

NUMERICAL DESCRIPTIONS OF COSMIC-RAY TRANSPORT

J.A. Earl

Dept. of Physics & Astronomy, Univ. of MD, College Park, MD 20742

J.R. Jokipii

Dept. of Planetary Sciences, Univ. of Arizona, Tucson, AZ 85721

ABSTRACT

The behavior of energetic particles in the solar system is described by a well known Fokker-Planck equation. Although analytic methods yield insight into the nature of its solutions, especially in the diffusion regime, calculations that go beyond diffusion are very complicated. Under these circumstances, numerical computations offer the only feasible way to obtain concrete results. However, the reliability of these calculations is of concern, because numerical methods are notorious for their errors and artifacts. To address this concern, the well known Milne problem of classical transport theory has been analyzed with the aid of three different numerical methods. These are:

1. The method of eigenfunctions in which the distribution function is approximated by a sum of eigenfunctions of the scattering operator. Its complexity limits the practicality of this approach, but it is closely related to the analytic and classical approaches.
2. Numerical solutions of a finite-difference equation. This is the most practical approach, but it is subject to subtle errors.
3. Direct simulation of the scattering and streaming of individual particles with the aid of Monte Carlo methods. The accuracy of this approach is limited by statistical considerations, but it is closely related to the physics.

If proper precautions are taken to ensure its validity, the second method gives results that are in precise quantitative agreement with those of the first. Results of Monte Carlo calculations are not accurate enough to define non-diffusive effects, but in the diffusion regime, they are consistent with diffusion theory.

1. Introduction. According to Weinberg and Wigner (1958), the Milne problem is "the touchstone of classical transport theory". It is in this traditional role as a test for new developments that this famous problem is addressed here. More specifically, the steady-state diffusion of particles to a free escape boundary from a planar source deep within a one-dimensional medium, which is the configuration assumed in the Milne problem, is treated under the further assumptions that the diffusing medium is magnetic turbulence superimposed upon a uniform guiding field and that the particles are cosmic rays whose pitch-angles

relative to the guiding field undergo scattering that is described by the Fokker-Planck formalism. Within this framework, the algebra needed to obtain analytic solutions in terms of eigenfunctions becomes manageable, because temporal variations are absent. These solutions, which embody non-diffusive effects, provide a convenient and well understood standard for judging the accuracy with which numerical methods describe these effects.

Some basic aspects of this picture are outlined within these proceedings in Paper SH4.1-4, and this paper deals with the same transport equation, invokes the same definitions and notation, and insofar as possible, uses the same numbers.

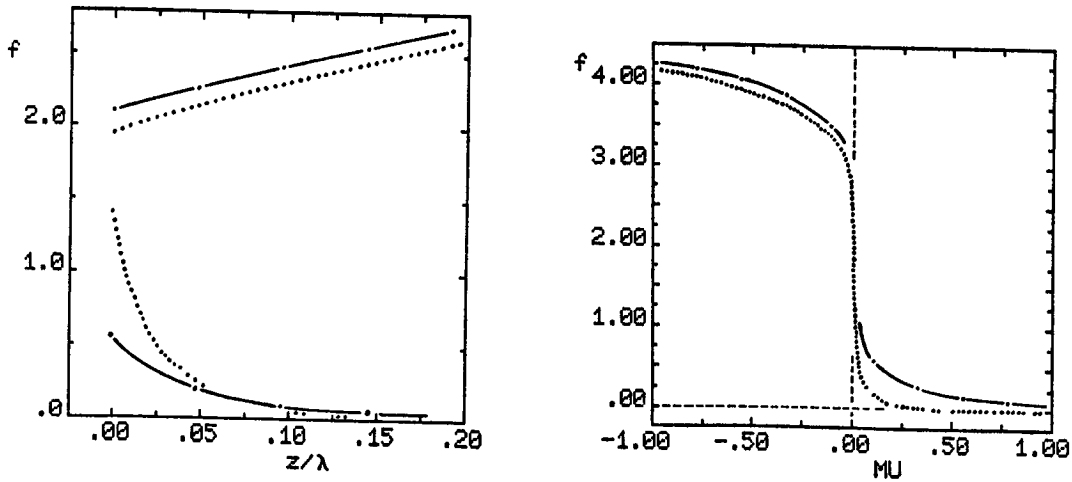
2. The Method of Eigenfunctions. This approach follows exactly the classical treatment in terms of spherical harmonics (Case and Zweifel, 1967, sec 8.4), except that the distribution function is expanded in terms of the scattering eigenfunctions tabulated by Bieber (1977) rather than in terms of Legendre polynomials. In the diffusion approximation, the distribution function $f\{z, \mu\}$ is approximated essentially by the first two eigenfunctions. In this case, the isotropic density is finite at the surface, and if it is extrapolated beyond the boundary along the same uniform slope that applies to diffusion deep within the medium, it reaches zero at a distance of $(2/3)\lambda$ outside the boundary. This result is exactly the one obtained from the classical P_1 approximation. To obtain a better description, assume that solutions of the form

$$T\{\mu\} \exp\{-z/\Lambda\}$$

can be expanded as a finite sum of four or more scattering eigenfunctions. Then the matrix form of the transport equation can be solved only for certain specific values of Λ that correspond to a distinct set of transient eigenfunctions. (Note that the term "transient", which derives from classical transport theory, has nothing to do with temporal variations.) These are summed in such a way that the returning intensity at the surface is as near as possible to zero over the outward facing hemisphere. In practice, only the first transient is significant. For the examples computed with $q = 1.8$, which are discussed below, the first characteristic length is $\Lambda = \lambda/40$.

3. Numerical Solutions of a Finite-Difference Equation. These computations invoke the same formulation and scattering operator as in Paper SH4.1-4, except that steady injection was accomplished by specifying at a fixed value of z a constant angular distribution. Starting from the diffusion solution, computations were carried out for a time sufficient to allow a particle moving in a straight line to traverse ten times the distance between injection point and boundary. At this time, the flux was essentially independent of z , which means that the solution was very close to the steady state. (See Earl, 1974b, eq. 24.)

In the figures below, results are plotted for two different values of the spatial increment, $\Delta z = \lambda/21$ (solid curves) and $\Delta z = \lambda/210$ (dotted curves), which are, respectively, substantially larger and



smaller than the characteristic length λ . Note that both increments are much less than the value $(3/4)\lambda$ derived by Kota et al. (1982) as a condition for validity of the diffusive picture. The analytical solution described above was computed separately, but it cannot be plotted, because its deviations from the dotted curves are smaller than the dots. In the figure at the left, the curves at the top give the isotropic density and those below give ten times the coefficient f_2 of the second scattering eigenfunction, which was obtained by expanding computed angular distributions as a series of four eigenfunctions. The surface density predicted at $z = 0$ by the solid curve is $\sim 8\%$ larger than expected. This discrepancy can be attributed to inaccuracies in the description of non-diffusive effects that can be seen in the curves below, which describe an even component that appears in the transient anisotropy, but not in the diffusion approximation. Here, the dotted curve follows exactly the expected exponential dependence characterized by $\Lambda = \lambda/40$, while the solid curve, whose spatial resolution is too coarse to describe this strong gradient, follows instead an exponential which has only 40% of the amplitude expected at the surface, and whose decay is characterized by $\Delta z = \lambda/21$.

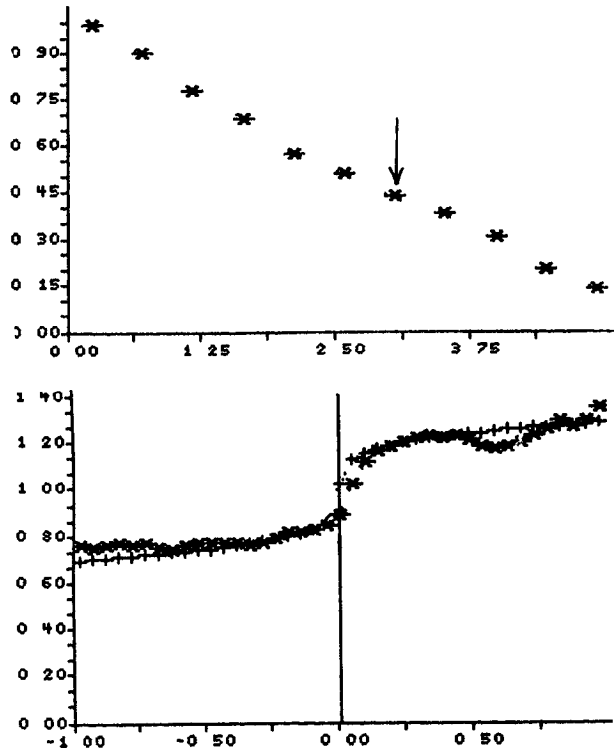
In the figure to the right, the intensity at the surface is plotted as a function of pitch-angle cosine μ . The eigenfunction solution is again indistinguishable from the dotted curve, whose rms deviation from expected values is 2.2%. Except for the point nearest to $\mu = 0$, the intensity predicted by the dotted curve within the hemisphere $\mu > 0$ is essentially zero, while in contrast, the solid curve, which refers to an inappropriate choice of Δz , exhibits here a small but finite intensity. Because scattering cannot turn back escaping particles beyond the boundary, no returning intensity is expected, and its degree of absence can be taken as a figure of merit for comparison of different calculations.

4. The Monte Carlo Method. The Fokker-Planck formalism derives from a statistical model in which scattering results from many tiny deflections of particle trajectories by turbulent fields. With the aid of

computers, these random deflections can be simulated directly, and the history of individual particles can be traced through space and time. Two crucial points of this approach are to choose the amplitudes of the deflections to be consistent with the Fokker-Planck coefficient ψ , and to adjust time increments so that the systematic drift in pitch-angle is compatible with the random spreading.

Because of the limitations on accuracy imposed on this method by statistical considerations, it is very difficult to study the non-diffusive effects considered above.

However, the figures to the right show results obtained in the diffusion regime from a Monte Carlo simulation of the Milne problem. Plotted above is the isotropic density as a function of distance, which shows the expected uniform slope downward to a finite density on the boundary at the right. The arrow here designates the point at which the angular distribution plotted below was obtained. This angular distribution is in good agreement with the one expected from diffusion theory, whose points are designated by + symbols.



5. Conclusions. The non-diffusive effects discussed above are miniscule, but the precision with which they are described numerically leaves little doubt that finite-difference calculations can give valid results, provided that their parameters are chosen appropriately. In the diffusion regime, Monte Carlo methods offer a useful alternative.

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