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**Nonlinear Equations for  
Dynamics of Pretwisted  
Beams Undergoing Small  
Strains and Large Rotations**

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NONLINEAR EQUATIONS OF DYNAMICS OF PRETWISTED BEAMS UNDERGOING SMALL STRAINS AND LARGE ROTATIONS

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The equations as listed are erroneous: eq. 23b (p. 10), eqs. 32 and 33 (p. 12), eq. 39 (p. 13), eq. 65 (p. 18), and eqs. A3 and A4 (p. 22). The corrected equations follow:

$$M_{\alpha} = \delta_{\alpha\beta} I_{\beta} \kappa_{\beta} + e_{\alpha\beta\gamma} \left[ E_{\beta} (s' - 1) + \frac{B_{\beta}}{2} (\kappa_{\beta} - \theta')^2 + D_{\beta} \theta' (\kappa_{\beta} - \theta') \right] \quad (23b)$$

$$\begin{aligned} \underline{a}^{MI} = & \underline{a}^{F*I} + \underline{\alpha}^{FI} \times (\underline{x}_3 + \underline{u}) + \underline{\omega}^{FI} \times [\underline{\omega}^{FI} \times (\underline{x}_3 + \underline{u})] + 2\underline{\omega}^{FI} \times \underline{F}_{\underline{u}} + \underline{F}_{\underline{u}} \\ & + \underline{\alpha}^{PI} \times \underline{\xi} + \underline{\omega}^{PI} \times (\underline{\omega}^{PI} \times \underline{\xi}) \end{aligned} \quad (32)$$

$$\begin{aligned} \underline{a}^{MI} = & \underline{a}^{F*I} + \underline{\alpha}^{FI} \times (\underline{x}_3 + \underline{u} + \underline{\xi}) + \underline{\omega}^{FI} \times [\underline{\omega}^{FI} \times (\underline{x}_3 + \underline{u} + \underline{\xi})] + 2\underline{\omega}^{FI} \times \underline{F}_{\underline{u}} + \underline{F}_{\underline{u}} \\ & + 2\underline{\omega}^{FI} \times (\underline{\omega}^{PF} \times \underline{\xi}) + \underline{\omega}^{PF} \times (\underline{\omega}^{PF} \times \underline{\xi}) + \underline{\alpha}^{PF} \times \underline{\xi} \end{aligned} \quad (33)$$

$$\begin{aligned} -\delta W_{\text{body}} = & \underline{b}_1^F \cdot \int_0^{\ell} \delta \underline{u}_1 \left\langle m \{ \underline{a}^{F*I} - \underline{g} + \underline{\alpha}^{FI} \times (\underline{x}_3 + \underline{u}) + \underline{\omega}^{FI} \times [\underline{\omega}^{FI} \times (\underline{x}_3 + \underline{u})] + 2\underline{\omega}^{FI} \right. \\ & \times \underline{F}_{\underline{u}} + \underline{F}_{\underline{u}} \} + \underline{\alpha}^{FI} \times \underline{m}_{\underline{\xi}} + \underline{\omega}^{FI} \times (\underline{\omega}^{FI} \times \underline{m}_{\underline{\xi}}) + 2\underline{\omega}^{FI} \cdot \underline{m}_{\underline{\xi}} \underline{\omega}^{PF} - 2\underline{\omega}^{FI} \cdot \underline{\omega}^{PF} \underline{m}_{\underline{\xi}} \\ & + \underline{\alpha}^{PF} \times \underline{m}_{\underline{\xi}} + \underline{\omega}^{PF} \times (\underline{\omega}^{PF} \times \underline{m}_{\underline{\xi}}) \rangle dx_3 + \underline{b}_1^P \cdot \int_0^{\ell} \left( \frac{\partial \kappa_1}{\partial u_{\alpha}^i} \delta u_{\alpha}^i + \delta_{31} \delta \theta_3 \right) \langle \underline{m}_{\underline{\xi}} \\ & \times \{ \underline{a}^{F*I} - \underline{g} + \underline{\alpha}^{FI} \times (\underline{x}_3 + \underline{u}) + \underline{\omega}^{FI} \times [\underline{\omega}^{FI} \times (\underline{x}_3 + \underline{u})] + 2\underline{\omega}^{FI} \times \underline{F}_{\underline{u}} + \underline{F}_{\underline{u}} \} \\ & + i_3 \underline{\alpha}^{FI} - \underline{\alpha}^{FI} \cdot \underline{i} + (\underline{\omega}^{FI} \cdot \underline{i}) \times \underline{\omega}^{FI} + 2(\underline{\omega}^{FI} \cdot \underline{i}) \times \underline{\omega}^{PF} + (\underline{\omega}^{PF} \cdot \underline{i}) \times \underline{\omega}^{PF} \\ & + i_3 \underline{\alpha}^{PF} - \underline{\alpha}^{PF} \cdot \underline{i} \rangle dx_3 \end{aligned} \quad (39)$$

$$\delta\psi_{Rj}^R(t) = \left[ \frac{\partial \kappa_i}{\partial u_i^\alpha} (0, t) \frac{\delta q_{\alpha 2}}{\ell} + \delta_{3i} \delta q_{41} \right] C_{ij} (0, t) \quad (65)$$

$$\phi_\alpha = (-2e_{\alpha\beta 3} C_{3\beta} + \phi_3 C_{3\alpha}) / (1 + C_{33}) \quad (A3)$$

$$\phi_\alpha = (-2e_{\alpha\beta 3} u_\beta^i + \phi_3 u_\alpha^i) / (1 + s^i + u_3^i) \quad (A4)$$

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NONLINEAR EQUATIONS FOR THE DYNAMICS OF PRETWISTED BEAMS  
UNDERGOING SMALL STRAINS AND LARGE ROTATIONS

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and  
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SUMMARY

Nonlinear beam kinematics are developed and applied to the dynamic analysis of a pretwisted, rotating beam element. The common practice of assuming moderate rotations caused by structural deformation in geometric nonlinear analyses of rotating beams has been abandoned in the present analysis. The kinematic relations that describe the orientation of the cross section during deformation are simplified by systematically ignoring the extensional strain compared to unity in those relations. Open cross section effects such as warping rigidity and dynamics are ignored, but other influences of warp are retained. The beam cross section is not allowed to deform in its own plane. Various means of implementation are discussed, including a finite element formulation. Numerical results obtained for nonlinear static problems show remarkable agreement with experiment.

1. INTRODUCTION

It is now widely recognized that aeroelastic analysis of helicopter rotor blades, particularly of hingeless and bearingless rotor blades, requires the incorporation of kinematical nonlinearity (ref. 1). The main reason for this requirement is that the stability and response of such systems depend strongly, in some cases, on the coupling between bending and torsion motion. This important coupling cannot be obtained accurately without consideration of the kinematical nonlinearities.

A brief history of the developments of nonlinear equations of motion for rotating beams prior to 1974 is given in reference 1. These developments are generally concerned with slender beams, with the effects of shear deformation ignored. Since 1974, the major contributions to this subject have been those of Kaza and Kvaternik (ref. 2), and Rosen and Friedmann (ref. 3). The equations of motion developed in references 1-3 are very similar. In references 1 and 3 an ordering scheme is used that limits the kinematical development to moderate rotations. In reference 2 the nonlinearities are limited to the second degree in the displacement variables. These two different methods of specifying "moderate rotations" in references 1-3 are virtually equivalent for this problem.

In a general-purpose analysis, a single set of equations is desirable - one that is valid for all values of the equation parameters, within some range. When moderate rotations are assumed, situations can easily arise in which the solution violates the assumption of moderate rotations. One example is the case of a thin beam for which

the ratio of bending stiffnesses is small compared to unity. In order to avoid this problem, certain ad hoc modifications to the equations of reference 3 were introduced and were necessary in order to produce the excellent correlation obtained in reference 4 with experimental data for large displacements of an end-loaded cantilever. For example, magnitudes of the beam bending and torsion stiffnesses had to be specified before the equations could be put into final form for solution. Ideally, the magnitude of parameters in the equations for general-purpose analyses should not influence the equations themselves. Such an ideal is evidently not present in the ordering schemes of references 1 and 3 or in any arbitrary a priori restriction to second-degree nonlinearity, as in reference 2.

There are other shortcomings in the equations in references 1-3. For example, the effects of pretwist are not treated rigorously. An improved treatment of pretwist effects is presented in reference 5 for a simplified problem involving only torsion and axial displacement. Additional work is required to incorporate those analysis techniques into a general, nonlinear, bending-torsion-extension analysis. In reference 6 exact nonlinear kinematical relationships are developed and additional insight is presented concerning relationships among the equations of references 1-3. Finally, in reference 7 it is shown that inconsistencies are virtually unavoidable in ordering schemes based on displacements and rotations when the magnitude of the torsion rigidity is small compared to bending stiffnesses.

Because of the problems with kinematical limitations in the above approaches, it seems appropriate to model the kinematics of a slender beam without resorting to an ordering scheme on rotations or to arbitrary restrictions on degree of nonlinearity allowed in expressions involving displacement. This method would avoid some of the limitations of previous analyses and circumvent nonrigorous modifications such as were found necessary in reference 4. In the present work, the development in reference 6 serves as a foundation along with some important observations from references 8 and 9. Rather than develop the partial differential equations of motion for this problem, the objective is to develop a statement of the principle of virtual work for dynamic analysis of a rotating beam element with Euler-Bernoulli kinematics. This statement will serve as the basis of a Ritz-type modal model for an entire rotor blade or of a single, finite element of a rotor blade without having to consider the partial differential equations and the natural boundary conditions, which are available from the statement, if needed.

Although the results presented here are for the case in which shear deformation is neglected, the necessary relations for including shear deformation are included as part of the development. Similarly, with the appropriate constitutive law, effects such as orthotropy or anisotropy could be incorporated. The initial curvature of the elastic axis and effects associated with open cross sections could also be incorporated. The detailed development of these topics is reserved for future extensions of the present analysis. The present development proceeds as follows: The beam kinematics are developed in section 2 in such a way that the necessity for an ordering scheme is obviated. This section includes the development of strain-displacement relations and direction cosines of the nominal cross-sectional plane. In section 3 generalized forces caused by internal loads are developed from a consideration of the strain energy. Expressions for generalized forces caused by inertial and gravitational loads are developed in section 4 based on the work done by these loads through a virtual displacement. Generalized forces caused by a general set of applied distributed loads are developed in section 5 similar to the development in section 4. A finite element implementation is discussed in section 6. Finally,

numerical results obtained from a finite element calculation of nonlinear static equilibrium are presented in section 7.

## 2. KINEMATICAL DEVELOPMENT

In this section, the kinematics for the beam element are developed starting with the rigid-body motion of the cross-section plane as described in reference 6. Cross-section warp is then superimposed on the rigid-body motion to obtain the final displacement field. Next, the strain-displacement relations are developed from the displacement field. The extensional strain is then assumed to be small compared to unity. This assumption is used to simplify the orientation description and moment strains considerably, without sacrificing accuracy.

### 2.1 Development of Displacement Field

First, consider a straight-beam element with the associated coordinate systems shown in figure 1. It is assumed that the motion of the frame  $F$  is known in an inertial frame  $I$ . A set of dextral axes  $x_i$ ,  $i = 1,2,3$  is assumed to originate at  $F^*$ , the origin of  $F$ . The  $x_3$ -axis lies along the elastic axis of the beam element. Each unit vector  $\underline{b}_i^F$ ,  $i = 1,2,3$  is parallel to the corresponding axis  $x_i$ . The beam is assumed to be pretwisted so that the local-beam cross section is rotated by an angle  $\theta$  about the  $x_3$ -axis at any point on the  $x_3$ -axis. At  $x_3 = 0$ ,  $\theta$  is defined to be zero so that the major axis is parallel to the  $x_2$ -axis and the minor axis is parallel to the  $x_1$ -axis. Consider an arbitrary material point in the beam prior to deformation, denoted by  $M_0$ . The position vector of  $M_0$  with respect to  $F^*$  is

$$\underline{R}^{M_0 F^*} = x_3 \underline{b}_3^F + (\xi_1 \cos \theta - \xi_2 \sin \theta) \underline{b}_1^F + (\xi_1 \sin \theta + \xi_2 \cos \theta) \underline{b}_2^F \quad (1)$$

Denote the same material point in the deformed beam by  $M$ . The position vector of  $M$  with respect to  $F^*$  is

$$\underline{R}^{M F^*} = x_3 \underline{b}_3^F + u_i \underline{b}_i^F + \xi_\alpha \underline{b}_\alpha^P + \Psi(x_3) \lambda(\xi_1, \xi_2) \underline{b}_3^P \quad (2)$$

where repeated Latin indices imply summation from 1 to 3 and repeated Greek indices imply summation from 1 to 2.

The axes  $\xi_1$  and  $\xi_2$  are along principal axes for the local cross section at a point  $P^*$  on the elastic axis and remain so during deformation. The unit vectors  $\underline{b}_\alpha^P$  are parallel to the cross-section principal axes at  $P^*$ , which is located at the origin of a dextral system  $P$ . The unit vector  $\underline{b}_3^P$  is defined as  $\underline{b}_1^P \times \underline{b}_2^P$  and is thus normal to the cross section. The displacements  $u_i$ ,  $i = 1,2,3$ , are along  $\underline{b}_i^F$ , respectively. A warp displacement field has been added vectorially to the rigid-body component of the position following Wempner (ref. 9). The warp amplitude is  $\Psi$ , and  $\lambda$  is the cross-section warp function.

The basis vectors for the  $P$  axes are denoted in equation (1) by

$$\underline{b}_i^P = C_{ij} \underline{b}_j^F \quad (3)$$

where  $C_{ij}$  is a  $3 \times 3$  matrix whose elements may be specified by any of several sets of parameters (ref. 10). All three-parameter descriptions of the rotation have inherent singularities. Classical Euler orientation angles (called body-two orientation angles in reference 10) have singularities at values of certain angles equal to zero. The Tait-Bryan orientation angles (called body-three orientation angles in reference 10) have all singularities at  $90^\circ$ . This fact makes them more amenable to descriptions of rotation caused by structural deformation than Euler angles, since the case of no rotation would correspond to no deformation and Euler angles are singular at that condition (ref. 6). To avoid singularities altogether, at least four parameters are required to describe the rotation. One of the best known descriptions is the set of Euler parameters. Euler parameters enable rotation to be described with four scalar quantities upon which the direction cosine matrix elements depend quadratically. It is possible to eliminate one of the Euler parameters algebraically and derive the Rodrigues parameters. There is a singularity at  $180^\circ$  of rotation in any direction, but this is rarely a problem in deformable structures. The direction cosines are simple ratios of quadratic polynomials in the Rodrigues parameters. Furthermore, the inverse operation (i.e., given the direction cosine matrix, find the parameters) is trivial compared to the same operation with orientation angles (ref. 10).

In this paper, both Tait-Bryan orientation angles and Rodrigues parameters are considered. The primary development is executed with the orientation angles because of the simplicity of the result and the familiarity of the method. Rotations free of singularity up to  $90^\circ$  are completely acceptable in helicopter blade applications. For applications in which the  $90^\circ$  restriction is unacceptable, the kinematical development for Rodrigues parameters is given in the appendix.

The direction cosines are here expressed in terms of Tait-Bryan orientation angles  $\theta_i$ ,  $i = 1, 2, 3$  defined as follows. Let basis vectors  $\underline{b}_i^P$  be introduced beginning with  $\underline{b}_1^P$  aligned with  $\underline{b}_1^F$ . Then perform sequential rotations  $\theta_i$  about  $\underline{b}_i^P$ ,  $i = 1, 2, 3$  until  $\underline{b}_i^P$  is aligned with the principal axes of the deformed beam  $\underline{\xi}_i$ ,  $i = 1, 2, 3$ , as shown in figure 2. Other sequences of rotations are possible, but here  $\theta_1 - \theta_2 - \theta_3$  is used. Direction cosines of the local, deformed-beam, cross-section principal axes  $P$  with respect to  $F$  when expressed in matrix form are

$$C = \begin{bmatrix} c_2 c_3 & s_3 c_1 + s_1 s_2 c_3 & s_3 s_1 - c_1 s_2 c_3 \\ -c_2 s_3 & c_3 c_1 - s_1 s_2 s_3 & c_3 s_1 + c_1 s_2 s_3 \\ s_2 & -s_1 c_2 & c_1 c_2 \end{bmatrix} \quad (4)$$

where  $c_i = \cos \theta_i$  and  $s_i = \sin \theta_i$ . The displacement field is now completely specified although, as pointed out in reference 6, two of the three angles  $\theta_1$  and  $\theta_2$  can be eliminated if shear deformation is neglected.

## 2.2 Development of Strain-Displacement Relations

Now that the displacement field is determined in terms of  $u_i$  and  $\theta_i$ , it is a straightforward matter to calculate strain-displacement relations. There are several strain measures that could be used in this type of analysis. Almansi strain was used by Hodges and Dowell (ref. 1) in their preliminary development, and Green strain has been used by almost everyone else. For infinitesimal strains, however, differences among the common definitions of strain are small and are ignored in this



development. The intent here is to develop a set of strain-displacement relations that will be linear in elongations and shears, but unrestricted in rotations up to changes in orientation where singularities are encountered. Thus, it is immaterial whether it is Green strain or Almansi strain that serves as the starting point since the relations will be simplified for small strains anyway.

The independent variables of the vectors  $\underline{R}^{M_0 F^*}$  and  $\underline{R}^{MF^*}$  (namely,  $\xi_1$ ,  $\xi_2$ , and  $x_3$ ) constitute a nonorthogonal curvilinear coordinate system identical to the one used in reference 5. The steps followed in reference 5 to derive a set of strain displacement relations from these vectors are as follows:

1. Obtain the covariant base vectors for the undeformed state from equation (1).
2. Obtain the covariant metric tensor for the undeformed state from step 1.
3. Obtain the contravariant metric tensor for the undeformed state from step 2.
4. Obtain the relationship between a local Cartesian coordinate system and step 1 from step 3.
5. Obtain the covariant base vectors for the deformed state from equation (2).
6. Obtain the covariant metric tensor for the deformed state from step 5.
7. Obtain the Green strain tensor from half the difference between step 6 and step 2.
8. Transform the Green strain tensor from step 7 to the local Cartesian coordinate system using step 4.
9. Ignore elongations and shears with respect to unity in all strain components (i.e., discard all squares and products of elongations and shears, leaving each strain component to be a linear combination of elongations and shears).

The details of the algebra, although lengthy, are straightforward and are omitted in this report. The engineering strain components resulting from these operations are

$$\left. \begin{aligned} \epsilon_{31} &= \bar{\epsilon}_{31} + (\lambda_1 - \xi_2)(\kappa_3 - \theta') \\ \epsilon_{32} &= \bar{\epsilon}_{32} + (\lambda_2 + \xi_1)(\kappa_3 - \theta') \\ \epsilon_{33} &= \bar{\epsilon}_{33} + \xi_2 \kappa_1 - \xi_1 \kappa_2 + (\xi_1^2 + \xi_2^2)(\kappa_3 - \theta')^2/2 \\ &\quad + (\xi_2 \lambda_1 - \xi_1 \lambda_2)(\kappa_3 - \theta')\theta' + \lambda(\kappa_3 - \theta')' \end{aligned} \right\} \quad (5)$$

The strains at the reference axis, referred to as force strains, are given by

$$\bar{\epsilon}_{3i} = C_{ij}(\delta_{3j} + u_j') - \delta_{3i} \quad (6)$$

where  $\delta_{ij}$  is the Kronecker symbol. The curvature-like quantities, referred to as moment strains, are given by

$$\kappa_i = \omega_{ij} \theta_j' \quad (7)$$

where, in matrix form,

$$\omega = \begin{bmatrix} c_2 c_3 & s_3 & 0 \\ -c_2 s_3 & c_3 & 0 \\ s_2 & 0 & 1 \end{bmatrix} \quad (8)$$

Derivatives of the warp function are denoted by  $\lambda_\alpha = \partial\lambda/\partial\xi_\alpha$ . A restricted warp amplitude  $\Psi = \kappa_3 - \theta_1'$  is assumed here instead of the more general formulation in terms of  $\Psi$  explicitly as in reference 5. This assumption results in the neglect of transverse shear in the outer fibers of the beam. It is interesting to note that the force and moment strains are very similar to those of Reissner (ref. 8) except that the ones above are expressed in terms of Tait-Bryan orientation angles instead of Rodrigues parameters as in reference 8. The moment strains have the dimensions of curvature, but differ from curvature by a factor of  $s'$ , where  $s$  is the length coordinate along the deformed-beam elastic axis (ref. 6) and  $()'$  denotes the derivative with respect to  $x_3$ . The moment strains also closely resemble those of Wempner (ref. 9) developed for small deformation of arbitrarily curved beams.

It is possible to eliminate  $\theta_1$  and  $\theta_2$  from the analysis if shear deformation is ignored, resulting in an Euler-Bernoulli beam model. For this case the vector tangent to the beam elastic axis,  $\partial R^{MF*}/\partial s|_{\xi_\alpha=0}$ , must remain normal to the local, beam cross section during deformation. Thus, from equation (2)

$$\partial R^{MF*}/\partial s|_{\xi_\alpha=0} = [\delta_{3i}(\partial x_3/\partial s) + \partial u_i/\partial s] b_i^F = b_3^P \quad (9)$$

By virtue of equations (3) and (9)

$$C_{3i} = \delta_{3i} \frac{\partial x_3}{\partial s} + \frac{\partial u_i}{\partial s} \quad (10)$$

Since  $C_{ij}$  is orthonormal,  $C_{3i}C_{3i} = 1$ . If all derivatives are expressed with respect to  $x_3$  instead of  $s$ , an expression for  $s'$  is obtained:

$$s' = [u_1'^2 + u_2'^2 + (1 + u_3')^2]^{1/2} \quad (11)$$

From equations (10) and (4)

$$s_2 = u_1'/s' \quad (12a)$$

$$-s_1 c_2 = u_2'/s' \quad (12b)$$

$$c_1 c_2 = (1 + u_3')/s' \quad (12c)$$

The angles  $\theta_1$  and  $\theta_2$  can be eliminated from  $C_{ij}$  with the following relations obtained from equations (12)

$$s_1 = \frac{-u_2'}{(s'^2 - u_1'^2)^{1/2}} \quad (13a)$$

$$c_1 = \frac{1 + u_3'}{(s'^2 - u_1'^2)^{1/2}} = \frac{(s'^2 - u_1'^2 - u_2'^2)^{1/2}}{(s'^2 - u_1'^2)^{1/2}} \quad (13b)$$

$$s_2 = \frac{u_1'}{s'} \quad (13c)$$

$$c_2 = \frac{(s'^2 - u_1'^2)^{1/2}}{s'} \quad (13d)$$

Equations (12), when substituted into the expressions for force strains, yields

$$\left. \begin{aligned} \bar{\epsilon}_{3\alpha} &= 0 \\ \bar{\epsilon}_{33} &= s' - 1 \end{aligned} \right\} \quad (14)$$

Considerably more algebra is required to eliminate  $\theta_1$  and  $\theta_2$  from the moment strains if the above method is used. It is relatively simple, however, to write them directly from reference 6 in terms of derivatives with respect to  $s$  where  $( )^+ = \partial/\partial s( )$ .

$$\frac{\kappa_1}{s'} = \frac{u_1^{++} s_3}{(1 - u_1^{+2})^{1/2}} - \frac{(1 - u_1^{+2})^{1/2} c_3}{(1 - u_1^{+2} - u_2^{+2})^{1/2}} \left[ u_2^{++} + \frac{u_1^+ u_2^+ u_1^{++}}{1 - u_1^{+2}} \right] \quad (15a)$$

$$\frac{\kappa_2}{s'} = \frac{u_1^{++} c_3}{(1 - u_1^{+2})^{1/2}} + \frac{(1 - u_1^{+2})^{1/2} s_3}{(1 - u_1^{+2} - u_2^{+2})^{1/2}} \left[ u_2^{++} + \frac{u_1^+ u_2^+ u_1^{++}}{1 - u_1^{+2}} \right] \quad (15b)$$

$$\frac{\kappa_3}{s'} = \theta_3^+ - \frac{u_1^+}{(1 - u_1^{+2} - u_2^{+2})^{1/2}} \left[ u_2^{++} + \frac{u_1^+ u_2^+ u_1^{++}}{1 - u_1^{+2}} \right] \quad (15c)$$

Equations (15) may be expressed in terms of  $( )'$  quantities instead of  $( )^+$  quantities by equation (11) and lengthy algebra. This is unnecessary, however, in light of the simplifications in the next section.

### 2.3 Small-Strain Simplification of Kinematics

Because the moment strains are complicated, it is useful to simplify them through the derivation of a small-strain approximation. This is accomplished by neglecting the longitudinal strain of the elastic axis with respect to unity in the direction cosines as well as the moment strains. The expression for the longitudinal force strain  $s' - 1$  is already linear in the elongation of the elastic axis; the moment strains that are to be obtained are then independent of elongation of the elastic axis (i.e., independent of  $s'$ ). It is interesting to note that Reissner's moment strains differ from curvatures by a factor of  $s'$  and are independent of  $s'$ . When  $\theta_1$  and  $\theta_2$  are eliminated, additional  $s'$  terms are introduced into the

equations through equations (13). Simply setting  $s' = 1$  in all places that it occurs in the moment strains and direction cosines removes the dependence of these quantities on  $\bar{\epsilon}_{33}$ .

Another way to view this approximation is to expand the strain in a Taylor series with elongations caused by stretching of the elastic axis as the small parameter. This separates elongation into components on and off the elastic axis. When only the terms linear in elongations are retained, the result is equivalent to having set  $s' = 1$  in all quantities except  $\bar{\epsilon}_{33}$ . Thus strain is considered small compared to unity as in the example given in reference 6, p. 32. It is claimed in reference 11 that this sort of approximation invalidates the strain for cases other than inextensional. In reality, a simplification of this sort cannot significantly affect the accuracy of the mathematical model as long as the strains are small relative to unity, which they must be for applications of Hooke's law. The resulting simplifications in the derivation and in the final equations are substantial.

If  $s'$  is set equal to 1 in equations (13) and (15), the result is

$$\kappa_1 = \frac{u_1'' s_3}{(1 - u_1'^2)^{1/2}} - \frac{(1 - u_1'^2)^{1/2} c_3}{(1 - u_1'^2 - u_2'^2)^{1/2}} \left[ u_2'' + \frac{u_1' u_2' u_1''}{1 - u_1'^2} \right] \quad (16a)$$

$$\kappa_2 = \frac{u_1'' c_3}{(1 - u_1'^2)^{1/2}} + \frac{(1 - u_1'^2)^{1/2} s_3}{(1 - u_1'^2 - u_2'^2)^{1/2}} \left[ u_2'' + \frac{u_1' u_2' u_1''}{1 - u_1'^2} \right] \quad (16b)$$

$$\kappa_3 = \theta_3' - \frac{u_1'}{(1 - u_1'^2 - u_2'^2)^{1/2}} \left[ u_2'' + \frac{u_1' u_2' u_1''}{1 - u_1'^2} \right] \quad (16c)$$

and

$$s_1 = \frac{-u_2'}{(1 - u_1'^2)^{1/2}} \quad (17a)$$

$$c_1 = \frac{(1 - u_1'^2 - u_2'^2)^{1/2}}{(1 - u_1'^2)^{1/2}} \quad (17b)$$

$$s_2 = u_1' \quad (17c)$$

$$c_2 = (1 - u_1'^2)^{1/2} \quad (17d)$$

Note that the sign of the square root quantities becomes ambiguous when  $\theta_1$  or  $\theta_2$  exceeds  $90^\circ$ . Thus, it is imperative in this formulation to restrict orientation angles to less than  $90^\circ$ . Also note that  $\kappa_i$  and  $C_{ij}$  are now independent of  $u_3'$ ;  $s'$  must not be set equal to unity in the force strain  $\bar{\epsilon}_{33} = s' - 1$ .

Geometric boundary conditions are determined from specified  $u_i$  and  $C_{ij}$  at  $x_3 = 0$  and  $x_3 = \ell$ . Natural boundary conditions may be obtained from the complete statement of the principle of virtual work as developed in the next three sections.

### 3. DEVELOPMENT OF GENERALIZED FORCES CAUSED BY INTERNAL LOADS FROM STRAIN ENERGY

In this section, the internal loads for the beam element are developed from the strain-displacement relations from section 2 and a conventional constitutive law. The beam is assumed to be of such a configuration that open-cross-section effects, such as warping *rigidity*, are negligible, which is justified for rotor-blade cross sections. However, the effects of warping are retained in other parts of the internal loads development. This assumption is helpful in a finite element context since enforcing kinematical boundary conditions on the warp displacement field at finite element nodes is not possible in the general case of beams being joined at arbitrary positions and orientations with respect to one another.

The virtual work on the internal loads is obtained from the variation of the strain energy  $U$  as in (ref. 9)

$$\delta U = \int_0^L \iint_A (E \epsilon_{33} \delta \epsilon_{33} + G \epsilon_{3\alpha} \delta \epsilon_{3\alpha}) d\xi_1 d\xi_2 dx_3 \quad (18)$$

where

$$\left. \begin{aligned} \delta \epsilon_{33} &= \delta s' + \xi_2 \delta \kappa_1 - \xi_1 \delta \kappa_2 + [(\xi_1^2 + \xi_2^2)(\kappa_3 - \theta') + (\xi_2 \lambda_1 - \xi_1 \lambda_2) \theta'] \delta \kappa_3 + \lambda \delta \kappa_3' \\ \delta \epsilon_{31} &= (\lambda_1 - \xi_2) \delta \kappa_3 \\ \delta \epsilon_{32} &= (\lambda_2 + \xi_1) \delta \kappa_3 \end{aligned} \right\} \quad (19)$$

The expressions are greatly simplified if the variations of force and moment strains are written as

$$\delta s' = \frac{\partial s'}{\partial u_1'} \delta u_1' \quad (20)$$

$$\delta \kappa_i = \frac{\partial \kappa_i}{\partial u_1''} \delta u_1'' + \delta_{i3} \delta \theta_3' + \frac{\partial \kappa_i}{\partial u_1'} \delta u_1' + e_{i\alpha 3} \kappa_\alpha \delta \theta_3 \quad (21)$$

where  $e_{ijk}$  is the Levi-Cevita permutation symbol and

$$\left. \begin{aligned} \frac{\partial s'}{\partial u_1'} &= \frac{\delta_{3i} + u_i'}{s'} \\ \frac{\partial \kappa_\alpha}{\partial u_1''} &= \frac{C_{\alpha 2}}{C_{33}} ; \quad \frac{\partial \kappa_3}{\partial u_1''} = \frac{-C_{31}^2 C_{32}}{C_{33}(1 - C_{31}^2)} ; \quad \frac{\partial \kappa_i}{\partial u_2''} = \frac{-C_{i1}}{C_{33}} \\ \frac{\partial \kappa_1}{\partial u_1'} &= \frac{u_1''}{C_{33}^2} \left[ -\frac{C_{22} C_{32}}{1 - C_{31}^2} + \frac{C_{12} C_{31}}{C_{33}} \right] - \frac{u_2'' C_{11} C_{31} C_{32}^2}{C_{33}^3 (1 - C_{31}^2)} \end{aligned} \right\} \quad (22)$$

$$\left. \begin{aligned}
\frac{\partial \kappa_2}{\partial u_1'} &= \frac{u_1''}{C_{33}^2} \left[ \frac{C_{12} C_{32}}{1 - C_{31}^2} + \frac{C_{22} C_{31}}{C_{33}} \right] - \frac{u_2'' C_{21} C_{31} C_{32}^2}{C_{33}^2 (1 - C_{31}^2)} \\
\frac{\partial \kappa_3}{\partial u_1'} &= \frac{-u_1'' C_{31} C_{32} (2C_{33}^2 + C_{31}^2 - C_{31}^4)}{C_{33}^3 (1 - C_{31}^2)^2} - \frac{u_2'' (1 - C_{32}^2)}{C_{33}^3} \\
\frac{\partial \kappa_i}{\partial u_2'} &= -\frac{C_{i1}}{C_{33}^3} (u_1'' C_{31} + u_2'' C_{32})
\end{aligned} \right\} \begin{array}{l} (22) \\ (cont) \end{array}$$

Next, the strains are substituted, equations (5) with  $\bar{\epsilon}_{3i}$  given by equation (14),  $\kappa_i$  by equations (16), and the virtual strains given in equations (19) into the strain energy by equations (18). The resulting expression can be arranged by terms that multiply  $\delta s'$ ,  $\delta \kappa_i$ , and  $\delta \kappa_3'$ . These coefficients are denoted by  $F_3$ ,  $M_i$ , and  $B$ , respectively, which are given by

$$\left. \begin{aligned}
F_3 &= E_0 (s' - 1) + E_2 \kappa_1 - E_1 \kappa_2 + \frac{I_3}{2} (\kappa_3 - \theta')^2 + D_0 \theta' (\kappa_3 - \theta') \\
M_\alpha &= \delta_{\alpha\beta} I_\beta \kappa_\beta + e_{\alpha\beta 3} \left[ E_\beta (s' - 1') + \frac{B_\beta}{2} (\kappa_3 - \theta')^2 + D_\beta \theta' (\kappa_3 - \theta') \right] \\
M_3 &= \left[ J + I_3 (s' - 1) - e_{\alpha\beta 3} B_\alpha \kappa_\beta + \frac{B_3}{2} (\kappa_3 - \theta')^2 + \frac{3D_3 \theta'}{2} (\kappa_3 - \theta') + D_4 \theta'^2 \right] \\
&\quad \times (\kappa_3 - \theta') + [D_0 (s' - 1) - e_{\alpha\beta 3} D_\alpha \kappa_\beta] \theta' \\
B &= S_1 (\kappa_3 - \theta') + S_2 \theta'
\end{aligned} \right\} (23)$$

These quantities are the stress resultants given in terms of the displacements. The distributed axial force is  $F_3$ ; local distributed moments along  $\underline{b}_i^P$  are denoted by  $M_i$ . The distributed bimoment is  $B$ , which may be neglected for the class of beams considered herein. The section properties used in equation (23) are defined by the following integrals over the cross section.

$$\left. \begin{aligned}
E_0 &\equiv \iint_A E \, d\xi_1 \, d\xi_2 \quad ; \quad D_0 \equiv \iint_A E \, e_{\alpha\beta 3} \lambda_\alpha \xi_\beta \, d\xi_1 \, d\xi_2 \\
E_\alpha &\equiv \iint_A E \xi_\alpha \, d\xi_1 \, d\xi_2 \quad ; \quad D_\alpha \equiv \iint_A E \xi_\alpha \, e_{\beta\gamma 3} \lambda_\beta \xi_\gamma \, d\xi_1 \, d\xi_2 \\
I_1 &\equiv \iint_A E \xi_3^2 \, d\xi_1 \, d\xi_2 \quad ; \quad D_3 \equiv \iint_A E \xi_\gamma \xi_\gamma \, e_{\alpha\beta 3} \lambda_\alpha \xi_\beta \, d\xi_1 \, d\xi_2
\end{aligned} \right\} (24)$$

$$\begin{aligned}
I_2 &\equiv \iint_A E \xi_1^2 d\xi_1 d\xi_2 & ; & & D_4 &\equiv \iint_A E (e_{\alpha\beta\gamma} \lambda_\alpha \xi_\beta)^2 d\xi_1 d\xi_2 \\
B_\alpha &\equiv \iint_A E \xi_\alpha \xi_\beta \xi_\beta d\xi_1 d\xi_2 & ; & & J &\equiv \iint_A G [(\xi_1 + \lambda_2)^2 + (\xi_2 - \lambda_1)^2] d\xi_1 d\xi_2 \\
B_3 &\equiv \iint_A E (\xi_\alpha \xi_\alpha)^2 d\xi_1 d\xi_2 & ; & & \iint_A E \xi_1 \xi_2 d\xi_1 d\xi_2 &\equiv 0 & ; & I_3 &= I_1 + I_2 \\
S_2 &= \iint_A E \lambda (\xi_2 \lambda_1 - \xi_1 \lambda_2) d\xi_1 d\xi_2 \\
S_1 &= \iint_A E \lambda \xi_\alpha \xi_\alpha d\xi_1 d\xi_2 \\
\iint_A E \lambda \xi_\alpha d\xi_1 d\xi_2 &= 0 & ; & & \iint_A E \lambda d\xi_1 d\xi_2 &\equiv 0
\end{aligned} \tag{24}$$

(cont)

The strain energy is then

$$\begin{aligned}
-\delta W_{\text{int}} = \delta U = \int_0^L &\left[ \left( F_3 \frac{\partial s'}{\partial u'_\alpha} + M_i \frac{\partial \kappa_i}{\partial u'_\alpha} \right) \delta u'_\alpha + F_3 \frac{\partial s'}{\partial u'_3} \delta u'_3 + e_{\alpha\beta\gamma} M_{\alpha\kappa\beta} \delta \theta_3 \right. \\
&\left. + M_i \frac{\partial \kappa_i}{\partial u''_\alpha} \delta u''_\alpha + M_3 \delta \theta_3 \right] dx_3
\end{aligned} \tag{25}$$

A suitable discretization of  $u_i$  and  $\theta_3$  is sufficient to obtain generalized forces in a Ritz-type formulation.

#### 4. GENERALIZED FORCES CAUSED BY INERTIAL AND GRAVITATIONAL LOADS

In this section the generalized forces caused by inertial and gravitational loads (i.e., body forces) are developed for the beam element. The effects of cross-section warp on the body forces are negligible and are not considered. It is assumed that the gravity field  $\underline{g}$ , the velocity of point  $F^*$  in an inertial frame  $\underline{v}^{F^*I}$ , the acceleration of point  $F^*$  in an inertial frame  $\underline{a}^{F^*I}$ , the angular velocity and angular acceleration of frame  $F$  in an inertial frame  $\underline{\omega}^{FI}$  and  $\underline{\alpha}^{FI}$ , respectively, are given and defined in frame  $F$  as

$$\left. \begin{aligned}
\underline{g} &= g_{Fi-i}^F & ; & & \underline{a}^{F^*I} &= a_{Fi-i}^{F^*I, F} & ; & & \underline{v}^{F^*I} &= v_{Fi-i}^{F^*I, F} \\
\underline{\omega}^{FI} &= \omega_{Fi-i}^{FI, F} & ; & & \underline{\alpha}^{FI} &= \alpha_{Fi-i}^{FI, F}
\end{aligned} \right\} \tag{26}$$

We must find the acceleration and virtual displacement of the point M where the position of M with respect to F\* is governed by

$$\underline{R}^{MF*} = \underline{x}_3 + \underline{u} + \underline{\xi} \quad (27)$$

where

$$\left. \begin{aligned} \underline{x}_3 &= x_3 \underline{b}_3^F \\ \underline{u} &= u_i \underline{b}_i^F \\ \underline{\xi} &= \xi_{\alpha} \underline{b}_{-\alpha}^P \end{aligned} \right\} \quad (28)$$

The velocity of M in an inertial frame is then

$$\underline{v}^{MI} = \underline{v}^{F*I} + \underline{\omega}^{FI} \times (\underline{x}_3 + \underline{u}) + \underline{F}\dot{\underline{u}} + \underline{\omega}^{PI} \times \underline{\xi} + \underline{P}\dot{\underline{\xi}} \quad (29)$$

where  $\underline{F}(\dot{\quad})$  is a time derivative in F. Cross-section distortion  $\underline{P}\dot{\underline{\xi}}$  is to be neglected. The angular velocity  $\underline{\omega}^{PI}$  can be written as

$$\underline{\omega}^{PI} = \underline{\omega}^{PF} + \underline{\omega}^{FI} \quad (30)$$

Differentiation yields the angular acceleration

$$\underline{\alpha}^{PI} = \underline{\alpha}^{FI} + \underline{\omega}^{FI} \times \underline{\omega}^{PF} + \underline{\alpha}^{PF} \quad (31)$$

Now the acceleration can be written in terms of previously developed expressions as

$$\begin{aligned} \underline{a}^{MI} &= \underline{a}^{F*I} + \underline{\alpha}^{FI} \times (\underline{x}_3 + \underline{u}) + \underline{\omega}^{FI} \times [\underline{\omega}^{FI} \times (\underline{x}_3 + \underline{u})] + 2\underline{\omega}^{FI} \times \underline{F}\dot{\underline{u}} + \underline{F}\ddot{\underline{u}} + \underline{\alpha}^{PI} \\ &\quad + \underline{\alpha}^{PI} \times \underline{\xi} + \underline{\omega}^{PI} \times (\underline{\omega}^{PI} \times \underline{\xi}) \end{aligned} \quad (32)$$

Substitution of equations (30) and (31) into (32) yields

$$\begin{aligned} \underline{a}^{MI} &= \underline{a}^{F*I} + \underline{\alpha}^{FI} \times (\underline{x}_3 + \underline{u} + \underline{\xi}) + \underline{\omega}^{FI} \times [\underline{\omega}^{FI} \times (\underline{x}_3 + \underline{u} + \underline{\xi})] + 2\underline{\omega}^{FI} \times \underline{F}\dot{\underline{u}} + \underline{F}\ddot{\underline{u}} \\ &\quad + 2\underline{\omega}^{FI} \times (\underline{\omega}^{PF} \times \underline{\xi}) + \underline{\omega}^{PF} \times (\underline{\omega}^{PF} \times \underline{\xi}) + \underline{\omega}^{PF} \times (\underline{\omega}^{FI} \times \underline{\xi}) + \underline{\alpha}^{PF} \times \underline{\xi} \end{aligned} \quad (33)$$

The virtual displacement is obtained by the replacement of  $\underline{F}(\dot{\quad})$  with  $\underline{F}\delta(\quad)$ , a variation in the frame F, in equation (29) and the substitution of  $\underline{\delta R}$  and  $\underline{\delta\psi}$  for  $\underline{v}$  and  $\underline{\omega}$ , respectively, in equations (29) and (30), as in reference 6, which yields

$$\underline{\delta R}^{MI} = \underline{\delta R}^{F*I} + \underline{\delta\psi}^{FI} \times (\underline{x}_3 + \underline{u} + \underline{\xi}) + \underline{F}\delta\underline{u} + \underline{\delta\psi}^{PF} \times \underline{\xi} \quad (34)$$

where  $\underline{\delta R}^{F*I} = \delta R_{F_i b_i^F}^{F_i b_i^F}$  and  $\underline{\delta\psi}^{FI} = \delta\psi_{F_i b_i^F}^{F_i b_i^F}$ . These quantities reflect any nonprescribed motion of the frame in inertial space. The angular velocity and virtual rotation of the cross-section axes P in F (ref. 6) can be written as



$$\underline{\omega}^{PF} = \left( \frac{\partial \kappa_i}{\partial u_\alpha''} \dot{u}'_\alpha + \delta_{3i} \dot{\theta}_3 \right) \underline{b}_i^P \quad (35)$$

and

$$\underline{\delta\psi}^{PF} = \left( \frac{\partial \kappa_i}{\partial u_\alpha''} \delta u'_\alpha + \delta_{3i} \delta \theta_3 \right) \underline{b}_i^P \quad (36)$$

The angular acceleration of P in F,  $\underline{\alpha}^{PF}$ , is simply the time derivative in F of  $\underline{\omega}^{PF}$

$$\underline{\alpha}^{PF} = \underline{F} \cdot \underline{\omega}^{PF} = \left( \frac{\partial \kappa_i}{\partial u_\alpha''} \ddot{u}'_\alpha + \delta_{3i} \ddot{\theta}_3 + \frac{\partial^2 \kappa_i}{\partial u_\alpha'' \partial u_\beta''} \dot{u}'_\alpha \dot{u}'_\beta \right) \underline{b}_i^P \quad (37)$$

The generalized forces are now found from the virtual work

$$\begin{aligned} -\delta W_{\text{body}} &= \int_0^\ell \iint_A \rho (\underline{a}^{MI} - \underline{g}) \cdot \underline{\delta R}^{MI} d\xi_1 d\xi_2 dx_3 \\ &= \int_0^\ell \iint_A \rho \{ (\underline{a}^{MI} - \underline{g}) \cdot (\underline{\delta R}^{F*I} + \underline{F} \underline{\delta u}) + [(\underline{x}_3 + \underline{u} + \underline{\xi}) \\ &\quad \times (\underline{a}^{MI} - \underline{g})] \cdot \underline{\delta\psi}^{FI} + [\underline{\xi} \times (\underline{a}^{MI} - \underline{g})] \cdot \underline{\delta\psi}^{PF} \} d\xi_1 d\xi_2 dx_3 \end{aligned} \quad (38)$$

The generalized forces associated with frame motion are useful in multibody/finite element applications in which it is necessary to couple bodies together that are moving relative to each other. These generalized forces are the coefficients of  $\underline{\delta R}^{FI}$  and  $\underline{\delta\psi}^{FI}$  and can be obtained in a straightforward manner; the calculation of these forces is left to the reader. For the generalized forces associated with  $\delta u_i$ ,  $\delta u'_\alpha$ , and  $\delta \theta_3$ , the necessary ingredients are suitable discretizations for  $u_i$  and  $\theta_3$  and the virtual work, which is given by

$$\begin{aligned} -\delta W_{\text{body}} &= \underline{b}_i^F \cdot \int_0^\ell \delta u_i \left\langle m \{ \underline{a}^{F*I} - \underline{g} + \underline{\alpha}^{FI} \times (\underline{x}_3 + \underline{u}) + \underline{\omega}^{FI} \times [\underline{\omega}^{FI} \times (\underline{x}_3 + \underline{u})] + 2\underline{\omega}^{FI} \right. \\ &\quad \times \underline{F} \cdot \underline{\dot{u}} + \underline{F} \cdot \underline{\ddot{u}} \} + \underline{\alpha}^{FI} \times \underline{m}_\xi + \underline{\omega}^{FI} \times (\underline{\omega}^{FI} \times \underline{m}_\xi) + 2\underline{\omega}^{FI} \cdot \underline{m}_\xi \underline{\omega}^{PF} - 2\underline{\omega}^{FI} \cdot \underline{\omega}^{PF} \underline{m}_\xi \\ &\quad + \underline{\alpha}^{PF} \times \underline{m}_\xi + \underline{\omega}^{PF} \times (\underline{\omega}^{PF} \times \underline{m}_\xi) + \underline{\omega}^{PF} \cdot \underline{m}_\xi \underline{\omega}^{FI} - \underline{\omega}^{PF} \cdot \underline{\omega}^{FI} \underline{m}_\xi \left. \right\rangle dx_3 \\ &+ \underline{b}_i^P \cdot \int_0^\ell \left( \frac{\partial \kappa_i}{\partial u_\alpha''} \delta u'_\alpha + \delta_{3i} \delta \theta_3 \right) \left\langle \underline{m}_\xi \times \{ \underline{a}^{F*I} - \underline{g} + \underline{\alpha}^{FI} \times (\underline{x}_3 + \underline{u}) + \underline{\omega}^{FI} \right. \\ &\quad \times [\underline{\omega}^{FI} \times (\underline{x}_3 + \underline{u})] + 2\underline{\omega}^{FI} \times \underline{F} \cdot \underline{\dot{u}} + \underline{F} \cdot \underline{\ddot{u}} \} + i_{3\alpha}^{FI} - \underline{\alpha}^{FI} \cdot \underline{i} + (\underline{\omega}^{FI} \cdot \underline{i}) \times \underline{\omega}^{FI} \\ &\quad + 2(\underline{\omega}^{FI} \cdot \underline{i}) \times \underline{\omega}^{PF} + (\underline{\omega}^{PF} \cdot \underline{i}) \times \underline{\omega}^{FI} + (\underline{\omega}^{PF} \cdot \underline{i}) \times \underline{\omega}^{PF} + i_{3\alpha}^{PF} - \underline{\alpha}^{PF} \cdot \underline{i} \left. \right\rangle dx_3 \end{aligned}$$

where the section integrals are defined as

$$\left. \begin{aligned} m &\equiv \iint_A \rho \, d\xi_1 \, d\xi_2 ; & i_3 &\equiv \iint_A \rho \underline{\xi} \cdot \underline{\xi} \, d\xi_1 \, d\xi_2 \\ \underline{m}_\xi &\equiv \iint_A \rho \underline{\xi} \, d\xi_1 \, d\xi_2 ; & \underline{i} &\equiv \iint_A \rho \underline{\xi} \underline{\xi} \, d\xi_1 \, d\xi_2 \end{aligned} \right\} \quad (40)$$

and

$$\underline{F} \dot{\underline{u}} = \dot{u}_i b_i^F ; \quad \underline{F} \ddot{\underline{u}} = \ddot{u}_i b_i^F \quad (41)$$

## 5. DISTRIBUTED APPLIED LOADS

In this section, generalized forces caused by distributed applied loads are simply stated for completeness. The applied force  $\underline{F}$  and moment  $\underline{M}$  act on the elastic axis so that the virtual work is

$$\begin{aligned} -\delta W_{\text{applied loads}} &= -\int_0^\ell \{ \underline{F} \cdot (\delta \underline{R}^{F*I} + \underline{F} \delta \underline{u}) + \underline{M} \cdot (\delta \underline{\psi}^{FI} + \delta \underline{\psi}^{PF}) \\ &\quad + [(\underline{x}_3 + \underline{u}) \times \underline{F}] \cdot \delta \underline{\psi}^{FI} \} dx_3 \end{aligned} \quad (42)$$

or

$$\begin{aligned} -\delta W_{\text{applied loads}} &= -\int_0^\ell \underline{M} \cdot \left( \frac{\partial \kappa_i}{\partial u_\alpha''} \delta u_\alpha' + \delta_{3i} \delta \theta_3 \right) b_i^P dx_3 - \int_0^\ell \underline{F} \cdot \underline{F} \delta \underline{u} dx_3 \\ &\quad - \delta \underline{R}^{F*I} \cdot \int_0^\ell \underline{F} dx_3 - \delta \underline{\psi}^{FI} \cdot \int_0^\ell [\underline{M} + (\underline{x}_3 + \underline{u}) \times \underline{F}] dx_3 \end{aligned} \quad (43)$$

As with the inertial loads, the generalized forces can be obtained from the virtual work and any suitable discretization of  $u_i$  and  $\theta_3$ .

## 6. VARIOUS SCHEMES OF IMPLEMENTATION

The total virtual work may be written as

$$\delta U - \delta W_{\text{body}} - \delta W_{\text{applied loads}} = 0 \quad (44)$$

where the various components of the total virtual work may be obtained from equations (25), (30), and (43). The kinematical variables  $u_i$ ,  $i = 1, 2, 3$ , and  $\theta_3$  may

be discretized in several ways. One way is to use a set of admissible functions in a Ritz-type analysis based on equation (44) where all the frame variables are prescribed and  $\frac{\delta R^F}{\delta R^F} I$  and  $\frac{\delta \psi^F}{\delta \psi^F} I$  are zero. A variation on that type of development would be to allow  $\frac{\delta R^F}{\delta R^F} I$  and  $\frac{\delta \psi^F}{\delta \psi^F} I$  to be nonzero, thus collecting frame (i.e., rigid body) forces and moments from equations (39) and (43) directly. In this case there would be prescribed and nonprescribed components of each of the kinematical quantities in equation (26). There are clearly other ways of using the frame motion to advantage.

If a finite-element discretization is used, it should be based on variable-order shape functions (refs. 12 and 13) to avoid poorly approximating the geometric stiffening term which, in this analysis, is calculated from the longitudinal strain  $s' - 1$ . If  $u_3$  is crudely approximated, this term will be inaccurate. It should be noted that in a redundant structure undergoing finite deformation, the geometric stiffening effect *must* be calculated from the strain-displacement relations.

In a finite-element implementation the frame motion and forces may be used to advantage when coupling elements together that are defined in different moving coordinate systems, such as at the interface between rotating and nonrotating components of a helicopter or at a hinge.

In figure 3, the beam element is shown with the frame F and two nodes R and T at the root and tip of the beam, respectively. The displacement and orientation of the beam cross section as a function of  $x_3$  is represented by the displacement and rotation of the nodes R and T and by a variable number of generalized coordinates that are kinematically uncoupled from nodal translations and rotations. The beam displacements at the root and tip, then, must be determined from the nodal displacements.

The bending displacements  $u_1$  and  $u_2$  are to be expanded in  $C^1$ -type functions  $\psi_i$ , and  $u_3$  and  $\theta_3$  in  $C^0$ -type functions  $\beta_i$ , namely

$$\left. \begin{aligned} u_\alpha &= \sum_{i=1}^{N_\alpha+1} q_{\alpha i}(t) \psi_i(x) \\ u_3 &= \sum_{i=1}^{N_3+1} q_{3i}(t) \beta_i(x) \\ \theta_3 &= \sum_{i=1}^{N_4+1} q_{4i}(t) \beta_i(x) \end{aligned} \right\} \quad (45)$$

where  $x = x_3/\ell$ ,  $0 \leq x \leq 1$ , and  $N_i$  and  $N_4$  are the orders of the shape function polynomials,  $N_\alpha \geq 3$  and  $N_3, N_4 \geq 1$ . The generalized coordinates  $q_{ji}$  with  $j = 1, 2, 3$  have the dimensions of length and  $q_{4i}$  is dimensionless. The boundary conditions on the functions are

$$\left. \begin{aligned} u_\alpha(0,t) &= q_{\alpha 1}(t) ; & u_\alpha(\ell,t) &= q_{\alpha 3}(t) \\ u_3(0,t) &= q_{31}(t) ; & u_3(\ell,t) &= q_{32}(t) \\ \theta_3(0,t) &= q_{41}(t) ; & \theta_3(\ell,t) &= q_{42}(t) \end{aligned} \right\} \quad (46)$$

and on the first derivatives

$$u'_\alpha(0,t) = q_{\alpha 2}(t)/\ell ; \quad u'_\alpha(\ell,t) = q_{\alpha 4}(t)/\ell \quad (47)$$

The standard linear displacement functions satisfy these conditions for  $\beta$ :

$$\beta_1 = 1 - x$$

$$\beta_2 = x$$

and the standard cubic displacement functions satisfy these conditions for  $\Psi$ :

$$\left. \begin{aligned} \Psi_1 &= 1 - 3x^2 + 2x^3 \\ \Psi_2 &= x - 2x^2 + 2x^3 \\ \Psi_3 &= 3x^2 - 2x^3 \\ \Psi_4 &= -x^2 + x^3 \end{aligned} \right\} \quad (49)$$

If  $N_\alpha$  exceeds 3 or  $N_3, N_4$  exceed 1, the following higher-order shape functions allow extra generalized coordinates to be introduced (ref. 13) but still to fulfill the above end conditions

$$\begin{aligned} \beta_{n+3} &= x(1-x)G_n(5,3,x) \\ \Psi_{n+5} &= x^2(1-x)^2G_n(9,5,x) \end{aligned} \quad n = 0, 1, \dots \quad (50)$$

where  $G_n$  is a Jacobi polynomial (ref. 14). At the ends, the displacement and rotation of the nodes are related to generalized coordinates from simple kinematics. Let

$$\underline{u}^R = u_{Ri}^R \underline{b}_{-i}^R ; \quad \underline{u}^T = u_{Ti}^T \underline{b}_{-i}^T \quad (51)$$

$$\underline{b}_{-i}^R = \underline{b}_{-i}^F ; \quad \underline{b}_{-i}^T = C_{ij}^{TR} \underline{b}_{-j}^R \quad (52)$$

where  $C^{TR}$  depends on the pretwist  $\theta(\ell)$  of the beam element at the tip of the element.

$$C^{TR} = \begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (53)$$

where  $s_\theta = \sin \theta(\ell)$  and  $c_\theta = \cos \theta(\ell)$ . At the root of the element, the displacements of the node and beam are identical

$$u_{Ri}^R \underline{b}_{-i}^R = q_{i1} \underline{b}_{-i}^F = q_{i1} \underline{b}_{-i}^R \quad (54)$$

or

$$u_{Ri}^R = q_{i1} \quad (55)$$

Similarly, the rotations must be the same. The direction cosines of  $R'$  with respect to its undeformed position  $R$ ,  $C^{R'R}$  are

$$C^{R'R}(t) = C(0,t) \quad (56)$$

At the tip, the displacements of the node and beam are identical

$$u_{Ti-i}^T b_i^T = q_{13} b_{-1}^F + q_{23} b_{-2}^F + q_{32} b_{-3}^F \quad (57)$$

or

$$\begin{Bmatrix} u_{T1}^T \\ u_{T2}^T \\ u_{T3}^T \end{Bmatrix} = C^{TR} \begin{Bmatrix} q_{13} \\ q_{23} \\ q_{32} \end{Bmatrix} \quad (58)$$

and the rotations of the tip node  $C^{T'R}$  are identical to those of the beam tip  $C(\ell,t)$

$$C^{T'R} = C^{T'T}(t)C^{TR} = C(\ell,t) \quad (59)$$

Equations (56) and (59) each represent nine equations, but only three are independent. Three preferred elements to equate are  $C_{21}$ ,  $C_{31}$ , and  $C_{32}$  since

$$\left. \begin{aligned} C_{21} &= -(1 - u_1'^2)^{1/2} \sin \theta_3 \\ C_{31} &= u_1' \\ C_{32} &= u_2' \end{aligned} \right\} \quad (60)$$

and are thus easily expressed in terms of  $q$ 's at the root and tip. The use of equations (46), (47), and (60) yields for the root, from equation (56)

$$\left. \begin{aligned} \frac{q_{\alpha 2}}{\ell} &= C_{3\alpha}^{R'R} \\ q_{41} &= -\sin^{-1} \left[ \frac{C_{21}^{R'R}}{(1 - C_{31}^{R'R^2})^{1/2}} \right] \end{aligned} \right\} \quad (61)$$

and for the tip, from equation (59)

$$\frac{q_{\alpha 4}}{\ell} = C_{3\alpha}^T R$$

$$q_{42} = -\sin^{-1} \left[ \frac{C_{21}^T R}{(1 - C_{31}^T R^2)^{1/2}} \right] \quad (62)$$

The forces and moments at the root and tip nodes must be determined from the generalized forces at the beam root and tip, respectively. First note that the virtual displacement and virtual rotation of the root node must be identical to those of the beam root

$$\left. \begin{aligned} \underline{\delta u}^R(t) &= \underline{\delta u}(0,t) \\ \underline{\delta \psi}^R(t) &= \underline{\delta \psi}(0,t) \end{aligned} \right\} \quad (63)$$

where  $\underline{\delta \psi} = \underline{\delta \psi}^{PF}$  (see eq. 36). Similarly at the tip

$$\left. \begin{aligned} \underline{\delta u}^T(t) &= \underline{\delta u}(\ell,t) \\ \underline{\delta \psi}^T(t) &= \underline{\delta \psi}(\ell,t) \end{aligned} \right\} \quad (64)$$

The relations for the virtual displacements are simple first variations of equations (55) and (58). The rotations are more involved. The use of equation (36) evaluated at the root yields

$$\delta \psi_{Ri}^R(t) = \left[ \frac{\partial \kappa_i}{\partial u_\alpha} (0,t) \frac{\delta q_{\alpha 2}}{\ell} + \delta_{3i} \delta q_{4:} \right] C_{ij}(0,t) \quad (65)$$

where  $\underline{\delta \psi}^R = \delta \psi_{Ri-i}^R$ . In matrix form

$$\begin{Bmatrix} \delta q_{12} \\ \delta q_{22} \\ \delta q_{41} \end{Bmatrix} = R(0,t) \begin{Bmatrix} \delta \psi_{R1}^R \\ \delta \psi_{R2}^R \\ \delta \psi_{R3}^R \end{Bmatrix} \quad (66)$$

where

$$R(x_3, t) = \begin{bmatrix} 0 & \ell C_{33} & -\ell C_{32} \\ -\ell C_{33} & 0 & \ell C_{31} \\ 0 & \frac{C_{32}}{1 - C_{31}^2} & \frac{C_{33}}{1 - C_{31}^2} \end{bmatrix} \quad (67)$$

A similar relation holds for the beam tip

$$\begin{Bmatrix} \delta q_{14} \\ \delta q_{24} \\ \delta q_{42} \end{Bmatrix} = R(\ell, t) C^{RT} \begin{Bmatrix} \delta \psi_{T1}^T \\ \delta \psi_{T2}^T \\ \delta \psi_{T3}^T \end{Bmatrix}$$

The virtual work of forces and moments at the root and tip nodes must equal the virtual work caused by generalized forces at the root and tip of the beam

$$\begin{aligned} \underline{F}^R \cdot \underline{\delta u}^R &= Q_{i1} \delta q_{i1} \\ \underline{M}^R \cdot \underline{\delta \psi}^R &= Q_{\alpha 2} \delta q_{\alpha 2} + Q_{41} \delta q_{41} \\ \underline{F}^T \cdot \underline{\delta u}^T &= Q_{\alpha 3} \delta q_{\alpha 3} + Q_{32} \delta q_{32} \\ \underline{M}^T \cdot \underline{\delta \psi}^T &= Q_{\alpha 4} \delta q_{\alpha 4} + Q_{42} \delta q_{42} \end{aligned} \quad (68)$$

so that

$$\begin{aligned} F_{Ri}^R &= Q_{i1} \\ \begin{Bmatrix} F_{T1}^T \\ F_{T2}^T \\ F_{T3}^T \end{Bmatrix} &= C^{TR} \begin{Bmatrix} Q_{13} \\ Q_{23} \\ Q_{32} \end{Bmatrix} \\ \begin{Bmatrix} M_{R1}^R \\ M_{R2}^R \\ M_{R3}^R \end{Bmatrix} &= R^T(0, t) \begin{Bmatrix} Q_{12} \\ Q_{22} \\ Q_{41} \end{Bmatrix} \\ \begin{Bmatrix} M_{T1}^T \\ M_{T2}^T \\ M_{T3}^T \end{Bmatrix} &= C^{TR} R^T(\ell, t) \begin{Bmatrix} Q_{14} \\ Q_{24} \\ Q_{42} \end{Bmatrix} \end{aligned} \quad (69)$$

where  $R^T$  is the transpose of  $R$  and the  $Q$ 's are the generalized forces from the discretization of the beam (i.e., coefficients of the  $\delta q$ 's).

The element mass matrix may be written explicitly from equation (39) by collecting coefficients of  $\underline{F}\ddot{u}$  and  $\underline{\alpha}^{PF}$ . After simplification, and ignoring rows and columns associated with frame degrees of freedom, the mass matrix can be obtained from

$$\begin{Bmatrix} \delta u_i \\ \delta u'_\alpha \\ \delta \theta_3 \end{Bmatrix}^T \begin{bmatrix} m\delta_{ij} & c_{ki} e_{kj\alpha} \frac{\partial \kappa_j}{\partial u''_\beta} m_\alpha & m_1 C_{2i} - m_2 C_{1i} \\ i_3 \frac{\partial \kappa_3}{\partial u''_\alpha} \frac{\partial \kappa_3}{\partial u''_\beta} + i_\gamma \frac{\partial \kappa_\gamma}{\partial u''_\alpha} \frac{\partial \kappa_\gamma}{\partial u''_\beta} & i_3 \frac{\partial \kappa_3}{\partial u''_\alpha} \\ \text{symmetric} & i_3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_j \\ \ddot{u}'_\beta \\ \ddot{\theta}_3 \end{Bmatrix} \quad (70)$$

To obtain the discretized matrix substitute the shape functions, equation (45), into equation (70) and integrate over the element using, for example, Gauss-Legendre quadrature. The dimension of the discretized matrix will depend on the number of internal degrees of freedom. Its contribution to the system in terms of nodal degrees of freedom is straightforward and can be left to the reader to determine. Similarly, elements of the gyroscopic matrix can be calculated.

These equations can be programmed either for finite-element computation as they are or written in a more explicit matrix form. When the equations are linearized about static equilibrium determined from nonlinear static equations, a convenient approach is to calculate the mass and gyroscopic matrices explicitly as above and solve for the stiffness matrix by numerically perturbing the total static generalized force.

## 7. RESULTS

In this section, two sets of numerical results are presented along with corresponding experimental data. The numerical results were obtained by exercising a preliminary version of a multibody/finite-element program presently under development in which the beam element formulation outlined in section 6 is implemented. The examples were set up for calculation using only the properties given in this report.

The first set of data concerns a cantilevered beam loaded with a tip weight (ref. 15). The properties are tabulated in table 1, and the experimental configuration is described in detail in reference 15. The beam was modeled with one element and a sufficiently large number of polynomials to achieve convergence. Results for transverse tip deflections  $u_1$  and  $u_2$  and tip rotation  $\theta_3$  are shown in figures 4-6 along with experimental data. The agreement is good, much better than in reference 15. This good agreement is achieved, however, without ad hoc modifications of the equations based on the values of beam stiffnesses as is done in reference 4.

The second set of data concerns a cantilevered rotating beam in axial airflow. The air flowing down through the rotor plane is induced by the thrust at collective pitch setting  $\theta_0$  for a two-bladed rotor. Bending and torsion moments were measured near the beam root for variable thrust settings. The program was set up to calculate air loads based on a quasi-steady aerodynamic formulation by Greenberg (ref. 16) and on uniform induced inflow determined from momentum theory. (The experimental configuration is described in a forthcoming publication by Sharpe (Sharpe, D. L.: An Experimental Investigation of the Flap-Lag-Torsion Aeroelastic Stability of a Small-Scale Hingeless Helicopter Rotor in Hover. NASA TP, to be published in 1985). The rotor properties are tabulated in tables 2 and 3.) The rotor blade was modeled with two uniform elastic segments and rigid masses for the



remainder of the structure. Results for a droop angle of  $0^\circ$  and precone angles of  $0^\circ$  and  $5^\circ$  are shown in figures 7 and 8. The agreement, again, is very good.

## CONCLUSIONS

Nonlinear beam kinematics for small strains and large rotations have been developed and applied to the dynamic analysis of a pretwisted, rotating beam element. There are no explicit restrictions on rotation caused by deformation in these equations—only the extensional strain of the elastic axis is required to be small relative to unity. The only restriction on the magnitudes of the orientation angles used in describing the cross-section orientation is that they remain less than  $90^\circ$ . For applications of the kinematics where larger rotations may be encountered, a method of overcoming the restriction on the magnitude of rotation, which utilizes Rodrigues parameters, is presented in the appendix.

In order to be applicable to all existing rotor/hub configurations in helicopters, the analysis needs to be incorporated into a hybrid multibody/finite-element program. This incorporation is under development. Useful future extensions include constitutive equations for composite beams and effects of shear deformation, warping rigidity, and initial curvature.

Ames Research Center  
National Aeronautics and Space Administration  
Moffett Field, California 94035, January 18, 1985

## APPENDIX

In this appendix the direction cosines and moment strains are expressed in terms of Rodrigues parameters. To remove the sign ambiguities of the development in the text, the direction cosines should be left in terms of  $u_i'$ ,  $i = 1, 2, 3$  so that

$$C_{3i} = (\delta_{3i} + u_i')/s' \quad (A1)$$

This is not necessary in the text because the use of orientation angles limits the rotations to be less than  $90^\circ$ ; furthermore, it makes the computation slightly more involved. For the use of Rodrigues parameters, we make use of similar relationships derived in references 8 and 10. The direction cosines in terms of Rodrigues parameters  $\phi_i$  are

$$C_{ij} = [(1 - \phi_\ell \phi_\ell / 4) \delta_{ij} + \phi_i \phi_j / 2 + e_{ijk} \phi_k] / (1 + \phi_m \phi_m / 4) \quad (A2)$$

After much algebraic manipulation,  $\phi_\alpha$  can be eliminated in terms of the third row of C

$$\phi_\alpha = (2e_{\alpha\beta 3} C_{3\beta} + \phi_3 C_{3\alpha}) / (1 + C_{33}) \quad (A3)$$

The use of equation (A1) then yields

$$\phi_\alpha = (2e_{\alpha\beta 3} u_\beta' + \phi_3 u_\alpha') / (1 + s' + u_3') \quad (A4)$$

which goes to infinity only when the beam rotations due to deformation reach  $180^\circ$ .

The moment strains, as simplified for a straight beam based on those of Reissner (ref. 8), are

$$\kappa_i = [(\delta_{ij} + e_{ijk} \phi_k / 2) \phi_k'] / (1 + \phi_\ell \phi_\ell / 4) \quad (A5)$$

which can be expressed in terms of  $u_i'$ ,  $\phi_3$ ,  $u_i''$ , and  $\phi_3'$  by differentiation and substitution of equation (A4) into equation (A5). It should be noted that the reference basis used by Reissner corresponds to the principal axes of the local undeformed beam, instead of the principal axes of the root used herein. Equation (A5) thus differs somewhat from Reissner's equation (49) in reference 8.

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TABLE 1.- CANTILEVERED BEAM LOADED WITH TIP WEIGHT

Property	Value	
	Metric	Standard
E	$7.2919 \times 10^{10} \text{ N/m}^2$	$(10.576 \times 10^6 \text{ lb/in.}^2)$
G	$3.022 \times 10^{10} \text{ N/m}^2$	$(4.383 \times 10^6 \text{ lb/in.}^2)$
$\rho$	$2807 \text{ kg/m}^3$	$(0.1014 \text{ lb/in.}^3)$
$I_1$		$(Ec^3t/12)$
$I_2$		$(Ect^3/12)$
c	1.270 cm	(0.4999 in.)
t	.3178 cm	(0.1251 in.)
L	50.76 cm	(19.985 in.)

TABLE 2.- ROTOR BLADE PROPERTIES (NACA 0012 AIRFOIL WITH TWO BLADES)

Rotor diameter, m . . . . .	1.923
Blade length (L), m . . . . .	.870
Hub offset, %R . . . . .	9.51
Chord, cm . . . . .	8.64
Taper . . . . .	0
Twist . . . . .	0
Maximum tip Reynolds number .	600,000

TABLE 3.- ROTOR BLADE STIFFNESS DISTRIBUTION

Inboard station, r/R	Outboard station, r/R	I <sub>2</sub> •		I <sub>1</sub> •		J•	
		10 <sup>3</sup> N-m <sup>2</sup>	(10 <sup>6</sup> lb-in. <sup>2</sup> )	10 <sup>3</sup> N-m <sup>2</sup>	(10 <sup>6</sup> lb-in. <sup>2</sup> )	10 <sup>3</sup> N-m <sup>2</sup>	(10 <sup>4</sup> lb-in. <sup>2</sup> )
0	0.0192	57.3831	(20.000)	57.3831	(20.0000)	562.4	(1960)
.0192	.0430	.4834	(.1635)	.6178	(.2153)	.0094	(0.0327)
.0430	.0456	62.8345	(21.9000)	62.8345	(21.9000)	562.4	(1960)
.0456	.0555	78.0409	(27.2000)	78.0409	(27.2000)	568.1	(1980)
.0555	.0608	52.2188	(18.2000)	52.2186	(18.2000)	208.9	(728.0)
.0608	.0634	.8607	(.30000)	86.9555	(30.3070)	51.6	(180.0)
.0634	.0951	.6955	(.24240)	62.6135	(21.8230)	47.6	(166.0)
.0951	1.000	.0160	(.00689)	.34378	(.11982)	.0508	(.1769)

TABLE 4.- BLADE MASS PROPERTY DISTRIBUTION

Inboard station, r/R	Outboard station, r/R	Mass/length, kg/m	Polar moment of inertia/length, kg-m <sup>2</sup> /m
0.0185	0.0215	5.214	--
.0215	.0374	.0214	--
.0374	.0407	5.418	5.827×10 <sup>-3</sup>
.0407	.0440	10.010	7.073×10 <sup>-3</sup>
.0440	.0456	12.745	4.715×10 <sup>-3</sup>
.0456	.0555	9.969	6.317×10 <sup>-3</sup>
.0555	.0608	5.265	1.468×10 <sup>-3</sup>
.0608	.0634	2.663	2.082×10 <sup>-3</sup>
.0634	.0951	2.429	2.082×10 <sup>-3</sup>
.0951	1.0000	.343	2.062×10 <sup>-4</sup>

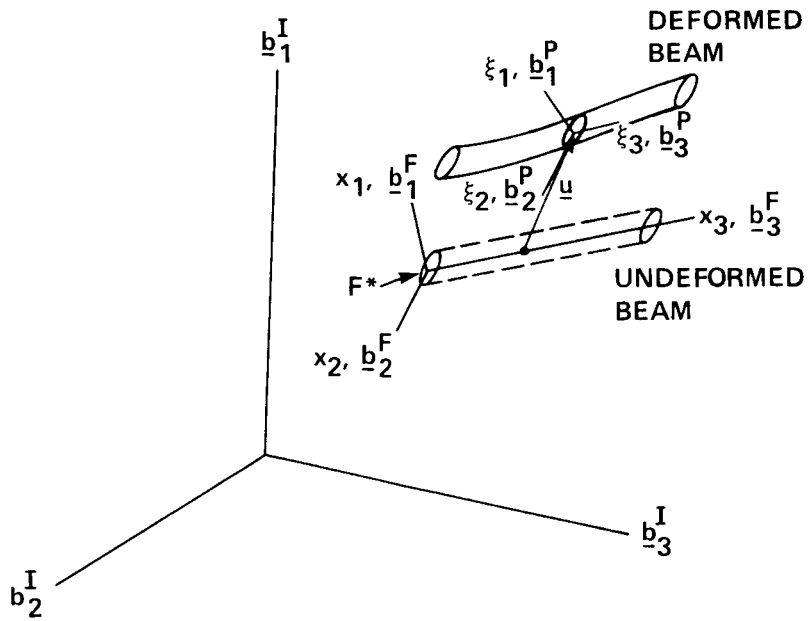


Figure 1.- Schematic of beam and associated coordinate systems.

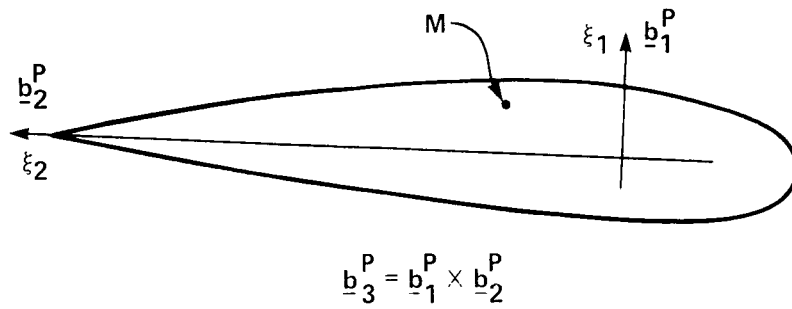


Figure 2.- Beam cross-section coordinate systems.

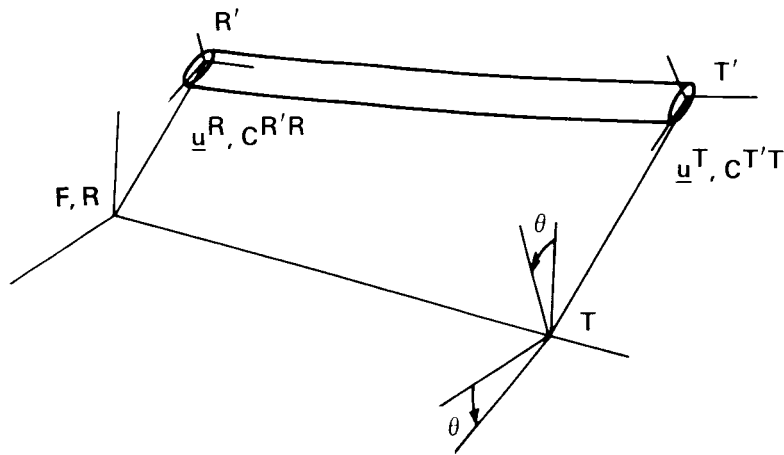


Figure 3.- The frame F, root node R, and tip node T for a beam element showing deflections  $\underline{u}^R$  and  $\underline{u}^T$  and change of orientation  $C^{R'R}$ ,  $C^{T'T}$  with respect to the initial positions and orientations.

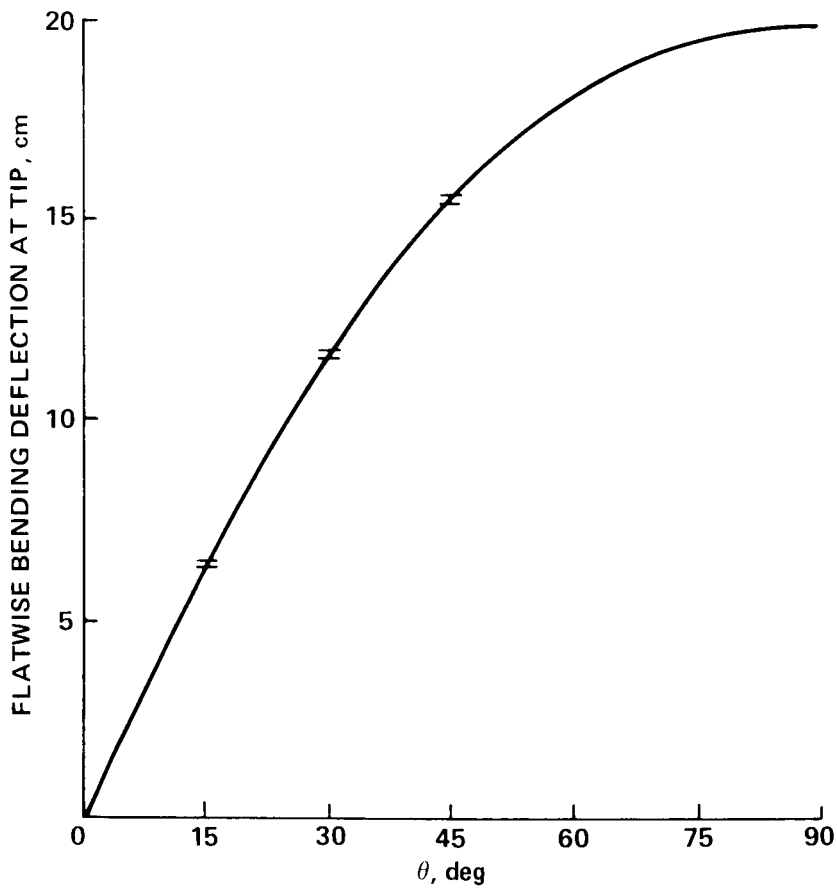


Figure 4.- Flatwise bending deflection at tip (cm).



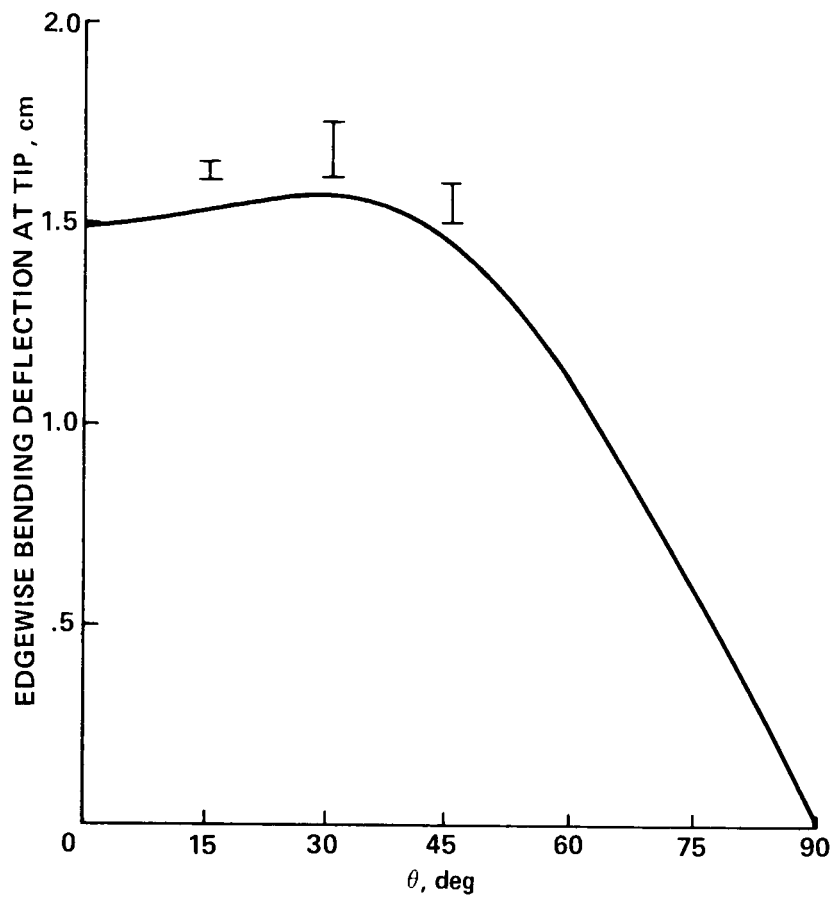


Figure 5.- Edgewise bending deflection at tip (cm).

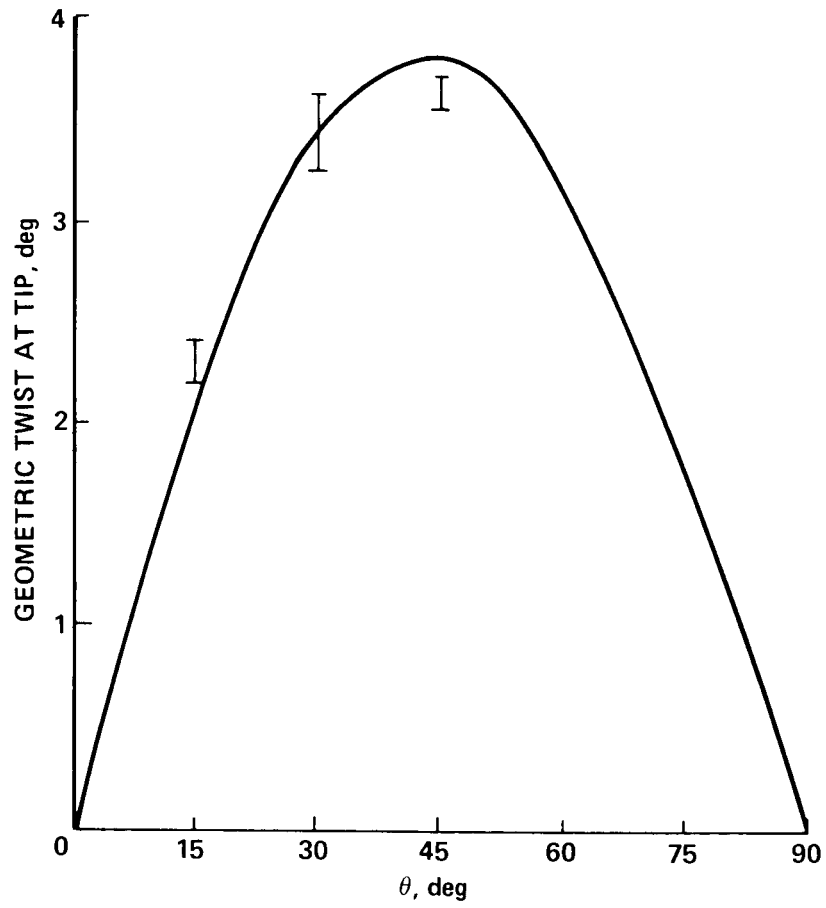


Figure 6.- Geometric twist at tip (deg).

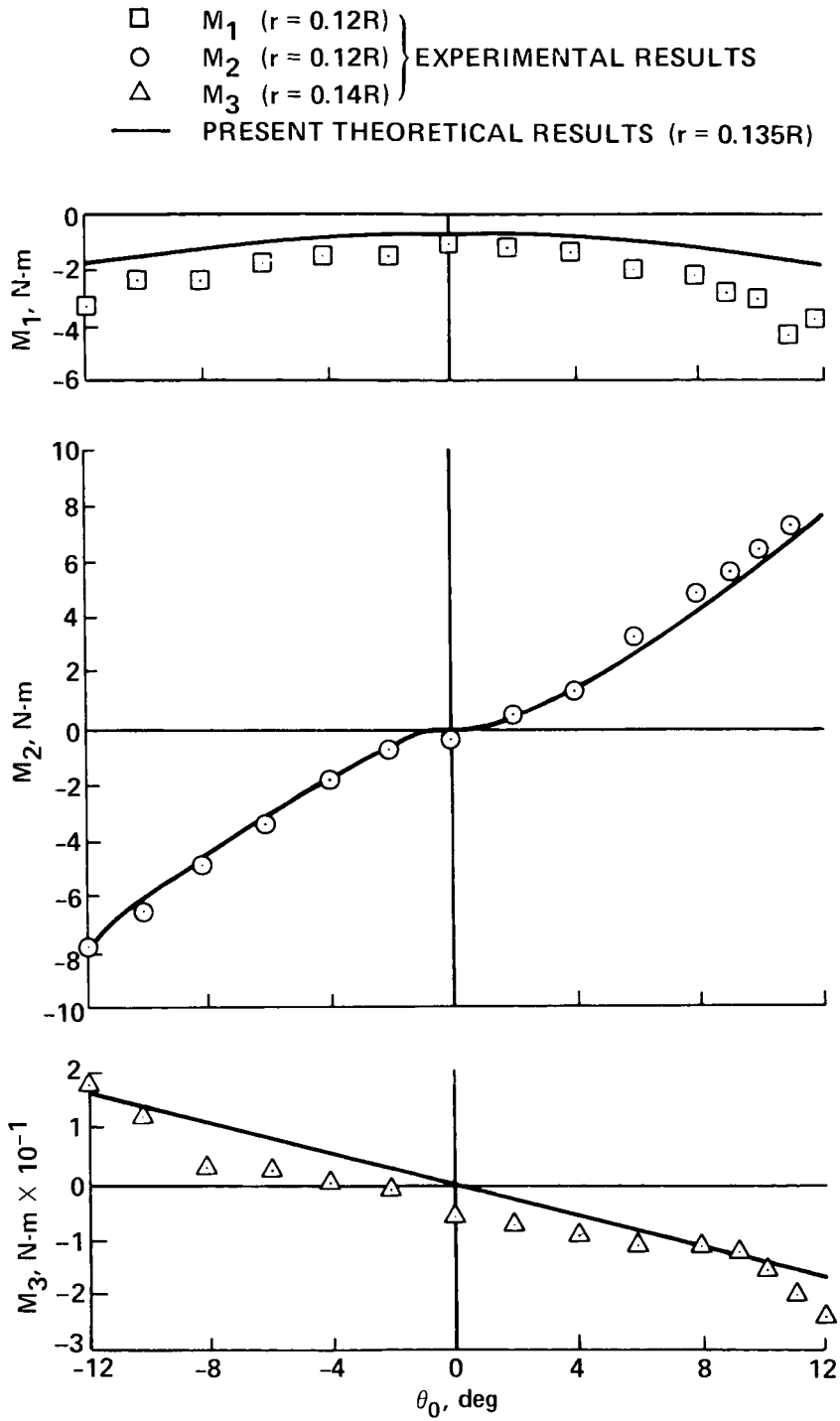


Figure 7.- Measured steady blade moments versus blade pitch angle: precone =  $0^\circ$ , droop =  $0^\circ$ , 1000 rpm, soft pitch flexure.

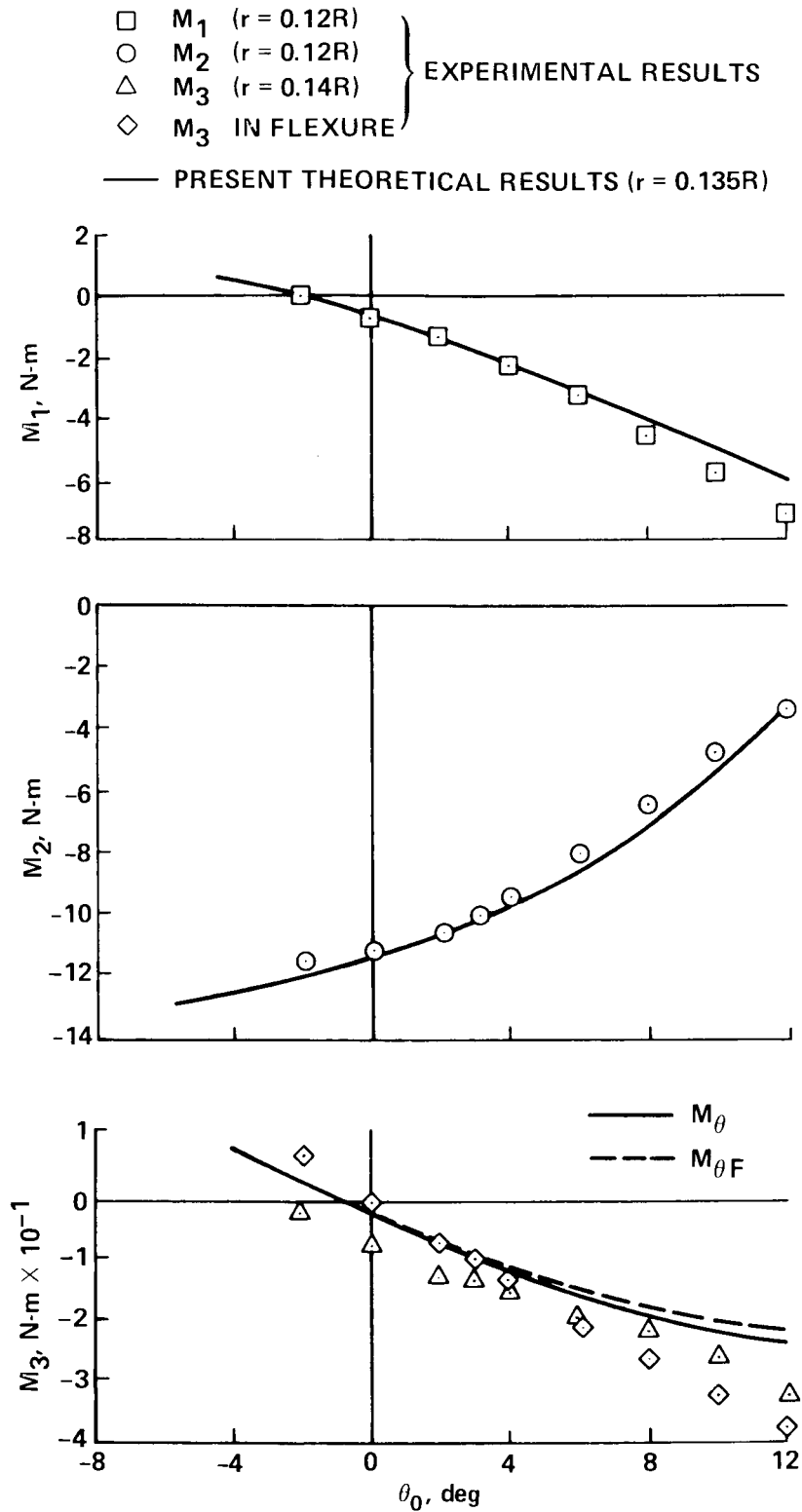


Figure 8.- Measured steady blade moments versus blade pitch angle: precone = 5°, droop = 0°, 1000 rpm, soft pitch flexure.

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16. Abstract  Nonlinear beam kinematics are developed and applied to the dynamic analysis of a pretwisted, rotating beam element. The common practice of assuming moderate rotations caused by structural deformation in geometric nonlinear analyses of rotating beams has been abandoned in the present analysis. The kinematic relations that describe the orientation of the cross section during deformation are simplified by systematically ignoring the extensional strain compared to unity in those relations. Open cross section effects such as warping rigidity and dynamics are ignored, but other influences of warp are retained. The beam cross section is not allowed to deform in its own plane. Various means of implementation are discussed, including a finite element formulation. Numerical results obtained for nonlinear static problems show remarkable agreement with experiment.					
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