# NUMERICAL AND ANALYTIC DESCRIPTIONS OF COSMIC-RAY TRANSPORT

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ABSTRACT

It is not trivial to solve the equations that describe charged particle transport with the aid of computers, fo. instabilities, inaccuracies, and subtle artifacts are well known afflictions of numerical analysis. Two specific points are:

1. To avoid gross inaccuracies, pitch-angle scattering must be treated with great care. In particular, slightly inappropriate numerical formulations give rise to mean free paths that are in error by large factors.

2. A previously unrecognized artifact, "numerical dispersion", is very similar to the physical phenomenon of dispersion. To avoid misinterpretations arising from this similarity, the spatial increment of the finite-difference grid must be a small fraction of the mean free path.

These points are illustrated by calculations based upon finite-difference approximations to the transport equation.

1. Introduction. The diffusive idealization, which has been almost universally invoked in discussions of cosmic-ray transport is easy to treat analytically. However, many observed phenomena give clear evidence for the presence of non-diffusive effects. One example is the so-called "scatter-free" propagation of kilovolt solar electrons which is inconsistent with diffusion, but which can readily be interpreted in terms of the coherent mode of propagation. Although the qualitative features of these effects have been outlined (Earl, 1974a, 1976), the theory is very complicated. Consequently, there is a need for reliable numerical computations which bypass these complexities and yield concrete results that are well suited for comparison with observations. This paper explores such methods within the limited context of rectilinear propagation of cosmic-ray along a uniform guiding field on which are superimposed random fields.

Under these circumstances, transport is described by

$$\frac{\partial f}{\partial s} + \mu\frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\,\psi\,\frac{\partial f}{\partial \mu} \qquad (1)$$

in which f is particle density in phase space, $\mu$ is the pitch-angle cosine, and z is distance parallel to the guiding field. The parameter s = Vt, where V is particle velocity, plays the role of a temporal variable. The coefficient of pitch-angle scattering is given by

$$\psi = \frac{(3/2\lambda)}{(2-q)(4-q)}\,(1-\mu^2)\,|\mu|^{q-1}, \qquad (2)$$

where $\lambda$ is the mean free path, and q is an index that measures the anisotropy of scattering.

Diffusion refers to a configuration of slow temporal variations of the anisotropy and weak spatial inhomogenities of the isotropic density $F_0$ (Earl, 1974b), which act as the source of a diffusive anisotropy given by

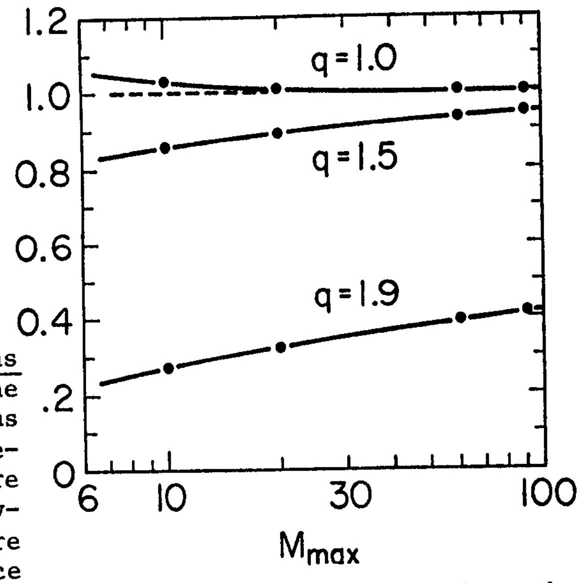
$$F_1 = -V \frac{\partial F_0}{\partial z} g, \qquad (3)$$

where

$$g\{\mu\} = [(4-q)\lambda/3]\ \mu|\mu|^{1-q} \qquad (4)$$

is a solution of

$$\frac{\partial}{\partial \mu} \psi \frac{\partial g}{\partial \mu} = -\mu. \qquad (5)$$




2. Finite-Difference Approximations to the Scattering Operator. In the discrete formulation, the continuous variables are replaced by a three-dimensional grid whose spacings are $\Delta z$, $\Delta\mu$, and $\Delta s = V\Delta t$, and the derivatives appearing in equation (1) are replaced by their finite-difference analogs. A test of the relationships implied by diffusion can be made without extensive calculations by replacing the differential operator in equation (5) by its finite version and solving numerically for the diffusive anisotropy. This procedure should give an answer consistent with equation (4), and if it does not, the effective mean free path is given by

$$\lambda = (3/2) \int_{-1}^{+1} \mu g \, d\mu. \qquad (6)$$

Kota et al. (1982) gave an explicit finite-difference form for the scattering operator. Although this derivation seems eminently reasonable from the standpoint of both mathematics and physics, the resulting scattering operator does not pass this test near the threshold at $q = 2$ for pure coherent propagation.

Mean free paths derived from this operator are given in the figure above, where the ratio of the actual value of $\lambda$ obtained from equation (6) to the nominal value that appears in equation (2) is plotted against the number $M = 2/\Delta\mu$ of increments in pitch-angle. The correspondence is close for $q = 1$ and fairly close for $q = 1.5$, but for $q = 1.9$, $\lambda$ is a factor of ~2.5 too small and shows little indication of converging to the nominal value as M increases. These inaccuracies are a consequence of the fact that weak scattering near $\mu = 0$ gives rise to large deviations from the analytic behavior. To avoid these deviations, I use a finite operator constructed so that equation (5) is satisfied exactly when g is described by the analytic expression. Far from $\mu = 0$, this operator behaves the same as Kota's operator, but it also gives the correct diffusive anisotropy and makes the actual mean free path identical to the nominal one.

3. Numerical Dispersion. In a configuration of weak scattering, particle bunches propagate coherently in a moving Gaussian profile given by

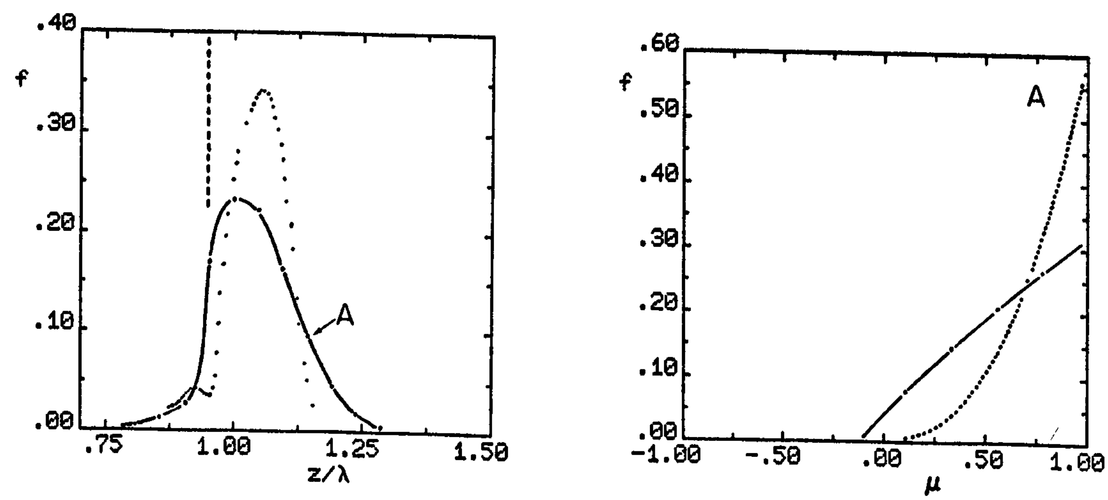
$$F_o = 1/(4\pi D_* t)^{1/2} \exp\{-(z - V_* t)^2/4D_* t\} \qquad (7)$$

where $V_*$ is a characteristic velocity, which corresponds to an angular distribution that is nearly isotropic in one hemisphere and zero in the other, and $D_*$ is the coefficient of dispersion, which describes the broadening of the Gaussian that arises from statistical fluctuations of individual velocities within this distribution. (Earl, 1974a).

The finite-difference implementation leads to a very similar effect which can be described by equation (7) with $D_*$ replaced by

$$D_{\dagger} = \frac{1}{4} V(\Delta z - \frac{1}{2}\Delta s) \qquad (8)$$

The total dispersion is a superposition of the physical effect described by $D_*$ and the numerical effect described by $D_{\dagger}$. To ensure that physics dominates, the condition $D_* \gg D_{\dagger}$ must be satisfied. In the examples presented below, where $q = 1.8$, this condition translates into a requirement that $\Delta z < \lambda/40$. This condition is similar in concept to the condition derived by Kota et al. for the validity of calculations of diffusion, which is $\Delta z < 3\lambda/4$. However, the required spatial resolution is much finer in the coherent regime that it is in the diffusion regime.




4. A Specific Example. To illustrate the points made above, two calculations were made which had identical values of $\lambda$ and $\Delta\mu = 0.22$, but which had values $\Delta z = \lambda/21$ (plotted as solid lines) and $\Delta z = \lambda/210$ (plotted as dotted lines) that are, respectively, well below and well above the required spatial resolution. In the figure to the left, snapshots of the density profiles at the instant $s = 0.19\lambda$, display the familiar coherent pulses followed by small but perceptible wakes. At this time, the dotted peak has moved a distance of $0.105\lambda$ to the right of the point indicated by a vertical dotted line at which a localized coherent angular distribution was injected at $s = 0$. This corresponds

to a coherent velocity $V_* = 0.55V$ that is in good agreement with the corresponding velocity of 0.53V tabulated by Bieber (1977). Similarly, the FWHM width of the pulse $0.135\lambda$ is in good agreement with the width $0.125\lambda$ expected from a combination of physical dispersion with $D_* = 0.0064\lambda V$ and numerical dispersion with $D_\dagger = 0.001\lambda V$.

The coherent velocity of the solid line peak is less than that of the dotted one and its dispersion, which is dominated by the numerical effect, is larger. To the right of point A, significant solid density appears in regions where the dotted density is zero. This discrepancy is relevant to descriptions of the early onset phase of solar particle events, for it means that the numerical dispersion can bring particles from the sun faster than the physics can, and this could lead to significant errors in the timing of flare events.

The angular distributions in the figure that appears at the right above add weight to this point. They refer to point A where the two profiles have equal densities and, consequently, can be directly compared without normalization. The dotted distribution is much more collimated in the forward direction near $\mu = 1$ than the solid one is. It is clear that any attempts to interpret observed angular distributions of the first few particles arriving in a flare event should take rigorous measures to exclude the effect of numerical dispersion.

5. Conclusions. Before non-diffusive transport can even be considered, the scattering operator must be carefully tailored to give valid results in the diffusion regime. Once this has been done, calculations that take proper account of numerical dispersion give results in good agreement with those expected from the theory of coherent propagation, but not with those calculated with inadequate spatial resolution. This situation can lead to insidious errors, for the invalid results appear to be qualitatively reasonable, and can be detected only by quantitative tests.

6. Acknowledgements. This research was supported by NASA under Grant NGR 21-002-066.

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