https://ntrs.nasa.gov/search.jsp?R=19850025835 2020-03-20T16:59:21+00:00Z

NASA-TP-2511 19850025835

**NASA Technical Paper 2511**

August 1985

# Space Station Rotational Equations of Motion

Mario H. Rheinfurth
and Stanley N. Carroll

NASA

3 1176 01316 2830

NASA Technical Paper 2511

1985

# Space Station Rotational Equations of Motion

Mario H. Rheinfurth
and Stanley N. Carroll

*George C. Marshall Space Flight Center*
*Marshall Space Flight Center, Alabama*

National Aeronautics
and Space Administratioı

Scientific and Technica
Information Branch

## TABLE OF CONTENTS



## LIST OF ILLUSTRATIONS

## DEFINITION OF SYMBOLS

| Symbol | Definition |
|---|---|
| $\underline{a}$ | relative acceleration of a mass element as observed in moving reference frame. |
| $A_p$ | coordinate transformation from the main body system to the appendage system. |
| $C_p{}^*$ | specially defined Coriolis matrix. Format is shown in body of report. |
| dm | element of mass. |
| $\underline{H}_p$ | angular momentum of rotating appendage about its mass center. |
| $\underline{H}_v$ | angular momentum of vehicle, including non-rotating appendages. |
| $\underline{H}_T$ | angular momentum of the total system about its mass center. |
| I | inertia matrix for the total system, a time varying quantity. |
| $I_p$ | inertia matrix for appendage |
| $\underline{l}_o$ | position vector of appendage mass center relative to the composite mass center. |
| $\underline{l}_p$ | position vector of appendage mass element relative to the appendage mass center. |
| $\underline{L}$ | vector sum of all external torques about the composite mass center. |
| $\underline{r}$ | position vector of mass element relative to composite mass center. |
| $\underline{r}_p$ | position vector of appendage mass element relative to composite mass center. |
| $\ddot{\underline{R}}$ | absolute acceleration of mass element. |
| $\ddot{\underline{R}}_o$ | absolute acceleration of the origin of the moving reference frame. |
| $\underline{v}$ | relative velocity of mass element observed in the moving reference frame. |
| $\alpha$ | rotation angle of appendage based on orbital position. |
| $\beta$ | rotation angle of appendage based on sun position. |
| $\underline{\Omega}$ | angular velocity vector of moving reference frame (main body) with respect to inertial space. |
| $\underline{\omega}_p$ | angular velocity vector of appendage reference frame with respect to the moving reference frame. |
| $\dot{\underline{\Omega}} = (\dot{\underline{\Omega}})_I$ | angular acceleration of the main body relative to inertial space. |
| $\dot{\underline{\omega}}_p = (\dot{\underline{\omega}}_p)_v$ | angular acceleration of an appendage relative to the main body. |

| | |
|---|---|
| $(\dot{\underline{H}})_I$ | time rate of change of vector H relative to inertial space. |
| $(\dot{\underline{H}})_v$ | time rate of change of vector H relative to the main body. |
| $\underline{\underline{E}}$ | unit dyadic. |
| $\underline{\underline{I}}$ | inertia dyadic for total system. |
| $\underline{\underline{I}}_p$ | inertia dyadic for appendage. |
| $\underline{\underline{C}}$ | Coriolis dyadic. |

TECHNICAL PAPER

# SPACE STATION ROTATIONAL EQUATIONS OF MOTION

## I. INTRODUCTION

This report derives the rotational equations of motion for a large space structure having appendages that can rotate relative to the main body. In particular, the equations were formulated for use in the Space Station Attitude Control and Stabilization Test Bed. The solar arrays and thermal radiators of the Space Station are required to maintain a specific alignment with the sun, whereas the main body of the Space Station rotates at orbital rate in an Earth pointing attitude. Thus the angular velocity of the appendages relative to the Space Station is approximately opposite to the orbital rate.

Two methods are given for the derivation of the rotational equations of motion. The first method uses D'Alembert's principle whereas the second method is based on the angular momentum concept. It is shown that the two formulations are dynamically equivalent although the resulting differential equations are significantly different concerning their mathematical structure. It should be noted that throughout the report the terms appendage, solar arrays or panels and thermal radiator are used interchangeably.

## II. DERIVATION OF ROTATIONAL EQUATIONS OF MOTION

### A. Method 1: D'Alembert's Principle

In this method, the rotational equations of motion are derived by applying D'Alembert's principle. The equations of motion are presented in a form suitable for attitude and stabilization studies of the Space Station. Its Power Tower configuration is schematically shown in Figure 1. The rotational equations will be written in terms of a moving reference frame, whose origin is located at the mass center of the total system and which rotates with the main body. In Figure 2 the XYZ system is fixed in inertial space and the xyz system is the moving reference (body) frame which translates and rotates relative to it. Notice that this body frame is, in general, not fixed in the main body but has a translational motion relative to it because of the change in location of the composite mass center relative to the main body due to internal moving components. However, the rotational rate of the body frame is identical to that of the main body at all times.

Now, D'Alembert's principle states that the vector sum of the moments of all inertial and external forces relative to any point is zero. Taking moments about the composite mass center of the Space Station, the following vector equation is obtained:

$$\underline{L} + \int \underline{r} \times (-\ddot{\underline{R}}\, dm) = 0 \quad . \tag{1}$$

The absolute acceleration $\ddot{\underline{R}}$ of each mass particle dm can be expressed in terms of quantities measured in the rotating reference frame by the well-known relation:

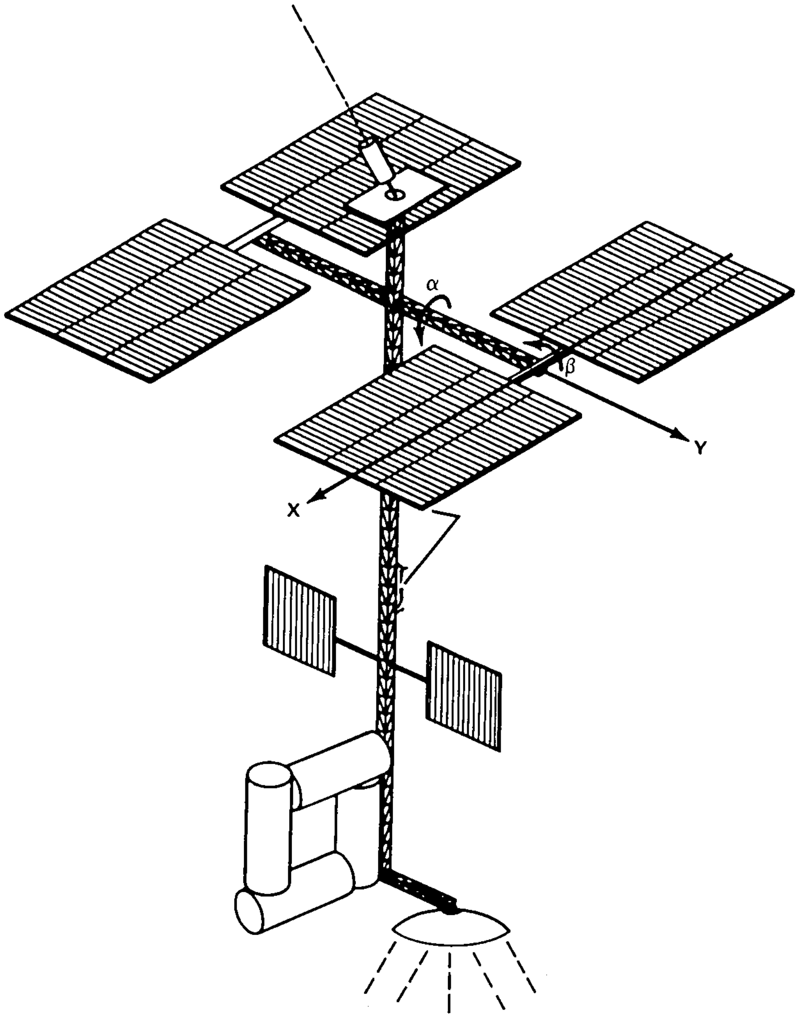


Figure 1. Space Station (Power Tower configuration).

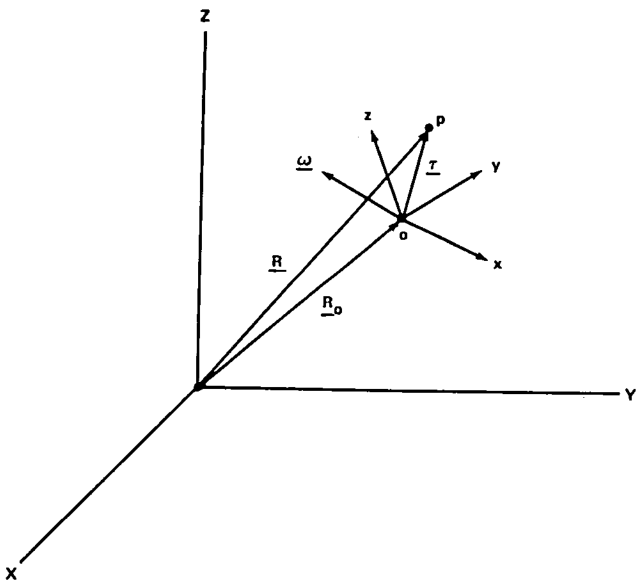


Figure 2. Inertial and moving reference frames.

$$\ddot{\underline{R}} = \ddot{\underline{R}}_o + (\dot{\underline{\Omega}} \times \underline{r}) + \underline{\Omega} \times (\underline{\Omega} \times r) + 2\,(\underline{\Omega} \times \underline{v}) + \underline{a} \quad . \tag{2}$$

The only components moving relative to the main body considered here are the solar panels. Their relative motion consists of angular rotations relative to the main body. The solar panels are assumed to be perfectly symmetric with respect to their axes of rotation. As a consequence, the mass center of the solar panels always remains at rest relative to the main body during their rotational motion. This restriction can be easily lifted if a higher fidelity model should be required later. The relative velocity $\underline{v}$ and relative acceleration $\underline{a}$ of a solar panel mass element are then given by

$$\underline{v} = \underline{\omega}_p \times \underline{l}_p \quad . \tag{3}$$

$$\underline{a} = \dot{\underline{\omega}}_p \times \; l_p + \underline{\omega}_p \times (\underline{\omega}_p \times \underline{l}_p) \quad . \tag{4}$$

The position vector $\underline{r}_p$ of a solar panel mass element with respect to the composite mass center can be expressed as:

$$\underline{r}_p = \underline{l}_o + \underline{l}_p \quad , \tag{5}$$

where $\underline{l}_o$ is the position vector of the solar panel mass center relative to the composite mass center.

Using equations (3) and (4), the five terms of equation (2) are substituted into equation (1). Now each term will be individually evaluated.

TERM I:

$$\int \underline{r} \times \ddot{\underline{R}}_o \, dm = \int \underline{r} dm \times \ddot{\underline{R}}_o = 0 \quad . \tag{I}$$

The vanishing of this term is caused by the fact that the origin of the body frame was chosen to coincide with the composite mass center at all times, or symbolically

$$\int \underline{r} \, dm = 0 \quad . \tag{6}$$

TERM II:

$$\int [\underline{r} \times (\dot{\underline{\Omega}} \times \underline{r})] \, dm = \underline{\underline{I}} \cdot \dot{\underline{\Omega}} \quad . \tag{II}$$

This term is brought into a vector-dyadic form by introducing the moment of inertia dyadic

$$\underline{\underline{I}} = \int [\underline{\underline{E}}\, r^2 - (\underline{r}\, \underline{r})]\ dm \tag{7}$$

where $\underline{\underline{E}}$ is the unit dyadic and the integration extends over the total system mass. Thus Term II is obtained.

TERM III:

$$\int \underline{r} \times [\underline{\Omega} \times (\underline{\Omega} \times \underline{r})]\ dm = \underline{\Omega} \times \int [\underline{r} \times (\underline{\Omega} \times \underline{r})]\ dm = \underline{\Omega} \times \underline{\underline{I}} \cdot \underline{\Omega} \quad . \tag{III}$$

To get Term III, use is made of the vector identity:

$$\underline{r} \times [\underline{\Omega} \times (\underline{\Omega} \times \underline{r})] = \underline{\Omega} \times [\underline{r} \times (\underline{\Omega} \times \underline{r})] \quad . \tag{8}$$

Term III is sometimes referred to as Euler coupling. But since it is the inertial torque obtained by summing up the moments of the centrifugal forces acting on each mass element it is more appropriate from a dynamic point of view to call it the centrifugal torque. Notice that the integration of Terms II and III extends over the total system including the articulated appendages. Moreover the moment of inertia dyadic relates to the instantaneous system configuration, and thus, changes with time whenever the appendages are in motion relative to the main body.

The last two terms of equation (2) reflect the influence of the rotating appendages on the attitude dynamics of the Space Station. When inserted into equation (1) the corresponding integration extends, of course, only over the appendage mass. As a reminder mathematical quantities pertaining to the appendages are identified by the subscript p.

TERM IV:

$$2\int \underline{r}_p \times [\underline{\Omega} \times (\underline{\omega}_p \times \underline{l}_p)]\ dm_p = 2\, \underline{\Omega} \cdot \underline{\underline{C}}_p \times \underline{\omega}_p \quad . \tag{IV}$$

As a preliminary step, equation (5) is used to obtain the result that

$$\int \underline{l}_p\ dm = 0 \quad . \tag{9}$$

Using equations (3), (4), (5) and (9), Term IV becomes

$$2\int \underline{r} \times (\underline{\Omega} \times \underline{v})\ dm_p = 2\int \underline{r}_p \times [\underline{\Omega} \times (\underline{\omega}_p \times \underline{l}_p)] dm_p = 2\int (\underline{l}_o \times \underline{l}_p) \times [\underline{\Omega} \times (\underline{\omega}_p \times \underline{l}_p)]\ dm_p$$

$$= 2\int \underline{l}_p \times [\underline{\Omega} \times (\underline{\omega}_p \times \underline{l}_p)] dm_p \quad . \tag{10}$$

Applying the vector triple product rule

$$\underline{\Omega} \times (\underline{\omega}_p \times \underline{l}_p) = (\underline{\Omega} \cdot \underline{l}_p)\, \underline{\omega}_p - (\underline{\Omega} \cdot \underline{\omega}_p)\, \underline{l}_p \tag{11}$$

equation (10) can be reformulated to obtain

$$2 \int \underline{l}_p \times [(\underline{\Omega} \cdot \underline{l}_p)\, \underline{\omega}_p - (\underline{\Omega} \cdot \underline{\omega}_p)\, \underline{l}_p] dm_p = 2 \int (\underline{\Omega} \cdot \underline{l}_p)\, (\underline{l}_p \times \underline{\omega}_p)\, dm_p \quad . \tag{12}$$

Finally the last term of equation (12) can be converted to vector-dyadic form by introducing the Coriolis dyadic

$$\underline{\underline{C}}_p = \int (\underline{l}_p\, \underline{l}_p)\, dm_p = (0.5 * \text{TR}\, \underline{\underline{I}}_p)\, \underline{\underline{E}} - \underline{\underline{I}}_p \quad . \tag{13}$$

where TR is the trace operator.

Note that Term IV represents the Coriolis torque (i.e., the moment of the Coriolis forces) acting on the appendage about its mass center.

<u>TERM V:</u>

$$\int (\underline{r}_p \times \underline{a})\, dm_p = \underline{\underline{I}}_p \cdot \dot{\underline{\omega}}_p + \underline{\omega}_p \times \underline{\underline{I}}_p \cdot \underline{\omega}_p \quad . \tag{V}$$

Using equation (5) and the vector identity (8), Term V is obtained.

This last term can also be converted to vector-dyadic form by defining the moment of inertia dyadic $\underline{\underline{I}}_p$ of the appendage about its center of mass. Taking steps analogous to the previous ones and using equation (4), the following equation is obtained:

$$\int (\underline{r}_p \times \underline{a})\, dm_p = \int \underline{r}_p \times [(\dot{\underline{\omega}}_p \times \underline{l}_p) + \underline{\omega}_p \, X \, (\underline{\omega}_p \times \underline{l}_p)]\, dm_p$$

$$= \int (\underline{l}_o + \underline{l}_p) \times [(\dot{\underline{\omega}}_p \times \underline{l}_p) + \underline{\omega}_p \times (\underline{\omega}_p \times \underline{l}_p)\, dm_p \quad . \tag{14}$$

Combining Terms (I) to (V) yields the rotational equation of motion in vector-dyadic notation as:

$$\underline{\underline{I}} \cdot \dot{\underline{\Omega}} + \underline{\Omega} \times \underline{\underline{I}} \cdot \underline{\Omega} + 2\, \underline{\Omega} \cdot \underline{\underline{C}}_p \times \underline{\omega}_p + \underline{\underline{I}}_p \cdot \dot{\underline{\omega}}_p + \underline{\omega}_p \times \underline{\underline{I}}_p \cdot \underline{\omega}_p = \underline{L} \quad . \tag{15}$$

Notice that the first two terms represent the well known Euler equations for rigid body rotation, whereas the last three terms reflect the contribution of an appendage rotating relative to the main body with angular velocity $\underline{\omega}_p$. If there are N rotating appendages, these terms are simply replaced by their appropriate sums, i.e.

$$\underline{\underline{I}} \cdot \dot{\underline{\Omega}} + \underline{\Omega} \times \underline{\underline{I}} \cdot \underline{\Omega} + 2 \sum_{p=1}^{N} \underline{\Omega} \cdot \underline{\underline{C}}_p \times \underline{\omega}_p + \sum_{p=1}^{N} \underline{\underline{I}}_p \cdot \dot{\underline{\omega}}_p + \sum_{p=1}^{N} \underline{\omega}_p \times \underline{\underline{I}}_p \cdot \underline{\omega}_p = \underline{L} \quad . \tag{16}$$

### B. Method 2: Angular Momentum Principle

The angular momentum principle states that the inertial time rate of change of the angular momentum of a dynamic system about its mass center is equal to the sum of all external torques (moments of the external forces) about its mass center. In equation form, this is given by:

$$\underline{L} = (\dot{\underline{H}}_T)_I \quad . \tag{17}$$

The letter subscript outside the parenthesis indicates the reference frame in which the time rate of change of the vector $H_T$ is observed. Using the known identity for vector differentiation the rotational equation of motion (17) can be written in the form

$$\underline{L} = (\dot{\underline{H}}_T)_v + \underline{\Omega} \times \underline{H}_T \quad . \tag{18}$$

It will now be shown that this equation is dynamically equivalent to the rotational equation of motion (15) obtained by applying D'Alembert's principle.

Before continuing with the equation development, a comment is in order regarding the usage of equation (17) to describe the rotational dynamics of a system. This form of equation has been used extensively in the past at MSFC for rigid body computer simulations in which Control Moment Gyros (CMG's) are used as means of attitude control. The angular momentum of the total system is then defined as the sum of the angular momentum of the main body and the CMG's about the mass center of the total system. Integration is performed directly on the term $(\dot{\underline{H}}_T)_v$ to obtain the total angular momentum. The desired angular velocity of the main body can then be calculated by subtracting the CMG angular momentum from the system angular momentum and premultiplying the resulting main body angular momentum by the inverse moment of inertia matrix. Because the rotating appendages of the Space Station are dynamically equivalent to CMG's the simple form of equation (18) is directly applicable here. The present method allows, therefore, to use the existing software with only minor modifications. Continuing with the objective to show dynamical equivalency of the two methods, the total angular momentum is defined as:

$$\underline{H}_T = \underline{H}_v + \underline{H}_p = \underline{\underline{I}} \cdot \underline{\Omega} + \underline{\underline{I}}_p \cdot \underline{\omega}_p \quad . \tag{19}$$

Differentiation of equation (19) with respect to time yields

$$(\dot{\underline{H}}_T)_v = \underline{\underline{I}} \cdot (\dot{\underline{\Omega}})_v + \underline{\underline{I}}_p \cdot (\dot{\underline{\omega}}_p)_v + (\dot{\underline{\underline{I}}})_v \cdot \underline{\Omega} + (\dot{\underline{\underline{I}}}_p)_v \cdot \underline{\omega}_p \quad . \tag{20}$$

Since the rotating appendages are the only components which cause the moment of inertia of the system to change with time:

$$(\dot{\underline{\underline{I}}})_v = (\dot{\underline{\underline{I}}}_p)_v \quad . \tag{21}$$

Regarding the differentiation of the two angular velocities $\underline{\Omega}$ and $\underline{\omega}_p$ in equation (20), it is noted that

$$(\dot{\underline{\Omega}})_v = (\dot{\underline{\Omega}})_I \qquad \text{and} \qquad (\dot{\underline{\omega}}_p)_v = (\dot{\underline{\omega}}_p)_p \quad . \tag{22}$$

The time rate of change of the moment of inertia dyadics can be expanded in a similar way as was done for the vector differentiation in equation (18) by using the known identity for dyadic differentiation in a rotating reference frame (Appendix B).

$$(\dot{\underline{\underline{I}}})_A = (\dot{\underline{\underline{I}}})_B + \underline{\omega}_{BA} \times \underline{\underline{I}} - \underline{\underline{I}} \times \underline{\omega}_{BA} \quad . \tag{23}$$

where $\underline{\omega}_{BA}$ is the angular velocity of frame B relative to frame A. Using equation (23), write:

$$(\dot{\underline{\underline{I}}}_p)_v = \underline{\omega}_p \times \underline{\underline{I}}_p - \underline{\underline{I}}_p \times \underline{\omega}_p \tag{24}$$

since $(\dot{\underline{\underline{I}}}_p)_p = 0$. Substituting equations (19), (20), (21), and (22) into the rotational equation (18) we obtain:

$$\underline{L} = \underline{\underline{I}} \cdot (\dot{\underline{\Omega}})_v + \underline{\underline{I}}_p \cdot (\dot{\underline{\omega}}_p)_p + (\dot{\underline{\underline{I}}}_p)_v \cdot (\underline{\Omega} + \underline{\omega}_p) + \underline{\Omega} \times (\underline{\underline{I}} \cdot \underline{\Omega} + \underline{\underline{I}}_p \cdot \underline{\omega}_p) \quad . \tag{25}$$

Then substituting equation (24) into equation (25) yields

$$\underline{L} = \underline{\underline{I}} \cdot (\dot{\underline{\Omega}})_v + \underline{\underline{I}}_p \cdot (\dot{\underline{\omega}}_p)_p + (\underline{\omega}_p \times \underline{\underline{I}}_p - \underline{\underline{I}}_p \times \underline{\omega}_p) \cdot (\underline{\Omega} + \underline{\omega}_p) + \underline{\Omega} \times (\underline{\underline{I}} \cdot \underline{\Omega} + \underline{\underline{I}}_p \cdot \underline{\omega}_p) \quad . \tag{26}$$

Since $\underline{\underline{I}}_p \times \underline{\omega}_p \cdot \underline{\omega}_p = 0$, the equation simplies further to the final form:

$$\underline{L} = \underline{\underline{I}} \cdot (\dot{\underline{\Omega}})_v + \underline{\underline{I}}_p \cdot (\dot{\underline{\omega}}_p)_p + \underline{\Omega} \times \underline{\underline{I}} \cdot \underline{\Omega} + \underline{\omega}_p \times \underline{\underline{I}}_p \cdot \underline{\omega}_p + [\underline{\omega}_p \times \underline{\underline{I}}_p \cdot \underline{\Omega} - (\underline{\underline{I}}_p \times \underline{\omega}_p) \cdot \underline{\Omega}$$

$$+ \underline{\Omega} \times \underline{\underline{I}}_p \cdot \underline{\omega}_p] \quad . \tag{27}$$

For dynamical equivalency of equation (27) and equation (15), the three terms in the square brackets of equation (27) must be identical to the term $2\underline{\Omega} \cdot \underline{\underline{C}}_p \times \underline{\omega}_p$. The proof of this identity is given in Appendix A. Thus, it has been demonstrated that the two methods of deriving the rotational equations of motion are equivalent in a dynamic sense.

## III. VECTOR DYADIC TO MATRIX TRANSFORMATION

For the purpose of computer programming, all vector dyadic equations have to be transformed to matrix equations. One of the major advantages of the vector-dyadic notation is, of course, its independence from any particular coordinate system. By contrast the matrix formulation requires the definition of a concomitant coordinate system.

The conversion from vector-dyadic to matrix notation is based upon the isomorphism (i.e., one-to-one correspondence) which exists between the two. In order to obtain these correspondences , two vectors are considered in an orthogonal coordinate system with unit vectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ along the axes such that

$$\underline{a} = a_1\,\underline{e}_1 + a_2\,\underline{e}_2 + a_3\,\underline{e}_3 \quad , \qquad \underline{b} = b_1\,\underline{e}_1 + b_2\,\underline{e}_2 + b_3\,\underline{e}_3 \tag{28}$$

where $a_i$ and $b_i$ ($i = 1,2,3$) are the components of the two vectors along the three axes. The easily verifiable isomorphisms are obtained:

<u>Scalar Product:</u>

$$\underline{a} \cdot \underline{b} \Leftrightarrow \underline{a}^T\,\underline{b} \quad .$$

<u>Vector Product:</u>

$$\underline{a} \times \underline{b} \Leftrightarrow \tilde{a}\,\underline{b}$$

where

$$\tilde{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

The inertia dyadic is similarly expressed in terms of the specified orthogonal coordinates as:

$$\underline{\underline{I}} = I_{11}\,\underline{e}_1\,\underline{e}_1 + I_{12}\,\underline{e}_1\,\underline{e}_2 + I_{13}\,\underline{e}_1\,\underline{e}_3 + I_{21}\,\underline{e}_2\,\underline{e}_1 + I_{22}\,\underline{e}_2\,\underline{e}_2 + I_{23}\,\underline{e}_2\,\underline{e}_3 + I_{31}\,\underline{e}_3\,\underline{e}_1$$

$$+ I_{32}\,\underline{e}_3\,\underline{e}_2 + I_{33}\,\underline{e}_3\,\underline{e}_3 \quad . \tag{30}$$

With this the following isomorphisms can also be established:

Dyadic Product (A):

$$\underline{\underline{I}} \cdot \underline{\omega} \Leftrightarrow I\,\underline{\omega}$$

where I is the moment of inertia matrix and $\underline{\omega}$ is a column matrix having the vector components of $\underline{\omega}$ as elements.

Dyadic Product (B):

$$\underline{\omega} \cdot \underline{\underline{I}} \Leftrightarrow \underline{\omega}^T\, I$$

where $\underline{\omega}^T$ is the transpose column matrix.

## IV. APPENDAGE ORIENTATION KINEMATICS

It was mentioned already that the matrix form of the equations of motion require the adoption of a specific reference frame (coordinate system). That is to say, all matrices and vectors appearing in the various terms of the following matrix equation have to refer to one and the same coordinate system, which in our case will naturally be the main body reference frame. Suppose that the direction cosine matrix specifying the relative orientation of an appendage to the main body is called $A_p$. Then the transformation of a column matrix (vector) $\underline{x}$ in the main body frame can be transformed to a corresponding vector $\underline{x}'$ in the appendage frame by the matrix relation

$$\underline{x}' = A_p\,\underline{x} \quad . \tag{31}$$

Likewise the transformation of a matrix I is accomplished by

$$I' = A_p\, I\, A_p^T \quad . \tag{32}$$

Here the primed quantities refer to the appendage coordinate system and the unprimed quantities refer to the main body coordinate system.

For the following matrix equation, however, the required transformation is just reversed, namely, the vectors and matrices pertaining to the appendages have to be transformed to the main body frame. The following relation is used for the vectors:

$$\underline{x} = A_p^T \, \underline{x}' \tag{33}$$

and the following relation is used for the matrices:

$$I = A_p^T \, I' \, A_p \quad . \tag{34}$$

The direction cosine matrix $A_p$ is defined in this report by the following sequence of rotations (Fig. 1).

1) A clockwise rotation $\alpha$ about the solar panel y-axis. The associated transformation matrix is

$$A_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad . \tag{35}$$

2) A clockwise rotation $\beta$ about the solar panel x-axis. The associated transformation matrix is

$$A_x(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \quad . \tag{36}$$

The total direction cosine matrix is then

$$A_p = A_x(\beta) \, A_y(\alpha) \quad . \tag{37}$$

$$A_p = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ \sin\alpha \ \sin\beta & \cos\beta & \cos\alpha \ \sin\beta \\ \sin\alpha \ \cos\beta & -\sin\beta & \cos\alpha \ \cos\beta \end{bmatrix} \quad . \tag{38}$$

It is also necessary to express the angular velocities $\dot{\alpha}$ and $\dot{\beta}$ of the solar panels in terms of components along the three axes of the solar panel coordinate system. This can be done by adding these angular velocities vectorially using the above direction cosine matrices. The result is given in matrix form as

$$\omega_p = A_x(\beta)\, A_y(\alpha) \begin{pmatrix} 0 \\ \dot{\alpha} \\ 0 \end{pmatrix} + A_x(\beta) \begin{pmatrix} \dot{\beta} \\ 0 \\ 0 \end{pmatrix} \quad . \tag{39}$$

From this equation the angular velocity components along the three axes of the solar panel coordinate system are obtained as

$$\omega_x = \dot{\beta} \quad , \quad \omega_y = \dot{\alpha}\cos(\beta) \quad , \quad \omega_z = -\dot{\alpha}\sin(\beta) \quad . \tag{40}$$

Having established these kinematical relationships, we can now transform the vector-dyadic equation (27) into a matrix equation using the isomophisms described in Section III as:

$$I\,\dot{\underline{\Omega}} + \tilde{\Omega}\, I\, \underline{\Omega} - \sum_p A_p^T\, \tilde{\omega}_p\, C_p^*\, A_p\, \underline{\Omega} + \sum_p A_p^T\, I_p\, \dot{\underline{\omega}}_p + \sum_p A_p^T\, \tilde{\omega}_p\, I_p\, \underline{\omega}_p = \underline{L} + \sum_p A_p^T\, \underline{L}_p \tag{41}$$

where $\underline{L}$ is the total torque acting on the main body and $\underline{L}_p$ is the total external torque acting on the p-th solar panel, properly transformed to the main body reference frame. The matrice $C_p^*$ is defined as (cf., equation 13):

$$C_p^* = \begin{bmatrix} J_{yp} + J_{zp} - J_{xp} & -2\,J_{xyp} & -2\,J_{xzp} \\ -2\,J_{xyp} & J_{xp} + J_{zp} - J_{yp} & -2\,J_{yzp} \\ -2\,J_{xzp} & -2\,J_{yzp} & J_{xp} + J_{yp} - J_{zp} \end{bmatrix} \tag{42}$$

where $J_{xp}$, $J_{yp}$, $J_{zp}$ are the moments of inertia of the p-th solar panel about the mass center and $J_{xyp}$, $J_{xzp}$, $J_{yzp}$ are the corresponding products of inertia.

## V. CONCLUSION

It was shown that the two methods for deriving the rotational equations of the Space Station with articulated appendages are dynamically equivalent. It is rather surprising that the resulting set of ordinary first order differential equations having such vastly different mathematical structure does indeed accurately represent the very same dynamical system. The equations based upon the angular momentum method are quite simple in structure and many existing rigid body computer programs can easily be modified to include the rotating appendages. On the other hand, the equations based upon D'Alembert's principle provide more insight into the dynamic origin of their individual terms, which can be of considerable benefit in trying to understand the dynamic behavior of the system. Moreover, further refinements such as mass center offset of the rotating appendages and their flexibility require only additional terms and no major restructuring of these equations. Ultimately, however, the choice of selecting the proper set of dynamic equations is largely a matter of personal predilection and experience with existing computer codes.

## APPENDIX A

The purpose of this appendix is to prove the identity

$$2\, \underline{\Omega} \cdot \underline{\underline{C}}_p \times \underline{\omega}_p = \underline{\omega}_p \times \underline{\underline{I}}_p \cdot \underline{\Omega} - \underline{\underline{I}}_p \times \underline{\omega}_p \cdot \underline{\Omega} + \underline{\Omega} \times \underline{\underline{I}}_p \cdot \underline{\omega}_p \quad . \tag{A-1}$$

As a preliminary step, it must first be shown that

$$\underline{\underline{I}}_p \times \underline{\omega}_p \cdot \underline{\Omega} = \underline{\underline{I}}_p \cdot \underline{\omega}_p \times \underline{\Omega} \quad . \tag{A-2}$$

To this end, it is assumed for a moment that the inertia dyadic is defined by the dyadic product of two vectors $\underline{a}$ and $\underline{b}$ as

$$\underline{\underline{I}}_p = \underline{a}\, \underline{b} \quad . \tag{A-3}$$

Substituting in equation (A-2) yields

$$(\underline{a}\, \underline{b}) \times \underline{\omega}_p \cdot \underline{\Omega} = \underline{a}\, (\underline{b} \times \underline{\omega}_p \cdot \underline{\Omega}) \quad . \tag{A-4}$$

Since the last term in the bracket represents a scalar triple product, the cross operation and dot operation can be interchanged, such that

$$\begin{aligned} \underline{a}\, (\underline{b} \times \underline{\omega}_p \cdot \underline{\Omega}) &= \underline{a}\, (\underline{b} \cdot \underline{\omega}_p \times \underline{\Omega}) \\ &= \underline{a}\, \underline{b} \cdot \underline{\omega}_p \times \underline{\Omega} \quad . \quad \text{Q.E.D.} \end{aligned} \tag{A-5}$$

To prove identity (A-1) it is best to return to the original definition of the moment of inertia. The first term on the right side of equation (A-1) can then be written as

$$\begin{aligned} \underline{\omega}_p \times \underline{\underline{I}}_p \cdot \underline{\Omega} &= \int \underline{\omega}_p \times [\underline{r} \times (\underline{\Omega} \times \underline{r})]\ dm \\ &= \int \underline{\omega}_p \times [(\underline{r} \cdot \underline{r})\, \underline{\Omega} - (\underline{r} \cdot \underline{\Omega})\, \underline{r}]\ dm \\ &= \int (\underline{r} \cdot \underline{r})\, (\underline{\omega}_p \times \underline{\Omega}) - (\underline{r} \cdot \underline{\Omega})\, (\underline{\omega}_p \times \underline{r})\ dm \quad . \end{aligned} \tag{A-6}$$

The last term of the right hand side of equation (A-1) is transformed likewise as:

$$\underline{\Omega} \times \underline{\underline{I}}_p \cdot \underline{\omega}_p = \int (\underline{r} \cdot \underline{r}) (\underline{\Omega} \times \underline{\omega}_p) - (\underline{r} \cdot \underline{\omega}_p) (\underline{\Omega} \times \underline{r}) \, dm \quad .$$

The second term of the right hand side of equation (A-1) is transformed as follows:

$$\begin{aligned} -\underline{\underline{I}}_p \cdot (\underline{\omega}_p \times \underline{\Omega}) = \underline{\underline{I}}_p \cdot (\underline{\Omega} \times \underline{\omega}_p) &= \int \left\{ \underline{r} \times [(\underline{\Omega} \times \underline{\omega}_p) \times \underline{r}] \right\} dm \\ &= -\int \left\{ \underline{r} \times [\underline{r} \times (\underline{\Omega} \times \underline{\omega}_p)] \right\} dm \\ &= -\int \left\{ \underline{r} \times [(\underline{r} \cdot \underline{\omega}_p) \underline{\Omega} - (\underline{r} \cdot \underline{\Omega}) \underline{\omega}_p] \right\} dm \\ &= \int \left\{ [(\underline{r} \cdot \underline{\Omega}) (\underline{r} \times \underline{\omega}_p) - (\underline{r} \cdot \underline{\omega}_p) (\underline{r} \times \underline{\Omega})] \right\} dm \quad . \end{aligned} \tag{A-8}$$

Adding the bottom lines of equations (A-6), (A-7), and (A-8) results in the cancellation of several terms with the final result:

$$\underline{\omega}_p \times \underline{\underline{I}}_p \cdot \underline{\Omega} - \underline{\underline{I}}_p \times \underline{\omega}_p \cdot \underline{\Omega} + \underline{\Omega} \times \underline{\underline{I}}_p \cdot \underline{\omega}_p = 2 \int (\underline{\Omega} \cdot \underline{r}) (\underline{r} \times \underline{\omega}_p) \, dm \quad . \tag{A-9}$$

The Coriolis term of the left hand side of equation (A-1) is now transformed, going back to the definition given in Term IV. Then

$$\begin{aligned} 2 \, \underline{\Omega} \cdot \underline{\underline{C}}_p \times \underline{\omega}_p &= 2 \int \underline{r} \times [\underline{\Omega} \times (\underline{\omega}_p \times \underline{r})] \, dm \\ &= 2 \int \underline{r} \times [(\underline{\Omega} \cdot \underline{r}) \underline{\omega}_p - (\underline{\Omega} \cdot \underline{\omega}_p) \underline{r} \, dm \\ &= 2 \int (\underline{\Omega} \cdot \underline{r}) (\underline{r} \times \underline{\omega}_p) \, dm \quad . \end{aligned} \tag{A-10}$$

It is seen that equation (A-10) is identical to equation (A-9). This completes the proof.

## APPENDIX B

This appendix develops the identity for the dyadic differentiation in rotating coordinate systems. Suppose that the dyadic $\underline{\underline{I}}$ is viewed by an observer in a coordinate system A and also by another observer in a coordinate system B, which rotates relative to system A. Let the rotation rate of system B relative to system A be defined by the angular velocity vector $\underline{\omega}_{AB}$. Then the following identity holds for the time derivative of a generic dyadic $\underline{\underline{I}}$:

$$(\dot{\underline{\underline{I}}})_A = (\dot{\underline{\underline{I}}})_B + \underline{\omega}_{AB} \times \underline{\underline{I}} - \underline{\underline{I}} \times \underline{\omega}_{AB} \quad . \tag{B-1}$$

To show this to be true let the dyadic $\underline{\underline{I}}$ be defined by the dyadic product of the two vectors $\underline{a}$ and $\underline{b}$, i.e.,

$$\underline{\underline{I}} = \underline{a}\ \underline{b} \quad . \tag{B-2}$$

Then the time derivative of $\underline{\underline{I}}$ as viewed in system A is:

$$(\dot{\underline{\underline{I}}})_A = (\dot{\underline{a}})_A\ \underline{b} + \underline{a}\ (\dot{\underline{b}})_A \quad . \tag{B-3}$$

Using the known identity for the vector differentiation in rotating coordinate systems

$$(\dot{\underline{a}})_A = (\dot{\underline{a}})_B + \underline{\omega}_{AB} \times \underline{a} \tag{B-4}$$

equation (B-3) can be written as:

$$\begin{aligned} (\dot{\underline{\underline{I}}})_A &= [(\dot{\underline{a}})_B + \omega_{AB} \times a]\ \underline{b} + a\ [(\dot{\underline{b}})_B + \underline{\omega}_{AB} \times \underline{b}] \\ &= [(\dot{\underline{a}})_B\ \underline{b} + \underline{a}\ (\dot{\underline{b}})_B] + \underline{\omega}_{AB} \times \underline{a}\ \underline{b} - \underline{a}\ \underline{b} \times \underline{\omega}_{AB} \quad . \end{aligned} \tag{B-5}$$

Note that the square bracket term of (B-5) is nothing other than the time derivative of the dyadic = $\underline{a}\ \underline{b}$ as viewed in system B. Thus the final result is obtained:

$$(\dot{\underline{\underline{I}}})_A = (\dot{\underline{\underline{I}}})_B + \underline{\omega}_{AB} \times \underline{\underline{I}} - \underline{\underline{I}} \times \underline{\omega}_{AB} \quad . \tag{B-6}$$

This completes the proof.

TECHNICAL REPORT STANDARD TITLE PAGE

| 1. REPORT NO. | 2. GOVERNMENT ACCESSION NO. | 3. RECIPIENT'S CATALOG NO. |
|---|---|---|
| NASA TP-2511 | | |

| 4. TITLE AND SUBTITLE | 5. REPORT DATE |
|---|---|
| Space Station Rotational Equations of Motion | August 1985 |
| | 6. PERFORMING ORGANIZATION CODE |
| 7. AUTHOR(S) | 8. PERFORMING ORGANIZATION REPORT # |
| Mario H. Rheinfurth and Stanley N. Carroll | |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS | 10. WORK UNIT NO. |
| George C. Marshall Space Flight Center<br>Marshall Space Flight Center, Alabama 35812 | M-497 |
| | 11. CONTRACT OR GRANT NO. |
| 12. SPONSORING AGENCY NAME AND ADDRESS | 13. TYPE OF REPORT & PERIOD COVERED |
| National Aeronautics and Space Administration<br>Washington, D.C. 20546 | Technical Paper |
| | 14. SPONSORING AGENCY CODE |

15. SUPPLEMENTARY NOTES

Prepared by Systems Dynamics Laboratory, Science and Engineering Directorate.

16. ABSTRACT

Dynamic equations of motion are developed which describe the rotational motion for a large space structure having rotating appendages. The presence of the appendages produce torque coupling terms which are dependent on the inertia properties of the appendages and the rotational rates for both the space structure and the appendages. These equations were formulated to incorporate into the Space Station Attitude Control and Stabilization Test Bed to accurately describe the influence rotating solar arrays and thermal radiators have on the dynamic behavior of the Space Station.

| 17. KEY WORDS | 18. DISTRIBUTION STATEMENT |
|---|---|
| Dynamics<br>Control<br>Rotating appendages<br>D'alemberts principle<br>Angular momentum | Unclassified – Unlimited<br><br>Subject Category 15 |

| 19. SECURITY CLASSIF. (of this report) | 20. SECURITY CLASSIF. (of this page) | 21. NO. OF PAGES | 22. PRICE |
|---|---|---|---|
| Unclassified | Unclassified | 20 | A02 |

MSFC - Form 3292 (May 1969)

For sale by National Technical Information Service, Springfield, Virginia 22161

NASA-Langley, 1985

National Aeronautics and
Space Administration

Washington, D.C.
20546

Official Business
Penalty for Private Use, $300

LANGLEY RESEARCH CENTER
3 1176 01316 2830

NASA

POSTMASTER: If Undeliverable (Section 158 Postal Manual) Do Not Return

DO NOT REMOVE SLIP FROM MATERIAL

Delete your name from this slip when returning material to the library.

| NAME | MS |
|---|---|
| ~~T. Holden~~ | ~~161~~ |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |
| | |

NASA Langley (Rev. May 1988) RIAD N-75