20

BASE ACCELERATION =  3 220000E+01

     NATURAL FREQUENCIES
N       OMEGA(N)**2  (SEC**-2)
1         1 655051E-01
2         1 059027E+02
3         1 142972E+03
4         5 012099E+03
5         1 470505E+04
```

EIGENFUNCTIONS (NON-NORMALIZED)

x	$y^{(1)}_{(x)}$	$y^{(2)}_{(x)}$	$y^{(3)}_{(x)}$	$y^{(4)}_{(x)}$	$y^{(5)}_{(x)}$
0 00000E+00	0 00000E+00	0 00000E+00	0 00000E+00	0 00000E+00	0 00000E+00
0 26800E+01	0.41721E-04	-0 80237E-03	0 25418E-02	-0 52092E-02	0 86654E-02
0 53600E+01	0.16641E-03	-0 31241E-02	0 96775E-02	-0 19296E-01	0 31340E-01
0 80400E+01	0 37331E-03	-0 68381E-02	0 20664E-01	-0 39984E-01	0 63036E-01
0 10720E+02	0 66154E-03	-0 11819E-01	0 34748E-01	-0 65064E-01	0 98844E-01
0 13400E+02	0 10302E-02	-0 17939E-01	0 51180E-01	-0 92400E-01	0 13415E+00
0 16080E+02	0 14782E-02	-0.25072E-01	0 69230E-01	-0.11993E+00	0 16493E+00
0 18760E+02	0 20047E-02	-0 33086E-01	0 88199E-01	-0 14574E+00	0 18787E+00
0 21440E+02	0 26085E-02	-0 41853E-01	0 10741E+00	-0.16814E+00	0 20046E+00
0 24120E+02	0 32885E-02	-0 51248E-01	0 12620E+00	-0.18570E+00	0 20108E+00
0 26800E+02	0 40435E-02	-0 61145E-01	0 14395E+00	-0 19733E+00	0 18908E+00
0 29480E+02	0 48722E-02	-0 71419E-01	0 16012E+00	-0 20219E+00	0 16494E+00
0 32160E+02	0 57733E-02	-0 81947E-01	0 17422E+00	-0 19980E+00	0 13012E+00
0 34840E+02	0 67456E-02	-0 92607E-01	0 18581E+00	-0 19001E+00	0 86888E-01
0 37520E+02	0 77875E-02	-0.10328E+00	0 19452E+00	-0 17307E+00	0 38085E-01
0 40200E+02	0 88976E-02	-0 11386E+00	0 20006E+00	-0 14959E+00	-0 13019E-01
0 42880E+02	0 10074E-01	-0 12424E+00	0 20222E+00	-0 12041E+00	-0 62886E-01
0 45560E+02	0 11317E-01	-0 13430E+00	0 20088E+00	-0 86620E-01	-0 10798E+00
0 48240E+02	0 12622E-01	-0 14394E+00	0 19600E+00	-0 49540E-01	-0 14514E+00
0 50920E+02	0 13990E-01	-0 15308E+00	0 18760E+00	-0 10676E-01	-0 17180E+00
0 53600E+02	0 15418E-01	-0 16161E+00	0 17579E+00	0 28371E-01	-0 18620E+00
0 56280E+02	0 16905E-01	-0 16947E+00	0 16076E+00	0 66022E-01	-0 18730E+00
0 58960E+02	0 18449E-01	-0 17656E+00	0 14279E+00	0 10078E+00	-0 17499E+00
0 61640E+02	0 20048E-01	-0 18283E+00	0 12219E+00	0 13123E+00	-0 15012E+00
0 64320E+02	0 21700E-01	-0 18819E+00	0 99360E-01	0 15610E+00	-0 11453E+00
0 67000E+02	0 23404E-01	-0 19261E+00	0 74706E-01	0 17434E+00	-0 70829E-01
0 69680E+02	0 25158E-01	-0 19602E+00	0 48708E-01	0 18521E+00	-0 22121E-01
0 72360E+02	0 26960E-01	-0 19840E+00	0 21882E-01	0 18832E+00	0 28218E-01
0 75040E+02	0 28808E-01	-0 19969E+00	-0 52418E-02	0 18356E+00	0 76647E-01
0 77720E+02	0 30700E-01	-0 19989E+00	-0 32143E-01	0 17111E+00	0 11965E+00
0 80400E+02	0 32634E-01	-0 19897E+00	-0 58310E-01	0 15144E+00	0 15405E+00
0 83080E+02	0 34608E-01	-0 19693E+00	-0 83221E-01	0 12541E+00	0 17745E+00
0 85760E+02	0 36620E-01	-0 19375E+00	-0 10634E+00	0 94224E-01	0 18846E+00
0 88440E+02	0 38669E-01	-0 18945E+00	-0 12725E+00	0 59078E-01	0 18605E+00
0 91120E+02	0 40752E-01	-0 18405E+00	-0 14555E+00	0 21375E-01	0 17023E+00
0 93800E+02	0 42868E-01	-0 17756E+00	-0 16086E+00	-0 17201E-01	0 14236E+00
0 96480E+02	0 45014E-01	-0 17001E+00	-0 17282E+00	-0 54938E-01	0 10464E+00
0 99160E+02	0 47188E-01	-0 16144E+00	-0 18125E+00	-0 90385E-01	0 59478E-01
0 10184E+03	0 49388E-01	-0 15190E+00	-0 18604E+00	-0 12225E+00	0 96639E-02
0 10452E+03	0 51612E-01	-0 14143E+00	-0 18701E+00	-0 14896E+00	-0 40771E-01
0 10720E+03	0 53859E-01	-0 13009E+00	-0 18410E+00	-0 16925E+00	-0 87855E-01
0 10988E+03	0 56126E-01	-0 11792E+00	-0 17739E+00	-0 18237E+00	-0 12853E+00
0 11256E+03	0 58411E-01	-0 10502E+00	-0 16705E+00	-0 18787E+00	-0 16008E+00
0 11524E+03	0 60712E-01	-0 91462E-01	-0 15338E+00	-0 18579E+00	-0 18089E+00
0 11792E+03	0 63026E-01	-0 77315E-01	-0 13660E+00	-0 17603E+00	-0 18911E+00
0 12060E+03	0 65353E-01	-0 62637E-01	-0 11690E+00	-0 15865E+00	-0 18335E+00
0 12328E+03	0 67690E-01	-0 47501E-01	-0 94618E-01	-0 13414E+00	-0 16359E+00
0 12596E+03	0 70036E-01	-0 32000E-01	-0 70237E-01	-0 10368E+00	-0 13153E+00
0 12864E+03	0 72387E-01	-0 16262E-01	-0 44470E-01	-0 69197E-01	-0 90955E-01
0 13132E+03	0 74741E-01	-0 37928E-03	-0 17859E-01	-0 32217E-01	-0 44939E-01
0 13400E+03	0 77096E-01	0 15543E-01	0 89914E-02	0 54750E-02	0 26399E-02

APPENDIX D

BUCKLING LOAD/ACCELERATION
PROGRAM LISTING

```
         C*************************************************************************
         C  THIS PROGRAM COMPUTES THE FIRST CRITICAL BUCKLING LOAD/ACCELERATION
         C  OF A UNIFORM CANTILEVERED BEAM WITH TIP BODY  THE "BUCKLING LOAD"
         C  CORRESPONDS TO THE BEAM ROOT AXIAL FORCE AND IS DIRECTLY PROPOR-
         C  TIONAL TO THE "BUCKLING ACCELERATION". THE TIP BODY MASS CENTER IS
         C  RESTRICTED TO LIE ALONG THE BEAM TIP TANGENT LINE ( WRITTEN BY
         C  JOEL STORCH & STEPHEN GATES  BASED UPON CSDL R-1675  OCTOBER 1983 )
         C*************************************************************************
         C
         C********************* INPUT/OUTPUT FILES ********************************
         C
         C  FILE#5: INPUT  NAMELIST INPUT DATA
         C  FILE#6: OUTPUT CRITICAL BUCKLING LOAD/ACCELERATION
         C
         C************** DESCRIPTION OF NAMELIST INPUT ITEMS *********************
         C
         C  "EI"  BEAM BENDING STIFFNESS
         C  "RHO" MASS PER UNIT LENGTH OF BEAM
         C  "L"   BEAM LENGTH
         C  "MT"  MASS OF TIP BODY
         C  "CT"  TIP BODY MASS CENTER OFFSET FROM BEAM ATTACHMENT POINT
         C
         C*************************************************************************
         C
ISN 0002       IMPLICIT REAL*8(A-H,O-Z)
ISN 0003       REAL*8 L,LAMS,MSTAR,MT
ISN 0004       DIMENSION LAMS(2)
ISN 0005       DATA LAMS/1 OD-8,1 87DO/
ISN 0006       COMMON ALPHA,BETA,C1,C2,C3,D1,D2
ISN 0007       EXTERNAL F
ISN 0008       NAMELIST /INPUT/ EI,L,RHO,MT,CT
ISN 0009       READ(5,INPUT)
ISN 0010       MSTAR=MT/(RHO*L)
ISN 0011       CSTAR=CT/L
         C
         C  ECHO PRINT INPUT DATA
         C
ISN 0012       WRITE(6,1) EI,RHO,L
ISN 0013     1 FORMAT(1H1,24X,'DATA FROM NAMELIST INPUT',//,'  EI = ',1PE13.6,2X
              +,' RHO = ',1PE13.6,2X,'   L = ',1PE13.6)
ISN 0014       WRITE(6,2) MT,CT
ISN 0015     2 FORMAT(1HO,'  MT = ',1PE13 6,2X,'  CT = ',1PE13.6,)
         C
         C  COMPUTE CONSTANTS
         C
ISN 0016       ALPHA=1 ODO+MSTAR
ISN 0017       BETA=MSTAR/DSQRT(1 ODO+1 ODO/MSTAR)
ISN 0018       C1=4 ODO/(3 ODO*ALPHA)
ISN 0019       C2=DSQRT(ALPHA)
ISN 0020       C3=2.ODO*C2/(3 ODO*BETA)
ISN 0021       C4=1 5DO*DSQRT(MSTAR)
ISN 0022       C5=2 ODO*DSQRT(MSTAR)/BETA
ISN 0023       D1=C4*CSTAR
```

```
ISN 0024          D2=C2-C5*CSTAR                                              00003000
ISN 0025          A=LAMS(1)                                                   00003200
ISN 0026          B=LAMS(2)                                                   00003300
ISN 0027          MAXFN=80                                                    00003400
            C                                                                 00003410
            C  COMPUTE FIRST ROOT OF BUCKLING CHARACTERISTIC EQUATION          00003420
            C                                                                 00003430
ISN 0028          CALL ZBRENT(F,1 0D-8,5,A,B,MAXFN,IER)                       00003500
ISN 0029          IF(IER  EQ. 0) GO TO 4                                      00003600
ISN 0031          WRITE(6,3) IER                                              00003700
ISN 0032      3   FORMAT(1H0,'ERROR IN ZBRENT ALGORITHM  IER = ',I3)          00003800
ISN 0033          STOP                                                        00004000
ISN 0034      4   PCR=(2.25D0*E /L**2)*B**2                                   00004100
ISN 0035          ACR=PCR/(RHO*  + MT)                                        00004200
ISN 0036          WRITE(6,5)PCR,ACR                                           00004300
ISN 0037      5   FORMAT(1H0,'FIRST CRITICAL BUCKLING LOAD = ',1PE13.6,/,     00004400
                 +'FIRST CRITICAL BUCKLING ACCELERATION = ',1PE13 6)          00004500
ISN 0038          STOP                                                        00005400
ISN 0039          END                                                         00005500


            C*****************  ***********************************************00005510
            C  THIS FUNCTION SUBPROGRAM EVALUATES THE BUCKLING CHARACTERISTIC  00005520
            C  EQUATION, EQ (3-12) EXPRESSED IN TERMS OF BESSEL FUNCTIONS OF   00005530
            C  POSITIVE ORDER.                                                 00005540
            C******************************************************************00005550
ISN 0002          FUNCTION F(LAM)                                             00005600
ISN 0003          IMPLICIT REAL*8(A-H,O-Z)                                    00005700
ISN 0004          REAL*8 LAM                                                  00005800
ISN 0005          COMMON ALPHA,BETA,C1,C2,C3,D1,D2                            00005900
ISN 0006          DIMENSION T(8),RJ(2),WK(4)                                  00006000
ISN 0007          A1=ALPHA*LAM                                                00006100
ISN 0008          A2=BETA*LAM                                                 00006200
            C                                                                 00006300
            C   COMPUTE BESSEL FUNCTIONS OF ORDERS 1/3,2/3,4/3,5/3            00006400
            C   AT ARGUEMENTS A1 AND A2                                       00006500
            C                                                                 00006600
ISN 0009          IS=0                                                        00006700
ISN 0010          DO 40 I=1,2                                                 00006800
ISN 0011          OR=DFLOAT(I)/3.0D0                                          00006900
ISN 0012          DO 30 NA=1,2                                                00007000
ISN 0013          IF(NA  EQ  1) ARG=A1                                        00007100
ISN 0015          IF(NA  EQ  2) ARG=A2                                        00007200
ISN 0017          CALL MMBSJR(ARG,OR,2,RJ,WK,IER)                             00007300
ISN 0018          IF( IER  EQ  0) GO TO 20                                    00007400
ISN 0020          WRITE(6,10) IER,ARG,OR                                      00007500
ISN 0021     10   FORMAT(1H0,5X,'ERROR IN EVALUATING BESSEL FUNCTION',3X,'IER=',  00007600
                 + I3,2X,'ARGUEMENT=',E13.5,2X,'ORDER=',F7 5)                 00007700
ISN 0022          STOP                                                        00007800
ISN 0023     20   IS=IS+1                                                     00007900
ISN 0024          T(IS)=RJ(1)                                                 00008000
ISN 0025          IS=IS+1                                                     00008100
ISN 0026          T(IS)=RJ(2)                                                 00008200
ISN 0027     30   CONTINUE                                                    00008300
ISN 0028     40   CONTINUE                                                    00008400
            C                                                                 00008500
            C  COMPUTE FUNCTION "F"                                           00008600
            C                                                                 00008700
ISN 0029          F=(C1/LAM*T(5)-T(6))*(C3/LAM*T(3)-C2*T(4)+D1*LAM*T(3))+     00008800
                 1 T(1)*(D2*T(7)+D1*LAM*T(8))                                 00008900
ISN 0030          RETURN                                                      00009000
ISN 0031          END                                                         00009100
```

SAMPLE INPUT/OUTPUT FOR BUCKLING PROGRAM

BEAM PARAMETERS

EI = 6 25 · 10^7 lb-ft^2

L = 134.0 ft

ρ = 0 4172 slug/ft

TIP BODY PARAMETERS

m = 155 28 slugs

c = 0 0 ft

NAMELIST INPUT

```
&INPUT EI=6.25E7,RHO= 4172,L=134 0,MT=155.28,CT=00.0,&END
```

PROGRAM OUTPUT

```
DATA FROM NAMELIST INPUT
EI =  6 250000E+07   RHO =  4 172000E-01     L =  1 340000E+02
MT =  1 552800E+02   CT =  0.000000E+00
FIRST CRITICAL BUCKLING LOAD =  1 054506E+04 lb
FIRST CRITICAL BUCKLING ACCELERATION =  4 993284E+01 ft/s^2
```

BEAM PARAMETERS

SAME AS ABOVE

TIP BODY PARAMETERS

m = 155 28 slugs

c = 67 0 ft

PROGRAM OUTPUT

```
DATA FROM NAMELIST INPUT
EI =  6 250000E+07   RHO =  4 172000E-01     L =  1.340000E+02
MT =  1 552800E+02   CT =  6.700C00E+01
FIRST CRITICAL BUCKLING LOAD =  5 261190E+03 lb
FIRST CrITICAL BUCKLING ACCELERATION =  2.491273E+01 ft/s^2
```

APPENDIX E

COMPARISON OF SELECTED NUMERICAL RESULTS

In this appendix we compare the numerical values for the fundamental natural frequencies obtained from four distinct numerical procedures. The results are for the case of a tip body with mass center along the beam tip tangent line and the particular parameters listed below.

Beam Parameters	Tip Body Parameters
$EI = 6.25 \cdot 10^7$ slug-ft^2	$m = 155.2$[illegible] slug
$\ell = 134.0$ ft	$I_t = 5.0 \quad 10^5$ slug-ft^2
$\rho = 0.4172$ slug/ft	$c = 67.0$ ft

The first two columns of natural frequencies are based upon the assumed modes formulation of Chapter 5, using the first 10 fixed base eigenfunctions of a clamped-free beam and clamped beam with tip body, respectively. The spline based Galerkin method used to generate the results of the third column, is described in Reference 2. The results of the fourth column are based upon the power series solution of Section 4.1, using six terms in each series.

Constant Axial Base Acceleration (ft/s^2)	Fundamental Natural Frequencies (s^{-2})			
	Assumed-Modes Method 10 Clamped--Free Eigenfunctions	Assumed-Modes Method 10 Clamped--Tip Body Eigenfunctions	Spline-Based Galerkin Method 10-cubic Splines	Power-Series Solution 6-Terms in Each Series
0.0	0.1329	0.1312	0.1312	0.1312
2.4913	0.1198	0.1181	0.1181	0.1181
4.9825	0.1067	0.1050	0.1050	0.1050
7.4738	0.0936	0.0919	0.0919	0.0919
9.9651	0.0805	0.0788	0.0788	0.0788
12.4564	0.0674	0.0657	0.0657	0.0657
14.9476	0.0543	0.0526	0.0526	0.0526
17.4389	0.0411	0.0395	0.0395	0.0395
19.9302	0.0280	0.0263	0.0263	0.0263
22.4215	0.0148	0.0132	0.0132	0.0132
24.9127	0.0017	0	0	0

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