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STUDY OF MESHING OF BEVELED GEARS WITH NORMALLY DECREASING ARC TEETH

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16. Abstract

The meshing of beveled gears is studied by two methods: the direct and inverse approaches. Gear wheels with teeth of equal height are studied, as well as wheels with normally-decreasing arc teeth. Different coordinate systems are utilized to plot the determining the rotation of the originating gear wheel and the meshing line of the gear wheel being cut. Matrices are used to determine the equations of the originating surfaces and the unit vectors of the normals to these originating surfaces. A calculation example is also given.

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STUDY OF MESHING OF BEVELED GEARS WITH NORMALLY DECREASING ARC TEETH

F. L. Litvin and Go Kay

We know of two versions of beveled gears, cut on machine tools /28* of the Gleason type: a) with teeth of equal height; b) with teeth of normally decreasing height.

The cutting scheme of the teeth of the former type of gear is based on meshing of the cut gear with a flat originating gear. The gear wheels, cut in this manner, are conjugate (the transmission of rotation takes place with a constant gear ratio), the surfaces of the teeth have point contact with different radii of the cutting heads, and the working line on the surface of the tooth — the geometric point of contact of this surface — has a direction perpendicular to the direction of the longitudinal line of the tooth [1].

The basis of the cutting scheme of gear wheels of the latter type is meshing of the cut gear wheel with a non-planar originating gear. The tooth surfaces of such gear wheels are not conjugate, and the transmission of rotation is accomplished with a variable gear ratio. The contact area is diagonal, since the working line on the side surface of the tooth of the gear wheel is not orthogonal to the longitudinal line of the tooth if the cutting of the teeth is carried out without special correction of the alignment. On the convex side of the gear wheel teeth, the working line begins at the tip of the inner part of the toothed ring and ends at the head of the outer part of the toothed ring. On the concave side of the teeth, the working line has an inverse nature. These characteristics of meshing are known in gear-cutting practice, and are examined in a number of studies [2,3].

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*Numbers in the margin indicate pagination in the foreign text.

The present study is devoted to the analytic study of meshing of gear wheels of the latter type, which are viewed as non-conjugate, for which we utilize the methods for solving direct and inverse problems, as set forth in the monograph of F. L. Litvin [1].

The authors obtained dependences which make it possible to achieve conjugation of the surfaces of the teeth with their contact at the midpoint. The results of the conducted study given below coincide with the experimental data which are well-known in the practice of gear cutting. The value of the utilized method of study consists, in the opinion of the authors, of the possibility of the objective evaluation of the utilized methods of correction of the adjustment of the machine tools and the determination of /29 the optimal magnitude of the adjustment parameters being corrected, which ensure the absence of obliqueness of the contact and minimal error of the gear ratio of the gear wheels. This method of study becomes especially effective with the utilization of mathematical computers for the calculations.

1. Direct and Inverse Problems of the Study of Gear Wheel Meshings

Known during the solution of the direct problem are the schematic of the toothed mechanism, the absolute movements of both gear wheels and the surface of the teeth of one of the wheels; it is necessary to determine the surface of the teeth of the other wheel.

Known during the solution of the inverse problem are the schematic of the toothed mechanism and the surfaces of the teeth of both wheels; it is necessary to find the law of movement in the form of a function which associates the positions of the gear wheels. At the same time, during the solution of this problem, in the case of point contact, one may find the working lines of the surfaces of the teeth of both wheels and the line of meshing.

In the system of coordinates, associated with the originating gear, let the equations of both originating surfaces Σ_1^p and Σ_2^p, utilized for forming the surfaces of the teeth of the beveled gear wheels with arc teeth, be known:

$$r_{u_1} = r_{u_1}(u_1,\ \vartheta_1);$$
$$r_{u_2} = r_{u_2}(u_2,\ \vartheta_2).$$

According to the method of solution of the direct problem, we will find the equations of the surfaces Σ_1 and Σ_2 of the teeth of the gear wheels, which envelop the originating surfaces, in the movable systems S_1 and S_2:

$$\begin{aligned} r_1 &= r_1(u_1,\ \vartheta_1,\ \psi_1); \\ f_1(u_1,\ \vartheta_1,\ \psi_1) &= 0 \end{aligned} \qquad (1)$$

and

$$\begin{aligned} r_2 &= r_2(u_2,\ \vartheta_2,\ \psi_2); \\ f_2(u_2,\ \vartheta_2,\ \psi_2) &= 0. \end{aligned} \qquad (2)$$

In these equations, ψ_1 and ψ_2 are the angles of rotation of the originating wheels, and the functions f_1 and f_2 are the equations of association between the parameters for the points of the characteristic. With a fixed value of ψ_1 (i=1,2), equations (1) and (2) are the equations of the lines of contact of the wheels being cut and the originating wheels in the systems S_1 and S_2; the locus of the points of the lines of contact form the surfaces of the teeth of the gear wheels in the systems S_1 and S_2.

In the next stage, which is solving the inverse problem, it is necessary to examine the meshing of the gear wheels with one another, and to find: a) the law of movement of the wheels; and b) the working lines on the surfaces of the teeth of both gear wheels.

In beveled toothed gearing, the wheels 1 and 2 execute rotary movements. Let rotation by an angle φ_1' be imparted to wheel 1 with its meshing with wheel 2. Then, the equations of the surface Σ_1 in the fixed system S_0 are written in the form: /30

$$\begin{aligned} r_0^{(1)} &= r_0^{(1)}(u_1,\ \vartheta_1,\ \psi_1,\ \varphi_1'); \\ f_1(u_1,\ \vartheta_1,\ \psi_1) &= 0. \end{aligned} \qquad (3)$$

As a result of rotation of wheel 1, wheel 2 rotates by an angle φ_2'. The surface Σ_2 in the fixed system S_0 is determined by the equations

$$\mathbf{r}_0^{(2)} = \mathbf{r}_0^{(2)}(u_2, \vartheta_2, \psi_2 \varphi_2');$$
$$f_2(u_2, \vartheta_2, \psi_2) = 0. \quad (4)$$

At the point of contact of the surfaces Σ_1 and Σ_2 in the fixed system S_0, they should have a common normal and

$$\mathbf{e}_0^{(1)} = \mathbf{e}_0^{(2)}, \quad (5)$$

where $e_0^{(1)}$ and $e_0^{(2)}$ are the unit vectors of the normal to the surfaces Σ_1 and Σ_2 in the fixed system S_0.

In order to determine the parameters u_1, ϑ_1, ψ_1, φ_1', u_2, ϑ_2, ψ_2 and φ_2', it is necessary to make use of the following system of equations:

$$\left.\begin{aligned} \bar{\mathbf{r}}_0^{(1)} &= \bar{\mathbf{r}}_0^{(2)}; \\ \bar{\mathbf{e}}_0^{(1)} &= \bar{\mathbf{e}}_0^{(2)}; \\ f_1(u_1, \vartheta_1, \psi_1) &= 0; \\ f_2(u_2, \vartheta_2, \psi_2) &= 0. \end{aligned}\right\} \quad (6)$$

In coordinate form, we will write the system of equations (6) as:

$$\left.\begin{aligned} x_0^{(1)} &= x_0^{(2)}; \\ y_0^{(1)} &= y_0^{(2)}; \\ z_0^{(1)} &= z_0^{(2)}; \\ e_{x_0}^{(1)} &= e_{x_0}^{(2)}; \\ e_{y_0}^{(1)} &= e_{y_0}^{(2)}; \\ e_{z_0}^{(1)} &= e_{z_0}^{(2)}; \\ f_1(u_1, \vartheta_1, \psi_1) &= 0; \\ f_2(u_2, \vartheta_2, \psi_2) &= 0. \end{aligned}\right\} \quad (7)$$

In system (7), there are only seven independent equations, since

$$[e_{x_0}^{(i)}]^2 + [e_{y_0}^{(i)}]^2 + [e_{z_0}^{(i)}]^2 = 1 \quad (i = 1,2).$$

During the solution of the system of equations (7), one of the parameters ψ_1 or ψ_2 may be considered fixed. If ψ_1 is considered as given, then, as a result of the solution of the system of equations

(7), the desired parameters u_1, ϑ_1, φ_1', u_2, ϑ_2, ψ_2, φ_2' will be found. By substituting the obtained values of the parameters, with varous ψ_1, into equations (1), (2) and (3) or (4), we will find the working lines and the meshing line on the surfaces Σ_1 and Σ_2.

The table of values of φ_1' and φ_2', in the form of the function $\varphi_2'=f(\varphi_1')$, represents the position function of the toothed gear mechanism being studied. After differentiation of the function $\varphi_2'=f(\varphi_1')$, we obtain the function of the gear ratio

$$i_{21} = \frac{df(\varphi_1')}{d\varphi_1'}.$$

2. Coordinate Systems Used /31

The originating wheel P_i is associated with the coordinate systems

$$x_{u_i},\ y_{u_i},\ z_{u_i}\ \&\ x_{p_i},\ y_{p_i},\ z_{p_i}\ \text{(Fig. 1—4)};$$

the system x_{u_i}, y_{u_i}, z_{u_i} is an auxiliary system, utilized for preliminary recording of the originating surface. Here and subsequently, i=1,2, since two originating surfaces, which do not coincide with one another, are utilized for cutting of wheels 1 and 2. The cutting head is a set of blades of rectilinear profile with a profile angle α_i, which form the originating beveled surface with their rotation

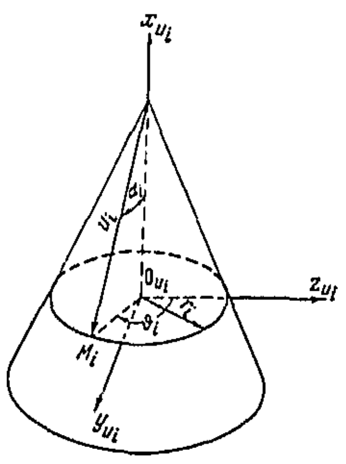

Fig. 1

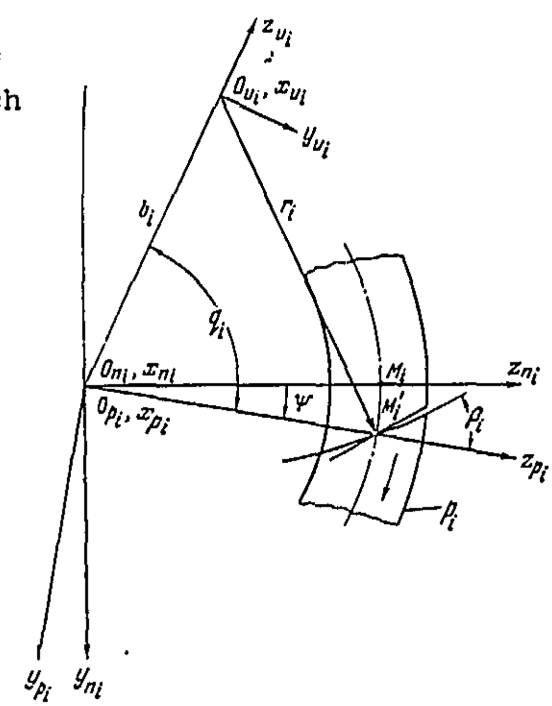

Fig. 2

around the axis x_{u_i}.

The coordinate system x_{n_i}, y_{n_i}, z_{n_i} (Fig. 2-4) is an auxiliary fixed coordinate system, in which the rotation of the originating wheel is prescribed. The plane $x_{n_i}=0$ is parallel to the plane which is tangential to the bevel of the recesses of the gear wheel being cut. With cutting of the teeth, the originating wheel rotates around the axis x_{n_i} (ψ_i is the angle of rotation of the originating wheel). The plane $x_{n_i}=0$ is called the adjusting plane, and the product radius r_i of the cutting head and the angle β_i, formed by the tangent to the longitudinal profile at the midpoint M_i with the axis z_{p_i}, are prescribed in it.

The fixed system of coordinates x_0, y_0, z_0 is utilized for determining the meshing line of the wheels being cut. The axis O_0z_0 coincides with the common generatrix of the initial bevels of the gear wheels; O_0 is the point of intersection of the axes of both wheels. The system x_0, y_0, z_0 differs from x_{n_i}, y_{n_i}, z_{n_i} in the rotation around y_{n_i} by an angle γ_i of the shank of the teeth of the wheel being cut and in the displacement of O_0, relative to O_{n_i}, by /32
$L\sin\gamma_i$; L is the average generatrix of the initial bevel of the wheels.

The coordinate system x_k, y_k, z_k (k=1 and 2) is associated with the wheel being cut (Fig. 3-6). Rotation of the wheel being cut is prescribed in the auxiliary fixed coordinate system x_{b_k}, y_{b_k}, z_{b_k} (k=1 and 2), the z_{b_k}-axis of which coincides with the z_k-axis.

3. Equations of Originating Surfaces /33

In the system x_{u_i}, y_{u_i}, z_{u_i}, the originating surface is determined by the following equations (Fig. 1):

$$\left.\begin{aligned} x_{u_i} &= r_i \operatorname{ctg} \alpha_i - u_i \cos \alpha_i; \\ y_{u_i} &= u_i \sin \alpha_i \sin \vartheta_i; \\ z_{u_i} &= u_i \sin \alpha_i \cos \vartheta_i, \end{aligned}\right\} \qquad (8)$$

where u_i, ϑ_i are independent parameters of the originating surface.

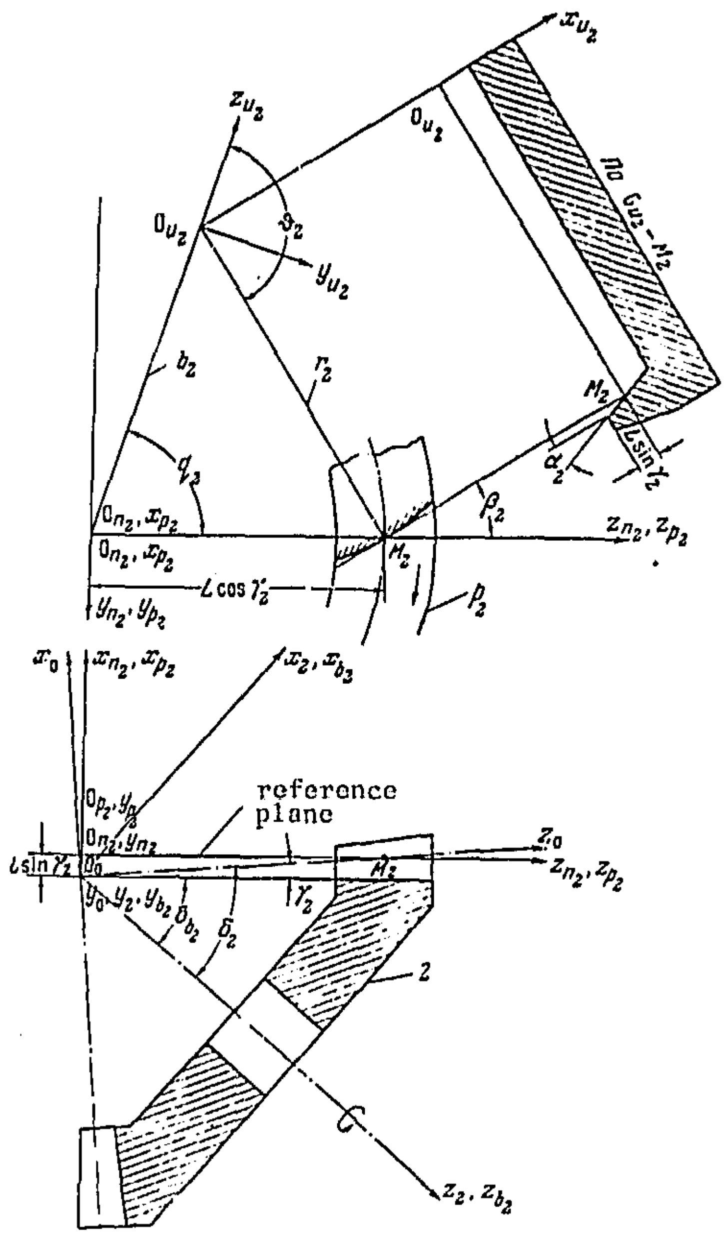

Fig. 3. Cutting diagram of convex side of tooth of wheel 2

The index i=1,2 is related to the first and second originating surfaces, respectively.

The unit vector of the normal to the originating surface (8) will have the form: /34

$$\mathbf{e}_{u_i} = k\left(\frac{\partial \mathbf{r}_{ui}}{\partial u_i} \times \frac{\partial \mathbf{r}_{ui}}{\partial \vartheta_i}\right), \qquad (9)$$

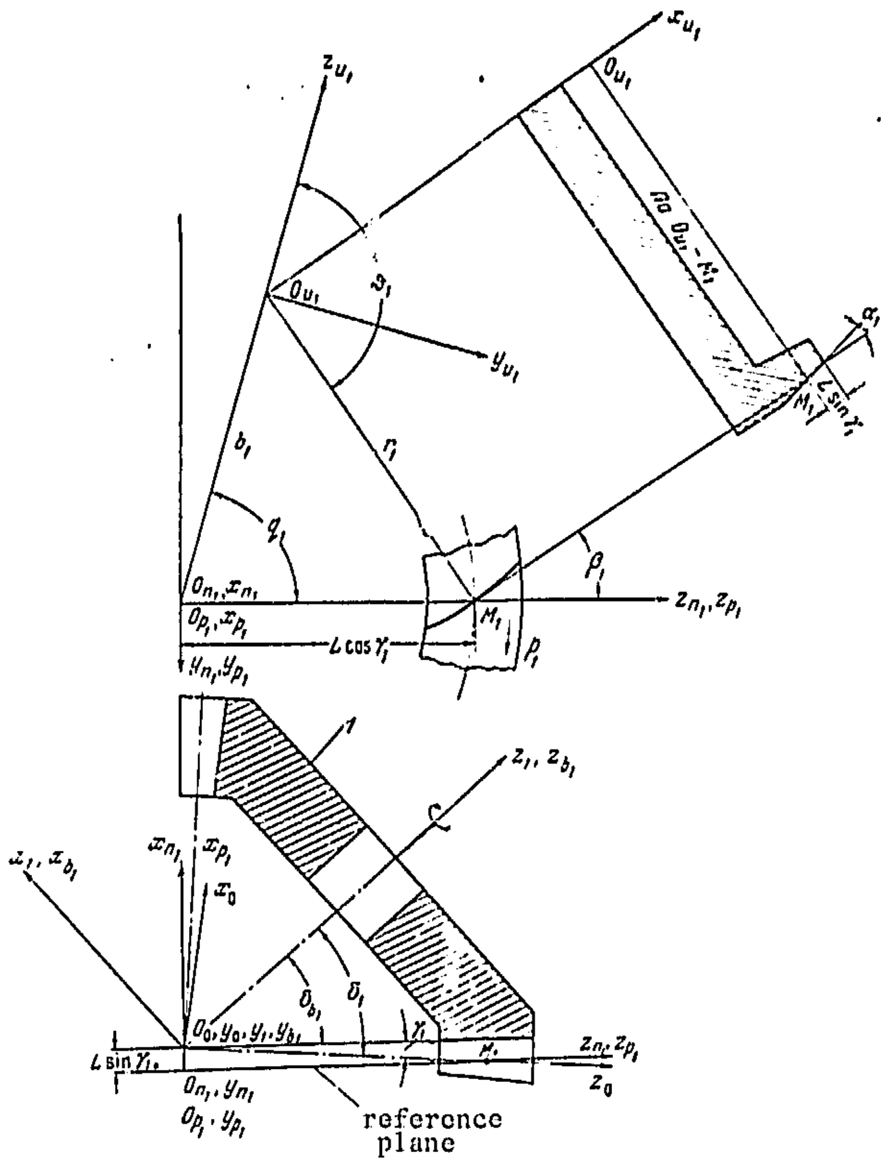

Fig. 4. Cutting diagram of concave side of tooth of wheel 1

where k is the norming factor, and its projections are expressed by the functions: /35

$$\left.\begin{aligned} e_{xu_i} &= \sin\alpha_i; \\ e_{yu_i} &= \cos\alpha_i \sin\vartheta_i; \\ e_{zu_i} &= \cos\alpha_i \cos\vartheta_i. \end{aligned}\right\} \quad (10)$$

Utilizing the matrix of transition from the system x_{u_i}, y_{u_i}, z_{u_i} to the fixed system x_{n_i}, y_{n_i}, z_{n_i} (Fig. 2):

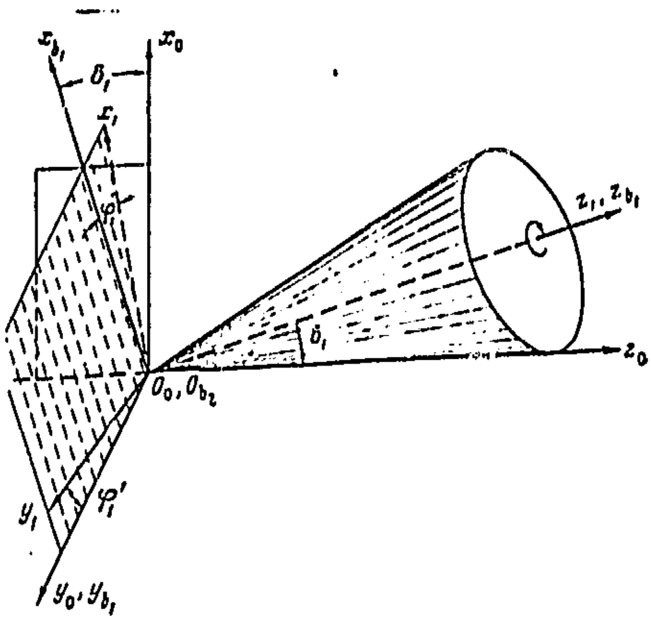

Fig. 5

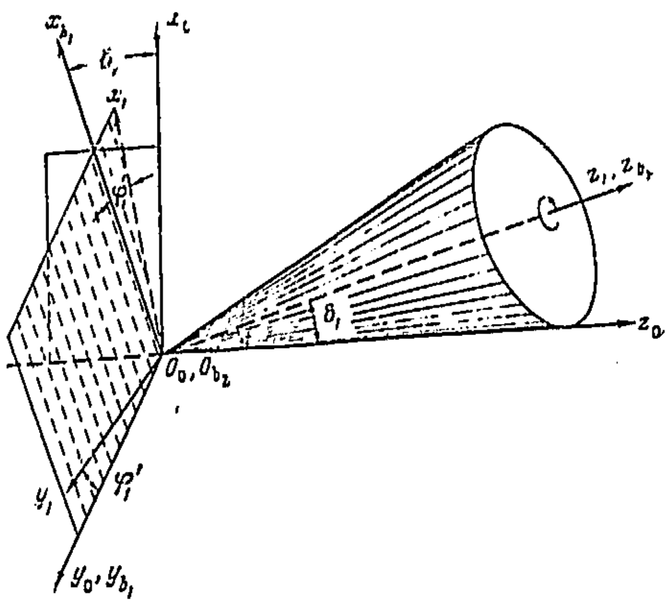

Fig. 6

$$M_{n_i n_i} = \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & \cos(q_i - \psi_i) & -\sin(q_i - \psi_i) & -b_i \sin(q_i - \psi_i) \\ 0 & \sin(q_i - \psi_i) & \cos(q_i - \psi_i) & b_i \cos(q_i - \psi_i) \\ 0 & 0 & 0 & 1 \end{matrix} \right\|, \quad (11)$$

we obtain the equations of the originating surface and the projections of the unit vector of the normal in the system x_{n_i}, y_{n_i}, z_{n_i}:

$$\left.\begin{aligned} x_{n_i} &= r_i \operatorname{ctg} \alpha_i - u_i \cos \alpha_i; \\ y_{n_i} &= u_i \sin \alpha_i \sin (\vartheta_i - q_i + \psi_i) - b_i \sin (q_i - \psi_i); \\ z_{n_i} &= u_i \sin \alpha_i \cos (\vartheta_i - q_i + \psi_i) + b_i \cos (q_i - \psi_i); \end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned} e_{xn_i} &= \sin \alpha_i; \\ e_{yn_i} &= \cos \alpha_i \sin (\vartheta_i - q_i + \psi_i); \\ e_{zn_i} &= \cos \alpha_i \cos (\vartheta_i - q_i + \psi_i), \end{aligned}\right\} \quad (13)$$

here, q_i and b_i are the mounting parameters of the cutting head.

The matrix of transition from the system x_{u_i}, y_{u_i}, z_{u_i} to the system x_0, y_0, z_0 is expressed thusly (Figs. 3 and 4):

$$M_{0u_i} = M_{0n_i} M_{n_i u_i} = \left\| \begin{matrix} \cos \gamma_i & \mp \sin \gamma_i \sin (q_i - \psi_i) & \mp \sin \gamma_i \cos (q_i - \psi_i) & \mp b_i \sin \gamma_i \cos (q_i - \psi_i) \pm L \sin \gamma_i \cos \gamma_i \\ 0 & \cos (q_i - \psi_i) & -\sin (q_i - \psi_i) & -b_i \sin (q_i - \psi_i) \\ \pm \sin \gamma_i & \cos \gamma_i \sin (q_i - \psi_i) & \cos \gamma_i \cos (q_i - \psi_i) & b_i \cos \gamma_i \cos (q_i - \psi_i) + L \sin^2 \gamma_i \\ 0 & 0 & 0 & 1 \end{matrix} \right\|. \quad (14)$$

Utilizing matrix (14), we obtain the equations of the originating surfaces and the unit vectors of the normals in the system x_0, y_0, z_0:

$$\left.\begin{aligned} x_0^{(p_i)} &= (r_i \operatorname{ctg} \alpha_i - u_i \cos \alpha_i \pm L \sin \gamma_i) \cos \gamma_i \mp \\ &\mp u_i \sin \alpha_i \sin \gamma_i \cos (\vartheta_i - q_i + \psi_i) \mp b_i \sin \gamma_i \cos (q_i - \psi_i); \\ y_0^{(p_i)} &= u_i \sin \alpha_i \sin (\vartheta_i - q_i + \psi_i) - b_i \sin (q_i - \psi_i); \\ z_0^{(p_i)} &= \pm (r_i \operatorname{ctg} \alpha_i - u_i \cos \alpha_i \pm L \sin \gamma_i) \sin \gamma_i + \\ &+ u_i \sin \alpha_i \cos \gamma_i \cos (\vartheta_i - q_i + \psi_i) + b_i \cos \gamma_i \cos (q_i - \psi_i); \end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned} e_{x_0}^{(p_i)} &= \sin \alpha_i \cos \gamma_i \mp \cos \alpha_i \sin \gamma_i \cos (\vartheta_i - q_i + \psi_i); \\ e_{y_0}^{(p_i)} &= \cos \alpha_i \sin (\vartheta_i - q_i + \psi_i), \\ e_{z_0}^{(p_i)} &= \pm \sin \alpha_i \sin \gamma_i + \cos \alpha_i \cos \gamma_i \cos (\vartheta_i - q_i + \psi_i). \end{aligned}\right\} \quad (16)$$

In equations (14), (15) and (16), the upper sign is related to /36
the case of cutting of the second wheel (i=2), and the lower sign is related to cutting of the first wheel (i=1).

Subsequently, we will also need the equations of the unit vectors of the normal to the theoretical originating surface, utilized for cutting of wheels with teeth of equal height, in the system x_0,

y_0, z_0.

By substituting $\gamma_1=0$ into equation (16), we obtain

$$\left.\begin{aligned} e_{x_0}^{(p)} &= \sin\alpha; \\ e_{y_0}^{(p)} &= \cos\alpha \sin(\vartheta - q + \psi); \\ e_{z_0}^{(p)} &= \cos\alpha \cos(\vartheta - q + \psi). \end{aligned}\right\} \qquad (17)$$

For the midpoint $\psi=0$ and $\vartheta=90°-\beta+q$; therefore, for this point, the projection of the unit vector of the normal is determined by the equations

$$\left.\begin{aligned} e_{x_0}^{(p)} &= \sin\alpha; \\ e_{y_0}^{(p)} &= \cos\alpha \cos\beta; \\ e_{z_0}^{(p)} &= \cos\alpha \sin\beta, \end{aligned}\right\} \qquad (18)$$

where α is the nominal angle of the profile of the cutter; β is the nominal angle of the spiral.

4. Surfaces of Teeth of Gear Wheels with Normally Decreasing Arc Teeth

The equations of the surfaces of the gear wheel teeth will be found by utilizing the method of solution of the direct problem.

The equations of association between the parameters u_2, ϑ_2, ψ_2 for the points of the characteristic (lines of contact of the originating surface and the surface being cut Σ_2^p and Σ_2) are obtained by making use of the condition that the normal, at the point of contact of the surfaces of the teeth, should pass through the axis of meshing — the instantaneous axis of rotation in the relative movement [1,4]. For our case, this axis is O_0z_0—the common generatrix of the axoids of the originating gear wheel and the gear wheel being cut. From Figure 3, it is evident that the axis of rotation of the originating wheel makes an angle with the axis of meshing on the machine tool which is equal to $90° - \gamma_2$. Therefore, the axoid of the originating gear wheel is a cone, and, according to the form of the axoid, the originating wheel is called beveled.

The axis of meshing O_0z_0 in the system x_{n2}, y_{n2}, z_{n2} is determined by the equations

$$Y_{n_2} = 0;$$
$$\frac{L \sin \gamma_2 + X_{n_2}}{Z_{n_2}} = \operatorname{tg} \gamma_2, \qquad (19)$$

where X_{n2}, Y_{n2}, Z_{n2} are the coordinates of the current point of the axis of meshing O_0z_0.

The normal to the originating surface (12), which intersects the axis of meshing, is determined by the equations

$$\frac{X_{n_2} - x_{n_2}}{e_{xn_2}} = \frac{Y_{n_2} - y_{n_2}}{e_{yn_2}} = \frac{Z_{n_2} - z_{n_2}}{e_{zn_2}}. \qquad (20)$$

Based on equations (12), (13), (19), we can use equation (20) /37
to find the following association between the parameters u_2, ϑ_2, ψ_2:

$$[u_2 - (r_2 \operatorname{ctg} \alpha_2 + L \sin \gamma_2) \cos \alpha_2] \sin (\vartheta_2 - q_2 + \psi_2) -$$
$$- b_2 \sin (q_2 - \psi_2) \sin \alpha_2 + b_2 \cos \alpha_2 \operatorname{tg} \gamma_2 \sin \vartheta_2 = 0. \qquad (21)$$

Having examined equations (8) and (12) together, we will determine the line of contact (characteristic) on the originating surface.

In order to obtain the equations of the surface of the teeth of gear wheel 2 in the system x_2, y_2, z_2, we will make use of the product of the matrices $M_{2b_2} M_{b_2 n_2}$, which expresses the transition from x_{n2}, y_{n2}, z_{n2} to x_2, y_2, z_2 (Fig. 3),

$$M_{2b_2} M_{b_2 n_2} = \left\| \begin{matrix} \cos \varphi_2 \cos \delta_{b_2} & \sin \varphi_2 & \cos \varphi_2 \sin \delta_{b_2} & L \sin \gamma_2 \cos \varphi_2 \cos \delta_{b_2} \\ -\sin \varphi_2 \cos \delta_{b_2} & \cos \varphi_2 & -\sin \varphi_2 \sin \delta_{b_2} & -L \sin \gamma_2 \sin \varphi_2 \cos \delta_{b_2} \\ -\sin \delta_{b_2} & 0 & \cos \delta_{b_2} & L \sin \gamma_2 \\ 0 & 0 & 0 & 1 \end{matrix} \right\|. \qquad (22)$$

Here, φ_2 is the angle of rotation of wheel 2 around the axis O_0z_0 with its meshing with the originating wheel, with

$$\varphi_2 = \psi_2 \frac{\cos\gamma_2}{\sin\delta_2}.$$

For a direct transition from x_{u_2}, y_{u_2}, z_{u_2} to x_2, y_2, z_2, one must make use of the product of the matrices $M_{2b_2}M_{b_2n_2}M_{n_2u_2}$; the matrix M_{n_2u} was represented earlier by expression (11). After transformations, we obtain

$$M_{2b_2}M_{b_2n_2}M_{n_2u_2} = \begin{Vmatrix} a_1^{(2)} & b_1^{(2)} & c_1^{(2)} & L\sin\gamma_2 a_1^{(2)} + b_2c_1^{(2)} \\ a_2^{(2)} & b_2^{(2)} & c_2^{(2)} & L\sin\gamma_2 a_2^{(2)} + b_2c_2^{(2)} \\ a_3^{(2)} & b_3^{(2)} & c_3^{(2)} & L\sin\gamma_2 a_3^{(2)} + b_2c_3^{(2)} \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \tag{23}$$

where

$$\begin{aligned}
a_1^{(2)} &= \cos\varphi_2\cos\delta_{b_2};\\
a_2^{(2)} &= -\sin\varphi_2\cos\delta_{b_2};\\
a_3^{(2)} &= -\sin\delta_{b_2};\\
b_1^{(2)} &= \sin\varphi_2\cos(q_2-\psi_2) + \cos\varphi_2\sin\delta_{b_2}\sin(q_2-\psi_2);\\
b_2^{(2)} &= \cos\varphi_2\cos(q_2-\psi_2) - \sin\varphi_2\sin\delta_{b_2}\sin(q_2-\psi_2);\\
b_3^{(2)} &= \cos\delta_{b_2}\sin(q_2-\psi_2);\\
c_1^{(2)} &= -\sin\varphi_2\sin(q_2-\psi_2) + \cos\varphi_2\sin\delta_{b_2}\cos(q_2-\psi_2);\\
c_2^{(2)} &= -\cos\varphi_2\sin(q_2-\psi_2) - \sin\varphi_2\sin\delta_{b_2}\cos(q_2-\psi_2);\\
c_3^{(2)} &= \cos\delta_{b_2}\cos(q_2-\psi_2).
\end{aligned}$$

Based on equations (8), (23) and (21), the surface of the teeth of wheel 2 is determined by the equations:

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$$\left.\begin{aligned}
x_2 &= (r_2\operatorname{ctg}\alpha_2 - u_2\cos\alpha_2 + L\sin\gamma_2)\,a_1^{(2)} + u_2\sin\alpha_2\sin\vartheta_2\, b_1^{(2)} + \\
&\quad + (u_2\sin\alpha_2\cos\vartheta_2 + b_2)\,c_1^{(2)};\\
y_2 &= (r_2\operatorname{ctg}\alpha_2 - u_2\cos\alpha_2 + L\sin\gamma_2)\,a_2^{(2)} + u_2\sin\alpha_2\sin\vartheta_2\, b_2^{(2)} + \\
&\quad + (u_2\sin\alpha_2\cos\vartheta_2 + b_2)\,c_2^{(2)};\\
z_2 &= (r_2\operatorname{ctg}\alpha_2 - u_2\cos\alpha_2 + L\sin\gamma_2)\,a_3^{(2)} + u_2\sin\alpha_2\sin\vartheta_2\, b_3^{(2)} + \\
&\quad + (u_2\sin\alpha_2\cos\vartheta_2 + b_2)\,c_3^{(2)};\\
&[u_2 - (r_2\operatorname{ctg}\alpha_2 + L\sin\gamma_2)\cos\alpha_2]\sin(\vartheta_2 - q_2 + \psi_2) + \\
&\quad + b_2[\cos\alpha_2\operatorname{tg}\gamma_2\sin\vartheta_2 - \sin\alpha_2\sin(q_2-\psi_2)] = 0.
\end{aligned}\right\} \tag{24}$$

Having used a similar means of derivation, the surface of the teeth of wheel 1 will be represented by the following equations:

$$
\left.
\begin{aligned}
x_1 &= (r_1 \operatorname{ctg}\alpha_1 - u_1\cos\alpha_1 - L\sin\gamma_1)\, a_1^{(1)} + u_1 \sin\alpha_1 \sin\vartheta_1 b_1^{(1)} + \\
&\quad + (u_1 \sin\alpha_1 \cos\vartheta_1 + b_1)\, c_1^{(1)};\\
y_1 &= (r_1 \operatorname{ctg}\alpha_1 - u_1\cos\alpha_1 - L\sin\gamma_1)\, a_2^{(1)} + u_1 \sin\alpha_1 \sin\vartheta_1 b_2^{(1)} + \\
&\quad + (u_1 \sin\alpha_1 \cos\vartheta_1 + b_1)\, c_2^{(1)};\\
z_1 &= (r_1 \operatorname{ctg}\alpha_1 - u_1\cos\alpha_1 - L\sin\gamma_1)\, a_3^{(1)} + u_1 \sin\alpha_1 \sin\vartheta_1 b_3^{(1)} + \\
&\quad + (u_1 \sin\alpha_1 \cos\vartheta_1 + b_1)\, c_3^{(1)};\\
&[-u_1 + (r_1 \operatorname{ctg}\alpha_1 - L\sin\gamma_1)\cos\alpha_1]\sin(\vartheta_1 - q_1 + \psi_1) + \\
&\quad + b_1[\cos\alpha_1 \operatorname{tg}\gamma_1 \sin\vartheta_1 + \sin\alpha_1 \sin(q_1 - \psi_1)] = 0.
\end{aligned}
\right\} \quad (25)
$$

Here

$$
\begin{aligned}
a_1^{(1)} &= \cos\varphi_1 \cos\delta_{b_1};\\
a_2^{(1)} &= \sin\varphi_1 \cos\delta_{b_1};\\
a_3^{(1)} &= \sin\delta_{b_1};\\
b_1^{(1)} &= -\sin\varphi_1 \cos(q_1 - \psi_1) - \cos\varphi_1 \sin\delta_{b_1} \sin(q_1 - \psi_1);\\
b_2^{(1)} &= \cos\varphi_1 \cos(q_1 - \psi_1) - \sin\varphi_1 \sin\delta_{b_1} \sin(q_1 - \psi_1);\\
b_3^{(1)} &= \cos\delta_{b_1} \sin(q_1 - \psi_1);\\
c_1^{(1)} &= \sin\varphi_1 \sin(q_1 - \psi_1) - \cos\varphi_1 \sin\delta_{b_1} \cos(q_1 - \psi_1);\\
c_2^{(1)} &= -\cos\varphi_1 \sin(q_1 - \psi_1) - \sin\varphi_1 \sin\delta_{b_1} \cos(q_1 - \psi_1);\\
c_3^{(1)} &= \cos\delta_{b_1} \cos(q_1 - \psi_1);\\
\varphi_1 &= \psi_1 \frac{\cos\gamma_1}{\sin\delta_1}.
\end{aligned}
$$

5. Selection of Parameters of Cutting Heads and Adjustment Parameters of Machine Tool

The selection of the parameters of the cutting heads and preliminary adjustment of the machine tool obeys the condition that the surfaces of the teeth of the gear wheels, with their contact at the midpoint, should be conjugate, i.e., with the transmission of rotation by the gear wheels, the instantaneous gear ratio will be equal to the prescribed ratio.

For this purpose, it is necessary that both of the originating surfaces and the surfaces of the teeth of the gear wheels contact one another at the midpoint M_i $(i=1,2)$ (Figs. 3 and 4), which represents the point of intersection of the axes $O_{n_i} z_{n_i}$ and $O_0 z_0$. The axis $O_0 z_0$ coincides with the common generatrix of the initial bevels of the beveled wheels, and with the absence of errors of the wheels, becomes their axis of rotation in relative movement. /39

Contact of the originating surface and the surface of the teeth of the wheel being cut at the midpoint M_i may be achieved by using adjustment of the generating chain of the machine tool. For this purpose, the gear ratio of the generating chain should be determined from the equation:

$$i_{p_l k} = \frac{\psi_l}{\varphi_k} = \frac{\sin \delta_l}{\cos \gamma_l} \quad (l = 1,\ 2,\ k = 1,\ 2). \tag{26}$$

What is more, if both of the originating surfaces contact one another at the point M_i, then, as is not difficult to see, this condition determines simultaneously that the surfaces of the teeth of the gear wheels will also contact one another at M_i.

For this purpose, it is necessary that the radius-vectors and the unit vectors of the normals at the point M_i are equal, i.e.,

$$\left.\begin{aligned} \mathbf{r}_0^{(p_1)} &= \mathbf{r}_0^{(p_2)}; \\ \mathbf{e}_0^{(p_1)} &= \mathbf{e}_0^{(p_2)}. \end{aligned}\right\} \tag{27}$$

Utilizing equations (15), (16) and (27) of the originating surfaces and the unit vectors of the normals in the system x_0, y_0, z_0, with $\psi_i=0$, we obtain:

$$\left.\begin{aligned} &(r_1 \operatorname{ctg} \alpha_1 - u_1 \cos \alpha_1 - L \sin \gamma_1) \cos \gamma_1 + u_1 \sin \alpha_1 \sin \gamma_1 \cos (\vartheta_1 - q_1) + \\ &\quad + b_1 \sin \gamma_1 \cos q_1 = (r_2 \operatorname{ctg} \alpha_2 - u_2 \cos \alpha_2 + L \sin \gamma_2) \cos \gamma_2 - \\ &\quad - u_2 \sin \alpha_2 \sin \gamma_2 \cos (\vartheta_2 - q_2) - b_2 \sin \gamma_2 \cos q_2; \\ &u_1 \sin \alpha_1 \sin (\vartheta_1 - q_1) - b_1 \sin q_1 = u_2 \sin \alpha_2 \sin (\vartheta_2 - q_2) - b_2 \sin q_2; \\ &-(r_1 \operatorname{ctg} \alpha_1 - u_1 \cos \alpha_1 - L \sin \gamma_1) \sin \gamma_1 + u_1 \sin \alpha_1 \cos \gamma_1 \cos (\vartheta_1 - q_1) + \\ &\quad + b_1 \cos \gamma_1 \cos q_1 = (r_2 \operatorname{ctg} \alpha_2 - u_2 \cos \alpha_2 + L \sin \gamma_2) \sin \gamma_2 + \\ &\quad + u_2 \sin \alpha_2 \cos \gamma_2 \cos (\vartheta_2 - q_2) + b_2 \cos \gamma_2 \cos q_2; \end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned} &\sin \alpha_1 \cos \gamma_1 + \cos \alpha_1 \sin \gamma_1 \cos (\vartheta_1 - q_1) = \sin \alpha_2 \cos \gamma_2 - \\ &\quad - \cos \alpha_2 \sin \gamma_2 \cos (\vartheta_2 - q_2); \\ &\cos \alpha_1 \sin (\vartheta_1 - q_1) = \cos \alpha_2 \sin (\vartheta_2 - q_2); \\ &-\sin \alpha_1 \sin \gamma_1 + \cos \alpha_1 \cos \gamma_1 (\vartheta_1 - q_1) = \sin \alpha_2 \sin \gamma_2 + \\ &\quad + \cos \alpha_2 \cos \gamma_2 \cos (\vartheta_2 - q_2). \end{aligned}\right\} \tag{29}$$

For the midpoint $M_i u_i = r_i/\sin\alpha_i$, $\vartheta_i = 90° - \beta_i + q_i$. After substitution of these values into equations (28), one may see that $r_0^{(p_1)} = r_0^{(p_2)}$.

Having substituted the above into equations (29), we obtain

$$\left.\begin{aligned}\sin\alpha_2\cos\gamma_2-\cos\alpha_2\sin\gamma_2\sin\beta_2&=\sin\alpha_1\cos\gamma_1+\cos\alpha_1\sin\gamma_1\sin\beta_1;\\ \cos\alpha_2\cos\beta_2&=\cos\alpha_2\cos\beta_2;\\ \sin\alpha_2\sin\gamma_2-\cos\alpha_2\cos\gamma_2\sin\beta_2&=-\sin\alpha_1\sin\gamma_1+\cos\alpha_1\cos\gamma_1\sin\beta_1.\end{aligned}\right\}\quad(30)$$

Contained in system (30) are four unknowns: α_1, α_2, β_1 and β_2; of the three equations of this system, however, only two are independent.

We will obtain the missing equations for determining the unknowns by utilizing the following conditions.

1. We will require that, at the midpoint, there will occur simultaneous contact of the three originating surfaces: two practical, utilized for cutting of the gear wheels using decreasing teeth, and one theoretical, utilized for cutting of wheels with teeth of equal height. For this purpose, proceeding from equations (18) and (30), we will obtain /40

$$\left.\begin{aligned}&\sin\alpha_2\cos\gamma_2-\cos\alpha_2\sin\gamma_2\sin\beta_2=\sin\alpha_1\cos\gamma_1+\\ &\quad+\cos\alpha_1\sin\gamma_1\sin\beta_1=\sin\alpha;\\ &\cos\alpha_2\cos\beta_2=\cos\alpha_1\cos\beta_1=\cos\alpha\cos\beta;\\ &\sin\alpha_2\sin\gamma_2+\cos\alpha_2\cos\gamma_2\sin\beta_2=\\ &=-\sin\alpha_1\sin\gamma_1+\cos\alpha_1\cos\gamma_1\sin\beta_1=\cos\alpha\sin\beta.\end{aligned}\right\}\quad(31)$$

In system (31), of the six equations, four are independent. After transformations, we will find the following four equations for determining α_1, α_2, β_1 and β_2:

$$\sin\alpha_i=\cos\gamma_i\sin\alpha\pm\sin\gamma_i\cos\alpha\sin\beta;\quad(32)$$

$$\cos\beta_i=\frac{\cos\alpha\cos 3}{\cos\alpha_i}.\quad(33)$$

Here and subsequently, the upper sign is related to the case of cutting of the second wehel (i=2), and the lower sign — to the case of the first wheel (i=1).

It is necessary to note that, with the given selection of the parameters α_1, α_2, β_1 and β_2, the angle of meshing on the surface of the teeth of the wheel being cut at the midpoint will be equal to the nominal value of the angle of meshing, i.e., 20°.

2. The second possibility of representation of the missing equations is based on the fact that the values of the angles α_1 and α_2 are calculated according to the known approximate formula, used in gear-cutting practice

$$\alpha_i = \alpha \pm \frac{1}{2} \sin \beta \, (\mathrm{tg}\, \gamma_2 + \mathrm{tg}\, \gamma_1) \frac{180^\circ}{\pi}. \tag{34}$$

The angles β_1 and β_2, after the determination of α_i, should be calculated from the equations in (30). After transformations, we obtain

$$\sin \beta_1 = \frac{\sin \alpha_2 - \sin \alpha_1 \cos (\gamma_2 + \gamma_1)}{\cos \alpha_1 \sin (\gamma_2 - \gamma_1)}; \tag{35}$$

$$\sin \beta_2 = \frac{\sin \alpha_2 \cos (\gamma_2 + \gamma_1) - \sin \alpha_1}{\cos \alpha_2 \sin (\gamma_2 - \gamma_1)}. \tag{36}$$

We would note that, in this case, the angle of meshing at the midpoint, strictly speaking, is no longer equal to 20°.

Calculation of the remaining parameters of the cutting heads and their setting is carried out according to the following formulas (Figs. 3 and 4):

$$r_i = r_u \mp \frac{W}{2} \mp L \sin \gamma_i \,\mathrm{tg}\, \alpha_i; \tag{37}$$

$$\mathrm{ctg}\, q_i = \frac{L \cos \gamma_i - r_i \sin \beta_i}{r_i \cos \beta_i}; \tag{38}$$

$$b_i = \frac{r_i \cos \beta_i}{\sin q_i}, \tag{39}$$

where r_u is the nominal radius of the cutting head; W is the set of the blades.

In gear-cutting practice, the following approximate dependences /41 [3,2] have been utilized for the calculation of β_i before now:

$$\beta_i = \beta \mp \gamma_i \, \mathrm{tg}\, \alpha \cos\beta,$$
$$\beta_i' = \beta \mp \mathrm{arc\,tg}\,(\mathrm{tg}\,\alpha \cos\beta \,\mathrm{tg}\,\gamma_i).$$

The equations given in the present paragraph for determining the parameters of the cutting heads and the data for adjusting the machine tool are more precise than those presently used in gear-cutting practice, namely:

a) in equation (38), the distance of the midpoint M_i from the axis $O_{p_i} x_{p_i}$ is taken as $L\cos\gamma_i$, rather than L, as had been used until now [2,3]; b) new dependences are obtained, which associate α_1, α_2, β_1 and β_2.

6. Determination of the Line of Meshing, the Working Lines on the Surfaces of the Teeth, the Position Function and the Gear Ratio of the Gear Wheels

During the solution of this problem, as indicated in paragraph 1, it is necessary to find the equations of the surfaces of the teeth of both gear wheels, and the projection of the unit vector of the normals to the surfaces in the fixed system x_0, y_0, z_0.

For this purpose, it is necessary to make use of the product of the matrices $M_{0b_2} M^*_{b_2 2}$, which expresses the transition from x_2, y_2, z_2 to x_0, y_0, z_0 (Fig. 5).

$$M_{0b_2} M^*_{b_2 2} = \left\| \begin{matrix} \cos\delta_2 \cos\varphi_2' & -\cos\delta_2 \sin\varphi_2' & -\sin\delta_2 & 0 \\ \sin\varphi_2' & \cos\varphi_2' & 0 & 0 \\ \sin\delta_2 \cos\varphi_2' & -\sin\delta_2 \sin\varphi_2' & \cos\delta_2 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\|. \qquad (40)$$

For a direct transition from x_{u_2}, y_{u_2}, z_{u_2} to x_0, y_0, z_0, one should utilize the product of the matrices $M_{0b_2} M^*_{b_2 2} M_{2b_2} M_{b_2 n_2} M_{n_2 u_2}$. The product of the matrices $M_{2b_2} M_{b_2 n_2} M_{n_2 u_2}$ was represented earlier by expression (23). After transformations, we obtain

$$M_{0b_1}M^{*}_{b_12}M_{2b_2}M_{b_2n_2}M_{n_2u_2} =$$

$$= \begin{Vmatrix} A_1^{(2)} & B_1^{(2)} & C_1^{(2)} & L\sin\gamma_2 A_1^{(2)} + b_2 C_1^{(2)} \\ A_2^{(2)} & B_2^{(2)} & C_2^{(2)} & L\sin\gamma_2 A_2^{(2)} + b_2 C_2^{(2)} \\ A_3^{(2)} & B_3^{(2)} & C_3^{(2)} & L\sin\gamma_2 A_3^{(2)} + b_2 C_3^{(2)} \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \qquad (41)$$

where

$$A_1^{(2)} = \cos\delta_2\cos(\varphi_2' - \varphi_2)\cos\delta_{b_2} + \sin\delta_2\sin\delta_{b_2};$$
$$A_2^{(2)} = \sin(\varphi_2' - \varphi_2)\cos\delta_{b_2};$$
$$A_3^{(2)} = \sin\delta_2\cos(\varphi_2' - \varphi_2)\cos\delta_{b_2} - \cos\delta_2\sin\delta_{b_2};$$
$$B_1^{(2)} = \cos\delta_2\cos(\varphi_2' - \varphi_2)\sin\delta_{b_2}\sin(q_2 - \psi_2) - $$
$$- \cos\delta_2\sin(\varphi_2' - \varphi_2)\cos(q_2 - \psi_2) - \sin\delta_2\cos\delta_{b_2}\sin(q_2 - \psi_2);$$
$$B_2^{(2)} = \sin(\varphi_2' - \varphi_2)\sin\delta_{b_2}\sin(q_2 - \psi_2) + \cos(\varphi_2' - \varphi_2)\cos(q_2 - \psi_2);$$

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$$B_3^{(2)} = \sin\delta_2\cos(\varphi_2' - \varphi_2)\sin\delta_{b_2}\sin(q_2 - \psi_2) -$$
$$- \sin\delta_2\sin(\varphi_2' - \varphi_2)\cos(q_2 - \psi_2) + \cos\delta_2\cos\delta_{b_2}\sin(q_2 - \psi_2);$$
$$C_1^{(2)} = \cos\delta_2\cos(\varphi_2' - \varphi_2)\sin\delta_{b_2}\cos(q_2 - \psi_2) +$$
$$+ \cos\delta_2\sin(\varphi_2' - \varphi_2)\sin(q_2 - \psi_2) - \sin\delta_2\cos\delta_{b_2}\cos(q_2 - \psi_2);$$
$$C_2^{(2)} = \sin(\varphi_2' - \varphi_2')\sin\delta_{b_2}\cos(q_2 - \psi_2) - \cos(\varphi_2' - \varphi_2)\sin(q_2 - \psi_2);$$
$$C_3^{(2)} = \sin\delta_2\cos(\varphi_2' - \varphi_2)\sin\delta_{b_2}\cos(q_2 - \psi_2) +$$
$$+ \sin\delta_2\sin(\varphi_2' - \varphi_2)\sin(q_2 - \psi_2) + \cos\delta_2\cos\delta_{b_2}\cos(q_2 - \psi_2).$$

Based on (8), (21), (41) and (10), (41), we obtain the equations of the surface of the teeth of wheel 2, and the projection of the unit vector of the normal $e_0^{(2)}$ in the system x_0, y_0, z_0

$$\left.\begin{aligned} x_0^{(2)} &= (r_2\operatorname{ctg}\alpha_2 - u_2\cos\alpha_2 + L\sin\gamma_2)A_1^{(2)} + u_2\sin\alpha_2\sin\vartheta_2 B_1^{(2)} + \\ &\quad + (u_2\sin\alpha_2\cos\vartheta_2 + b_2)C_1^{(2)}; \\ y_0^{(2)} &= (r_2\operatorname{ctg}\alpha_2 - u_2\cos\alpha_2 + L\sin\gamma_2)A_2^{(2)} + u_2\sin\alpha_2\sin\vartheta_2 B_2^{(2)} + \\ &\quad + (u_2\sin\alpha_2\cos\vartheta_2 + b_2)C_2^{(2)}; \\ z_0^{(2)} &= (r_2\operatorname{ctg}\alpha_2 - u_2\cos\alpha_2 + L\sin\gamma_2)A_3^{(2)} + u_2\sin\alpha_2\sin\vartheta_2 B_3^{(2)} + \\ &\quad + (u_2\sin\alpha_2\cos\vartheta_2 + b_2)C_3^{(2)}; \\ &[u_2 - (r_2\operatorname{ctg}\alpha_2 + L\sin\gamma_2)\cos\alpha_2]\sin(\vartheta_2 - q_2 + \psi_2) + \\ &\quad + b_2[\cos\alpha_2\operatorname{tg}\gamma_2\sin\vartheta_2 - \sin\alpha_2\sin(q_2 - \psi_2)] = 0; \end{aligned}\right\} \quad (42)$$

$$\left.\begin{aligned} e_{x_0}^{(2)} &= \sin\alpha_2 A_1^{(2)} + \cos\alpha_2\sin\vartheta_2 B_1^{(2)} + \cos\alpha_2\cos\vartheta_2 C_1^{(2)}; \\ e_{y_0}^{(2)} &= \sin\alpha_2 A_2^{(2)} + \cos\alpha_2\sin\vartheta_2 B_2^{(2)} + \cos\alpha_2\cos\vartheta_2 C_2^{(2)}; \\ e_{z_0}^{(2)} &= \sin\alpha_2 A_3^{(2)} + \cos\alpha_2\sin\vartheta_2 B_3^{(2)} + \cos\alpha_2\cos\vartheta_2 C_3^{(2)}. \end{aligned}\right\} \quad (43)$$

One may similarly determine the surface of the teeth and the projection of the unit vector of the normal for wheel 1, which are expressed by the equations

$$\left.\begin{aligned}
x_0^{(1)} &= (r_1 \operatorname{ctg}\alpha_1 - u_1\cos\alpha_1 - L\sin\gamma_1)\,A_1^{(1)} + u_1\sin\alpha_1\sin\vartheta_1 B_1^{(1)} + \\
&\quad + (u_1\sin\alpha_1\cos\vartheta_1 + b_1)\,C_1^{(1)};\\
y_0^{(1)} &= (r_1 \operatorname{ctg}\alpha_1 - u_1\cos\alpha_1 - L\sin\gamma_1)\,A_2^{(1)} + u_1\sin\alpha_1\sin\vartheta_1 B_2^{(1)} + \\
&\quad + (u_1\sin\alpha_1\cos\vartheta_1 + b_1)\,C_2^{(1)};\\
z_0^{(1)} &= (r_1 \operatorname{ctg}\alpha_1 - u_1\cos\alpha_1 - L\sin\gamma_1)\,A_3^{(1)} + u_1\sin\alpha_1\sin\vartheta_1 B_3^{(1)} + \\
&\quad + (u_1\sin\alpha_1\cos\vartheta_1 + b_1)\,C_3^{(1)};\\
&[-u_1 + (r_1\operatorname{ctg}\alpha_1 - L\sin\gamma_1)\cos\alpha_1]\sin(\vartheta_1 - q_1 + \psi_1) + \\
&\quad + b_1[\cos\alpha_1\operatorname{tg}\gamma_1\sin\vartheta_1 + \sin\alpha_1\sin(q_1 - \psi_1)] = 0
\end{aligned}\right\} \quad (44)$$

and

$$\left.\begin{aligned}
e_{x_0}^{(1)} &= \sin\alpha_1 A_1^{(1)} + \cos\alpha_1\sin\vartheta_1 B_1^{(1)} + \cos\alpha_1\cos\vartheta_1 C_1^{(1)};\\
e_{y_0}^{(1)} &= \sin\alpha_1 A_2^{(1)} + \cos\alpha_1\sin\vartheta_1 B_2^{(1)} + \cos\alpha_1\cos\vartheta_1 C_2^{(1)};\\
e_{z_0}^{(1)} &= \sin\alpha_1 A_3^{(1)} + \cos\alpha_1\sin\vartheta_1 B_3^{(1)} + \cos\alpha_1\cos\vartheta_1 C_3^{(1)};
\end{aligned}\right\} \quad (45)$$

In these equations /43

$$\begin{aligned}
A_1^{(1)} &= \cos\delta_1\cos(\varphi_1' - \varphi_1)\cos\delta_{b_1} + \sin\delta_1\sin\delta_{b_1},\\
A_2^{(1)} &= -\sin(\varphi_1' - \varphi_1)\cos\delta_{b_1};\\
A_3^{(1)} &= -\sin\delta_1\cos(\varphi_1' - \varphi_1)\cos\delta_{b_1} + \cos\delta_1\sin\delta_{b_1};\\
B_1^{(1)} &= -\cos\delta_1\cos(\varphi_1' - \varphi_1)\sin\delta_{b_1}\sin(q_1 - \psi_1) + \\
&\quad + \cos\delta_1\sin(\varphi_1' - \varphi_1)\cos(q_1 - \psi_1) + \sin\delta_1\cos\delta_{b_1}\sin(q_1 - \psi_1);\\
B_2^{(1)} &= \sin(\varphi_1' - \varphi_1)\sin\delta_{b_1}\sin(q_1 - \psi_1) + \cos(\varphi_1' - \varphi_1)\cos(q_1 - \psi_1);\\
B_3^{(1)} &= \sin\delta_1\cos(\varphi_1' - \varphi_1)\sin\delta_{b_1}\sin(q_1 - \psi_1) - \\
&\quad - \sin\delta_1\sin(\varphi_1' - \varphi_1)\cos(q_1 - \varphi_1) + \cos\delta_1\cos\delta_{b_1}\sin(q_1 - \psi_1);\\
C_1^{(1)} &= -\cos\delta_1\cos(\varphi_1' - \varphi_1)\sin\delta_{b_1}\cos(q_1 - \psi_1) - \\
&\quad - \cos\delta_1\sin(\varphi_1' - \varphi_1)\sin(q_1 - \psi_1) + \sin\delta_1\cos\delta_{b_1}\cos(q_1 - \psi_1);\\
C_2^{(1)} &= \sin(\varphi_1' - \varphi_1)\sin\delta_{b_1}\cos(q_1 - \psi_1) - \cos(\varphi_1' - \varphi_1)\sin(q_1 - \psi_1);\\
C_3^{(1)} &= \sin\delta_1\cos(\varphi_1' - \varphi_1)\sin\delta_{b_1}\cos(q_1 - \psi_1) + \\
&\quad + \sin\delta_1\sin(\varphi_1' - \varphi_1)\sin(q_1 - \psi_1) + \cos\delta_1\cos\delta_{b_1}\cos(q_1 - \psi_1).
\end{aligned}$$

In order to determine the point of contact of the surfaces of both wheels, it is necessary to make use of the system of equations

$$\begin{aligned}
x_0^{(1)} &= x_0^{(2)};\\
y_0^{(1)} &= y_0^{(2)};\\
z_0^{(1)} &= z_0^{(2)};
\end{aligned}$$

$$\left.\begin{aligned}
&e_{x_0}^{(1)} = e_{x_0}^{(2)};\\
&e_{y_0}^{(1)} = e_{y_0}^{(2)};\\
&e_{z_0}^{(1)} = e_{z_0}^{(2)};\\
&[-u_1 + (r_1 \operatorname{ctg}\alpha_1 - L\sin\gamma_1)\cos\alpha_1]\sin(\vartheta_1 - q_1 + \psi_1) +\\
&\quad + b_1[\cos\alpha_1 \operatorname{tg}\gamma_1 \sin\vartheta_1 - \sin\alpha_1 \sin(q_1 - \psi_1)] = 0;\\
&[u_2 - (r_2 \operatorname{ctg}\alpha_2 + L\sin\gamma_2)\cos\alpha_2]\sin(\vartheta_2 - q_2 + \psi_2) +\\
&\quad + b_2[\cos\alpha_2 \operatorname{tg}\gamma_2 \sin\vartheta_2 - \sin\alpha_2 \sin(q_2 - \psi_2)] = 0.
\end{aligned}\right\} \quad (46)$$

From analytic geometry (5), it is common knowledge that the coefficients of formulas of transformation of rectangular coordinates are associated by the following equations:

$$\left.\begin{aligned}
&[A_1^{(i)}]^2 + [A_2^{(i)}]^2 + [A_3^{(i)}]^2 = 1;\\
&[B_1^{(i)}]^2 + [B_2^{(i)}]^2 + [B_3^{(i)}]^2 = 1;\\
&[C_1^{(i)}]^2 + [C_2^{(i)}]^2 + [C_3^{(i)}]^2 = 1;
\end{aligned}\right\} \quad (47)$$

$$\left.\begin{aligned}
&A_1^{(i)}B_1^{(i)} + A_2^{(i)}B_2^{(i)} + A_3^{(i)}B_3^{(i)} = 0;\\
&B_1^{(i)}C_1^{(i)} + B_2^{(i)}C_2^{(i)} + B_3^{(i)}C_3^{(i)} = 0;\\
&C_1^{(i)}A_1^{(i)} + C_2^{(i)}A_2^{(i)} + C_3^{(i)}A_3^{(i)} = 0
\end{aligned}\right\} \quad (48)$$

or /44

$$\left.\begin{aligned}
&[A_1^{(i)}]^2 + [B_1^{(i)}]^2 + [C_1^{(i)}]^2 = 1;\\
&[A_2^{(i)}]^2 + [B_2^{(i)}]^2 + [C_2^{(i)}]^2 = 1;\\
&[A_3^{(i)}]^2 + [B_3^{(i)}]^2 + [C_3^{(i)}]^2 = 1;
\end{aligned}\right\} \quad (49)$$

$$\left.\begin{aligned}
&A_1^{(i)}A_2^{(i)} + B_1^{(i)}B_2^{(i)} + C_1^{(i)}C_2^{(i)} = 0;\\
&A_2^{(i)}A_3^{(i)} + B_2^{(i)}B_3^{(i)} + C_2^{(i)}C_3^{(i)} = 0;\\
&A_3^{(i)}A_1^{(i)} + B_3^{(i)}B_1^{(i)} + C_3^{(i)}C_1^{(i)} = 0.
\end{aligned}\right\} \quad (50)$$

In these equations, i=1,2.

From equations (46), it is evident that

$$[x_0^{(1)}]^2 + [y_0^{(1)}] + [z_0^{(1)}]^2 = [x_0^{(2)}]^2 + [y_0^{(2)}]^2 + [z_0^{(2)}]^2 \quad (51)$$

and

$$e_{x_0}^{(1)}x_0^{(1)} + e_{y_0}^{(1)}y_0^{(1)} + e_{z_0}^{(1)}z_0^{(1)} = e_{x_0}^{(2)}x_0^{(2)} + e_{y_0}^{(2)}y_0^{(2)} + e_{z_0}^{(2)}z_0^{(2)}. \quad (52)$$

By substituting expressions (42), (44) into (51), expressions (42), (43), (44), (45) into (52), and taking dependences (47), (48) into account, after transformations, we obtain

$$(r_1 \operatorname{ctg}\alpha_1 - u_1 \cos\alpha_1 - L\sin\gamma_1)^2 + (u_1 \sin\alpha_1 \sin\vartheta_1)^2 + (u_1 \sin\alpha_1 \cos\vartheta_1 + b_1)^2 =$$
$$= (r_2 \operatorname{ctg}\alpha_2 - u_2\cos\alpha_2 + L\sin\gamma_2)^2 + (u_2 \sin\alpha_2 \sin\vartheta_2)^2 +$$
$$+ (u_2 \sin\alpha_2 \cos\vartheta_2 + b_2)^2; \quad (53)$$
$$\sin\alpha_1 (r_1 \operatorname{ctg}\alpha_1 - L\sin\gamma_1) + b_1 \cos\alpha_1 \cos\vartheta_1 =$$
$$= \sin\alpha_2 (r_2 \operatorname{ctg}\alpha_2 + L\sin\gamma_2) + b_2\cos\alpha_2\cos\vartheta_2. \quad (54)$$

In order to determine the parameters u_1, ϑ_1, φ_1', u_2, ϑ_2, ψ_2, φ_2', utilizing dependences (53), (54) and (39), we will obtain the following system of equations:

$$\left.\begin{aligned}
&[-u_1 + (r_1 \operatorname{ctg}\alpha_1 - L\sin\gamma_1)\cos\alpha_1]\sin(\vartheta_1 - q_1 + \psi_1) + \\
&+ b_1[\cos\alpha_1 \operatorname{tg}\gamma_1 \sin\vartheta_1 + \sin\alpha_1 \sin(q_1 - \psi_1)] = 0; && \text{(a)}\\
&\sin\alpha_1 (r_1 \operatorname{ctg}\alpha_1 - L\sin\gamma_1) + b_1\cos\alpha_1\cos\vartheta_1 = \\
&= \sin\alpha_2 (r_2 \operatorname{ctg}\alpha_2 + L\sin\gamma_2) + b_2\cos\alpha_2\cos\vartheta_2; && \text{(b)}\\
&(r_1 \operatorname{ctg}\alpha_1 - u_1\cos\alpha_1 - L\sin\gamma_1)^2 + (u_1\sin\alpha_1\sin\vartheta_1)^2 + \\
&+ (u_1 \sin\alpha_1\cos\vartheta_1 + b_1)^2 = (r_2\operatorname{ctg}\alpha_2 - u_2\cos\alpha_2 + L\sin\gamma_2)^2 + \\
&+ (u_2\sin\alpha_2\sin\vartheta_2)^2 + (u_2\sin\alpha_2\cos\vartheta_2 + b_2)^2; && \text{(c)}\\
&[u_2 - (r_2\operatorname{ctg}\alpha_2 + L\sin\gamma_2)\cos\alpha_2]\sin(\vartheta_2 - q_2 + \psi_2) + \\
&+ b_2[\cos\alpha_2 \operatorname{tg}\gamma_2\sin\vartheta_2 - \sin\alpha_2\sin(q_2 - \psi_2)] = 0; && \text{(d)}\\
&\sin\alpha_1 A_1^{(1)} + \cos\alpha_1 \sin\vartheta_1 B_1^{(1)} + \cos\alpha_1\cos\vartheta_1 C_1^{(1)} = \\
&= \sin\alpha_2 A_1^{(2)} + \cos\alpha_2\sin\vartheta_2 B_1^{2} + \cos\alpha_2\cos\vartheta_2 C_1^{(2)}; && \text{(e)}\\
&\sin\alpha_1 A_3^{(1)} + \cos\alpha_1\sin\vartheta_1 B_3^{(1)} + \cos\alpha_1\cos\vartheta_1 C_3^{(1)} = \\
&= \sin\alpha_2 A_3^{(2)} + \cos\alpha_2\sin\vartheta_2 B_3^{(2)} + \cos\alpha_2\cos\vartheta_2 C_3^{(2)}; && \text{(f)}\\
&(r_1\operatorname{ctg}\alpha_1 - u_1\cos\alpha_1 - L\sin\gamma_1) A_2^{(1)} + u_1\sin\alpha_1\sin\vartheta_1 B_2^{(1)} + \\
&+ (u_1\sin\alpha_1\cos\vartheta_1 + b_1) C_2^{(1)} = (r_2\operatorname{ctg}\alpha_2 - u_2\cos\alpha_2 + \\
&+ L\sin\gamma_2) A_2^{(2)} + u_2\sin\alpha_2\sin\vartheta_2 B_2^{(2)} + (u_2\sin\alpha_2\cos\vartheta_2 + b_2) C_2^{(2)}, && \text{(g)}
\end{aligned}\right\} \quad (55)$$

where ψ_1 is a fixed parameter, and u_1, ϑ_1, $(\varphi_1 - \varphi_1')$, u_2, ϑ_2, ψ_2 and $(\varphi_2 - \varphi_2')$ are the desired parameters.

The system of equations (55) is nonlinear, and it is necessary /45 to utilize the method of sequential approximations for its solution. With a fixed value of ψ_1, prescribing the value of ϑ_1, we will find u_1 and ϑ_2 from equations (55a) and (55b), and then, knowing the values of u_1, ϑ_1 and ϑ_2, we will determine u_2 from (55c); by substituting u_2, ϑ_2 into equation (55d), we will obtain ψ_2 Knowing ϑ_1, ψ_1, ϑ_2 and ψ_2, and solving (55e) and (55f) together, we will find $(\varphi_1' - \varphi_1)$ and $(\varphi_2' - \varphi_2)$. If, with a fixed value of ψ_1, the obtained magnitudes of u_1, ϑ_1, φ_1', u_2, ϑ_2, ψ_2, φ_2' do not satisfy the calculation process set forth above, then values of u_1, ϑ_1, φ_1', u_2, ϑ_2, ψ_2,

φ_2', which satisfy the system of equations in (55), will not yet be found.

By substituting the obtained values into equations (44), (24) and (25), we will find the coordinates of the point of the line of meshing, and the point of contact on the surfaces of the teeth of wheels 1 and 2.

By utilizing the obtained data of φ_1' and φ_2', one may compile the position function of the wheels: $\varphi_2'=f(\varphi_1')$ in tabular form. Their instantaneous gear ratio may be determined by two means: a) differentiation of the position function; b) determination of the position of the instantaneous axis of rotation of the gear wheels. In the latter case, by utilizing the coordinates of the point of contact of the surfaces of the teeth in the fixed system S_0 (coordinates of the point of the meshing line) and the projection of the unit vector of the normal at this very same point, it is necessary, by extending the normal, to determine the point M of intersection of the normal with the plane of the axes of the gear wheels. The line O_0M, where O_0 is the point of intersection of the axes of the wheels, is the instantaneous axis of rotation of the wheels in relative movement.

7. Calculation Example

Examined herein is the case of cutting of gear wheels with a gear ratio $i_{12}=1$, in which, as is common knowledge, the area of contact has a sharply-pronounced diagonal nature. The direction of the teeth on wheel 1 is to the right, and on wheel 2 — to the left; the number of teeth $Z_1=Z_2=20$, the face modulus $m_s=10$ mm, the nominal value of $\beta=35°$, the angle of meshing $\alpha=20°$, and the width of the ring B=40 mm. The nominal radius of the cutting head $r_u=152.4$ mm, and the set of the blades W=1.524 mm. The method of cutting of the teeth is one-sided.

Utilizing geometric calculation of the ENIMS system [2], and the calculation formulas for adjustment of the machine tool, set forth in section 5, we will obtain the following values of the parameters of meshing of the gear wheels, the cutting heads and the adjustment of the machine tool:

	Wheel 1	Wheel 2
Average generatrix L, mm	121,4214	121,4214
Angle of initial bevel δ_i, degrees	45	45
Shank angle γ_i, degrees.	4,125308	4,125308
Angle of inner bevel δ_{bi}, degrees.	40,87460	40,87460
Blade angle α_i, degrees.	14,62072	22,37028
Angle of spiral in reference plane β_i, degrees . .	36,24563	33,78168
Generating radius r_i, mm	155,9378	148,0431
Angular setting q_i, mm	77,05435	72,50271
Radial setting b_i, mm.	120,0421	120,0173
Gear ratio of generating chain $i_{p_i k}$.	0,7089436	0,7089436

With the utilization of the system of equations in (55), the following results are obtained. /46

Angle of rotation ψ_1 of originating wheel P_1, °.	−5	0	+5
Parameters of originating surface Σ_1^p:			
ϑ_1, degrees.	133,30	130,81	128,25
u_1, mm	517,24	514,88	512,45
Parameters of originating surface Σ_2^p:			
ϑ_2, degrees.	131,21	128,72	126,16
u_2, mm	392,21	388,08	385,68
Angle of rotation ψ_2 of originating wheel P_2, °.	−5,05	0	5,014
Angle of rotation φ_1' of wheel 1, degrees	−7,353	0	—
Angle of rotation φ_2' of wheel 2, degrees	−7,28	0	—
Coordinates of point of meshing line, mm:			
$x_0^{(1)}$	−2,6	0,00	2,7
$y_0^{(1)}$	−6,1	0,00	6,5
$z_0^{(1)}$	116,4	121,42	126,2
Coordinates of point of contact of surface of teeth of wheel 1, mm:			
x_1	−84,2	−85,86	—
y_1	4,7	0,00	—
z_1	80,5	85,86	—
Coordinates of point of contact on surface of teeth of wheel 2, mm:			
x_2	80,6	85,86	—
y_2	4,2	0,00	—
z_2	84,1	85,86	—

Presented in Figure 7,a is the projection of the working line of the surface of the teeth of wheel 1 on the plane $x_1O_0z_1$, and in Figure 7,b — the projection of the working line of the surface of the teeth of wheel 2 on the plane $x_2O_0z_2$.

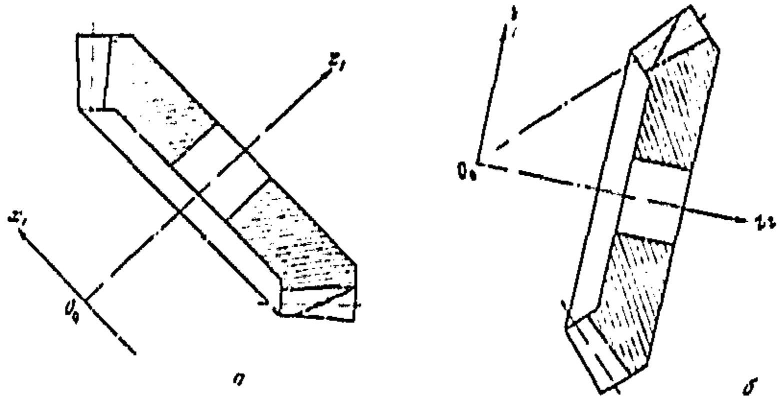

Fig. 7. Projections of working lines of surfaces of teeth of gear wheels

The enumerated calculations and constructions (Fig. 7) indicate that, with the recommended method of calculation of the parameters of the cutting heads and the adjustment of the machine tool, the working lines pass through the designated midpoints of the surfaces of the teeth, but the contact area has a diagonal form, while /47 the gear ratio of the wheels is not constant. The indicated defects may be decreased by correction of the adjustment of the machine tool, but this should be the subject of a special study.

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