WHY DO LEAKY-BOX MODELS WORK SO FINE?

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ABSTRACT

By introducing the concept of the age distribution of cosmic rays it is possible to decouple the spatial from the momentum transport; and simple leaky-box type equations result. The influence of spatial inhomogeneities, geometries and source distributions enters the spatially homogeneous, infinite (i.e. leaky box) problem through appropriate mean lifetimes. A precise prescription of how to obtain these mean lifetimes: for comparison with data measured in the vicinity of the solar system they have to be calculated from the age distribution at the solar system.

1. Introduction. The propagation of cosmic ray particles in the Galaxy at large momenta, $p > 10$ GeV/c, where spatial diffusion in partially random magnetic fields dominates convection and adiabatic deceleration in the galactic wind, is described by the steady-state transport equation (e.g. Schlickeiser 1983)

$$\nabla \cdot [K(\underset{\sim}{r},p)\,\nabla f] + L_p f + q(\underset{\sim}{r})\,Q(p) = 0 \qquad (1),$$

where f is the particle's phase space density, $K(\underset{\sim}{r},p) = \kappa(p)\,\underset{\sim}{K}(\underset{\sim}{r})$ the spatial diffusion tensor, with $\kappa(p)$ chosen dimensionless without loss of generality, and L_p is the momentum operator

$$L_p \equiv p^{-2}\frac{d}{dp}\left[p^2 D(p)\frac{d}{dp} - p^2\,\dot{p}_{gain} - p^2\,\dot{p}_{loss}\right] - \frac{1}{\tau_c} \qquad (2),$$

describing first ($\dot{p}_{gain}$) and second ($D(p)$) order Fermi acceleration as well as continuous ($\dot{p}_{loss}$) and catastrophic (τ_c) momentum loss processes. $q(\underset{\sim}{r})\,Q(p)$ represents sources and sinks of particles.

2. Decoupling of spatial and momentum transport. If the parameters $D(p)$, $\dot{p}_{gain}$, $\dot{p}_{loss}$ and τ_c do not depend on position ($\underset{\sim}{r}$), equation (1) is solved by the convolution (Lerche and Schlickeiser 1985a)

$$f(p,\underset{\sim}{r}) \equiv \int_0^\infty dy\; G(p,t)\; N(\underset{\sim}{r},y) \qquad (3),$$

with

$$\frac{1}{\kappa(p)}\,L_p\,G(p,y) - \frac{\partial G(p,y)}{\partial y} = 0 \qquad (4)$$

and $G(p,y)$ satisfying given momentum boundary conditions and

$$\lim_{y\to\infty} G(p,y) = 0 \quad (5)$$

$$G(p,y=0) = Q(p)/\kappa(p) \quad (6).$$

$N(\underset{\sim}{r},y)$ satisfies given spatial boundary conditions, and

$$\underset{\sim}{\nabla}\cdot[K(\underset{\sim}{r})\,\underset{\sim}{\nabla}\,N(\underset{\sim}{r},y)] = \frac{\partial N(\underset{\sim}{r},y)}{\partial y} \quad (7),$$

$$N(\underset{\sim}{r},y=0) = q(\underset{\sim}{r}) \quad (8),$$

and $N(\underset{\sim}{r},y)$ is normalized as

$$\int_0^\infty dy\, N(\underset{\sim}{r},y) = 1 \quad (9).$$

$N(\underset{\sim}{r},y)$ is commonly referred to as the age distribution of cosmic rays.

The solution of the system of equations (4) - (8) would yield via (3) the particle's momentum spectrum $f(p,\underset{\sim}{r})$ at all positions $\underset{\sim}{r}$ in the Galaxy. However, particle flux measurements can only be obtained at the position of the solar system. In order to compare with these observations we would infer from the general solution $f(p,\underset{\sim}{r})$ the momentum spectrum at the solar system,

$$H(p) \equiv f(p,\underset{\sim}{r}_s) = \int_0^\infty dy\, G(p,y)\, N(\underset{\sim}{r}_s,y) \quad (10)$$

by setting $\underset{\sim}{r} = \underset{\sim}{r}_s$ in (3).

3. Leaky-box equations. However, instead of following this scheme to solve the two partial differential equations (4) and (7), it is much easier to derive from (4) a simpler equation for the momentum spectrum H(p) at the solar system from the known cosmic ray age distribution $N(\underset{\sim}{r}_s,y)$ at the solar system. Integrate equation (4) after multiplying it with $N(\underset{\sim}{r}_s,y)$ to obtain

$$\frac{1}{\kappa(p)} L_p \int_0^\infty dy\, G(p,y)\, N(\underset{\sim}{r}_s,y) = \int_0^\infty dy\, \frac{\partial G(p,y)}{\partial y}\, N(\underset{\sim}{r}_s,y)$$

$$= -\frac{Q(p)\, N(\underset{\sim}{r}_s,y=0)}{\kappa(p)} - \int_0^\infty dy\, G(p,y)\, \frac{\partial N(\underset{\sim}{r}_s,y)}{\partial y} \quad (11),$$

where the right hand side has been integrated by parts taking into account (5) and (6). With (10) equation (11) reads

$$L_p H(p) + \kappa(p) \int_0^\infty dy\, G(p,y)\, \frac{\partial N(\underset{\sim}{r}_s,y)}{\partial y} = -Q(p)\, N(\underset{\sim}{r}_s,y=0) \quad (12)$$

which is an ordinary differential equation in momentum.

If the age distribution at the solar system is a pure exponential,

$$N(\underset{\sim}{r}_s,y) = \frac{1}{\langle T\rangle(\underset{\sim}{r}_s)} \exp(-y/\langle T\rangle(\underset{\sim}{r}_s)) \qquad (13),$$

one infers $N(\underset{\sim}{r}_s,y=0) = 1/\langle T\rangle(\underset{\sim}{r}_s)$ and $\partial N(\underset{\sim}{r}_s,y)/\partial y = (-1/\langle T\rangle(\underset{\sim}{r}_s))\, N(\underset{\sim}{r}_s,y)$, so that (12) attains the form of a simple leaky-box equation

$$L_p H(p) - \frac{\kappa(p)\, H(p)}{\langle T\rangle(\underset{\sim}{r}_s)} = - \frac{Q(p)}{\langle T\rangle(\underset{\sim}{r}_s)} \qquad (14),$$

where the whole spatial problem enters through the constant parameter $\langle T\rangle(\underset{\sim}{r}_s)$ characterizing the exponential age distribution (13). It has to be emphasized that equation (14) is only valid at the position of the solar system. If for some reason, one wants to calculate the momentum spectrum of particles at some other place in the Galaxy, say $\underset{\sim}{r}_i$, one has to calculate new parameters $\langle T\rangle(\underset{\sim}{r}_i)$ from the respective age distribution $N(\underset{\sim}{r}_i,y)$.

But is the age distribution of cosmic rays arriving at the solar system pure exponential, as assumed in (13) and (14)? As shown by Lerche and Schlickeiser (1985b), in a variety of circumstances the age distribution at the solar system can be expanded in a series of exponentials:

$$N(\underset{\sim}{r}_s,y) = \sum_{m=1}^{\infty} \frac{c_m}{\langle T\rangle_m} \exp(-y/\langle T\rangle_m) \qquad (15).$$

The convolution of $N(\underset{\sim}{r}_s,y)$ from (15) with equation (4) yields for particle momentum spectrum near the solar system

$$H(p) = \sum_{m=1}^{\infty} c_m H_m(p) \qquad (16),$$

where $H_m(p)$ is the solution of equation (14) with $\langle T\rangle_m$ playing the role of $\langle T\rangle(\underset{\sim}{r}_s)$. So in general, $H(p)$ is attained by an (infinite) sum of leaky-box solutions. Very often it suffices to consider only the $m = 1$ contribution, when the first eigenvalue dominates the behaviour of the age distribution.

4. Conclusions. In studies of the transport and acceleration of cosmic energetic particles the age distribution plays the same role as the escape probability in studies of photon propagation through hot plasma (see e.g. Sunyaev and Titarchuk 1980). The influence of spatial inhomogeneities, geometries and source distributions can be estimated using the solution for the spatially homogeneous infinite problem (leaky-box equation (14)) with the appropriate escape lifetimes $\langle T\rangle$. A precise prescription of how to obtain these escape lifetimes has been given: namely, they have to be calculated from the age distribution at the spatial position of the observer.

References

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