THE DISPERSIVE EVOLUTION OF CHARGED-PARTICLE BUNCHES IN RANDOM MAGNETIC FIELDS

J.A. Earl

Dept. of Physics & Astronomy, Univ. of MD, College Park, MD 20742

ABSTRACT

Shortly after a strongly anisotropic beam of charged particles is injected along a guiding magnetic field on which is superimposed a small random component, the particle density can be represented by a Gaussian profile whose center moves with the coherent velocity V_* and whose width increases with time at a rate controlled by the coefficient of dispersion D_*. Both parameters depend upon the mean free path λ, which characterizes scattering by the random fields, and the focusing length L, which characterizes spatial variations of the guiding field. These dependences are known explicitly for V_*. Formulae for D_* are available only in the limits of very weak and very strong focusing. This paper presents a new expression for D_*, which spans this gap.

1. Introduction. The equation which describes particle transport along a guiding field under the combined influences of scattering by magnetic turbulence and focusing by spatial inhomogenieties of the guiding field is

$$\frac{\partial f}{\partial t} + \mu V \frac{\partial f}{\partial z} = \frac{1}{2} e^{G} \frac{\partial}{\partial \mu} \phi e^{-G} \frac{\partial f}{\partial \mu}, \qquad (1)$$

where f is particle density in phase space, z is distance along the guiding field, μ is the pitch-angle cosine and V is particle velocity. If the coefficient of pitch-angle scattering is given by the standard form

$$\phi = \frac{3(V/\lambda)}{(2-q)(4-q)} (1-\mu^2) |\mu|^{q-1}, \qquad (2)$$

where q is an index that measures the anisotropy of scattering, then the odd function G is given by

$$G\{z,\mu\} = -\frac{V}{B}\frac{\partial B}{\partial z} \int_0^{\mu} \frac{1-\nu^2}{\phi\{z,\nu\}} d\nu = \frac{(4-q)}{3} \frac{\lambda}{L} \mu |\mu|^{1-q}, \qquad (3)$$

where L is the focusing length and B is the guiding field. Throughout this paper both λ and L will be assumed to be constant.

Even with these assumptions, it is not possible to solve the transport equation in closed form. However, there are approximations that provide considerable insight into phenomena, such as those outlined in the abstract, that do not fit into the familiar picture of purely diffusive transport. A first-order analysis (Earl, 1981; Kunstmann, 1979) leads to simplified transport equations which describe coherent propagation in terms of infinitely narrow pulses, but which do not include the dispersive spreading of these disturbances. This paper presents a

second-order analysis which includes this effect.

In the context of new possibilities for numerical solutions opened up by advances in computer technology, the phenomenon of dispersion takes on special significance, for there is an artifact, which I call "numerical dispersion", and which can obscure the physical effect. (See Paper SH 4.1-4) Numerical dispersion is an artifact of the Boltzmann operator that appears on the left hand side of equation (1). Consequently, it is not affected by focusing. To avoid inaccuracies, it is necessary to specify sufficiently fine spatial resolution in the numerical implementation so that the numerical dispersion coefficient is much smaller than the physical one. This paper provides the quantitative knowledge of D_* needed to make this specification.

2. The Coefficient of Dispersion. The second approximation to the distribution function takes a form

$$f = F_{\#}\{z,t\} + Be^{G}H_{\S}\{z,t\} + W\{\mu\}\frac{\partial F_{\#}}{\partial z} - Be^{G}W\{-\mu\}\frac{\partial H_{\S}}{\partial z}, \qquad (4)$$

in which the first two terms represent supercoherent and pseudodiffusive components that appeared in the first-order approximation and the second two are anisotropic components that are proportional to the spatial gradients of $F_{\#}$ and $H_{\S}$. Both of these additions involve the same function $W\{\mu\}$, but the sign of its argument is different in the two new terms. This relationship reflects a fundamental symmetry of focused transport that I call the "principle of complementarity". When equation (4) is substituted in the transport equation, and when the first-order transport equations are satisfied, the following equation for W is obtained:

$$\frac{1}{2}\frac{\partial}{\partial\mu}e^{-G}\frac{\partial W\{\mu\}}{\partial\mu} - \frac{V_{\#}K}{L(K^2-1)}W\{-\mu\} = \left(\mu V + \frac{V_{\#}^2K}{(K^2-1)}\right)e^{-G} - \frac{V_{\#}K}{(K^2-1)}, \qquad (5)$$

where

$$K = \frac{1}{2}\int_{-1}^{+1}e^{G}\,d\mu, \qquad (6)$$

is a normalization constant that goes from $K = 1$ in the limit of weak focusing ($\lambda/L \ll 1$) to $K = \infty$ in the limit of strong focusing, and

$$V_{\#} = \frac{V}{2K}\int_{1}^{+1}\mu e^{G}\,d\mu, \qquad (7)$$

is a characteristic velocity that I call the pseudodiffusive velocity. Note that this is not the coherent velocity V_*, which is given by

$$V_* = \frac{V_{\#}K}{(K^2-1)^{1/2}}. \qquad (8)$$

Although the presence of the term in $W\{-\mu\}$ complicates equation (5), it can be solved for W with the aid of the method of

eigenfunctions. In fact, because this term is relatively small, W can be evaluated fairly accurately by a simple double integration.

When the additional terms in f are taken into account, the second-order transport equations take the forms

$$\frac{\partial F_{\#}}{\partial t} - V_{\#}\frac{\partial F_{\#}}{\partial z} + \frac{B}{K}\frac{\partial H_{\$}}{\partial t} = D_{*}\frac{\partial^2 F_{\#}}{\partial z^2}, \qquad (9)$$

and

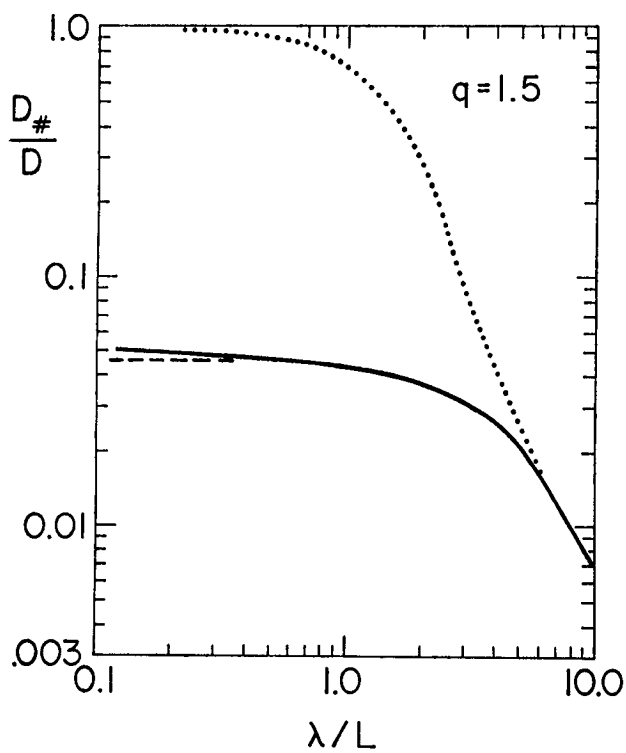
$$\frac{\partial H_{\$}}{\partial t} + V_{\#}\frac{\partial H_{\$}}{\partial z} + \frac{1}{BK}\frac{\partial F_{\#}}{\partial t} = D_{*}\frac{\partial^2 H_{\$}}{\partial z^2}, \qquad (10)$$

in which the left hand sides reproduce the first-order transport equations and the right hand sides are second-order terms. Here, the coefficient of dispersion, given by

$$D_{*} = -\frac{1}{2K}\int_{-1}^{+1} d\mu \left[\left(\mu V + \frac{V_{\#}K^2}{K^2-1}\right) e^{-G} - \frac{V_{\#}K}{K^2-1}\right] W\{\mu\}, \qquad (11)$$

determines the size of the new terms, which involve the second spatial derivatives of the supercoherent and pseudodiffusive components. They describe the dispersive aspect of coherent propagation, in which the delta function pulses of the first-order description are more accurately described as Gaussian disturbances whose centers move with the coherent velocity V_{*} while their widths increase with time at a rate controlled by the coefficient D_{*}.

In the figure, separate curves show how D_{*} depends upon (λ/L) when the function $W\{\mu\}$ is evaluated in three different ways. The dashed curve gives the exact result, in which the term in $W\{-\mu\}$ is retained in equation (5). The solid curve gives a much simpler result obtained from an approximate evaluation in which this term is ignored. For most applications, in which its ~20% deviation from the exact result is not important,



this approximation is adequate. In contrast, the dotted curve, which gives a result obtained by Kunstmann (1979) and by Bieber (1977) deviates by a factor of ~20 at small values of (λ/L). These results grew out of an analysis which describes correctly the strong focusing regime, but which does not take into account the coupling between components embodied in equations (9) and (10). Unless focusing is very strong, this neglect leads to a large overestimate of the dispersive effect.

3. Summary. The coefficient of dispersion given by equation (11) is a basic parameter needed to implement numerical solutions of the transport equation. In this context, where great accuracy is not required, the limitations of the present analysis to constant λ and L are not important.

4. Acknowledgements. This research was supported by the National Aeronautics and Space Administration through grant NGR-21-002-066.

References

Bieber, J.W. 1977, Ph.D. thesis, University of Maryland.
Earl, J.A. 1981, Ap.J., 251, 739.
Kunstmann, J. 1979, Ap.J., 229, 812.