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NONLINEAR PROBLEMS IN FLIGHT DYNAMICS INVOLVING AERODYNAMIC BIFURCATIONS

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SUMMARY

Aerodynamic bifurcation is defined as the replacement of an unstable equilibrium flow by a new stable equilibrium flow at a critical value of a parameter. A mathematical model of the aerodynamic contribution to the aircraft's equations of motion is amended to accommodate aerodynamic bifurcations. Important bifurcations such as, for example, the onset of large-scale vortex-shedding are defined. The amended mathematical model is capable of incorporating various forms of aerodynamic responses, including those associated with dynamic stall of airfoils.

1. INTRODUCTION

In recent years we have become increasingly aware of the connections which link our approach to nonlinear problems in flight dynamics to the body of theory underlying the rapidly growing field known as "nonlinear dynamical systems." In Ref. 1, we attempted to identify the connections explicitly within a framework designed to facilitate study of the aerodynamic contribution to the flight-dynamics system. The framework is composed of parallel observational and mathematical components. A study of observations of fluid flows singled out four important elements for the observational component. First, flow patterns have definite structure (e.g., three-dimensional vortical structures). Second, structures undergo systematic changes with variations in parameters such as angle of attack or Reynolds number. Third, under some conditions the changes lead to chaos (e.g., turbulence). Fourth, the structures have definite spatial and temporal scales. Parallel elements of the mathematical component feature the language of topology and concepts drawn from dynamical-systems theory. The premises of the theoretical framework are: (1) Structures are describable in topological terms. (2) Changes in structure are describable by bifurcation theory. (3) Chaos is describable by the theory of strange attractors and fractals. (4) Scales are describable by group theory ideas. Recently (Ref. 2), we proposed that the framework may be useful as well for the understanding and the potential modeling of turbulent flows.

Here, we shall focus on one element of the framework, namely, change, and its representation by means of bifurcation theory. Bifurcation theory has two principal realms of application in flight-dynamics studies. First, when the form of the aerodynamic contribution to the aircraft's equations of motion is known, and the equations of motion are of the appropriate form, bifurcation theory provides a useful means of analyzing the aircraft's motion, particularly in the vicinity of stability boundaries. This first realm was the subject of study in Ref. 3, and the study is pursued further in Ref. 4 at this conference. The second realm, which will be the subject of this paper, is concerned with the mathematical modeling of the aerodynamic contribution itself.

Reference 5, a companion paper of Ref. 1, was a reconsideration of an approach to mathematical modeling of the aerodynamic characteristics, in which nonlinear indicial responses are used as a basis for arriving at the form of the aerodynamic response to arbitrary motions. Without modification, validity of the approach rests on a fundamental assumption that the indicial response exists and is unique in every incremental step of the motion. The question, "When does the fundamental assumption fail?" led to the recognition that the assumption may be invalidated through the mechanism of flow instability. On exceeding a critical value of a parameter (e.g., angle of attack), the flow field corresponding to the equilibrium state of an aerodynamic response may become unstable to small disturbances. The replacement of the unstable state by a new stable equilibrium state satisfying the same boundary conditions implies failure of the uniqueness condition, and may imply failure of the existence condition as well. Thus, the equilibrium state undergoes a bifurcation, and the circumstances are precisely those that bifurcation theory is designed to address. Although the requisite amendments were not carried out in Ref. 5, it was concluded that bifurcation theory would provide the means to extend the mathematical model so as to acknowledge the occurrence of aerodynamic bifurcation phenomena. As a consequence, the model would have the potential of accounting for those critical points in maneuvers where sudden and dramatic changes in flow structure may occur.

In this paper, we shall try to validate the conclusion of Ref. 5, and demonstrate how the mathematical model can be amended to accommodate the occurrence of aerodynamic bifurcations. A number of important aerodynamic bifurcations will be defined and some of their potential effects on dynamical systems illustrated. Following the analysis presented in Ref. 5, we shall show how the approach to the modeling involving nonlinear indicial responses can be made compatible with an approach based on the use of nonlinear algebraic functional expansions. The latter approach, which has the virtue of concreteness, was formulated originally (Ref. 6) to address nonlinear problems in the field of electrical circuits and systems. Its adaptation here exemplifies the community of interests developing across various disciplines through the unifying medium of research in nonlinear dynamical systems.

2. MODELING INFLUENCE OF PAST MOTION BY PULSES

The following is a slightly modified version of the analysis originally presented in Ref. 5. Again, for clarity, the two-dimensional wing is adopted as illustration, although the results will have more general bearing.

Let the wing move away from a coordinate system whose origin is fixed in space at the center of gravity at a time $\xi = 0$. The distance traveled by the center of gravity along the flightpath is measured by a coordinate s . Let the center of gravity move at constant velocity V_0 , so that the trace of its path, plotted against time ξ , is a straight line. This is shown on Fig. 1. The wing is allowed to undergo changes only in the angle of attack α , where α is the angle between the velocity vector and the wing chord line. Projections of the leading and trailing edges of the wing onto the plane containing the velocity vector are maximum when $\alpha = 0$. These maximum projections also trace out straight lines on Fig. 1, parallel to the trace of the center of gravity.

As illustrated in Fig. 1, let the angle of attack α be zero for all time ξ except at $\xi = \xi_1$, where a pulse occurs of amplitude $\alpha(\xi_1)$ and of infinitesimal duration $\Delta\xi_1$. Consider the response to the pulse at ξ_1 of the lift coefficient C_L at a measuring time t subsequent to ξ_1 . The lift response at t will be a function of the elapsed time $t - \xi_1$ and the amplitude of the pulse $\alpha(\xi_1)$. If we assume that there will be a range of α ($0 < \alpha < \alpha_1$) in which the dependence of C_L on $\alpha(\xi_1)$ is analytic, then we can write the response in C_L as a Taylor-series expansion in $\alpha(\xi_1)$ of the form

$$C_L(t) = C_{L}(t - \xi_1, \alpha(\xi_1))_{dir} = \sum_n a_n(t - \xi_1) [\alpha(\xi_1)]^n \Delta\xi_1 \quad (1)$$

The first of the forms in Eq. (1) will be used in the subsequent analysis to distinguish between direct (subscript *dir*) and interference (subscript *int*) effects.

Now let us consider the response in C_L at the measuring time t to a pair of pulses located at ξ_1 and ξ_2 with $\xi_1, \xi_2 < t$. Here, in addition to the direct influence of each of the pulses acting as if in isolation, the interference between the pulses also will influence the lift. The interference effect can be written in a form resembling a product of responses to single pulses

$$C_{L_{int,2}} = \sum_{m,n} b_{mn}(t - \xi_1, t - \xi_2) [\alpha(\xi_1)]^m [\alpha(\xi_2)]^n \Delta\xi_1 \Delta\xi_2 \quad (2)$$

where the subscript (*int,2*) means "interference between a pair of pulses." The form vanishes properly with the vanishing of either of the pulses and retains the analytic dependence on angle of attack. With the addition of the direct influence of the two pulses, the lift coefficient at time t takes the form

$$C_L(t) = C_{L}(t - \xi_1, \alpha(\xi_1))_{dir} + C_{L}(t - \xi_2, \alpha(\xi_2))_{dir} + C_{L}(t)_{int,2} \quad (3)$$

The process of adding pulses can be continued indefinitely in the same way. At the next stage, the interference between triplets of pulses must be considered as well as that between pairs. Going to the limit of a continuous distribution of pulses starting at time $\xi = 0$ yields a summation of multiple integrals having the form

$$C_L(t) = C_{L_{dir}} + C_{L_{int,2}} + C_{L_{int,3}} + \dots \quad (4)$$

with

$$C_{L_{dir}} = \sum_n \int_0^t a_n(t - \xi_1) [\alpha(\xi_1)]^n d\xi_1 \quad (5)$$

$$C_{L_{int,2}} = \sum_{m,n} \int_0^t [\alpha(\xi_2)]^m d\xi_2 \int_0^{\xi_2} b_{mn}(t - \xi_1, t - \xi_2) [\alpha(\xi_1)]^n d\xi_1 \quad (6)$$

$$C_{L_{int,3}} = \sum_{m,n,p} \int_0^t [\alpha(\xi_3)]^m d\xi_3 \int_0^{\xi_3} [\alpha(\xi_2)]^n d\xi_2 \times \int_0^{\xi_2} c_{mnp}(t - \xi_1, t - \xi_2, t - \xi_3) [\alpha(\xi_1)]^p d\xi_1 \quad (7)$$

⋮

Written as a nonlinear functional expansion, Eq. (4) represents the lift coefficient at time t in response to an arbitrary variation of α over the time interval zero to t . The form of Eq. (4) confirms an important point made in Ref. 6. It will be noted that a partial summation consisting of the leading term from each of Eqs. (5), (6), . . . forms a Volterra series (Refs. 6,7). The fact that there are terms left over confirms that the a priori adoption of a Volterra series to represent the lift coefficient would have been insufficiently general to accommodate the Taylor series form of the dependence on angle of attack.

3. FORMATION OF INDICIAL RESPONSE

Given Eq. (4), one can now use it to form the indicial response in accordance with the definition in Ref. 8. To indicate the form of the result, it will suffice to consider terms in Eq. (4) only through the series representing $C_{L_{int,2}}$. Two motions need to be considered. In the first, the wing undergoes the motion under study $\alpha(\xi)$ from time zero up to a time $\xi = \tau$, where $\tau < t$. Subsequent to τ , α is held constant at $\alpha(\tau)$. Thus, in Eqs. (4), (5), and (6),

$$\left. \begin{aligned} \alpha_1(\xi) &= \alpha(\xi) ; & 0 < \xi < \tau \\ &= \alpha(\tau) ; & \xi \geq \tau \end{aligned} \right\} \quad (8)$$

The direct and interference contributions to $C_L(t)$ take the form

$$\left. \begin{aligned} C_{L_{dir}} &= \sum_n \int_0^\tau a_n(t - \xi_1) [\alpha(\xi_1)]^n d\xi_1 + \sum_n [\alpha(\tau)]^n \int_\tau^t a_n(t - \xi_1) d\xi_1 \\ C_{L_{int,2}} &= \sum_{m,n} \int_0^\tau [\alpha(\xi_2)]^m d\xi_2 \int_0^{\xi_2} b_{mn}(t - \xi_1, t - \xi_2) [\alpha(\xi_1)]^n d\xi_1 \\ &+ \sum_{m,n} [\alpha(\tau)]^m \int_\tau^t d\xi_2 \int_0^\tau b_{mn}(t - \xi_1, t - \xi_2) [\alpha(\xi_1)]^n d\xi_1 \\ &+ \sum_{m,n} [\alpha(\tau)]^{m+n} \int_\tau^t d\xi_2 \int_\tau^{\xi_2} b_{mn}(t - \xi_1, t - \xi_2) d\xi_1 \end{aligned} \right\} \quad (9)$$

In functional notation (Ref. 7), the lift response to the same motion is

$$C_L(t) = C_L[\alpha_1(\xi)] = C_L[\alpha(\xi); t, \tau] \quad (10)$$

where the notation in the first form should be understood to mean that $C_L(t)$ is a functional of the variation $\alpha_1(\xi)$ over the interval $0 < \xi < t$. The second form, which is defined in Ref. 8, is designed to make more explicit that $\alpha_1(\xi)$ represents the motion under study $\alpha(\xi)$ over the interval $0 < \xi < \tau$, but is constrained to remain constant at $\alpha(\tau)$ for $\tau \leq \xi \leq t$. Thus, $C_L(t)$ is both a functional of $\alpha(\xi)$ and a function of t and τ .

In the second motion, the wing undergoes the same angle-of-attack history $\alpha(\xi)$ up to time τ . Subsequent to τ , the angle of attack is again held constant, but is given an incremental step change $\Delta\alpha$ over its previous value of $\alpha(\tau)$. Thus, in the second motion,

$$\left. \begin{aligned} \alpha_2(\xi) &= \alpha(\xi) ; & 0 < \xi < \tau \\ &= \alpha(\tau) + \Delta\alpha ; & \xi \geq \tau \end{aligned} \right\} \quad (11)$$

The direct and interference contributions to $C_L(t)$ become

$$\left. \begin{aligned} C_{L_{dir}} &= \sum_n \int_0^\tau a_n(t - \xi_1) [\alpha(\xi_1)]^n d\xi_1 + \sum_n [\alpha(\tau) + \Delta\alpha]^n \int_\tau^t a_n(t - \xi_1) d\xi_1 \\ C_{L_{int,2}} &= \sum_{m,n} \int_0^\tau [\alpha(\xi_2)]^m d\xi_2 \int_0^{\xi_2} b_{mn}(t - \xi_1, t - \xi_2) [\alpha(\xi_1)]^n d\xi_1 \\ &+ \sum_{m,n} [\alpha(\tau) + \Delta\alpha]^m \int_\tau^t d\xi_2 \int_0^\tau b_{mn}(t - \xi_1, t - \xi_2) [\alpha(\xi_1)]^n d\xi_1 \\ &+ \sum_{m,n} [\alpha(\tau) + \Delta\alpha]^{m+n} \int_\tau^t d\xi_2 \int_\tau^{\xi_2} b_{mn}(t - \xi_1, t - \xi_2) d\xi_1 \end{aligned} \right\} \quad (12)$$

Again, written as a functional, the lift response to the motion $\alpha_2(\xi)$ is

$$C_L(t) = C_L[\alpha_2(\xi)] \quad (13)$$

The indicial lift response is formed by taking the difference between Eqs. (9) and (12) (or between Eqs. (10) and (13)), dividing by the incremental step $\Delta\alpha$, and going to the limit as $\Delta\alpha \rightarrow 0$. Existence of the limit is ensured by the analyticity of the functional dependence on $\alpha(\xi)$. The result is

$$\begin{aligned}
\lim_{\Delta\alpha \rightarrow 0} \frac{\Delta C_L(t)}{\Delta\alpha} &= \lim_{\Delta\alpha \rightarrow 0} \left(\frac{C_L[\alpha_2(\xi)] - C_L[\alpha_1(\xi)]}{\Delta\alpha} \right) \\
&= \sum_n n[\alpha(\tau)]^{n-1} \int_{\tau}^t a_n(t - \xi_1) d\xi_1 \\
&\quad + \sum_{m,n} m[\alpha(\tau)]^{m-1} \int_{\tau}^t d\xi_2 \int_0^{\tau} b_{mn}(t - \xi_1, t - \xi_2) [\alpha(\xi_1)]^n d\xi_1 \\
&\quad + \sum_{m,n} (m+n)[\alpha(\tau)]^{m+n-1} \int_{\tau}^t d\xi_2 \int_{\tau}^{\xi_2} b_{mn}(t - \xi_1, t - \xi_2) d\xi_1 \quad (14)
\end{aligned}$$

Eq. (14) reveals the form of the indicial lift response to a step change in angle of attack in terms of functional expansions. The first and third terms on the right-hand side of Eq. (14) do not depend on the past motion, but only on the level of the angle of attack at which the step was made. The second term depends on the past motion, since $\alpha(\xi_1)$ with $0 < \xi_1 < \tau$ appears within the integral. The leading term of this past dependence has the form

$$\int_{\tau}^t d\xi_2 \int_0^{\tau} b_{11}(t - \xi_1, t - \xi_2) \alpha(\xi_1) d\xi_1$$

Dependence on the past thus arises from interference effects between pulses prior to τ , the origin of the step, and perturbation pulses of $O(\Delta\alpha)$ originating subsequent to τ . In the general case, and just as before (Ref. 8), the indicial response is itself a functional.

In fact, in formal terms (see Ref. 9 for a useful account of the essentials) the operations involved in forming the indicial response amount to taking a Fréchet derivative of the functional $C_L[\alpha_1(\xi)]$. To see this, we adopt the notation of Ref. 9, letting

$$h = \epsilon \eta \quad (15)$$

where we set

$$\left. \begin{aligned}
\epsilon &= \Delta\alpha \\
\eta &= 0; \quad 0 < \xi < \tau \\
&= 1; \quad \xi \geq \tau
\end{aligned} \right\} \quad (16)$$

so that

$$\alpha_2(\xi) = \alpha_1(\xi) + \epsilon \eta \quad (17)$$

Then, following Ref. 9, we have

$$\begin{aligned}
\lim_{\Delta\alpha \rightarrow 0} \frac{\Delta C_L(t)}{\Delta\alpha} &= \lim_{\epsilon \rightarrow 0} \left(\frac{C_L[\alpha_1(\xi) + \epsilon \eta] - C_L[\alpha_1(\xi)]}{\epsilon} \right) \\
&= \frac{d}{d\epsilon} C_L[\alpha_1(\xi) + \epsilon \eta] \Big|_{\epsilon=0} \\
&= C_L'[\alpha_1(\xi)] \eta \quad (18)
\end{aligned}$$

The equivalence of the operations yielding Eqs. (14) and (18) should be evident. Finally, to conform with the notation of Ref. 8, we set

$$\lim_{\Delta\alpha \rightarrow 0} \frac{\Delta C_L(t)}{\Delta\alpha} = C_L'[\alpha_1(\xi)] \eta = C_{L_\alpha}[\alpha(\xi); t, \tau] \quad (19)$$

where the indication of a separate dependence on t and τ rather than on elapsed time $t - \tau$ alone should be noted. It can be easily verified that the first and third terms in Eq. (14), which depend only on $\alpha(\tau)$, indicate a dependence on $t - \tau$ alone; however, as a consequence of its dependence on the past motion, the second term cannot be cast as a function of $t - \tau$ alone.

4. GENERALIZED SUPERPOSITION INTEGRAL

Just as before, Eq. (19) can be used to form a generalized superposition integral for the response in C_L to an arbitrary angle-of-attack variation. The result is

$$C_L(t) = C_L(t; \alpha(0)) + \int_0^t C_{L_\alpha}[\alpha(\xi); t, \tau] \frac{d\alpha}{d\tau} d\tau \quad (20)$$

with

$$C_L(t; \alpha(0)) = C_L[\alpha(\xi); t, 0] \quad (21)$$

It is important to understand the meaning of the first term in Eq. (20). It is the contribution to the lift at time t caused by the flow field existent at the initial instant $\xi = 0$. It would be the only contribution if the angle-of-attack variation subsequent to $\xi = 0$ had remained constant at $\alpha(0)$. A reading of the notation of the second form for this term in Eq. (21) (cf. the definitions following Eq. (10)) provides a more complete interpretation: The first term in Eq. (20) is the lift at time t due to an angle-of-attack variation $\alpha(\xi)$ which is equal to the motion history prior to $\xi = 0$, and which is constrained at $\xi = \tau = 0$ to remain constant at $\alpha(0)$ for all $\xi \geq \tau = 0$. by substituting Eq. (14) for $C_L[\alpha(\xi); t, \tau]$ in the integral term in Eq. (20) and carrying out the integration, one will verify that the form of Eq. (4) is restored through terms of the series representing $C_{L_{int,2}}$ (i.e., through Eqs. (5) and (6)). We conclude first that an approach to modeling based on nonlinear indicial responses and generalized superposition integrals is compatible with one based on nonlinear functional expansions. Second, validity of the approach rests on the assumption of Fréchet differentiability of the lift response $C_L[\alpha(\xi); t, \tau]$ over the interval $0 < \tau < t$. Invalidation of this assumption may signal the occurrence of an aerodynamic bifurcation.

5. CONNECTION WITH BIFURCATION THEORY

Consider the first of the two maneuvers involved in the formation of the indicial response, that is, the maneuver (cf. Eqs. (8) and (10)) to which the lift response $C_L[\alpha(\xi); t, \tau]$ corresponds. The angle of attack attains a constant value $\alpha(\tau)$ subsequent to τ , and it is reasonable to expect that the flow field at the subsequent time t will approach an equilibrium state that corresponds to this fixed boundary condition as the elapsed time $t - \tau \rightarrow \infty$. Heretofore (Ref. 8), the analysis has contained an implicit assumption that as the flow field approached the equilibrium state it became time-invariant, which meant that the corresponding lift coefficient C_L approached a unique constant value $C_L(\infty; \alpha(\tau))$ as $t - \tau \rightarrow \infty$. As long as this was true, it was reasonable to expect that an incremental change in $\alpha(\tau)$ of $O(\Delta\alpha)$ would result in an incremental change in $C_L(\infty; \alpha(\tau))$, likewise of $O(\Delta\alpha)$. We now recognize that at least two conditions must be satisfied in order that this be true: (1) The change of $O(\Delta\alpha)$ must not result in a loss of analyticity in the dependence of $C_L[\alpha(\xi); t, \tau]$ on $\alpha(\xi)$. (2) The time-invariant equilibrium flow represented by $C_L(\infty; \alpha(\tau))$ must be asymptotically stable to small perturbations. We deal with the second condition here, assuming for the moment that the first condition has been satisfied. Although exceptions to sufficiency can be envisioned, we shall restrict our study to the class of flows for which asymptotic stability of the equilibrium state is not only necessary, but also sufficient, to ensure that $C_L[\alpha(\xi); t, \tau]$ will be Fréchet differentiable.

It can happen, however, that as $\alpha(\tau)$ is increased in small increments, a critical value of $\alpha(\tau)$ can be reached at which the stationary equilibrium flow represented by $C_L(\infty; \alpha(\tau))$ will no longer be stable to small perturbations. Loss of asymptotic stability signals loss of Fréchet differentiability at the critical value of $\alpha(\tau)$. With the stationary equilibrium state asymptotically unstable, the system will seek a new equilibrium state, and of those that are possible will select one that can remain stable to small perturbations. This is precisely the situation that bifurcation theory is designed to address. Bifurcation theory provides a means of classifying and characterizing the properties of the new equilibrium states that can arise when the given equilibrium state becomes unstable. We call the replacement of an unstable equilibrium state by a new stable equilibrium state an aerodynamic bifurcation. We will now show how the mathematical model Eq. (20) can be amended to acknowledge its occurrence.

Referring to Fig. 2, let us assume that the angle-of-attack variation under study $\alpha(\tau)$ passes through a critical value α_c at a value of $\tau = \tau_c$ within the interval of interest $0 < \tau < t$. If the angle-of-attack variation $\alpha(\tau)$ were constrained to remain constant at α_c for all $\tau > \tau_c$, the corresponding lift response $C_L[\alpha(\xi); t, \tau_c]$ would begin to depart from an initially infinitesimally close neighboring response, such that $C_L[\alpha(\xi); t, \tau_c]$ would not be Fréchet differentiable. Thus, we cannot allow the integration in Eq. (20) to proceed beyond τ_c without acknowledging the loss of Fréchet differentiability. Let us assume that for all other angles of attack within the range $\alpha(0) < \alpha < \alpha(t)$ the lift responses represent asymptotically stable flows permitting Fréchet differentiation. Then we can isolate the critical state by stopping the integration in Eq. (20) just short of τ_c and starting again just beyond τ_c . Within the isolated interval $\tau_c - \epsilon < \tau < \tau_c + \epsilon$, we acknowledge the aerodynamic bifurcation by allowing the solution to change discretely to a new equilibrium state. Thus,

$$C_L(t) = C_L(t; \alpha(0)) + \int_0^{\tau_c - \epsilon} C_{L_\alpha}[\alpha(\xi); t, \tau] \frac{d\alpha}{d\tau} d\tau + \int_{\tau_c + \epsilon}^t C_{L_\alpha}[\alpha(\xi); t, \tau] \frac{d\alpha}{d\tau} d\tau + \Delta C_L(t; \alpha_c) \quad (22)$$

where

$$\Delta C_L(t; \alpha_c) = C_L[\alpha(\xi); t, \tau_c + \epsilon] - C_L[\alpha(\xi); t, \tau_c - \epsilon] \quad (23)$$

This procedure was introduced earlier (Ref. 8) to acknowledge the occurrence of mean or constant values of the lift coefficient in the equilibrium state $C_L(\infty; \alpha(\tau))$ which could exhibit the discontinuous

and double-valued behavior characteristic of hysteresis. This was a case in which the first of the two conditions just stipulated, namely, analytic dependence of $C_L[\alpha(\xi);t,\tau]$ on α , was violated. The particular way in which this occurred is common in aerodynamics: The variation with α of the lift coefficient in the equilibrium state $C_L(=;\alpha(\tau))$ develops a fold at a critical value of α , α_c , so that the slope of $C_L(=;\alpha(\tau))$ versus α becomes infinite there. A jump in $C_L(=;\alpha(\tau))$ necessarily ensues to the upper branch of the folded curve with an infinitesimal increase in α beyond α_c , and hysteresis follows on the return route. We show this as sketch (a) in Fig. 3 along with, for comparison, sketches (b) and (c) of the two most common forms of aerodynamic bifurcation. These sketches represent bifurcation to another time-invariant equilibrium state. The distinction between the class represented by sketch (a) (the class studied in Ref. 8) and the examples of aerodynamic bifurcations should be noted. In sketch (a) there is only one solution curve; no new branches resulting from instability appear. Discontinuous and double-valued behavior result merely from the fold that develops in the solution curve. In the case of supercritical bifurcation, sketch (b), a new branch of stable solutions appears which is connected to the old branch at the critical point α_c . Beyond α_c , two branches of solutions exist satisfying the same boundary conditions, of which one is stable and the other is unstable. Supercritical bifurcation solutions start from zero and are initially infinitesimal as α increases beyond α_c , so that Eq. (23) would be zero. Supercritical bifurcation is not an important mechanism by which time-invariant equilibrium states exchange stability. The bifurcation can be easily superseded by smooth alternative solution curves through a variety of means (cf. Refs. 10-12). In subcritical bifurcation, sketch (c), the bifurcation solution curve that branches away from the now unstable branch is itself unstable. If it turns back, as indicated in sketch (c), the upper branch will be stable. A discontinuous jump in $C_L(=;\alpha(\tau))$ results as α increases beyond α_c , and hysteresis follows on the return route, just as in sketch (a). Here, of course, Eq. (23) will have a value.

Figure 3 has had the important function of reminding us that there are at least two mechanisms which can invalidate the conditions permitting Fréchet differentiation of the lift response $C_L[\alpha(\xi);t,\tau]$. There may be others. Joseph (Refs. 11,12) notes, for example, that isolated solution branches of flows in the equilibrium state can exist which are not connected to other solution branches through bifurcation. Later, we shall entertain the possibility that the initial flow as well as the stability of flow in the equilibrium state may influence Fréchet differentiability. A theory of the means of invalidating Fréchet differentiability of the lift response offers the possibility of an alternative to bifurcation theory. Inasmuch as it would incorporate the latter, it would be a potentially more inclusive way of classifying aerodynamic phenomena that are important in flight-dynamics applications.

6. QUALIFICATIONS

Before proceeding with the study of aerodynamic bifurcations, we must introduce some important qualifications. Our current understanding of fluid-dynamic bifurcations stems almost entirely from results of analytical and computational studies of laminar flows governed by the incompressible Navier-Stokes equations. Within these confines, studies have been further limited to flows in bounded domains. Boundary conditions have been simple enough to permit reduction of the linear stability problem (the necessary first step in the study of bifurcations) to the form of ordinary differential equations (e.g., the Orr-Sommerfeld equations). The principal findings on which our current understanding is based are summarized in Refs. 10-13. Qualifications are necessary when we attempt to extrapolate this body of knowledge to the study of the aerodynamic bifurcations that occur under full-scale flight conditions. These involve turbulent compressible flows which, if averaging and modeling are introduced, will be governed not by the compressible Navier-Stokes equations but by evolution equations of unknown forms. Further, the domains of the flows are typically unbounded, rather than bounded. Finally, the boundary conditions, reflecting the geometries typical of modern aircraft, ordinarily will not allow reduction of the linear stability problem to the level of ordinary differential equations.

The situation requires us to stipulate a list of presuppositions which will be the basis for the extrapolations we make. The list is inspired by one that Lanford compiled to justify the study of turbulence within a framework based on the incompressible Navier-Stokes equations (Ref. 14). Our list will merely supplement Lanford's, taking additional account of turbulence modeling, compressibility, and the necessity to consider flows in unbounded domains. The issue of boundary conditions is not addressed on the assumption that their increased complexity does no more than increase the role of numerical computations.

6.1 Turbulence Modeling

Let us consider the flow field to which a typical lift response $C_L[\alpha(\xi);t,\tau]$ corresponds. We assume that the flow field at time $\xi = \tau$ is given (to within a certain accuracy), and we must determine the subsequent flow at time t . Inasmuch as $\alpha(\xi)$ remains constant at $\alpha(\tau)$ for all $\xi = t > \tau$, we have fixed boundary conditions. This form of the fluid-dynamics problem is known as an initial boundary value problem (IBVP; cf. Ref. 11). We assume that any given realization of the IBVP is governed with sufficient accuracy by the compressible Navier-Stokes equations. Paralleling Ref. 14, in the absence of proof we simply postulate that a solution of the IBVP would exist for all $\xi = t \geq \tau$ and would be unique. However, the Reynolds number is sufficiently high so that a solution would exhibit chaotic behavior, implying a sensitive dependence on the initial conditions. Thus, small perturbations of the initial conditions would cause repetitions of the given motion to result in a set of motions having stochastic properties.

The situation calls for the application of some form of averaging of the compressible Navier-Stokes equations to suppress the appearance of the small-scale chaotic structures. After averaging, the repository of solutions of the resulting equations must remain faithful to that of the unaveraged equations, neither allowing uncharacteristic solution forms nor losing essential ones. The necessity of closing the averaged equations in conformity with this requirement introduces the turbulence modeling problem. The convergence of interests this situation represents for the fields of research concerned with turbulence modeling and with the modeling of aerodynamic responses has been noted in Ref. 5. In Ref. 2, we have tried to show how the theoretical framework underlying the deterministic approach to the study of turbulence might assist the modeling effort, particularly at this juncture, where it is important to give adequate representation of bifurcation phenomena. We cannot go further here in specifying modeling requirements except to postulate the one feature that must be retained in common with the original Navier-Stokes equations: Whatever the form of modeling adopted, it must remain possible to cast the modeled equations in the form of autonomous evolution equations, namely,

$$\dot{\vec{u}}_t = H(\vec{u}, \lambda) \quad (24)$$

where \vec{u} is the velocity vector and λ is a parameter (e.g., angle of attack, Reynolds number, Mach number). We believe that the premise is a reasonable one, involving no more in principle than the use of orthogonal projections (Refs. 15,16) to eliminate gradient terms (e.g., the pressure term) from the modeled equations.

6.2 Compressibility

We shall assume that modeled equations of turbulent flows resulting from averaging the compressible Navier-Stokes equations remain at least as well-behaved as the original equations and their incompressible counterparts. Of course, the new parameters introduced by compressibility (e.g., Mach number) will themselves have critical values at which additional bifurcations should be expected.

6.3 Unbounded Domains

Let us return to the IBVP represented by $C_L[\alpha(\xi); t, \tau]$ and assume that the premises of the previous sections are now in force. Henceforth, the equations governing the flow field are assumed to be modeled equations of turbulent flow, of known form, and satisfying the form of Eq. (24) by suitable projections.

The issue concerning the absence of bounds on the flow domain arises in evaluating the stability of the flow field in the equilibrium state. We say that the equilibrium state is reached in the limit as $t - \tau \rightarrow \infty$, so that transient effects associated with the initial conditions have vanished. A study of the stability of the equilibrium flow must be undertaken first, since bifurcation of the equilibrium state will not occur unless a critical value of the parameter in question (here, α) exists at which the flow field in the equilibrium state becomes unstable. Stability is evaluated by determining whether an infinitesimal perturbation of the equilibrium flow decays or grows with time. The perturbation's fate can be determined from a linear spectral problem, the governing equations being obtained from a linearized perturbation of the equations governing the equilibrium flow. Typically, with a bounded domain, it can be shown that solution of the spectral problem yields a countably infinite set of isolated eigenvalues in a complex plane. Stability is indicated if all of the eigenvalues lie in the left-hand plane. On the other hand, little is known about the spectrum of eigenvalues for unbounded domains. In the case of flows in unbounded domains governed by the incompressible Navier-Stokes equations, it is believed (Refs. 17,18) that an incomplete set of discrete eigenvalues exists which again controls stability, and that there is additionally a continuous spectrum having negative real part (i.e., lying in the left half of the complex plane). We postulate that what is believed to be the case for flows in unbounded domains governed by the incompressible Navier-Stokes equations is in fact the case for our problem.

6.4 Conclusions

Accepting the last premise (which implies accepting all of the previous ones) leads to several conclusions concerning the role of bifurcation theory in the modeling of aerodynamic responses. First, since we have postulated that stability of the equilibrium state continues to be controlled by the discrete part of the eigenvalue spectrum alone, entailing no change from the rules governing bounded domains, we conclude that the role of bifurcation theory regarding the equilibrium state should carry over intact to flows in unbounded domains. That is, all of the instability mechanisms that have been uncovered by studies of flows in bounded domains governed by the incompressible Navier-Stokes equations should have counterparts in flows in unbounded domains governed by modeled turbulent flow equations. With the onset of instability of the equilibrium state, bifurcation theory again should be capable of classifying and characterizing the properties of the new stable equilibrium states that can arise to replace the unstable one.

On the other hand, concerned as it is only with the equilibrium state, bifurcation theory does not suffice to completely resolve the IBVP from which the indicial lift response is derived. We note that when the eigenvalue spectrum lies in the left-half plane, indicating stability of the equilibrium state, the IBVP for an infinitesimal step perturbation of the boundary condition $\alpha(\xi); \xi \geq \tau$ is a linear problem. In fact, the transient flow field from which the indicial response $C_L[\alpha(\xi); t, \tau]$ is derived can be obtained from a suitable superposition of the eigensolutions (including both the discrete and continuous parts) of the linear spectral problem. The superposition is made determinate by the necessity of matching a prescribed flow field at the initial instant $\xi = \tau$. This is how the dependence on the past

motion is expressed, which makes $C_L[\alpha(\xi);t,\tau]$ a nonlinear functional even though derivable from linearized perturbation equations. Thus, recalling the derivation by means of functional expansions, we affirm that so long as stability of the equilibrium state ensures that $C_L[\alpha(\xi);t,\tau]$ will be Fréchet differentiable, the differential $\Delta\alpha C_L[\alpha(\xi);t,\tau]$ can be determined from a linear problem. With the onset of instability, however, the perturbation flow no longer can be obtained from the solution of the linear spectral problem, since at least one of the discrete eigensolutions would grow without bounds. Thus, the incremental change in lift coefficient given by Eq. (23) must be determined from the fully nonlinear perturbation flow equations.

7. AERODYNAMIC BIFURCATIONS

In Ref. 1, we postulated six major subdivisions in the form of aerodynamic force and moment responses by means of a set of sketches, and these are reproduced in Fig. 4. Although the ordinate f_a in each of the sketches can be made to represent a variety of characteristics, it will be convenient here to let f_a represent the lift coefficient in the equilibrium state. We now briefly outline how modeling of the aerodynamic responses incorporating the six forms of lift coefficient may be effected by means of the amendment we have introduced to accommodate aerodynamic bifurcations.

7.1 Major Subdivisions

First, it should be clear that aerodynamic systems represented by sketches (a) and (b) in Fig. 4, involving linear or nonlinear single-valued lift variations, do not require modification of the treatment contained in, for example, Ref. 8. In formal terms, these systems can be characterized as having aerodynamic responses $C_L[\alpha(\xi);t,\tau]$ that are Fréchet differentiable over the entire angle-of-attack range of interest.

Sketch (c) in Fig. 4 introduces a class of multivalued solutions of the flow in the equilibrium state. Mean or constant values of the lift coefficient in the equilibrium state exhibit the discontinuous and double-valued behavior characteristic of hysteresis. As we have already noted, such cases were treated in Ref. 8 by a procedure similar to the one described here. The class includes both solution curves having folds, and subcritical exchanges of stability between time-invariant equilibrium states.

Sketches (d) and (e) in Fig. 4 introduce bifurcations that result in time-dependent equilibrium states. Accommodating these phenomena within the mathematical model will require the use of Eqs. (22) and (23). Hopf bifurcation is indicated in sketch (d), wherein a formerly stable time-invariant equilibrium state is replaced by a time-varying periodic equilibrium state. The difference in lift responses resulting from these two forms of equilibrium states is reflected in Eq. (23). Hopf bifurcations are common in aerodynamics. Physically, the usual origin of such a large-scale oscillatory state is the onset of vortex-shedding. Of the many examples, we cite here stall on airfoils when the angle of attack exceeds a critical value (Refs. 19,20) and the wake of the flow past a cylinder when the Reynolds number exceeds 50 (Ref. 21). The equilibrium state resulting from a Hopf bifurcation is often succeeded by bifurcation to a quasi-periodic equilibrium state with further increase of the parameter. To accommodate this and subsequent bifurcations, the integration regime in Eq. (22) must be interrupted at each such occurrence, and an interval of $O(\epsilon)$ inserted in which a discrete change, such as Eq. (23), to a new equilibrium state is allowed. The quasi-periodic state may be succeeded by an aperiodic (chaotic) equilibrium state. As noted in sketch (e) in Fig. 4, appearance of the latter state signals the presence of a "strange attractor." The slender body of revolution may undergo such a sequence of bifurcations as angle of attack is increased incrementally toward 90° . This example, among others, is discussed at length in Ref. 1. Several additional examples of bifurcation sequences relevant to aerodynamic applications are contained in Refs. 2 and 22.

The last subdivision in Fig. 4, sketch (f), representative of aerodynamic systems having strong rate dependence, goes beyond the previous ones in that more is at issue in the modeling than accommodating bifurcations of the equilibrium state. An adequate representation of the aerodynamic system in this category of problem represents a severe test for any mathematical model. We believe that the oscillating airfoil in what is referred to as dynamic stall is exemplary in characterizing the nature of the problem in this category. Fortunately, a comprehensive set of experimental results exists for the problem, reported in a series of papers by McAllister, Carr, and McCroskey (Refs. 19,20). Included are excellent flow-visualization pictures which reveal many features of the physical mechanisms involved. In the following section we refer to these results and briefly demonstrate the extent to which our amended model, Eqs. (22) and (23), is capable of capturing their essentials.

7.2 Dynamic Stall

Figure 5, reproduced from Ref. 1, is representative of some of the principal results of Refs. 19 and 20. Here, angle of attack α is defined geometrically rather than kinematically as before. Flow-visualization pictures (obtained by means of the hydrogen-bubble technique, Ref. 19) of the airfoil oscillating about the quarter-chord clearly show that as the airfoil passes through a given angle of attack ($\alpha = 15^\circ$) with α increasing ($\dot{\alpha} > 0$), the flow has a very different character than it has as the airfoil passes through the same angle of attack with α decreasing ($\dot{\alpha} < 0$). The plots of C_L versus α at the bottom of Fig. 5 (Ref. 20) reveal that the lift variations likewise are very different, depending on whether α is increasing or decreasing. With α increasing, the instantaneous value of the lift

coefficient can significantly exceed its static value (more precisely, in our terms, the mean value of the lift response in the equilibrium state), while the opposite is true with α decreasing. Finally, the differences between lift coefficient with α increasing and decreasing are dramatically enhanced as the reduced frequency k is increased incrementally. Increasing k may be interpreted as increasing the magnitude of $\dot{\alpha}$, since with $\alpha = \alpha_m + \alpha_o \sin \omega t$ (Fig. 5) and $k = \omega c / 2V_o$, we have $|\dot{\alpha}c / 2V_o|_{\max} = \alpha_o k$.

Thus, both sign and magnitude of $\dot{\alpha}$ significantly influence the experimental results. These are important new features that a mathematical model ought to be capable of capturing.

Let us first take up the issue of aerodynamic bifurcation in the equilibrium state for the airfoil under study. As the angle of attack is increased incrementally, a critical value of α is reached ($\alpha < 10^\circ$) at which separation of the flow begins, signaled by the appearance of a singular point in skin friction on the upper surface near the leading edge. With an infinitesimal increase in α , the singular point splits to form a half-saddle point of separation followed by a half-saddle point of reattachment, creating the familiar leading-edge separation bubble. The onset of separation is designated a structural bifurcation, inasmuch as it signals a change in the topological structure of the flow (Refs. 1,22). The change occurs uniquely (the original flow structure no longer represents a possible solution of the flow equations) and the flow remains steady. At an angle of attack of about 12° , an inflexional flow profile within the bubble becomes unstable, and periodic vortex-shedding begins. This is a Hopf bifurcation and available evidence suggests that it is supercritical. Figure 6 is a bifurcation diagram of the event with amplitude of the periodic lift fluctuation, designated $C_L(\omega; \alpha(\tau))$, as ordinate. The mean value of the lift coefficient in the equilibrium state is shown in the same figure. To conform with the notation of Ref. 5, we designate it by $\bar{C}_L(\omega; \alpha(\tau))$. Beyond $\alpha = \alpha_c = 12^\circ$, the diversion of some of the available energy to maintain the fluctuation causes $\bar{C}_L(\omega; \alpha(\tau))$ to begin to drop away from the values it would have taken (dashed curve) if the original stationary flow had remained stable. Its variation with α is that of the measurements for the curves labeled "static" in the plots of C_L versus α in Fig. 5. In the absence of contrary evidence, we assume that no further aerodynamic bifurcations occur over the remaining range of angle of attack. Thus, Eqs. (22) and (23) should apply to the oscillatory motion illustrated in Fig. 5, with Eq. (23) required at $\alpha = \alpha_c = 12^\circ$.

Now let us examine the behavior of a typical lift response $C_L[\alpha(\xi); t, \tau]$ for the part of a cycle in which α is increasing ($\dot{\alpha} > 0$), and let us choose $\alpha(\tau)$ to be somewhat larger than $\alpha_c = 12^\circ$. During the portion of the motion ($0 < \xi < \tau$) before $\alpha(\xi)$ is constrained to remain constant at $\alpha(\tau)$, the conditions correspond to those depicted via flow visualization in the left-hand photograph of Fig. 5. Let us focus attention on the flow component parallel to the airfoil surface at a station near the leading edge at time ξ just prior to $\xi = \tau$. We choose to observe the flow in the same reference frame as the one in which the camera photographing the flow was situated. The origin of coordinates is fixed to the airfoil's axis of rotation, but, rather than rotating with the airfoil, the coordinates remain stationary relative to the oncoming wind. Within this reference frame, the instantaneous velocity profile at the fixed station may be viewed as having two components, as shown in Fig. 7(a). The first profile is the time-invariant mean profile which would exist alone were the airfoil to remain stationary at the instantaneous position $\alpha(\xi) = \alpha(\tau)$. It has a region of reversed flow typical of the velocity profiles within the separation bubble in the equilibrium state. The second profile represents velocity induced by the motion of the airfoil. Its magnitude at the airfoil surface is directly proportional to the angular velocity $\dot{\alpha}(\xi)$, and in particular, since $\xi = \tau$, it is proportional to $\dot{\alpha}(\tau)$. It is clear that with $\dot{\alpha}(\tau) > 0$, this contribution tends to diminish the region of separated flow, and more so as $\dot{\alpha}(\tau)$ is increased. The situation begins to change as soon as ξ exceeds τ , for then α is constrained to remain constant at $\alpha(\tau)$. As shown in Fig. 7(b), while the first profile remains the same, the second profile immediately changes, since the velocity at the wall must go to zero as soon as the airfoil stops moving. With increasing values of ξ this profile decays progressively, so that the zone of separated flow associated with the first profile gradually begins to appear. After a lapse of time, a separation profile emerges that is unstable to small perturbations. In response, the periodic fluctuation characteristic of vortex-shedding begins, and its amplitude grows toward the constant value corresponding to the amplitude of fluctuations in the equilibrium state at the chosen value of $\alpha(\tau) > \alpha_c$. We show this schematically in Fig. 8. The interval of ξ during which the onset of fluctuations is delayed is denoted by T ; clearly, the duration of T is directly proportional to the magnitude of $\dot{\alpha}(\tau)$. During the interval T , in which the flow over the upper surface of the airfoil remains more or less attached, the mean value of the lift coefficient tends toward a value representative of the attached-flow condition. A conservative estimate for it would be the value obtained from a computation based on inviscid flow theory; we indicate it on Fig. 8 as an upper bound (dashed line). After the fluctuation begins, the mean value of the lift coefficient begins to decline toward the value it reaches in the equilibrium state. Thus, with $\dot{\alpha}(\tau) > 0$, the instantaneous mean value of the lift coefficient can overshoot its "static" value; the extent depends directly on the length of the interval T , which in turn depends directly on the magnitude of $\dot{\alpha}(\tau)$.

The behavior of the analogous lift response $C_L[\alpha(\xi); t, \tau]$ for the part of a cycle in which α is decreasing ($\dot{\alpha} < 0$) is very much the reverse of the behavior just described. Let $\alpha(\tau)$ have the same value as before ($\alpha(\tau) > \alpha_c$), so that over the portion of the motion $0 < \xi < \tau$, conditions correspond to those depicted in the right-hand photograph of Fig. 5. In sharp contrast to the previous case with $\dot{\alpha} > 0$, the velocity profile induced by the airfoil's motion drastically extends the separated flow regime, particularly while the airfoil is moving, and the more so as $\dot{\alpha}(\tau)$ is increased negatively. After ξ exceeds τ , whereupon α is constrained to remain constant at $\alpha(\tau)$, the violent fluctuations begin to die down as the second profile dies away. In contrast to the previous case, there is no interval in which the airfoil is free of fluctuations. The time it takes for the violent fluctuations to die down to the level

representative of the equilibrium state is directly proportional to the magnitude of $|\dot{\alpha}(\tau)|$. During this interval, the mean value of the lift coefficient must be less than its "static" value; the extent of the difference again depends directly on the magnitude of $|\dot{\alpha}(\tau)|$. (We note in passing that large values of $|\dot{\alpha}(\tau)|$ may represent a class where Fréchet differentiability could depend on more than stability of flow in the equilibrium state. It may depend as well on the nature of the initial flow. The possibility will require further study.)

This explanation of events suggested by the form of modeling we have adopted would appear to be in full accord with experimental observations. With one potential reservation, we conclude that a mathematical model at the level of Eqs. (22) and (23) will be capable of incorporating the events we have described.

7.3 Applications to Dynamical Systems

We have seen that the amended mathematical model should be capable of describing the aerodynamic characteristics in all six forms of aerodynamic force (or moment) responses illustrated in Fig. 4. The forms involving aerodynamic bifurcations that result in time-dependent equilibrium states (sketches (d)-(f), Fig. 4) are of particular interest. Applied to problems involving dynamical maneuvers of aircraft, these forms will require extensive and novel experimentation. Problems in this category can be divided into two subclasses.

The first subclass of problems involves oscillatory motions about an equilibrium state at which a bifurcation occurs. This was the case in the dynamic stall experiment studied in the last section. Similar cases need to be studied in which the oscillatory motions are free, rather than forced, to determine how vortex-shedding frequencies may be modulated by the system's natural frequencies, causing, for example, "frequency lock-in." Another potentially important question is whether vortex-shedding (from, e.g., three-dimensional analogs of the airfoil's leading-edge separation bubble) may be a source of wing rock. Here, the phase relations between vortex-shedding on the two wing panels would appear to be a controlling factor.

The second subclass of problems involves oscillatory motions that remain within a space free of critical values of parameters. Here, simplifications in the modeling can be effected as previously (Ref. 8) in connection with time-invariant equilibrium states. Dynamical systems within this subclass of problems involving vortex-shedding from obstacles have inspired a large collection of literature under the category of "vortex-induced oscillations." Excellent surveys of the field have been published by Sarpkaya (Ref. 23) and very recently by Bearman (Ref. 24). The archetypal problem is the flexibly mounted cylinder immersed in a uniform oncoming stream. Applied to this problem, our approach to modeling yields a differential equation of the form

$$\ddot{h} + 2\mu\dot{h} + \kappa^2 h = B(\dot{h})\sin(\omega(\dot{h})t) \quad (25)$$

where the quantities h, μ, κ characterizing the spring-mass system are illustrated in Fig. 9(a). The forcing term, which expresses the contribution of the fluctuating lift due to vortex-shedding, is novel in that the amplitude B and frequency ω are coupled to the velocity \dot{h} on the left-hand side. Our modeling approach provides an explanation as follows: Amplitude B is a function of instantaneous Reynolds number (Fig. 9(b)), which makes it a function of total velocity $\sqrt{U_\infty^2 + \dot{h}^2}$. Strouhal number k , the dimensionless frequency of the fluctuating lift in the equilibrium state, is a constant, independent of Reynolds number in the range of velocities of interest (Fig. 9(b)). This makes ω , the actual frequency, a function of \dot{h} through

$$k = \frac{\omega d}{U_{\text{tot}}} = \frac{\omega d}{\sqrt{U_\infty^2 + \dot{h}^2}} = \text{const.} \quad (26)$$

We have found that Eq. (25) has a rich repertory of solutions, and appears to be capable of capturing distinctive features of the system's behavior (frequency lock-in, amplitude jumps, hysteresis) that careful experiments have revealed (Refs. 23,24). The equation's richness is easier to understand if we transform it into a set of autonomous first-order equations. There are three such equations, a number which suffices to permit solutions having "chaotic" behavior. We believe that study of simple systems such as this one can teach us how to perceive what otherwise might seem bizarre or even random behavior on the part of more complicated dynamical systems such as maneuvering aircraft.

8. CONCLUDING REMARKS

This paper is a continuation of our effort to provide a consistent formulation and theoretical method for studying nonlinear problems in flight dynamics. We have proposed a framework having parallel observational and mathematical components to facilitate study of the aerodynamic contribution to the flight-dynamics system. The observational component consists of the elements structure, change, chaos, and scale. These are complemented by elements of a mathematical framework featuring the language of topology and concepts drawn from the theory of nonlinear dynamical systems. Here, particular emphasis has been placed on the element of change and its mathematical representation by means of bifurcation theory.

Of the two principal realms of application of bifurcation theory to flight-dynamics studies, we have concentrated attention on its role in the mathematical modeling of the aerodynamic contribution to the

aircraft's equations of motion. Our study stressed a number of themes which led to the following conclusions:

1. Studies in nonlinear flight dynamics can be enhanced by linking them to the common features of parallel studies in various allied fields. The unifying medium is the rapidly growing body of theory underlying research in nonlinear dynamical systems. Exemplifying this, our approach to modeling, involving nonlinear indicial responses and generalized superposition integrals, was made compatible with one originating in the field of electrical circuits and systems, based on the use of nonlinear functional expansions. Operations involved in forming the indicial response were found to be equivalent to Fréchet differentiation. Invalidation of Fréchet differentiability proved to be the common element signaling the necessity of amending the mathematical model to accommodate potential discontinuous aerodynamic behavior. Aerodynamic bifurcation is one means of invalidating Fréchet differentiability.
2. Aerodynamic bifurcation can be defined as the replacement of an unstable equilibrium flow by a new stable equilibrium flow at a critical value of a parameter. Acceptance of a list of presuppositions permitted extrapolating our understanding of aerodynamic bifurcations from a framework involving bounded laminar flows governed by the incompressible Navier-Stokes equations to one involving unbounded turbulent flows governed by modeled evolution equations based on the compressible Navier-Stokes equations. The issue of aerodynamic bifurcations represents a convergence of interests for the fields of turbulence modeling and the modeling of aerodynamic responses. Amended to accommodate aerodynamic bifurcations, with one potential reservation the mathematical model should be capable of describing the aerodynamic characteristics of all six major subdivisions that we have postulated for the aerodynamic response.
3. Aerodynamic bifurcation is one of at least two means of invalidating Fréchet differentiability of the aerodynamic response. A theory for enumerating these means offers a possible alternative to bifurcation theory. Inasmuch as it would incorporate the latter, it would be a potentially more inclusive way of classifying aerodynamic phenomena that are important in flight-dynamics applications.

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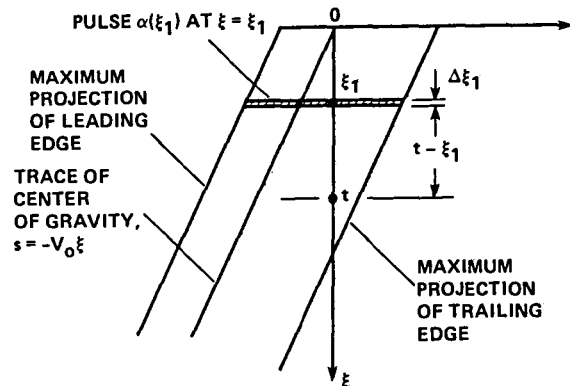


Fig. 1 Boundary conditions for lift at time t due to pulse in angle of attack at time ξ_1 .

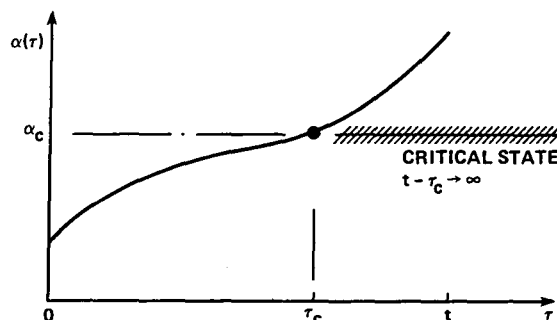


Fig. 2 Passage of angle of attack through a critical value α_c at which equilibrium flow becomes unstable.

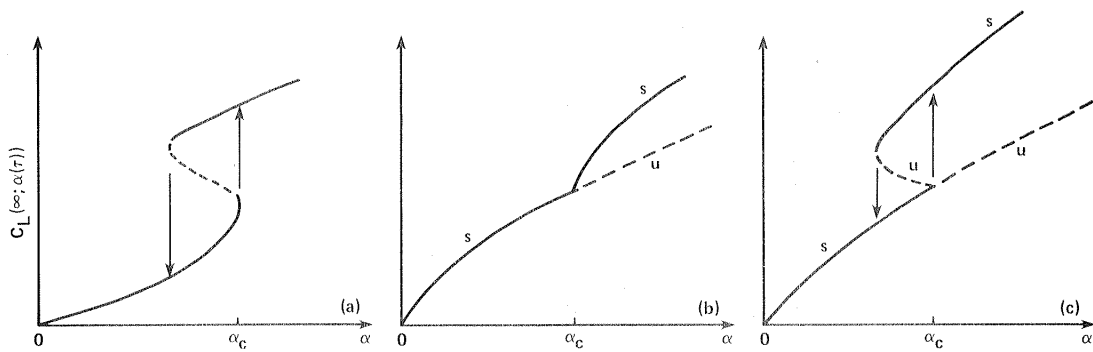


Fig. 3 Comparison of forms of time-invariant lift coefficients in the equilibrium state. (a) Fold; no bifurcation. (b) Supercritical bifurcation. (c) Subcritical bifurcation.

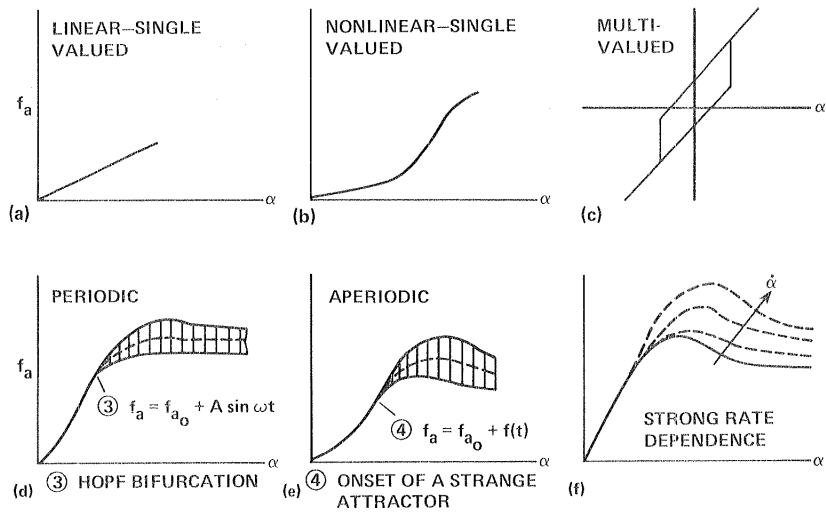
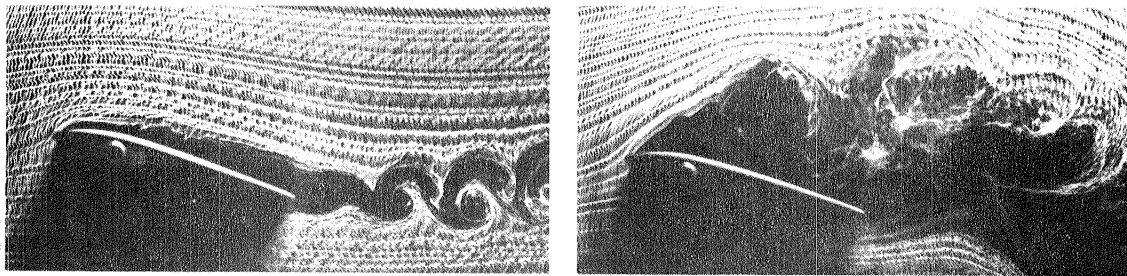


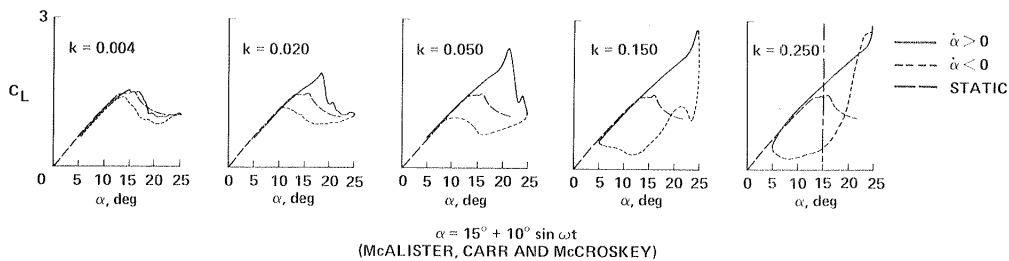
Fig. 4 Major subdivisions of aerodynamic forces and moments.



$\alpha = 15^\circ, \dot{\alpha} > 0, k = 0.25$

$\alpha = 10^\circ + 10^\circ \sin \omega t$
(McALISTER AND CARR)

$\alpha = 15^\circ, \dot{\alpha} < 0, k = 0.25$



$\alpha = 15^\circ + 10^\circ \sin \omega t$
(McALISTER, CARR AND McCROSKEY)

Fig. 5 Aerodynamics of an oscillating airfoil: dynamic stall.

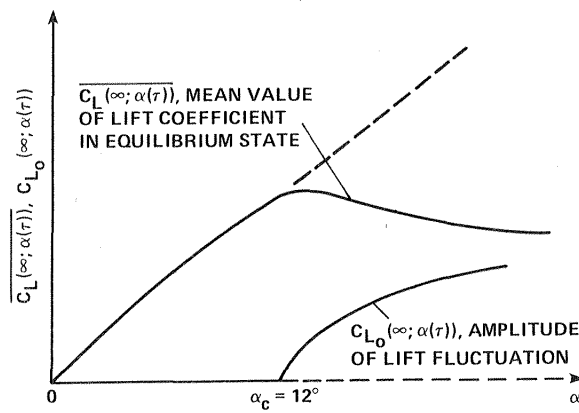


Fig. 6 Bifurcation diagram for amplitude of lift fluctuation.

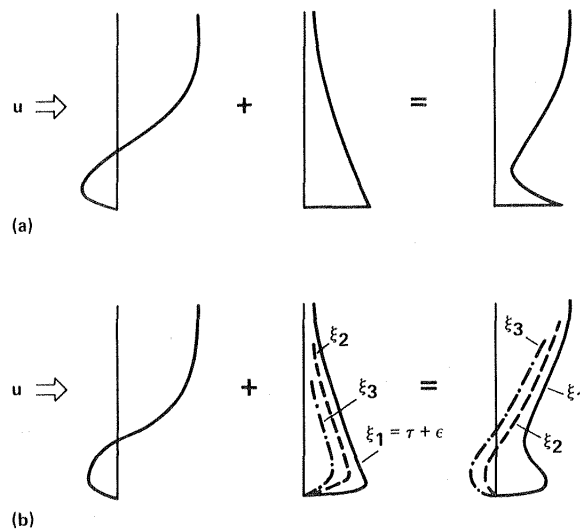


Fig. 7 Velocity profiles near leading edge. (a) $\xi < \tau$. (b) $\xi > \tau$.

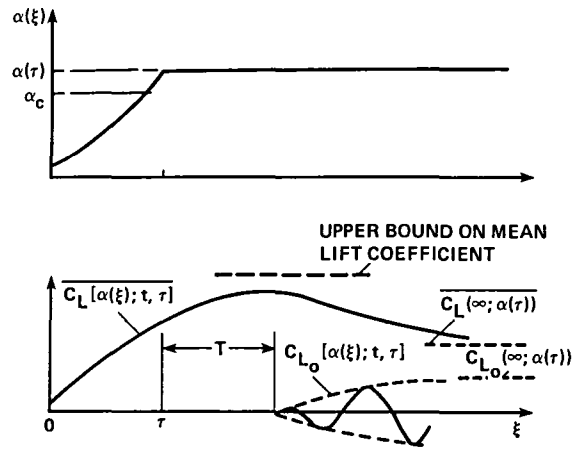
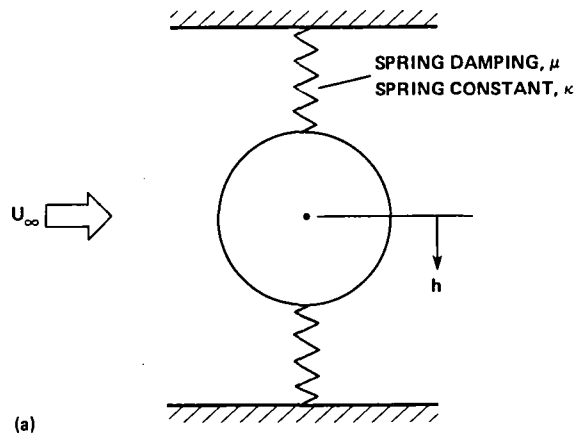
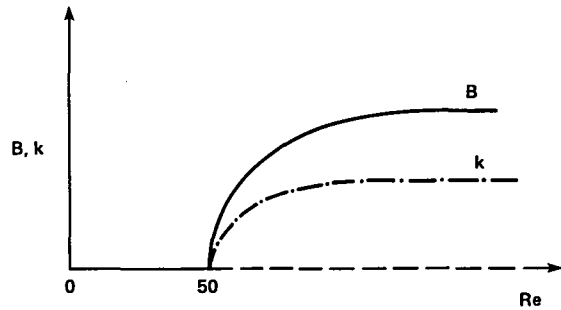


Fig. 8 Mean and fluctuating lift responses.



(a)



(b)

Fig. 9 Flexibly mounted cylinder. (a) Spring-mass system. (b) Bifurcation diagram for cylinder in uniform stream.

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16. Abstract Aerodynamic bifurcation is defined as the replacement of an unstable equilibrium flow by a new stable equilibrium flow at a critical value of a parameter. A mathematical model of the aerodynamic contribution to the aircraft's equations of motion is amended to accommodate aerodynamic bifurcations. Important bifurcations such as, for example, the onset of large-scale vortex-shedding are defined. The amended mathematical model is capable of incorporating various forms of aerodynamic responses, including those associated with dynamic stall of airfoils.					
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