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A FIRST-ORDER TIME-DOMAIN GREEN'S FUNCTION APPROACH TO SUPERSONIC UNSTEADY FLOW

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FOREWORD

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ABSTRACT

In this report a time-domain Green's Function Method for unsteady supersonic potential flow around complex aircraft configurations is presented.

We focus here on the supersonic range wherein the linear potential flow assumption is valid. In this range the effects of the nonlinear terms in the unsteady supersonic compressible velocity potential equation are negligible and therefore these terms will be omitted in this report.

The Green's function method is employed in order to convert the potential-flow differential equation into an integral one. This integral equation is then discretized, in space through standard finite-element technique, and in time through finite-difference, to yield a linear algebraic system of equations relating the unknown potential to its prescribed co-normalwash (boundary condition) on the surface of the aircraft. The arbitrary complex aircraft configuration (e.g., finite-thickness wing, wing-body-tail) is discretized into hyperboloidal (twisted quadrilateral) panels. The potential and co-normalwash are assumed to vary linearly within each panel. Consistent with the spatial linear (first-order) finite-element approximations, the potential and co-normalwash are assumed to vary linearly in time.

The long range goal of our research is to develop a comprehensive theory for unsteady supersonic potential aerodynamics which is capable of yielding accurate results even in the low supersonic (i.e., high transonic) range.

LIST OF SYMBOLS

a_∞	speed of sound
$\overline{a}_1$	contravariant base vector, see equation (25)
$\overline{a}_2$	contravariant base vector, see equation (25)
β	$(M_\infty^2 - 1)^{1/2}$
$E(\overline{P})$	domain function, see equation (15)
$\delta(\overline{p})$	Dirac delta function
δ_{jk}	Kronecker delta
F_{ik}	finite-element shape function
G	Green's function
H	Heaviside function
k	reduced frequency, $\omega\ell/U_\infty$
ℓ	reference length
M_∞	free stream Mach number U_∞/a_∞
$\overline{n}$	unit normal to σ_B
$\overline{N}$	unit normal to Σ_B
N_e	total number of elements
N_n	total number of nodes
ω	circular frequency
$\overline{p}$	point having coordinates x, y, z
$\overline{p}_*$	control point, (x_*, y_*, z_*)
$\overline{P}$	point having coordinates X, Y, Z
$\overline{P}_*$	control point, (X_*, Y_*, Z_*)
$p.f.$	Hadamard finite part
$\overline{R}$	$\overline{P} - \overline{P}_*$
R'	hyperbolic radius, see equation (11)
t	time
T	nondimensional time, $a_\infty\beta t/\ell$
U_∞	velocity of undisturbed flow
x, y, z	space coordinates
X, Y, Z	nondimensional Prandtl-Glauert coordinates $X = x/\beta\ell, Y = y/\ell, Z = z/\ell$
σ	surface of body in x, y, z space
Σ	surface of body in X, Y, Z space
Σ_i	surface of element i in X, Y, Z space
ϕ	perturbation velocity potential

Φ nondimensional perturbation velocity potential, $\phi/U_\infty \ell$

ψ co-normalwash in x, y, z space

Ψ co-normalwash in X, Y, Z space

Operators

∇_o^2 $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
Laplace operator in the physical space

$\nabla \circ \nabla$ $\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} - \frac{\partial^2}{\partial Z^2}$

$\frac{d}{dt}$ $\frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x}$

$\circ$ Supersonic dot product, see equation (5)

1. INTRODUCTION

In this report we demonstrate how the Green's Function Method of Potential Aerodynamics may be implemented in the time domain so as to enable it to handle unsteady supersonic flow around complex aircraft configurations.

We focus here on the supersonic range wherein the linear potential flow assumption is valid. In this range the effects of the nonlinear terms in the unsteady supersonic compressible velocity potential equation are negligible, and therefore these terms will be omitted in this report.

The Green's function method (Ref. 1) is employed in order to convert the potential-flow differential equation into an integral one. This integral equation is then discretized, in space through standard finite-element technique (Refs. 2 and 3), and in time through finite-difference (Ref. 4.), to yield a linear algebraic system of equations relating the unknown potential to its prescribed co-normalwash on the surface of the aircraft. The arbitrary complex aircraft configuration (e.g., finite-thickness wing, wing-body-tail) is discretized into hyperboloidal (twisted quadrilateral) panels. The potential and co-normalwash are assumed to vary linearly within each panel. Consistent with the spatial linear (first-order) finite- element approximations, the potential and co-normalwash are assumed to vary linearly in time.

A frequency-domain first-order Green's function formulation for linear oscillatory supersonic flow has been developed recently (Ref. 5). Presented here is the corresponding theory formulated in the time-domain setting.

The long range goal of our research is to develop a comprehensive theory for unsteady supersonic potential aerodynamics for the low supersonic (i.e., high transonic) range.

1.1 A Brief Description of the Green's Function Method

Before getting into the specifics of this report we begin with a brief description of the Green's Function Method. This method applies to the equation of the perturbation velocity potential. The potential function Φ at any point $\overline{P}_*$ in the flow field is given by an integral of terms containing the value of the potential and its co-normal derivative on the surface, σ, surrounding the body and its wake. An integral equation for the potential on the surface of the body is obtained by letting the point $\overline{P}_*$ † approach a point on the surface. With this method, the wake is a natural by-product and is treated as a layer of doublets. It may be noted that the integral equation does not require that the boundary condition on the co-normalwash be satisfied, but rather makes use of the continuity of the potential as the control point approaches the surface σ. The tangency boundary conditions are automatically satisfied by the type of representation obtained with the Green's Function Method.

In current applications, the surface of the aircraft is divided into small quadrilateral elements. Each element is replaced by a paraboloidal hyperboloid surface defined by the four corners of the element. In this process the continuity of the surface is maintained but discontinuities in the slopes are introduced. The aircraft wake, on the other hand, is divided into strips parallel to the streamlines. These wake strips originate from the trailing edge and extend to infinity downstream. It should be noted that integrals over these wake strips may be carried out in an analogous way to their subsonic counterpart (see Refs. 3 and 6).

In the 0^{th} order theory, the unknown Φ (in the Prandtl-Glauert Space) is assumed to be constant within each element, while in the 1st order theory Φ is taken in the form $\Phi = \Phi_o + \xi\Phi_1 + \eta\Phi_2 + \xi\eta\Phi_3$ where ξ and η are local element-wise surface coordinates, and the coefficients $\Phi_o, \ldots, \Phi_3$ are chosen to interpolate the Φ values at the four corners of the element. In either situation the integral equation is approximated by a system of algebraic equations. This system of algebraic equations is then solved by standard numerical methods. It has been found (see Ref. 7) that in the supersonic range at least a 1-st order theory is required in order to yield a nonsingular set of algebraic equations due to a numerical rather than physical anomaly. The numerical implementation presented in this report employs 1^{st} order panels.

2. UNSTEADY SUPERSONIC FLOW

Our point of departure is the linearized equation for the unsteady potential compressible aerodynamic flow

$$\nabla_o^2\phi - \frac{1}{a_\infty^2}\frac{d^2\phi}{dt^2} = 0 \tag{1}$$

where ∇_o^2 is the Laplace operator in the physical space while ϕ is the perturbation potential. Choosing a frame of reference such that the undisturbed flow has velocity U_∞ in the

† The bar is used herein to indicate vector quantities.

direction of the positive x-axis, the linearized total time derivative is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x}. \tag{2}$$

Introducing the generalized Prandtl-Glauert transformation

$$X = x/\beta l,\ Y = y/l,\ Z = z/l,\ T = a_\infty \beta t/l,\ \Phi = \phi/U_\infty l \tag{3}$$

where l is a characteristic length, $M_\infty = U_\infty/a_\infty$ and $\beta = (M_\infty^2 - 1)^{1/2}$, Eq. (1) yields*

$$\nabla \circ \nabla \Phi + \beta^2 \frac{\partial^2 \Phi}{\partial T^2} + 2M_\infty \frac{\partial^2 \Phi}{\partial X \partial T} = 0 \tag{4}$$

where $\circ$ stands for the supersonic dot-product defined as

$$\bar{a} \circ \bar{b} = a_x b_x - a_y b_y - a_z b_z \tag{5}$$

where $\bar{a}$ and $\bar{b}$ are two arbitrary vectors. Thus the operator $\nabla \circ \nabla$ stands for

$$\nabla \circ \nabla = \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} - \frac{\partial^2}{\partial Z^2} \tag{6}$$

Note that $\bar{a} \circ \bar{a}$ is not necessarily positive. We define the 'super-norm' of a vector $\bar{a}$ as

$$a' = \|\bar{a}\| = |\bar{a} \circ \bar{a}|^{1/2} \tag{7}$$

and will use this notation in later sections of this report.

2.1 Supersonic Integral Equation

In order to obtain the Supersonic Green's function integral equation we proceed as follows: With $\overline{P}$ and $\overline{P}_*$ representing the sending and receiving points respectively, the Green's function G for Equation (10) satisfies

$$\nabla \circ \nabla G + \beta^2 \frac{\partial^2 G}{\partial T^2} + 2M_\infty \frac{\partial^2 G}{\partial X \partial T} = \delta(\overline{P} - \overline{P}_*,\ T - T_*) \tag{8}$$

$$G = 0 \text{ at } \infty$$

one well known solution of which is given by (see Ref. 7)

$$G = \frac{H}{4\pi R'} \left(\delta_{\Theta^+} + \delta_{\Theta^-}\right) \tag{9}$$

where

$$H(\overline{P}, \overline{P}_*) = \begin{cases} 1 & \text{if } X - X_* \leq 0 \text{ and } \overline{R} \circ \overline{R} \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

* See Appendix C for derivation.

Here we define

$$\overline{R} = \overline{P} - \overline{P}_* = (X - X_*)\bar{i} + (Y - Y_*)\bar{j} + (Z - Z_*)\bar{k}$$
$$R' = \|\overline{R}\| = |(X - X_*)^2 - (Y - Y_*)^2 - (Z - Z_*)^2|^{1/2} \tag{11}$$

while

$$\delta_{\Theta^\pm} = \delta(T - T_* + \Theta^\pm) \tag{12}$$

with

$$\Theta^\pm = M_\infty(X_* - X) \pm R' \tag{13}$$

Here $\{\overline{P}|H(\overline{P},\overline{P}_*) = 1\}$ defines the zone of influence, or Mach forecone, with vertex at $\overline{P}_*$.

Next for a closed bounded surface Σ bounding a volume V, we define the domain function

$$E(\overline{P}_*) = \begin{cases} 1 & \text{if } \overline{P}_* \notin V \\ 0 & \text{if } \overline{P}_* \in V \end{cases} \tag{14}$$

Note that for $\overline{P}_*$ on Σ the function $E(\overline{P}_*)$ will measure the so-called supersonic solid angle of Σ at $\overline{P}_*$ (see Ref. 7 for details). Hence $E(\overline{P}_*)$ satisfies the notation

$$E(\overline{P}_*) = \oiint_\Sigma \overline{N} \circ \nabla\left(\frac{H}{4\pi R'}\right) d\Sigma + 1 \tag{15}$$

where $\overline{N}$ is the outward unit normal to Σ. Applying the Green's Function Method, with the Green's Function G given by Eq. (9), it can be shown (see Ref. 7 for details) that Eq. (8) may be integrated to yield

$$4\pi E(\overline{P}_*)\Phi(\overline{P}_*, T_*) = \oiint_\Sigma([\Psi]^{\Theta^+} + [\Psi]^{\Theta^-})\left(\frac{-H}{R'}\right)d\Sigma +$$
$$\oiint_\Sigma([\Phi]^{\Theta^+} + [\Phi]^{\Theta^-})\frac{\partial}{\partial N^c}\left(\frac{H}{R'}\right)d\Sigma -$$
$$\oiint_\Sigma\left\{\left[\frac{\partial\Phi}{\partial T}\right]^{\Theta^+}\frac{\partial\hat{\Theta}^+}{\partial N^c} - \left[\frac{\partial\Phi}{\partial T}\right]^{\Theta^-}\frac{\partial\hat{\Theta}^-}{\partial N^c}\right\}\frac{H}{R'}d\Sigma \tag{16}$$

where the co-normalwash Ψ is defined as

$$\Psi = \nabla\Phi \circ \overline{N} = \partial\Phi/\partial N^c \tag{17}$$

with the conormal $\overline{N}^c$ defined as

$$\overline{N}^c = -N_x\bar{i} + N_y\bar{j} + N_z\bar{k} \tag{18}$$

so that

$$\frac{\partial}{\partial N^c} = -N_x\frac{\partial}{\partial X} + N_y\frac{\partial}{\partial Y} + N_z\frac{\partial}{\partial Z} \tag{19}$$

Furthermore

$$\hat{\Theta}^\pm = M_\infty(X - X_*) \pm R' \tag{20}$$

It should be noted here that Σ includes the surfaces of the aircraft and its wake. Throughout this report, for simplicity, the condition of supersonic trailing edge will be assumed. Thus, in the absence of the wake, Σ represents the aircraft surface. However, the general method employed here is not limited to the supersonic trailing-edge condition. The treatment of the wake when trailing edges are subsonic is analogous to the time-domain subsonic formulations and hence the reader is referred to Ref. 4 for further details.

3. NUMERICAL FORMULATION

In this section, a space and time discretization procedure will be introduced in order to approximate the integral equation by a linear algebraic system of time-delay difference equations. Solving this time-dependent system in a step-by-step manner yields the desired time-dependent perturbation velocity potential solution on the aircraft surface. Once the velocity potential is known at a given time step, the pressure coefficient may be computed through Bernoulli's Theorem.

3.1 Finite Element Formulation

Assuming that the surface Σ is divided into N_e small finite elements Σ_i, Equation (16) yields

$$\begin{aligned} 4\pi E(\overline{P}_*)\Phi(\overline{P}_*,T_*) = \sum_{i=1}^{N_e}\Bigg\{ & \iint_{\Sigma_i} \left([\Psi]^{\Theta^+} + [\Psi]^{\Theta^-}\right)\left(\frac{-H}{R'}\right) d\Sigma \\ & + \sum_{i=1}^{N_e} \iint_{\Sigma_i} \left([\Phi]^{\Theta^+} + [\Phi]^{\Theta^-}\right)\frac{\partial}{\partial N^c}\left(\frac{H}{R'}\right) d\Sigma \\ + \sum_{i=1}^{N_e} & \iint_{\Sigma_i} \left(\left[\frac{\partial \Phi}{\partial T}\right]^{\Theta^+} \frac{\partial \hat{\Theta}^+}{\partial N^c} + \left[\frac{\partial \Phi}{\partial T}\right]^{\Theta^-} \frac{\partial \hat{\Theta}^-}{\partial N^c}\right)\left(\frac{-H}{R'}\right) d\Sigma \Bigg\} \end{aligned} \tag{21}$$

Each surface element Σ_i is approximated by a hyperbolic paraboloid given in the form

$$\overline{P} = \overline{P}_c + \overline{P}_1\xi + \overline{P}_2\eta + \overline{P}_3\xi\eta \tag{22}$$

where $\overline{P}_c, \overline{P}_1, \overline{P}_2$ and $\overline{P}_3$ are obtained in terms of the locations of the four corner points as (See Fig. 1)

$$\begin{pmatrix} \overline{P}_c \\ \overline{P}_1 \\ \overline{P}_2 \\ \overline{P}_3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} \overline{P}_{+-} \\ \overline{P}_{++} \\ \overline{P}_{-+} \\ \overline{P}_{--} \end{pmatrix} \tag{23}$$

Here, $\overline{P}_{+-}$, for instance, refers to the element corner for which $\xi = +1$ and $\eta = -1$. The other corners, $\overline{P}_{++}$, $\overline{P}_{-+}$ and $\overline{P}_{--}$ are defined similarly. It may be noted that the surface defined according to Eq. (22) is continuous since adjacent elements have in common the straight line connecting the two common corner points.

3.2 Surface Geometry for Hyperboloidal Elements

We note that the geometry of the hyperboloidal element is a particular case of the general equation for a surface in three- dimensional Euclidean space which is given by

$$\overline{P} = \overline{P}(\xi, \eta) \tag{24}$$

where ξ and η are generalized curvilinear coordinates on the surface. For a hyperboloidal surface, the two basis vectors $\overline{a}_1$ and $\overline{a}_2$ are given by

$$\begin{aligned} \overline{a}_1 &= \frac{\partial \overline{P}}{\partial \xi} = \overline{P}_1 + \eta \overline{P}_3 \\ \overline{a}_2 &= \frac{\partial \overline{P}}{\partial \eta} = \overline{P}_2 + \xi \overline{P}_3 \end{aligned} \tag{25}$$

(See equation (22).

The unit normal to the surface is given by

$$\overline{N} = \overline{a}_1 \times \overline{a}_2 / |\overline{a}_1 \times \overline{a}_2| \tag{26}$$

and is directed according to the right hand rule such that the normal points outward from the surface (see Fig. 2). The surface element $d\Sigma$ is given by

$$d\Sigma = |\overline{a}_1 d\xi \times \overline{a}_2 d\eta|$$

or

$$d\Sigma = |\overline{a}_1 \times \overline{a}_2| d\xi d\eta \tag{27}$$

3.3 First Order Space Discretization

In what follows we shall take the potential function $\Phi(\xi, \eta, \tau)$ over a surface element, say Σ_i, at a fixed time τ, as

$$\begin{aligned} \Phi^i(\xi, \eta, \tau) = [&(1+\xi)(1-\eta)\Phi_{+-}(\tau) + (1+\xi)(1+\eta)\Phi_{++}(\tau) \\ &+ (1-\xi)(1+\eta)\Phi_{-+}(\tau) + (1-\xi)(1-\eta)\Phi_{--}(\tau)]/4 \end{aligned} \tag{28}$$

Equation (28) expresses the values of Φ^i at any point (ξ, η, τ) of Σ_i in terms of the values of Φ at the four corner points–also at time τ.

More generally, consider the first-order *global* shape function with the following definition:

$$F_{ik}(\xi, \eta) = \begin{cases} (1+\xi)(1-\eta)/4 & \text{if node } k \text{ coincides with corner } +- \text{ of element } i \\ (1+\xi)(1+\eta)/4 & \text{if node } k \text{ coincides with corner } ++ \text{ of element } i \\ (1+\xi)(1+\eta)/4 & \text{if node } k \text{ coincides with corner } -+ \text{ of element } i \\ (1-\xi)(1-\eta)/4 & \text{if node } k \text{ coincides with corner } -- \text{ of element } i \\ 0 & \text{otherwise.} \end{cases}$$

(29)

With F_{ik} defined by Equation (29), Equation (28) may be rewritten as

$$\Phi^i(\xi,\eta,\tau) = \sum_{k=1}^{N_n} F_{ik}(\xi,\eta)\Phi_k(\tau) \tag{30}$$

where N_n is the total number of nodes on the surface Σ, and Φ_k denotes Φ at the k^{th} node.

Similarly, the supersonic co-normalwash $\Psi(= \overline{N} \circ \nabla\Phi)$ may be represented by (See Ref. 5)

$$\Psi^i(\xi,\eta,\tau) = \sum_{k=1}^{N_n} F_{ik}(\xi,\eta)\Psi_k(\tau) \tag{31}$$

The same first-order finite-element approximation, Equation (29), has been employed for Ψ. Note that if Φ is approximated by the 1^{st} order finite-element expression while Ψ is represented by a 0^{th} order formula, $\Psi^i(\xi,\eta,\tau) = \Psi^i(0,0,\tau) = \text{const.}$, a mixed type formulation would result. In subsequent portions of this report, Φ and Ψ will be taken 1^{st} order.

3.4 Numerical Approximation of the Integral Equation

Due to the hyperbolic nature of the governing unsteady supersonic velocity potential equation, the Heaviside function appears in the resulting supersonic Green's function integral formulation, (eqs. (16) and (21)). The appearance of the Heaviside function $H(\overline{P},\overline{P}_*)$ under the integral restricts the integration to that portion of the panel that is within the Mach Forecone with vertex at $\overline{P}_*$.

Note here that the potential, Φ, and the co-normalwash, Ψ, under the integral are both confined to the Mach Forecone region. A Taylor expansion in both space and time may be written for Φ and Ψ so that they become explicitly dependent on the space and time variables. The spatial derivatives of the Taylor expansion for $\Phi(\text{or}\Psi)$ may be computed through the first-order finite- element representation introduced in the preceding subsection. The temporal derivatives, on the other hand, may be approximated by a backward finite-difference in time. With $\Phi(\Psi)$ expressed as an explicit function of time, the constraint that $\Phi(\Psi)$ varies within the Forecone may be imposed, through the Heaviside function, in a straightforward manner.

We remark that the time variable includes a delay term which is a function of the spatial variables (see Eq. (13)). The kernels of the integrals are essentially of the type $(R')^{-n}$ (where n equals 1 for source and n equals 3 for doublet). We adopt the tactic of treating R', in addition to ξ and η, as independent variable. Thus, we express Φ and Ψ via Taylor expansion in these variables. Explicitly, let us rewrite the time delay expression as

$$T - \Theta^{\pm} = \tau(\xi,\eta,T) \pm R' \tag{32}$$

where

$$\tau(\xi,\eta,T) = T - M_\infty[X_*(\xi,\eta) - X(\xi,\eta)] \tag{33}$$

Thus, Φ may be expanded in ξ,η and R' around a point "o" which is the geometric center of the portion of the element that is in the Mach Forecone, i.e.,

$$\begin{aligned}
\Phi(\xi,\eta,\tau \pm R') \simeq \Phi(\xi_o,\eta_o,\tau_o \pm R'_o) + \left(\frac{\partial\Phi}{\partial\xi} + \frac{\partial\Phi}{\partial\tau}\frac{\partial\tau}{\partial\xi}\right)\Big|_o(\xi-\xi_o) \\
+\left(\frac{\partial\Phi}{\partial\eta} + \frac{\partial\Phi}{\partial\tau}\frac{\partial\tau}{\partial\eta}\right)\Big|_o(\eta-\eta_o) \pm \frac{\partial\Phi}{\partial\tau}\Big|_o(R'-R'_o) \\
+\left(\frac{\partial^2\Phi}{\partial\xi\partial\eta} + \frac{\partial^2\Phi}{\partial\xi\partial\tau}\frac{\partial\tau}{\partial\eta} + \frac{\partial^2\Phi}{\partial\eta\partial\tau}\frac{\partial\tau}{\partial\xi} + \frac{\partial^2\Phi}{\partial\tau^2}\frac{\partial\tau}{\partial\xi}\frac{\partial\tau}{\partial\eta}\right)\Big|_o(\xi-\xi_o)(\eta-\eta_o) \\
+\frac{1}{2}\frac{\partial^2\Phi}{\partial\tau^2}|_o(R'-R'_o)^2
\end{aligned} \tag{34}$$

where the subscript o denotes evaluation at the geometric center of the portion of the element that is in the Mach Forecone while

$$\frac{\partial\tau}{\partial\xi} = M_\infty(X_1 + X_3\eta) \tag{35}$$

and

$$\frac{\partial\tau}{\partial\eta} = M_\infty(X_2 + X_3\xi) \tag{36}$$

with

$$X_1 = \frac{\partial\overline{P}}{\partial\xi}\cdot\bar{i} = (\overline{P}_1 + \overline{P}_3\eta)\cdot\bar{i} \tag{37}$$

$$X_2 = \frac{\partial\overline{P}}{\partial\eta}\cdot\bar{i} = (\overline{P}_2 + \overline{P}_3\xi)\cdot\bar{i} \tag{38}$$

$$X_3 = \frac{\partial^2\overline{P}}{\partial\xi\partial\eta}\cdot i = \overline{P}_3\cdot i \tag{39}$$

Expansion (34) is 2^{nd} order accurate. Consistent with the finite-element approximation for the space variables, the 2^{nd} order spatial derivative terms, i.e., $\partial^2\Phi/\partial\xi^2$ and $\partial^2\Phi/\partial\eta^2$, and higher order spatial derivatives, vanish and hence it is sufficient to include up to 2^{nd} order terms only in Eq. (34).

Next. recall in Section 3.3 that

$$\Phi^i(\xi,\eta,\tau) = \sum_{k=1}^{N_n} F_{ik}(\xi,\eta)\Phi_k(\tau)$$

where $F_{ik}(\xi,\eta,\tau)$ is given by Eq. (29) and τ is a prescribed time. Hence we may compute the spatial derivatives of Φ at the point o as

$$\frac{\partial\Phi^i(\xi_o,\eta_o,\tau_o \pm R'_o)}{\partial\xi} = \sum_{k=1}^{N_n}\frac{\partial F_{ik}(\xi_o,\eta_o)}{\partial\xi}\Phi_k(\tau_o \pm R'_o) \tag{40}$$

$$\frac{\partial \Phi^i(\xi_o,\eta_o,\tau_o \pm R'_o)}{\partial \eta} = \sum_{k=1}^{N_n} \frac{\partial F_{ik}(\xi_o,\eta_o)}{\partial \eta} \Phi_k(\tau_o \pm R'_o) \tag{41}$$

$$\frac{\partial^2 \Phi^i(\xi_o,\eta_o,\tau_o \pm R'_o)}{\partial \eta} = \sum_{k=1}^{N_n} \frac{\partial^2 F_{ik}(\xi_o,\eta_o)}{\partial \xi \partial \eta} \Phi_k(\tau_o \pm R'_o) \tag{42}$$

$$\frac{\partial^2 \Phi^i(\xi_o,\eta_o,\tau_o \pm R'_o)}{\partial \xi \partial \tau} = \sum_{k=1}^{N_n} \frac{\partial F_{ik}(\xi_o,\eta_o)}{\partial \xi} \frac{\partial \Phi_k(\tau_o \pm R'_o)}{\partial \tau} \tag{43}$$

$$\frac{\partial^2 \Phi^i(\xi_o,\eta_o,\tau_o \pm R'_o)}{\partial \eta \partial \tau} = \sum_{k=1}^{N_n} \frac{\partial F_{ik}(\xi_o,\eta_o)}{\partial \eta} \frac{\partial \Phi_k(\tau_o \pm R'_o)}{\partial \tau} \tag{44}$$

Making use of Equations (40) through (44) with $\overline{P}_*$ coinciding with the node j, Equation (21) becomes

$$\begin{aligned} 4\pi E_j \Phi_j(T) = \sum_{k=1}^{N_n} \Big\{ & \tilde{B}_{jk}[\Psi_k(T-\Theta^+_{jk}) + \Psi_k(T-\Theta^-_{jk})] \\ & + \hat{B}_{jk}[\dot{\Psi}_k(T-\Theta^+_{jk}) + \dot{\Psi}_k(T-\Theta^-_{jk})] \\ & + \check{B}_{jk}[\ddot{\Psi}_k(T-\Theta^+_{jk}) + \ddot{\Psi}_k(T-\Theta^-_{jk})] \\ & + \tilde{C}_{jk}[\Phi_k(T-\Theta^+_{jk}) + \Phi_k(T-\Theta^-_{jk})] \\ & + \hat{C}_{jk}[\dot{\Phi}_k(T-\Theta^+_{jk}) + \dot{\Phi}_k(T-\Theta^-_{jk})] \\ & + \check{C}_{jk}[\ddot{\Phi}_k(T-\Theta^+_{jk}) + \ddot{\Phi}_k(T-\Theta^-_{jk})] \\ & + \hat{D}_{jk}[\dot{\Phi}_k(T-\Theta^+_{jk}) + \dot{\Phi}_k(T-\Theta^-_{jk})] \\ & + \check{D}_{jk}[\ddot{\Phi}_k(T-\Theta^+_{jk}) + \ddot{\Phi}_k(T-\Theta^-_{jk})] \Big\} \end{aligned} \tag{45}$$

where Θ^+_{jk} and Θ^-_{jk} are understood, for notational simplicity, to be evaluated at the point o and

$$\begin{aligned} \tilde{B}_{jk} = \sum_{i=1}^{N_e} \iint_{\Sigma_i} [F^o_{ik} + \frac{\partial F^o_{ik}}{\partial \xi}(\xi - \xi_o) + \frac{\partial F^o_{ik}}{\partial \eta}(\eta - \eta_o) + \\ \frac{\partial^2 F^o_{ik}}{\partial \xi \partial \eta}(\xi_o - \xi_o)(\eta - \eta_o)]\left(\frac{-H}{R'}\right) d\Sigma \end{aligned} \tag{46a}$$

$$\begin{aligned} \hat{B}_{jk} = \sum_{i=1}^{N_e} \iint_{\Sigma_i} [F^o_{ik} \frac{\partial \tau}{\partial \xi}\Big|_o (\xi - \xi_o) + F^o_{ik} \frac{\partial \tau}{\partial \eta}\Big|_o (\eta - \eta_o) \\ + \left(\frac{\partial F^o_{ik}}{\partial \xi} \frac{\partial \tau}{\partial \eta}\Big|_o + \frac{\partial F^o_{ik}}{\partial \eta} \frac{\partial \tau}{\partial \xi}\Big|_o\right)(\xi - \xi_o)(\eta - \eta_o)]\left(\frac{-H}{R'}\right) d\Sigma \\ + (R' - R'_o)\left(\frac{-H}{R'}\right) d\Sigma \end{aligned} \tag{46b}$$

$$\check{B}_{jk} = \sum_{i=1}^{N_e} \iint_{\Sigma_i} (R' - R'_o)^2 \left(\frac{-H}{R'}\right) d\Sigma \tag{46c}$$

$$\tilde{C}_{jk} = \sum_{i=1}^{N_e} \iint_{\Sigma_i} [F_{ik}^o + \frac{\partial F_{ik}^o}{\partial \xi}(\xi - \xi_o) + \frac{\partial F_{ik}^o}{\partial \eta}(\eta - \eta_o)$$

$$+ \frac{\partial^2 F_{ik}^o}{\partial \xi \partial \eta}(\xi - \xi_o)(\eta - \eta_o)] \frac{\partial}{\partial N^c}\left(\frac{H}{R'}\right) d\Sigma \tag{47a}$$

$$\hat{C}_{jk} = \sum_{i=1}^{N_e} \iint_{\Sigma_i} [F_{ik}^o \frac{\partial \tau}{\partial \xi}\bigg|_o (\xi - \xi_o) + F_{ik}^o \frac{\partial \tau}{\partial \eta}\bigg|_o (\eta - \eta_o)$$
$$+ \left(\frac{\partial F_{ik}^o}{\partial \xi}\frac{\partial \tau}{\partial \eta}\bigg|_o + \frac{\partial F_{ik}^o}{\partial \eta}\frac{\partial \tau}{\partial \xi}\bigg|_o\right)(\xi - \xi_o)(\eta - \eta_o)] \tag{47b}$$
$$+ (R' - R'_o)\frac{\partial}{\partial N^c}\left(\frac{-H}{R'}\right) d\Sigma$$

$$\check{C}_{jk} = \sum_{i=1}^{N_e} \iint_{\Sigma_i} (R' - R'_o)^2 \frac{\partial}{\partial N^c}\left(\frac{H}{R'}\right) d\Sigma \tag{47c}$$

$$\hat{D}_{jk} = \sum_{i=1}^{N_e} \iint_{\Sigma_i} \left(\frac{-H}{R'}\right) \frac{\partial R'}{\partial N^c} d\Sigma \tag{48a}$$

$$\check{D}_{jk} = \sum_{i=1}^{N_e} \iint_{\Sigma_i} (R' - R'_o)\left(\frac{-H}{R'}\right)\left(\frac{\partial R'}{\partial N^c}\right) d\Sigma \tag{48b}$$

where F_{ik}^o, for instance, indicates F_{ik} evaluated at (ξ_o, η_o).

Equation (45), with the coefficients defined by Equations (46) through (48), is the time-dependent aerodynamic operator relating the potential to its co-normalwash at N_n nodal locations. These integral coefficients will be evaluated by a hybrid analytical-numerical scheme. More explicitly, each of these integrals will be integrated analytically in the ξ-direction followed by a Gaussian Quadrature numerical integration procedure in the η direction. Discussion of these integrals will be presented in Section 4.

In Equation (45), the spatial variables have been discretized by a 1^{st} order finite-element approximation. Following the procedure introduced in Ref. 4, discretization of the time-dependent functions, Φ and Ψ, will be discussed in the following.

First, the time variable may be written as

$$T = (n + \alpha)\Delta T \text{ where } n \text{ is an integer and } 0 \leq \alpha < 1 \tag{49}$$

Next, a continuous function $g(T)$ with T given by Equation (49) may be approximated as

$$g(T) = g[(n + \alpha)\Delta T] \cong (1 - \alpha)g(n\Delta T) + \alpha g[(n + 1)\Delta T] \tag{50}$$

Setting $T = n\Delta T$ and using Equations (46) through (50), Equation (45) may be written as

$$
\begin{aligned}
4\pi E_j \Phi_j(n) = \sum_{k=1}^{N_n} \tilde{B}_{jk}\Big\{&(1-\alpha^+_{jk})\Psi_k(n-m^+_{jk}) + (1-\alpha^-_{jk})\Psi_k(n-m^-_{jk}) \\
&+ \alpha^+_{jk}\Psi_k(n-m^+_{jk}-1) + \alpha^-_{jk}\Psi_k(n-m^-_{jk}-1)\Big\} \\
&+ \hat{B}_{jk}\Big\{(1-\alpha^+_{jk})\dot{\Psi}_k(n-m^+_{jk}) + (1-\alpha^-_{jk})\dot{\Psi}_k(n-m^-_{jk}) \\
&+ \alpha^+_{jk}\dot{\Psi}_k(n-m^+_{jk}-1) + \alpha^-_{jk}\dot{\Psi}_k(n-m^-_{jk}-1)\Big\} \\
&+ \check{B}_{jk}\Big\{(1-\alpha^+_{jk})\ddot{\Psi}_k(n-m^+_{jk}) + (1-\alpha^-_{jk})\ddot{\Psi}_k(n-m^-_{jk}) \\
&+ \alpha^+_{jk}\ddot{\Psi}_k(n-m^+_{jk}-1) + \alpha^-_{jk}\ddot{\Psi}_k(n-m^-_{jk}-1)\Big\} \\
&+ \tilde{C}_{jk}\Big\{(1-\alpha^+_{jk})\Phi_k(n-m^+_{jk}) + (1-\alpha^-_{jk})\Phi_k(n-m^-_{jk}) \\
&+ \alpha^+_{jk}\Phi_k(n-m^+_{jk}-1) + \alpha^-_{jk}\Phi_k(n-m^-_{jk}-1)\Big\} \\
&+ \hat{C}_{jk}\Big\{(1-\alpha^+_{jk})\dot{\Phi}_k(n-m^+_{jk}) + (1-\alpha^-_{jk})\dot{\Phi}_k(n-m^-_{jk}) \\
&+ \alpha^+_{jk}\dot{\Phi}_k(n-m^+_{jk}-1) + \alpha^-_{jk}\dot{\Phi}_k(n-m^-_{jk}-1)\Big\} \\
&+ \check{C}_{jk}\Big\{(1-\alpha^+_{jk})\ddot{\Phi}_k(n-m^+_{jk}) + (1-\alpha^-_{jk})\ddot{\Phi}_k(n-m^-_{jk}) \\
&+ \alpha^+_{jk}\ddot{\Phi}_k(n-m^+_{jk}-1) + \alpha^-_{jk}\ddot{\Phi}_k(n-m^-_{jk}-1)\Big\} \\
&+ \hat{D}_{jk}\Big\{(1-\alpha^+_{jk})\dot{\Phi}_k(n-m^+_{jk}) + (1-\alpha^-_{jk})\dot{\Phi}_k(n-m^-_{jk}) \\
&+ \alpha^+_{jk}\dot{\Phi}_k(n-m^+_{jk}-1) + \alpha^-_{jk}\dot{\Phi}_k(n-m^-_{jk}-1)\Big\} \\
&+ \check{D}_{jk}\Big\{(1-\alpha^+_{jk})\ddot{\Phi}_k(n-m^+_{jk}) + (1-\alpha^-_{jk})\ddot{\Phi}_k(n-m^-_{jk}) \\
&+ \alpha^+_{jk}\ddot{\Phi}_k(n-m^+_{jk}-1) + \alpha^-_{jk}\ddot{\Phi}_k(n-m^-_{jk}-1)\Big\}
\end{aligned}
\tag{51}
$$

where ΔT has been dropped for notational simplicity (i.e., $\Phi(n\Delta T) = \Phi(n)$) and

$$\Theta^\pm_{jk} = (m^\pm_{jk} + \alpha^\pm_{jk})\Delta T \tag{52}$$

On the other hand, the time derivative of Φ may be evaluated by finite-difference as

$$
\begin{aligned}
\frac{\partial \Phi^i_k(\tau_\pm)}{\partial \tau} &= [\Phi_k(\tau_\pm) - \Phi_k(\tau_\pm - \Delta T)]/\Delta T; \\
\frac{\partial^2 \Phi^i_k(\tau_\pm)}{\partial \tau^2} &= [\Phi_k(\tau_\pm) - 2\Phi_k(\tau_\pm - \Delta T) + \Phi_k(\tau_\pm - 2\Delta T)]/\Delta T^2
\end{aligned}
\tag{53}
$$

where $\tau_\pm = \tau_o \pm R'_o$ and ΔT is an arbitrary small time increment.

Note that if the time increment is sufficiently large, the integer delay-time index (for instance, $m^{\pm}_{jk}$) might be zero so that at the present time step an unknown would appear on the right hand side of Equation (51). The co-normalwash on the other hand is known (prescribed) at all times—including the current time step.

In order to express $\Phi_k(n)$ explicitly (in terms of $\Phi_k(n - m^{\pm}_{jk})$), re-define the integral coefficients so that

$$\begin{aligned}
B^{1+}_{jk} &= (1-\alpha^{+}_{jk})(\tilde{B}_{jk} + \hat{B}_{jk}/\Delta T + \check{B}_{jk}/\Delta T^2) \\
B^{1-}_{jk} &= (1-\alpha^{-}_{jk})(\tilde{B}_{jk} + \hat{B}_{jk}/\Delta T + \check{B}_{jk}/\Delta T^2) \\
B^{2+}_{jk} &= \tilde{B}_{jk}\alpha^{+}_{jk} - \hat{B}_{jk}(1-2\alpha^{+}_{jk})/\Delta T - \check{B}_{jk}(2-3\alpha^{+}_{jk})/\Delta T^2 \\
B^{2-}_{jk} &= \tilde{B}_{jk}\alpha^{-}_{jk} - \hat{B}_{jk}(1-2\alpha^{-}_{jk})/\Delta T - \check{B}_{jk}(2-3\alpha^{-}_{jk})/\Delta T^2 \\
B^{3+}_{jk} &= -\hat{B}_{jk}\alpha^{+}_{jk}/\Delta T + \check{B}_{jk}(1-3\alpha^{+}_{jk})/\Delta T^2 \\
B^{3-}_{jk} &= -\hat{B}_{jk}\alpha^{-}_{jk}/\Delta T + \check{B}_{jk}(1-3\alpha^{-}_{jk})/\Delta T^2 \\
B^{4+}_{jk} &= \check{B}_{jk}\alpha^{+}_{jk}/\Delta T^2 \\
B^{4-}_{jk} &= \check{B}_{jk}\alpha^{-}_{jk}/\Delta T^2
\end{aligned} \tag{54}$$

$$\begin{aligned}
C^{o+}_{jk} &= \begin{cases} (1-\alpha^{+}_{jk})[\tilde{C}_{jk} + (\hat{C}_{jk}+\hat{D}_{jk})/\Delta T + (\check{C}_{jk}+\check{D}_{jk})/\Delta T^2] & \text{if } m^{+}_{jk} = 0 \\ 0 & \text{if } m^{+}_{jk} \neq 0 \end{cases} \\
C^{o-}_{jk} &= \begin{cases} (1-\alpha^{-}_{jk})[\tilde{C}_{jk} + (\hat{C}_{jk}+\hat{D}_{jk})/\Delta T + (\check{C}_{jk}+\check{D}_{jk})/\Delta T^2] & \text{if } m^{-}_{jk} = 0 \\ 0 & \text{if } m^{-}_{jk} \neq 0 \end{cases} \\
C^{1+}_{jk} &= \begin{cases} (1-\alpha^{+}_{jk})[\tilde{C}_{jk} + (\hat{C}_{jk}+\hat{D}_{jk})/\Delta T + (\check{C}_{jk}+\check{D}_{jk})/\Delta T^2] & \text{if } m^{+}_{jk} \neq 0 \\ 0 & \text{if } m^{+}_{jk} = 0 \end{cases} \\
C^{1-}_{jk} &= \begin{cases} (1-\alpha^{-}_{jk})[\tilde{C}_{jk} + (\hat{C}_{jk}+\hat{D}_{jk})/\Delta T + (\check{C}_{jk}+\check{D}_{jk})/\Delta T^2] & \text{if } m^{-}_{jk} \neq 0 \\ 0 & \text{if } m^{-}_{jk} = 0 \end{cases} \\
C^{2+}_{jk} &= \tilde{C}_{jk}\alpha^{+}_{jk} - (\hat{C}_{jk}+\hat{D}_{jk})(1-2\alpha^{+}_{jk})/\Delta T - (\check{C}_{jk}+\check{D}_{jk})(2-3\alpha^{+}_{jk})/\Delta T^2 \\
C^{2-}_{jk} &= \tilde{C}_{jk}\alpha^{-}_{jk} - (\hat{C}_{jk}+\hat{D}_{jk})(1-2\alpha^{-}_{jk})/\Delta T - (\check{C}_{jk}+\check{D}_{jk})(2-3\alpha^{-}_{jk})/\Delta T^2 \\
C^{3+}_{jk} &= (\hat{C}_{jk}+\hat{D}_{jk})(-\alpha^{+}_{jk}/\Delta T) + (\check{C}_{jk}+\check{D}_{jk})(1-3\alpha^{+}_{jk})/\Delta T^2 \\
C^{3-}_{jk} &= (\hat{C}_{jk}+\hat{D}_{jk})(-\alpha^{-}_{jk}/\Delta T) + (\check{C}_{jk}+\check{D}_{jk})(1-3\alpha^{-}_{jk})/\Delta T^2 \\
C^{4+}_{jk} &= (\check{C}_{jk}+\check{D}_{jk})\alpha^{+}_{jk}/\Delta t^2 \\
C^{4-}_{jk} &= (\check{C}_{jk}+\check{D}_{jk})\alpha^{-}_{jk}/\Delta t^2
\end{aligned} \tag{55}$$

With equations (54) and (55), Equation (51) becomes

$$\sum_{k=1}^{Nn} A_{jk}\Phi_k(n) = b_j(n) \tag{56}$$

with

$$A_{jk} = E_{jk} - C_{jk}^{o+} - C_{jk}^{o-} \tag{57}$$

$$b_j(n) = \sum_{k=1}^{N_n} \sum_{i=1}^{4} \Big\{ B_{jk}^{i+} \Psi_k(n - m_{jk}^{+} - i + 1) + B_{jk}^{i-} \Psi_k(n - m_{jk}^{-} - i + 1) +$$

$$C_{jk}^{i+} \Phi_k(n - m_{jk}^{+} - i + 1) + C_{jk}^{i-} \Phi_k(n - m_{jk}^{-} - i + 1) \Big\}$$

where E_{jk} is defined according to Eq. (15).

Thus, solving the algebraic system of equations (56) yields the solution Φ, at current time step n. Note that the right hand side of Equation (56) is completely determined since it includes only Φ at preceding time steps.

4. PANEL INTEGRALS FOR THE SOURCE AND DOUBLET COEFFICIENTS

The supersonic unsteady Green's Function numerical formulation, Equation (45), Sec. 3, involves essentially two types of integrals. The source integrals (Equations (46a-c)) possess an integrable singularity on the Mach forecone, so they may be interpreted in a classical sense. The doublet integrals (Equations (47a-c)), which involve a derivative of the Green's Function and hence must be viewed in distribution sense in order to be properly interpreted.

With this in mind, we shall formulate the procedure for calculating the supersonic coefficients. This procedure is described in detail in Ref. 5. With the exception of the two additional integrals arising in the process of introducing the time-domain numerical formulation, (Equations 48a and b) the remaining integral calculating procedure is presented in Ref. 5. Therefore, for completeness, it is sufficient to give only a summary of this procedure here.

As mentioned earlier, once the coefficients in Equations (45) through (48) are computed, the potential at time t may be given in terms of the potential and co-normalwash at preceding times through the algebraic relation in Equation (56). Due to the complexity of these surface integrals, it is impossible to obtain closed-form analytical expression for them. As in Ref. 5, we shall follow the tactic of computing these double (surface) integrals, in ξ and η, analytically in the ξ direction and numerically in the η direction.

4.1 Some Definite Integrals

In what follows we shall allow the η integral to be evaluated numerically and will analytically compute the ξ integral.

By using equation (29) of Section 3.2, we deduce that

$$|\bar{a}_1 \times \bar{a}_2| = |(\bar{P}_1 + \eta \bar{P}_3) \times (\bar{P}_2 + \xi \bar{P}_3)| \tag{58}$$

We now make the approximation

$$|\bar{a}_1(\eta) \times \bar{a}_2(\xi)| \sim A(\eta) + B(\eta)\xi \tag{59}$$

where

$$A(\eta) = |\bar{a}_1(\eta) \times \bar{a}_2(0)| \tag{60}$$

and

$$B(\eta) = \frac{|\bar{a}_1(\eta) \times \bar{a}_2(1)| - |\bar{a}_1(\eta) \times \bar{a}_2(-1)|}{2} \tag{61}$$

Making use of Equation (59) and recalling the relationship (27), the integrals (46) through (48) may be reduced to the consideration of the following ξ-integrals:

$$\begin{aligned} \alpha_m(\eta) &= \text{p.f.} \int_{-1}^{+1} \xi^m \frac{H}{(R')^3} d\xi \\ \beta_m(\eta) &= \int_{-1}^{+1} \xi^m \frac{H}{R'} d\xi \end{aligned} \tag{62}$$

$$\gamma_m(\eta) = \int_{-1}^{+1} \xi^m H R' d\xi$$

$$\Delta_m(\eta) = \text{p.f.} \int_{-1}^{1} \xi^m \frac{H}{(R')^2} d\xi$$

and

$$\kappa_m(\eta) = \int_{-1}^{1} \xi^m H d\xi$$

for $m = 0, 1, 2$

The $\beta_m(\eta)$, $\gamma_m(\eta)$ and $\kappa_m(\eta)$ are convergent integrals while the 'p.f.' in the $\alpha_m(\eta)$ and $\Delta_m(\eta)$ integrals indicate that these integrals must be interpreted in the sense of the *Hadamard Finite Part* in order to assume a finite value.

4.2 Some Indefinite Integrals

In this section we shall explicitly obtain the indefinite integrals

$$\begin{aligned} \hat{\alpha}_m(\xi, \eta) &= \int \xi^m \frac{d\xi}{R'^3} \\ \hat{\beta}_m(\xi, \eta) &= \int \xi^m \frac{d\xi}{R'} \end{aligned} \tag{63}$$

$$\hat{\gamma}_m(\xi, \eta) = \int \xi^m R' d\xi$$

$$\hat{\Delta}_m(\xi, \eta) = \int \xi^m \frac{d\xi}{R'^2}$$

and

$$\hat{\kappa}(\xi, \eta) = \int \xi^m d\xi$$

for $m = 0, 1, 2$

which occur in the process of evaluating the first order supersonic source and doublet coefficients.

For convenience we write

$$R' = \|\overline{R}\| = (a\xi^2 + b\xi + c)^{1/2} \tag{64}$$

where

$$\begin{aligned} a &= \overline{a}_1 \circ \overline{a}_1 \\ b &= 2(\overline{P}_0 + \eta\overline{P}_2) \circ \overline{a}_1 \\ c &= (\overline{P}_0 + \eta\overline{P}_2) \circ (\overline{P}_0 + \eta\overline{P}_2) \end{aligned} \tag{65}$$

We also let

$$d = b^2 - 4ac = -4(\overline{R} \times \overline{a}_1 \circ \overline{R} \times \overline{a}_1) \tag{66}$$

From standard integral tables we obtain

$$\hat{\alpha}_0(\xi,\eta) = \int \frac{d\xi}{R'^3} = \begin{cases} -(4a\xi + 2b)/dR', & \text{for } d \neq 0 \\ -1/R'(2a\xi + b), & \text{for } d = 0,\ a > 0 \\ \xi c^{-3/2} & \text{for } d = 0,\ a = 0,\ c > 0 \\ 0 & \text{otherwise} \end{cases} \tag{67}$$

$$\hat{\alpha}_1(\xi,\eta) = \int \frac{\xi d\xi}{R'^3} = \begin{cases} -(aR')^{-1} - \Big(b/2a\Big)\hat{\alpha}_o & \text{for } a \neq 0,\ d \neq 0 \\ 2R'/b^2 + \Big(2c/b^2\Big)/R' & \text{for } a = 0,\ d \neq 0 \\ -(aR')^{-1} - (b/2a)\hat{\alpha}_o & \text{for } a > 0,\ d = 0 \\ \xi^2 c^{-3/2}/2 & \text{for } a = 0,\ b = 0,\ c > 0 \\ 0 & \text{otherwise} \end{cases} \tag{68}$$

A third basic integral we shall need is

$$\hat{\beta}_o(\xi,\eta) = \int \frac{d\xi}{R'} = \begin{cases} ln|(2R'\sqrt{a} + 2a\xi + b)/\sqrt{a}|/\sqrt{a} & \text{for } a > 0 \\ & \quad d \neq 0 \\ -\tan^{-1}\Big((2a\xi + b)/2\sqrt{-a}R'\Big)/\sqrt{-a} & \text{for } a < 0 \\ & \quad d \neq 0 \\ ln|2R'\sqrt{a} + 2a\xi + b|/\sqrt{a} & \text{for } a > 0 \\ & \quad d = 0 \\ 2R'/b & \text{for } a = 0 \\ & \quad d \neq 0 \\ \xi/\sqrt{c} & \text{for } a = 0 \\ & \quad d = 0, c > 0 \\ 0 & \text{for otherwise} \end{cases} \tag{69}$$

In terms of $\hat{\alpha}_o, \hat{\alpha}_1$ and $\hat{\beta}_o$ above we are able to express the integrals $\hat{\alpha}_2, \hat{\beta}_1, \hat{\gamma}_o$

$$\hat{\alpha}_2(\xi,\eta) = \int \frac{\xi^2 d\xi}{R'^3} = \begin{cases} (\hat{\beta}_o - b\hat{\alpha}_1 - c\hat{\alpha}_o)/a & \text{for } a \neq 0 \\ \Big[(2/3)R'^3 - 4cR' - 2c^2/R'\Big]/b^3 & \text{for } a = 0 \\ & \quad b \neq 0 \\ \xi^3 c^{-3/2}/3 & \text{for } a = 0 \\ & \quad b = 0, c > 0 \\ 0 & \text{otherwise} \end{cases} \tag{70}$$

$$\hat{\beta}_1(\xi,\eta) = \int \frac{\xi d\xi}{R'} = \begin{cases} 2R' - \hat{\beta}_o b/2a & \text{for } a \neq 0 \\ (2/3)R'^3/b^2 - (2/b^2)R'c & \text{for } a = 0 \\ & \quad b \neq 0 \\ \xi^2 c^{-1/2}/2 & \text{for } a = 0 \\ & \quad b = 0, c > 0 \\ 0 & \text{otherwise} \end{cases} \tag{71}$$

$$\hat{\gamma}_o(\xi,\eta) = \int R' d\xi = \begin{cases} (2a\xi + b)R'/4a + (4ac - b^2)\hat{\beta}_o/8a & \text{for } a \neq 0 \\ & \quad d \neq 0 \\ (2/3b)(b\xi + c)^{3/2} & \text{for } a = 0 \\ & \quad d \neq 0 \\ \sqrt{c}\xi & \text{for } a = 0 \\ & \quad b = 0, c > 0 \\ 0 & \text{otherwise} \end{cases} \tag{72}$$

In terms of $\hat{\beta}_o, \hat{\beta}_1$ and $\hat{\gamma}_o$ we obtain $\hat{\gamma}_1, \hat{\beta}_2$ and $\hat{\gamma}_2$

$$\hat{\gamma}_1(\xi,\eta) = \int \xi R' d\xi = \begin{cases} [(2/3)R'^3 - b\hat{\gamma}_o]/2a & \text{for } a \neq 0 \\ [(2/5)R'^5 - (2c/3)R'^3]/b^2 & \begin{array}{l}\text{for } a = 0 \\ \quad b \neq 0\end{array} \\ \xi^2\sqrt{c}/2 & \begin{array}{l}\text{for } a = 0 \\ \quad b = 0, c > 0\end{array} \\ 0 & \text{otherwise} \end{cases} \tag{73}$$

$$\hat{\beta}_2(\xi,\eta) = \int \frac{\xi^2 d\xi}{R'} = \begin{cases} (\xi/2a)R' - (3b/4a)\hat{\beta}_1 - (c/2a)\hat{\beta}_o & \text{for } a \neq 0 \\ (2/5)(b\xi + c)^{5/2}/b^3 - (2c/b)\hat{\beta}_1 - (c^2/b^2)\hat{\beta}_o & \begin{array}{l}\text{for } a = 0 \\ \quad b \neq 0\end{array} \\ \xi^2/3\sqrt{c} & \begin{array}{l}\text{for } a = 0 \\ \quad b = 0 \\ \quad c > 0\end{array} \\ 0 & \text{otherwise} \end{cases} \tag{74}$$

$$\hat{\gamma}_2(\xi,\eta) = \int \xi^2 R' d\xi = \begin{cases} (\xi - 5b/6a)R'^3/4a + (5b^2 - 4ac)\hat{\gamma}_o/16a^2 & \text{for } a \neq 0 \\ [(2/7)R'^7 - (4/5)cR'^5 + (2/3)c^2R'^3]/b^3 & \begin{array}{l}\text{for } a = 0 \\ \quad b \neq 0\end{array} \\ \xi^3\sqrt{c}/3 & \begin{array}{l}\text{for } a = 0 \\ \quad b = 0, c > 0\end{array} \\ 0 & \quad\text{otherwise} \end{cases} \tag{75}$$

Next, consider the basic integral

$$\hat{\Delta}_m(\xi,\eta) = \int \frac{\xi^m}{R'^2} d\xi = \int \frac{\xi^m d\xi}{a\xi^2 + b\xi + c} \quad \text{for } m = 0, 1, 2$$

From standard tables we find

$$\hat{\Delta}_o(\xi,\eta) = \begin{cases} \log[(2a\xi + b - \sqrt{a})/(2a\xi + b + \sqrt{d})]/\sqrt{d} & \begin{array}{l}\text{for } a > 0 \\ \quad d > 0\end{array} \\ 2\tan^{-1}[(2a\xi + b)/\sqrt{-d}]/\sqrt{-d} & \begin{array}{l}\text{for } a > 0 \\ \quad d < 0\end{array} \\ -2/(2a\xi + b) & \begin{array}{l}\text{for } a > 0 \\ \quad d = 0\end{array} \\ \log|b\xi + c|/b & \begin{array}{l}\text{for } a = 0 \\ \quad b \neq 0\end{array} \\ \xi/c & \begin{array}{l}\text{for } a = 0 \\ \quad b = 0, \\ \quad c \neq 0\end{array} \end{cases} \tag{76}$$

$$\hat{\Delta}_1(\xi,\eta) = \begin{cases} (\log|a\xi^2+b\xi+c| - b\hat{\Delta}_o)/2a & \text{for } a \neq 0 \\ \xi/b - c\log|b\xi+c|/b^2 & \text{for } a = 0 \\ & \quad b \neq 0 \\ \xi^2/2c & \text{for } a = 0 \\ & \quad b = 0, c \neq 0 \end{cases} \tag{77}$$

and

$$\hat{\Delta}_2(\xi,\eta) = \begin{cases} (\xi - b\hat{\Delta}_1 - c\hat{\Delta}_o)/a & \text{for } a \neq 0 \\ \xi^2/2b - c\hat{\Delta}_1/b & \text{for } a = 0 \\ & \quad b \neq 0 \\ \xi^2/3c & \text{for } a = 0 \\ & \quad b = 0 \end{cases} \tag{78}$$

Note also

$$\begin{aligned} \hat{\kappa}_o &= \int d\xi = \xi \\ \hat{\kappa}_1 &= \int \xi d\xi = \xi^2/2 \\ \hat{\kappa}_2 &= \int \xi^2 d\xi = \xi^3/3 \end{aligned} \tag{79}$$

In sections 4.3 and 4.4 we shall see how the indefinite integrals $\hat{\alpha}_m$, $\hat{\beta}_m$ $\hat{\gamma}_m$, $\hat{\Delta}_m$ and $\hat{\kappa}_m$ given above are utilized in evaluating $\alpha_m(\eta), \beta_m(\eta)$, $\gamma_m(\eta)$, $\hat{\Delta}_m(\eta)$ and $\kappa_m(\eta)$ for $m = 0, 1$ and 2.

4.3 EVALUATION OF THE INTEGRALS FOR FULL PANELS

In the case of a full panel, i.e. one in which $\{(\xi,\eta)| -1 \leq \xi \leq +1, -1 \leq \eta \leq +1\}$ lies entirely within the open Mach forecone $= \{\overline{R}|X - X_* < 0 \text{ and } \overline{R} \circ \overline{R} > 0\}$ the evaluation of the α_m, β_m and γ_m follows easily. In this situation these integrals are convergent and the Hadamard Finite part is not needed.

The indefinite integrals $\hat{\alpha}_m, \hat{\beta}_m, \hat{\gamma}_m, \hat{\Delta}_m$ and $\hat{\kappa}_m$ given above are to be utilized in evaluating $\alpha_m(\eta), \beta_m(\eta)$, $\gamma_m(\eta)$, $\Delta_m(\eta)$ and $\kappa_m(\eta)$ for $m = 0, 1, 2$.

$$\begin{aligned} \alpha_m(\eta) &= \hat{\alpha}_m(1,\eta) - \hat{\alpha}_m(-1,\eta) \\ \beta_m(\eta) &= \hat{\beta}_m(1,\eta) - \hat{\beta}_m(-1,\eta) \\ \gamma_m(\eta) &= \hat{\gamma}_m(1,\eta) - \hat{\gamma}_m(-1,\eta) \\ \Delta_m(\eta) &= \hat{\Delta}_m(1,\eta) - \hat{\Delta}_m(-1,\eta) \\ \kappa_m(\eta) &= \hat{\kappa}_m(1,\eta) - \hat{\kappa}_m(-1,\eta) \end{aligned} \tag{80}$$

$$\text{for } m = 0, 1, 2$$

In this situation the $\alpha_m(\eta), \beta_m(\eta), \gamma_m(\eta)$, $\Delta_m(\eta)$ and $\kappa_m(\eta)$ are analytic functions of η for $-1 \leq \eta \leq +1$, and the numerical computation of the definite integrals involving these functions and appearing in Section 4 may be carried out by a standard numerical integration such as Gaussian quadrature.

4.4 Evaluation of the Integrals for Partial Panels

In the situation where a panel lies partially within the Mach forecone the evaluation of $\alpha_m(\eta), \beta_m(\eta), \gamma_m(\eta), \Delta_m(\eta)$ and $\kappa_m(\eta)$ takes a bit more doing. We note that integrals $\hat{\alpha}_m$, $m = 0, 1, 2$ are singular on the Mach cone $= \{R | X - X_* \leq 0, \overline{R} \circ \overline{R} = 0\}$. Thus, for ξ_* such that $R(\xi_*, \eta)$ lies on the Mach cone, we must evaluate $\hat{\alpha}_m(\xi_*, \eta)$ in accordance with the *Hadamard Finite Part*. We obtain

$$\text{p.f. } \hat{\alpha}_m(\xi_*, \eta) = 0, \quad m = 0, 1 \tag{81}$$

$$\text{p.f. } \hat{\Delta}_m(\xi_*, \eta) = 0, \quad m = 0, 1, 2$$

On the contrary, the integrals $\hat{\beta}_m$, $\hat{\gamma}_m$ and $\hat{\kappa}_m$, $m = 0, 1, 2$, are not singular at (ξ_*, η) so in order to calculate the value of these integrals at that point it is enough to plug that point into the expressions for these integrals. There are a few provisos however. A problem will occur in calculating $\hat{\beta}_o(\xi_*, \eta)$ from expression (69) in the case $a < 0$ and $d \neq 0$, since $R'(\xi_*, \eta) = 0$. However the identity

$$\hat{\beta}_o(\xi, \eta) = -\frac{1}{\sqrt{-a}} \tan^{-1}\left(\frac{2a\xi + b}{2\sqrt{-a}R'}\right) = -\frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{2a\xi + b}{\sqrt{d}}\right) \tag{82}$$

for $a < 0$, $d \neq 0$ together with the fact that $2a\xi_* + b = \pm d$ and $d > 0$ for ξ_* with $\overline{R}(\xi_*, \eta)$ on the Mach cone shows that

$$\hat{\beta}_o(\xi_*, \eta) = -\frac{\pi}{2\sqrt{-a}} \text{ sgn } (2a\xi_* + b)$$

$$\text{for } a < 0, \; d \neq 0 \tag{83}$$

In addition, we show, in Appendix A, that

$$\text{p.f. } \hat{\alpha}_2(\xi_*, \eta) = \begin{cases} \frac{\pi}{2(-a)^{3/2}} \text{ sgn } (2a\xi_* + b) & \text{for } a < 0, d = 0 \\ 0 & \text{otherwise} \end{cases} \tag{84}$$

Since the expressions $\hat{\beta}_1, \hat{\gamma}_0, \hat{\gamma}_1, \hat{\beta}_2$ and $\hat{\gamma}_2$ are all given in terms of $\hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}_o$ and R', there is no difficulty in evaluating these functions at ξ_* with $\overline{R}(\xi_*, \eta)$ on the Mach cone via use of Equations (70)—(84) where applicable.

We are now ready to investigate our integrals for a fixed η with $-1 \leq \eta \leq 1$. We look at the intersection of the interval $-1 \leq \xi \leq +1$ with the Mach forecone $\{\overline{R} | \overline{R} \circ \overline{R} \geq 0, \; X - X_* \leq 0\}$. Four cases may occur:

(i) The intersection is empty.

(ii) The intersection is a closed interval $[\xi_\ell, \xi_u]$ with $\xi_\ell \leq \xi_u$.
(See Figure 3.)

(iii) The intersection is a single point ξ_o with $|\xi_o| = 1$ but $\overline{R}(\xi_o, \eta) \circ \overline{a}_1 \neq 0$.
(See Figure 4.)

(iv) The intersection is a single point ξ_o with

(a) $-1 < \xi_o < 1$ and $\overline{R}(\xi_o, \eta) \circ \overline{a}_1 = 0$ (See Figure 5a.)

(b) $|\xi_o| = 1$ and $\overline{R}(\xi_o, \eta) \circ \overline{a}_1 = 0$ (See Figure 5b.)

A point (ξ_o, η_o) where $X - X_* < 0$, $\overline{R} \circ \overline{R} = 0$ and $\overline{R} \circ \overline{a}_1 = 0$ is called a *critical point* if $-1 \leq \xi_o \leq 1$, $-1 \leq \eta_o \leq 1$, i.e., (ξ_o, η_o) is a critical point in Case (iv) above.

Case (i)

In this case we define

$$\begin{aligned} \alpha_m(\eta) = 0, \ \beta_m(\eta) = 0, \ \gamma_m(\eta) = 0 \\ \Delta_m(\eta) = 0 \text{ and } \kappa_m(\eta) = 0 \end{aligned} \tag{85}$$

for $m = 0, 1, 2$

Case (ii)

In this case we define

$$\begin{aligned} \alpha_m(\eta) &= \hat{\alpha}_m(\xi_u, \eta) - \hat{\alpha}_m(\xi_\ell, \eta) \\ \beta_m(\eta) &= \hat{\beta}_m(\xi_u, \eta) - \hat{\beta}_m(\xi_\ell, \eta) \\ \gamma_m(\eta) &= \hat{\gamma}_m(\xi_u, \eta) - \hat{\gamma}_m(\xi_\ell, \eta) \\ \Delta_m(\eta) &= \hat{\Delta}_m(\xi_n, \eta) - \hat{\Delta}_m(\xi_\ell, \eta) \\ \kappa_m(\eta) &= \hat{\kappa}_m(\xi_n, \eta) - \hat{\kappa}_m(\xi_\ell, \eta) \end{aligned} \tag{86}$$

for $m = 0, 1, 2$

where $\hat{\alpha}_m$, $\hat{\beta}_m$, $\hat{\gamma}_m$, $\hat{\Delta}_m$ and $\hat{\kappa}_m$ are given as in Section 5. We remark however that if either ξ_u or ξ_ℓ or both lie on the Mach cone the evaluation of the $\hat{\alpha}_m$, $\hat{\beta}_m$, and $\hat{\Delta}_m$ for such ξ_* must follow (81), (83) and (84) of this section where applicable.

Case (iii)

In this situation, we have a limiting situation where either $\xi_\ell \to +1$ or $\xi_u \to -1$. The functions $\hat{\alpha}_m(\xi, \eta)$, $\hat{\beta}_m(\xi, \eta)$ and $\hat{\gamma}_m(\xi, \eta)$ are continuous at such a point and thus we find that

$$\begin{aligned} \alpha_m(\eta) = 0 \\ \beta_m(\eta) = 0 \end{aligned}$$

$$\gamma_m(\eta) = 0 \tag{87}$$
$$\Delta_m(\eta) = 0$$

and

$$\kappa_m(\eta) = 0$$

for $m = 0, 1, 2$

Case (iv) - (ξ_o, η_o) is a Critical Point

Assume at first that $-1 < \eta_o < +1$. Then the equation $\overline{R} \circ \overline{R} = a\xi^2 + b\xi + c = 0$ possesses a double root at $\xi_o = -b/2a$ with $-1 < \xi_o < 1$, provided $a < 0$. At this point, $d = 0$. (Note: If $a = \overline{a}_1 \circ \overline{a}_1 > 0$, we cannot have a point on the Mach cone with $\overline{R} \circ \overline{a}_1 = 0$ unless $|\overline{R}| = 0$ identically.) Now in accordance with Eq. (A.19) of Appendix A, we find that in this case $(a < 0,\ d = 0)$

$$\begin{aligned} \beta_o(\eta_o) &= \frac{\pi}{\sqrt{-a}} \\ \beta_1(\eta_o) &= \frac{\pi}{\sqrt{-a}} \xi_o \\ \beta_2(\eta_o) &= \frac{\pi}{\sqrt{-a}} \xi_o^2 \end{aligned} \tag{88}$$

while $\gamma_m(\eta_o) = 0$ and $\kappa_m(\eta_o) = 0$ for $m = 0, 1, 2$.

The $\alpha_m(\eta)$, $m = 0, 1, 2$ behave in a more complicated manner. These expressions may be given in the form

$$\alpha_m(\eta) = \alpha_m^{\text{reg}}(\eta) + \alpha_m^{\text{sp}} \delta(\eta - \eta_o) \qquad \text{for } m = 0, 1, 2$$

where:

$$\begin{aligned} \alpha_o^{\text{reg}}(\eta_o) &= 0 \\ \alpha_1^{\text{reg}}(\eta_o) &= 0 \end{aligned} \tag{89}$$

and

$$\alpha_2^{\text{reg}}(\eta_o) = -\frac{\pi}{(-a)^{3/2}}$$

and the special distributional contribution to α_o, α_1, and α_2 is given by $\alpha_m^{\text{sp}} \delta(\eta - \eta_o)$ where:

$$\alpha_m^{\text{sp}} = \frac{-\pi \xi_o^m}{|\overline{R} \cdot \overline{a}_1 \times \overline{a}_2|} \tag{90}$$

for $m = 0, 1, 2$

In the cases where $-1 < \eta_o < 1$ and $|\xi_o| = 1$ the only change with the above is that the special contributions α_o^{sp}, α_1^{sp}, and α_2^{sp} are divided in half.

Thus

$$\alpha_m^{\text{sp}} = -\frac{\pi}{2} \frac{\xi_o^m}{|\overline{R} \cdot \overline{a}_1 \times \overline{a}_2|} \tag{91}$$

We have not yet considered the situation where $|\eta_o| = 1$. Here if $|\xi_o| < 1$ we set

$$\alpha_m^{\mathrm{sp}} = \begin{cases} 0 & \text{if } \eta_o \overline{R} \circ \overline{a}_2 > 0 \\ -\xi_o^m \pi / |\overline{R} \cdot \overline{a}_1 \times \overline{a}_2| & \text{if } \eta_o \overline{R} \circ \overline{a}_2 < 0 \end{cases} \tag{92}$$

for $m = 0, 1, 2$

and if both $|\eta_o| = 1$ and $|\xi_o| = 1$ we set

$$\alpha_m^{\mathrm{sp}} = \begin{cases} 0 & \text{if } \eta_o \overline{R} \circ \overline{a}_2 > 0 \\ -\frac{1}{2} \xi_o^m \pi / |\overline{R} \cdot \overline{a}_1 \times \overline{a}_2| & \text{if } \eta_o \overline{R} \circ \overline{a}_2 < 0 \end{cases} \tag{93}$$

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APPENDIX A—CRITICAL POINT AND SPECIAL SINGULARITY

In this appendix we shall study singular integrals of the form

$$\text{p.f.} \int \frac{S(\xi,\eta)}{R'^3} d\xi = F_s(\eta) \tag{A.1}$$

where $S(\xi,\eta)$ is a polynomial in ξ of degree ≤ 1.

We focus on the situation where $a = \bar{a}_1 \circ \bar{a}_1 < 0$ in the neighborhood of a point $(\xi_o, \eta_o), -1 < \xi_o < +1, -1 < \eta_o < +1$ where $\overline{R} \circ \overline{R} = 0$, $\overline{R} \circ \bar{a}_1 = 0$ and $X - X_* < 0$.

We shall find that this integral (A.1) exists only in distribution sense as a function of η and in fact takes the 'value'

$$F_s(\eta) = -\frac{S(\xi_o,\eta_o)}{|\overline{R} \cdot \bar{a}_1 \times \bar{a}_2|} \bigg|_{\substack{\xi=\xi_o \\ \eta=\eta_o}} \pi\delta(\eta - \eta_o) \tag{A.2}$$

Proof: We study in detail the prototype situation of Fig. 6 where $\overline{R} \circ \bar{a}_2 > 0$ at (ξ_o, η_o) and the Mach cone intersects $\eta = +1$ at two ξ values both with $|\xi| < 1$. The situation with $\overline{R} \circ \bar{a}_2 < 0$ at (ξ_o, η_o) with the cone inverted may be handled by analogy.

We note that for $\eta_1 < \eta_o$ the cone does not interact the line $\eta = \eta_1$ so that clearly for such η_1, $F_s(\eta_1) = 0$.

Further for $\eta_1 > \eta_o$ the integral (A.1) has a 3/2-order singularity at two distinct ξ values on the line $\eta = \eta_1$. By definition of the Hadamard Finite Part it follows that $F_s(\eta_1) = 0$ here again.

In order to demonstrate (A.2) we therefore need only show that

$$\int_{-1}^{+1} F_s(\eta) d\eta = -\frac{S(\xi_o,\eta_o)\pi}{|\overline{R} \cdot \bar{a}_1 \times \bar{a}_2|} \bigg|_{\substack{\xi=\xi_o \\ \eta=\eta_o}} \tag{A.3}$$

In fact if we denote by

$$W_s(\eta) = \text{p.f.} \int_{\eta_o}^{\eta} \int_{\text{allowable } \xi} \frac{S(\xi,\eta)}{R'^3} d\xi \, d\eta \tag{A.4}$$

for $\eta > \eta_o$

it is clear from previous remarks about $F_s(\eta)$ that

1: $W_s(\eta)$ will be independent of η for $\eta > \eta_o$, and that

2:

$$W_s(+1) = \int_{1}^{+1} F_s(\eta) \, d\eta$$

so that it suffices for us to prove that for any $\eta > \eta_o$

$$W_s(\eta) = \left. \frac{S(\xi_o, \eta_o)\pi}{|\overline{R} \cdot \ddot{a}_1 \times \ddot{a}_2|} \right|_{\substack{\xi = \xi_o \\ \eta = \eta_o}} \tag{A.5}$$

Interchanging order of integration in (A.4) we find that for any $\eta_1 > \eta_o$

$$W_s(\eta_1) = \text{ p.f. } \int\limits_{\text{allowable } \xi} \int\limits_{\eta_o}^{\eta_1} \frac{S(\xi,\eta)}{R'^3}\, d\eta\, d\xi \tag{A.6}$$

We now denote by $V(\xi, \eta_1)$ the integral

$$V(\xi, \eta_1) = \text{ p.f. } \int\limits_{\eta_o}^{\eta_1} \frac{S(\xi,\eta)}{R'^3}\, d\eta \tag{A.7}$$

for $\eta_1 > \eta_o$

We have previously seen that

$$\int_\xi V(\xi, \eta) d\xi = W_s(\eta) = W_s$$

$$\overline{R} \circ \overline{R} > 0$$

$$X - X_* < 0$$

is independent of η for $\eta > \eta_o$.

Let us now evaluate the integral (A.7) explicitly.

We note that for any fixed $\eta > \eta_o$ we may write

$$R' = \sqrt{-a}\,\sqrt{(\xi_u - \xi)(\xi - \xi_\ell)}$$

in the situation of Fig. 6. Here $\xi_\ell(\eta)$ and $\xi_u(\eta)$ are roots of $|\overline{R}| = 0$ with $\xi_\ell(\eta) < \xi_u(\eta)$.

If we now write

$$V(\xi, \eta) = \frac{U(\xi, \eta)}{[(\xi_u - \xi)(\xi - \xi_\ell)]^{1/2}}$$

we find that

$$W_s = W_s(\eta) = \int_{\xi_\ell}^{\xi_u} \frac{U(\xi,\eta) d\xi}{[(\xi_u - \xi)(\xi - \xi_\ell)]^{1/2}} \tag{A.8}$$

Next making the change of variable $\xi = (\xi_u - \xi_\ell)\rho/2 + (\xi_u + \xi_\ell)/2$ in (A.8) we obtain

$$W_s = W_s(\eta) = \int\limits_{-1}^{+1} \frac{U[(\xi_u - \xi_\ell)\rho/2 + (\xi_u + \xi_\ell)/2, \eta] d\rho}{(1 - \rho^2)^{1/2}} \tag{A.9}$$

From (A.7) we have seen that for $\eta > \eta_o$ the integral (A.9) is independent of η. Now letting $\eta \downarrow \eta_o$ we of course have $\xi_u(\eta) \downarrow \xi_o$ and $\xi_\ell(\eta) \uparrow \xi_o$ and then obtain that

$$W_s = U(\xi_o, \eta_o) \int_{-1}^{1} \frac{1}{(1-\rho^2)^{1/2}} = \pi U(\xi_o, \eta_o) \tag{A.10}$$

Next we compute $U(\xi_o, \eta_o)$ in terms of $S(\xi_o, \eta_o)$.

We note that

$$\frac{R'V(\xi,\eta)}{|\bar{a}_1|} = U(\xi,\eta)$$

so setting $\xi = \xi_o$ and using L'Hôpital's Rule as $\eta \downarrow \eta_o$ we obtain after simplification

$$U(\xi_o, \eta_o) = \lim_{\eta \to \eta_o} \frac{S(\xi_o, \eta_o)}{-|\bar{a}_1|(\bar{R} \circ \bar{a}_2)}$$

and then from (A.10) we have

$$W_s = \frac{S(\xi_o, \eta_o)\pi}{-|\bar{a}_1|(\bar{R} \circ \bar{a}_2)} \tag{A.11}$$

Now in the situation as pictured in Fig. 6 $\bar{R} \circ \bar{a}_2 > 0$. In the situation where the Mach cone is inverted, $\bar{R} \circ \bar{a}_2 < 0$ at (ξ_o, η_o), (see Fig. 7) but $\eta_2 < \eta_o$ in (A.6) causing a double sign reversal so that we obtain in either case

$$\text{p.f.} \int \frac{S(\xi,\eta)}{R'^3} d\xi = -\frac{S(\xi_o, \eta_o)}{|\bar{a}_1|(\bar{R} \circ \bar{a}_2)}\Bigg|_{\substack{\xi=\xi_o \\ \eta=\eta_o}} \pi\delta(\eta - \eta_o) \tag{A.12}$$

To complete our proof we note from Appendix B that at a point $(\xi_o.\eta_o)$ with $\bar{R} \circ \bar{R} = 0$ and $\bar{R} \circ \bar{a}_1 = 0$ we have that

$$|\bar{R} \cdot \bar{a}_1 \times \bar{a}_2| = |\bar{a}_1||\bar{R} \circ \bar{a}_2| \tag{A.13}$$

so that finally from (A.12)

$$F_s(\eta) = \text{ p.f.} \int \frac{S(\xi,\eta)}{R'^3} d\xi = -\frac{S(\xi_o, \eta_o)}{|\bar{R} \cdot \bar{a}_1 \times \bar{a}_2|}\Bigg|_{\substack{\xi=\xi_o \\ \eta=\eta_o}} \pi\delta(\eta - \eta_o) \tag{A.14}$$

which completes the proof of (A.2).

We are now prepared to relate the foregoing to the evaluation of the $\alpha_m(\eta)$ for $m = 0, 1, 2$ at a critical point (ξ_o, η_o). For simplicity we assume $-1 < \xi_o < 1$ and $-1 < \eta_o < 1$ in our discussion.

We recall that the $\alpha_m(\eta)$ are defined as

$$\alpha_m(\eta) = \text{ p.f.} \int \frac{\xi^m}{R'^3}\, d\xi \quad \text{for } m = 0, 1, 2$$

In the cases $m = 0$ or $m = 1$, ξ^m will of course be a polynomial of degree ≤ 1 so that from item (A.2) we can immediately say that with $S(\xi_o, \eta_o) = \xi_o^m$

$$\alpha_m(\eta) = -\left.\frac{\xi_o^m \pi}{|\overline{R} \cdot \overline{a}_1 \times \overline{a}_2|}\right|_{\substack{\xi=\xi_o \\ \eta=\eta_o}} \delta(\eta - \eta_o)$$

for $m = 0$ or 1

In the notation of Section 7 we set $\alpha_m^{\text{reg}}(\eta_o) = 0$ and

$$\alpha_m^{\text{sp}} = \left.\frac{-\pi \xi_o^m}{|\overline{R} \cdot \overline{a}_1 \times \overline{a}_2|}\right|_{\substack{\xi=\xi_o \\ \eta=\eta_o}} \quad \text{for } m = 0 \text{ or } 1 \tag{A.15}$$

In the case $m = 2$ we are dealing with

$$\alpha_2(\eta) = \text{ p.f. } \int \frac{\xi^2}{R'^3} d\xi$$

This integral contains both a regular and a singular component. To isolate them let us note that if $R' = (a\xi^2 + b\xi + c)^{1/2}$ then $\xi^2 = R'^2/a + S(\xi, \eta)$ where $S(\xi, \eta)$ is a polynomial in ξ of degree ≤ 1. Note that $\xi_o^2 = S(\xi_o, \eta_o)$.

Thus we may write

$$\alpha_2(\eta) = \frac{1}{a} \int \frac{1}{R'} d\xi + \text{ p.f. } \int \frac{S(\xi, \eta)}{R'^3} d\xi \tag{A.16}$$

Now the second integral on the right of (A.16) will equal

$$\left.\frac{-S(\xi_o, \eta_o)\pi}{|\overline{R} \cdot \overline{a}_1 \times \overline{a}_2|}\right|_{\substack{\xi=\xi_o \\ \eta=\eta_o}} \delta(\eta - \eta_o)$$

by (A.2) but since $\xi_o^2 = S(\xi_o, \eta_o)$ this integral must equal

$$\left.\frac{-\pi \xi_o^2}{|\overline{R} \cdot \overline{a}_1 \times \overline{a}_2|}\right|_{\substack{\xi=\xi_o \\ \eta=\eta_o}} \tag{A.17}$$

and we set

$$\alpha_2^{\text{sp}} = \left.\frac{-\pi \xi_o^2}{|\overline{R} \cdot \overline{a}_1 \times \overline{a}_2|}\right|_{\substack{\xi=\xi_o \\ \eta=\eta_o}}$$

as we have stated in Eq. (83).

To proceed we must have

$$\alpha_2^{\text{reg}}(\eta_o) = \frac{1}{a} \int \frac{1}{R'} d\xi \ .$$

But this integral equals

$$\frac{1}{a}\beta_o(\eta_o)$$

in our notation of Section 7.

To show

$$\alpha_2^{\text{reg}}(\eta_o) = \frac{\pi}{(-a)^{3/2}}$$

it suffices to show that

$$\beta_o(\eta_o) = \frac{\pi}{\sqrt{-a}}$$

We proceed with the latter. In fact we compute $\beta_m(\eta)$ for $m = 0,1,2$ at once. Suppose the Mach cone cuts our panel as in Fig. 6 and we consider $\eta > \eta_o$. Then with the notation $R' = \sqrt{-a}\,\sqrt{(\xi_u - \xi)(\xi - \xi_\ell)}$ we see that

$$\beta_m(\eta) = \frac{1}{\sqrt{-a}} \int_{\xi_\ell}^{\xi_u} \frac{\xi^m}{\sqrt{(\xi_u - \xi)(\xi - \xi_\ell)}} d\xi \qquad (A.18)$$

for $m = 0,1,2$

The change of variable $\xi = (\xi_u + \xi_\ell)/2 + (\xi_u - \xi_\ell)\rho/2$ transforms A.18 into

$$\beta_m(\eta) = \frac{1}{\sqrt{-a}} \int_{-1}^{+1} \frac{[(\xi_u + \xi_\ell)/2 + (\xi_u - \xi_\ell)\rho/2]^m}{\sqrt{1-\rho^2}}\, d\rho \quad \text{for} \quad m = 0,1,2$$

Now letting $\eta \downarrow \eta_o$ we have that $\xi_u \downarrow \xi_o$ and $\xi_\ell \uparrow \xi_o$ so that in the limit

$$\beta_m(\eta_o) = \frac{\xi_o^m}{\sqrt{-a}} \int_{-1}^{+1} \frac{d\rho}{\sqrt{1-\rho^2}} = \frac{\pi\,\xi_o^m}{\sqrt{-a}} \qquad (A.19)$$

for $m = 0,1,2$

An almost identical argument shows that

$$\gamma_m(\eta) = \sqrt{-a} \int_{\xi_\ell}^{\xi_u} \xi^m \sqrt{(\xi_u - \xi)(\xi - \xi_\ell)}\, d\xi$$

for $m = 0,1,2$

transforms into

$$\gamma_m(\eta) = \sqrt{-a} \int_{-1}^{+1} [(\xi_u + \xi_\ell)/2 + (\xi_u - \xi_\ell)\rho/2]^m\, [(\xi_u - \xi_\ell)/2]^2 \sqrt{1-\rho^2}\, d\rho$$

for $m = 0,1,2$

so that as $\eta \downarrow \eta_o$, $\xi_u \downarrow \xi_o$, $\xi_\ell \uparrow \xi_o$ we obtain

$$\gamma_m(\eta_o) = 0 \qquad \text{for } m = 0,1,2 \qquad (A.20)$$

Finally, consideration of $\Delta_m(\eta)$ at a critical point yields, in a similar way

$$\Delta_m(\eta_o) = 0 \quad \text{for} \quad m = 0,1,2 \qquad (A.21)$$

APPENDIX B.—A LEMMA CONCERNING SUPERDOT PRODUCT

In this brief appendix we prove an elementary lemma concerning the superdot product. This lemma comes into play in proving formula (A.2) of Appendix A.

LEMMA: Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be three vectors in R^3. Assume that (i) $\bar{a} \circ \bar{a} = 0$. (ii) $\bar{a} \circ \bar{b} = 0$ and (iii) $\bar{b} \circ \bar{b} \leq 0$. Then

$$|\bar{a} \cdot \bar{b} \times \bar{c}| = \sqrt{-\bar{b} \circ \bar{b}}\ |\bar{a} \circ \bar{c}|. \tag{B.1}$$

We refer the reader to Ref. 3 for definition and properties of the superdot product.

PROOF: Without loss of generality we may assume coordinates have been rotated so that $a_z = 0$. We may assume $|\bar{a}| \neq 0$.

Then a takes the form

$$\bar{a} = a_x \bar{i} + a_y \bar{j} \text{ with } a_x^2 = a_y^2 \neq 0 \text{ from (i)}.$$

Let us write

$$\bar{b} = b_x \bar{i} + b_y \bar{j} + b_z \bar{k}$$

then $0 = a_x b_x - a_y b_y$ from (ii) or equivalently

$$a_x b_x = a_y b_y \tag{B.2}$$

We now proceed by cases.

Case 1: If $a_x = a_y \neq 0$ then from (B.2) it follows that $b_x = b_y$. Then

$$\begin{aligned}\bar{a} \cdot \bar{b} \times \bar{c} = {} & \text{Det} \begin{pmatrix} a_x & a_y & 0 \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix} \\ & = a_x(b_y c_z - b_z c_y) - a_y(b_x c_z - b_z c_x) \\ & = (a_x c_x - a_y c_y) b_z \end{aligned}$$

Thus $\bar{a} \cdot \bar{b} \times \bar{c} = (\bar{a} \circ \bar{c}) b_z$ and $|\bar{a} \cdot \bar{b} \times \bar{c}| = |\bar{a} \circ \bar{c}| \sqrt{-\bar{b} \circ \bar{b}}$ in this case.

Case 2: Here $a_x = -a_y \neq 0$.

It follows from (B.2) that $b_x = -b_y$

Then again

$$\bar{a} \cdot \bar{b} \times \bar{c} = \text{Det} \begin{pmatrix} a_x & a_y & 0 \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix}$$

$$
\begin{aligned}
&= a_x(b_y c_z - b_z c_y) - a_y(b_x c_z - b_z c_x) \\
&= (a_x c_x - a_y c_y) - b_z \\
&= (\bar{a} \circ \bar{c})(-b_z)
\end{aligned}
$$

and so

$$|\bar{a} \cdot \bar{b} \times \bar{c}| = |\bar{a} \circ \bar{c}|\sqrt{-\bar{b} \circ \bar{b}} \text{ once again.}$$

APPENDIX C—DERIVATION OF THE SUPERSONIC OSCILLATORY P.D.E.

We begin with the linearized potential flow equation:

$$\nabla_o^2 \phi - \frac{1}{a_\infty^2}\left(\frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x}\right)^2 \phi = 0 \tag{C.1}$$

Passing to Prandtl-Glauert coordinates after the introduction of scaled variables as indicated in equation (3) we proceed as follows:

$$
\begin{aligned}
\nabla_o^2 \phi - \frac{1}{a_\infty^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{2U_\infty}{a_\infty^2}\frac{\partial^2 \phi}{\partial t \partial x} - \frac{U_\infty^2}{a_\infty^2}\frac{\partial^2 \phi}{\partial x^2} &= 0 \\
-\beta^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{a_\infty^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{2M_\infty}{a_\infty}\frac{\partial^2 \phi}{\partial t \partial x} &= 0
\end{aligned}
$$

$$
\begin{aligned}
&-\beta^2 \frac{\partial^2 \Phi}{\partial (\beta l X)^2} + \frac{\partial^2 \Phi}{\partial (lY)^2} + \frac{\partial^2 \Phi}{\partial (lZ)^2} \\
&-\frac{1}{a_\infty^2}\frac{\partial^2 \Phi}{\partial (lT/a_\infty \beta)^2} - \frac{2M_\infty}{a_\infty}\frac{\partial^2 \Phi}{\partial (lT/a_\infty \beta)\partial(\beta l X)} = 0
\end{aligned}
$$

which results after simplification in

$$\frac{\partial^2 \Phi}{\partial X^2} - \frac{\partial^2 \Phi}{\partial Y^2} - \frac{\partial^2 \Phi}{\partial Z^2} + \beta^2 \frac{\partial^2 \Phi}{\partial T^2} + 2M_\infty \frac{\partial^2 \Phi}{\partial X \partial T} = 0 \tag{C.2}$$

Then equation (C.2) can be written as

$$\nabla \circ \nabla \Phi + \beta^2 \frac{\partial^2 \Phi}{\partial T^2} + 2M_\infty \frac{\partial^2 \Phi}{\partial X \partial T} = 0 \tag{C.3}$$

which is Eq. (4) of the text.

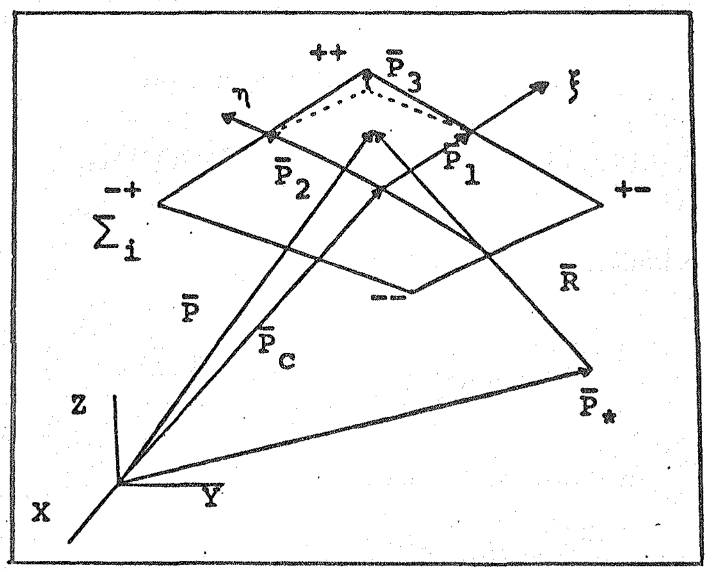

Fig. 1 Geometry of the hyperboloidal element

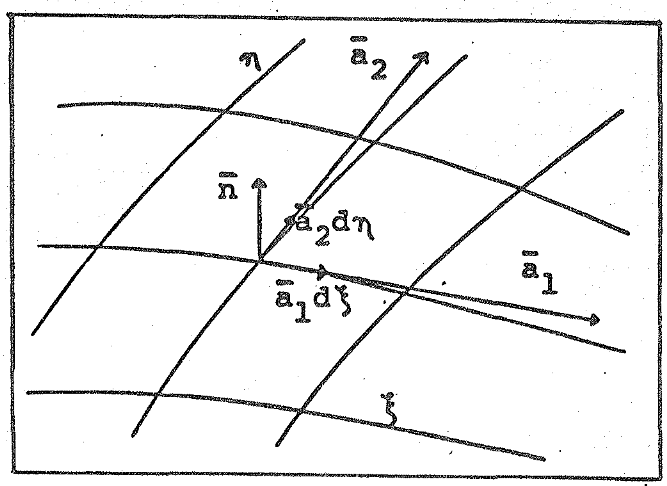

Fig. 2 Surface geometry

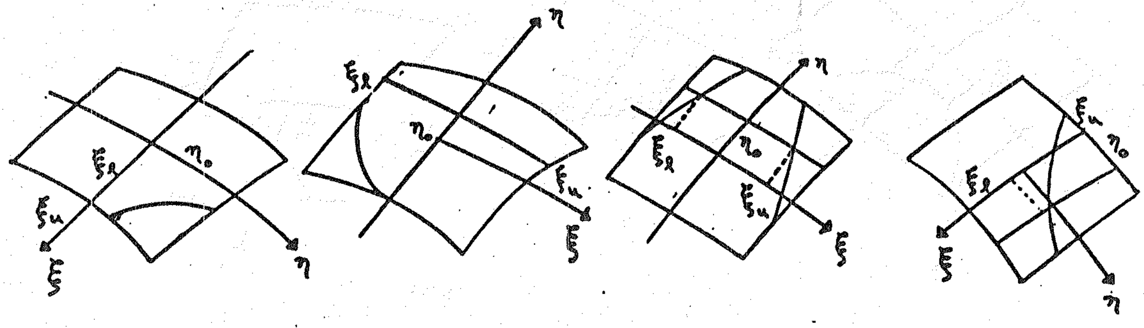

Fig. 3 Illustration of case (ii)

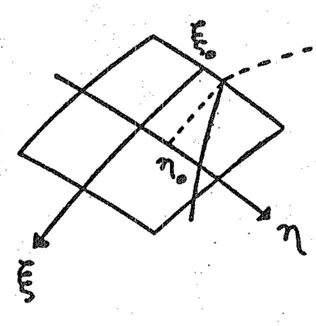

Fig. 4 Illustration of case (iii)

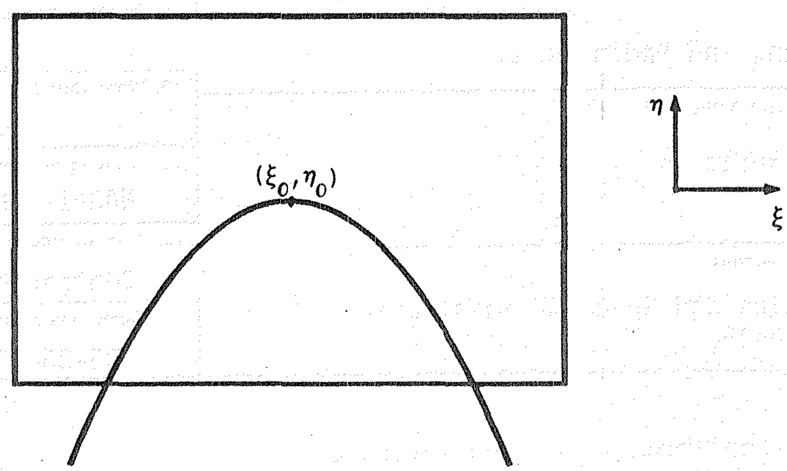

Fig. 7 Case of critical point, with $\bar{R} \circ \bar{a}_2 < 0$

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16. Abstract

In this report a time-domain Green's Function Method for unsteady supersonic potential flow around complex aircraft configurations is presented.

We focus here on the supersonic range wherein the linear potential flow assumption is valid. In this range the effects of the nonlinear terms in the unsteady supersonic compressible velocity potential equation are negligible and therefore these terms will be omitted in this report.

The Green's function method is employed in order to convert the potential-flow differential equation into an integral one. This integral equation is then discretized, in space through standard finite-element technique, and in time through finite-difference, to yield a linear algebraic system of equations relating the unknown potential to its prescribed co-normalwash (boundary condition) on the surface of the aircraft. The arbitrary complex aircraft configuration (e.g., finite-thickness wing, wing-body-tail) is discretized into hyperboloidal (twisted quadrilateral) panels. The potential and co-normalwash are assumed to vary linearly within each panel. Consistent with the spatial linear (first-order) finite-element approximations, the potential and co-normalwash are assumed to vary linearly in time.

The long range goal of our research is to develop a comprehensive theory for unsteady supersonic potential aerodynamics which is capable of yielding accurate results even in the low supersonic (i.e., high transonic) range.

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