INSTABILITIES IN DECELERATING SUPERSONIC FLOWS WITH APPLICATIONS TO COSMIC RAY SHOCKS

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ABSTRACT. We investigate the nature of instabilities in cosmic ray shocks by using two distinct models for the shock wave. For wavelengths which are short relative to the thickness of the shock wave, the shock is treated as a smoothly decelerating flow, and an appropriate JWKB type expansion is used to describe the perturbations to the flow. In this, the short wavelength regime, the presence of "squeezing" and an effective "g" renders strong cosmic ray shocks unstable in a way which is similar to instabilities in other supersonic flows, such as in de Laval nozzle flow or a heat conduction dominated shock wave. In the long wavelength limit, where the shock is treated as a discontinuous transition, we derive a "stability function" which, if negative, corresponds to unstable disturbances growing exponentially in time. In this case, we find that if the cosmic ray fluid is relativistic ($\gamma_c = 4/3$) and the background plasma ideal ($\gamma = 5/3$), then strong shocks are unstable.

1. INTRODUCTION. We examine the stability of cosmic ray shocks (Axford et al [1977], Drury and Völk [1981], Axford et al [1982], Völk [1984], Drury [1984], Dorfi [1984]) in both short and long wavelength regimes. The short wavelength instability in the two-fluid description of a cosmic ray shock may be shown to be similar to the instabilities in other supersonic decelerating flows. Erpenbeck, [1962], established that discontinuous transitions (shocks) are always stable for ideal fluids. We generalize his results, and show that the relativistic two-fluid model considered here is unstable.

2. THE GOVERNING EQUATIONS. The equations which describe the self-consistent interaction between a background thermal plasma and a relativistic cosmic ray fluid are given by

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\,(\rho\underline{u}) = 0; \qquad \text{(continuity)} \qquad (2.1a)$$

$$\rho\frac{D\underline{u}}{Dt} + \nabla\,(p + p_c) = 0; \qquad \text{(momentum)} \qquad (2.1b)$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u + \frac{P}{\gamma-1}\right) + \mathrm{div}\left[\rho\underline{u}\left(\tfrac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{P}{\rho}\right) + \frac{\gamma_c}{\gamma_c-1}p_c\underline{u} - \frac{\kappa}{\gamma_c-1}\nabla p_c\right] = 0, \qquad \text{(energy)} \qquad (2.1c)$$

where ρ, p, p_c, and $\underline{u}$ denote gas density, gas pressure, cosmic ray pressure and gas velocity respectively, and κ an effective cosmic ray diffusion coefficient. These equations are supplemented by the cosmic ray energy equation

$$\frac{\partial}{\partial t}\left(\frac{p_c}{\gamma_c-1}\right) + \frac{1}{\gamma_c-1}\,\mathrm{div}\,[\gamma_c p_c\underline{u} - \kappa\nabla p_c] = \underline{u}.\nabla p_c, \qquad (2.2)$$

which, together with 2.1(a,b), enables 2.1(c) to be expressed in the adiabatic form

$$\frac{D}{Dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0, \tag{2.3}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}.\nabla$ is the convective derivative. The derivation and discussion of these equations may be found in, for example, Drury, [1983].

A perturbation analysis of (2.1), (2.2) and (2.3) shows that short wavelength perturbations propagate at the plasma sound speed, c, where

$$c = \sqrt{\gamma p/\rho}, \tag{2.4}$$

and hence decouple from the cosmic ray fluid. Conversely, long wavelength disturbances propagate at the mixed sound speed,

$$\sqrt{a_c^2 + c^2} = \sqrt{(\gamma_c p_c + \gamma_p)/\rho}, \tag{2.5}$$

Ptsukin, [1981], $a_c \equiv$ cosmic ray fluid sound speed.

A discussion of the shock structure problem may be found in Axford et al, [1982], and Drury and Völk, [1981], McKenzie and Völk [1982], Völk et al [1983].

3. THE SHORT WAVELENGTH INSTABILITY. For the sake of completeness we present a brief modified account of the instability discussed by Drury [1984]. By considering waves whose wavelengths are short relative to the thickness of the shock wave, one may treat the transition as a continuous decelerating flow. If one views the decelerating flow as an "effective gravitational force", the stability problem has the character of a "generalized Rayleigh-Taylor" type problem for a disturbance propagating through the system. The introduction of compressive perturbations $\delta\psi = (\rho, p, p_c, u)$ on the background state $\psi_o = (\rho_o, p_o, p_{co}, u_o)$ enables equations (2.1) and (2.3) to be linearized. We introduce a shock thickness parameter L, related to the decelerating flow u_o, by defining

$$\frac{1}{L} = \frac{-1}{u_o}\frac{du_o}{dx},$$

where $\frac{du_o}{dx}$ is the flow velocity profile. On either side of the shock, u_o is constant, and hence $\frac{1}{L} = 0$. By using the steady state conservation laws (2.1) and (2.3), one can express the gradients of the remaining background variables ρ_o, p_o, p_{co}, as functions of L and the Mach number of the flow. We restrict the analysis to velocity profiles which vary slowly in space, and thus need only consider terms which are $O(\frac{1}{L})$. This is tantamount to performing a WKBJ expansion in space. The use of this expansion, and the search for plane wave solutions whose amplitudes vary harmonically, reduces (2.1) - (2.3) to the following dispersion equation;

$$\omega' + \frac{iu_o}{L} = \frac{k^2(a_c^2 + \frac{ic_o^2}{kL}(M^2-1))}{(\omega' - i\kappa k^2 + \frac{i\gamma_c u_o}{L})} - \frac{iu_o^2(k + \frac{i}{L})}{L\,(\omega' + \frac{iu_o}{L})} + \frac{c_o^2k^2\,(1 + \frac{i}{kL})}{(\omega' + \frac{i\gamma u_o}{L})}, \tag{3.2}$$

where $\omega' = \omega - u_o k$ in the Doppler shifted frequency, and k the wave number. In the absence of a shock, the dispersion relation reduces to

$$\omega' = 0; \quad \omega' = \pm c_o k. \quad \text{(entropy-vorticity; sound waves)}$$

A perturbation analysis of (3.2), about these frequencies, indicates that in the presence of a decelerating flow,

$$Im\,\omega' = -\frac{u_o}{L}, \tag{3.3}$$

for an entropy wave, and for a sound wave

$$Im\,\omega' = \frac{a_c^2}{2\kappa} - \frac{c_o^2}{2L}[M^2 - (\gamma+1)\,M - 1], \tag{3.4}$$

which, since we consider solutions of the form $\exp i\omega t$, implies instability for sufficiently large Mach numbers M.

4. THE LONG WAVELENGTH INSTABILITY. In this section we generalize and extend the results of D'Iakov, [1958], Erpenbeck, [1962 a,b], and McKenzie and Westphal, [1968] to include the effect of a cosmic ray fluid. We derive the transmission coefficient of an acoustic wave incident on an oblique shock, and examine the singularities of this transmission function. In the case of a supersonic-subsonic transition it is straightforward to show that, if the shock is perturbed, then three waves diverge from the shock (and thus the problem is well posed). After perturbing the generalized Rankine-Hugoniot boundary conditions

$$[\rho u_n] = 0; \quad \text{(continuity)} \tag{4.1a}$$

$$[p + p_c + \rho u_n] = 0; \quad \text{(normal momentum)} \tag{4.1b}$$

$$[\underline{u}_t] = 0; \quad \text{(tangential momentum)} \tag{4.1c}$$

$$[\tfrac{1}{2}u_n^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} + \frac{\gamma_c}{\gamma_c-1}\frac{p_c}{\rho}] = 0, \quad \text{(energy)} \tag{4.1d}$$

where n and t represent directions normal and tangential to the shock, one may derive the transmission coefficient for an acoustic wave,

$$\frac{\delta p_2}{\delta p_1} = \frac{\alpha_3(1 - \frac{u_y k_y}{\omega})\,N_1 + \beta_3\beta_1 N_2 + \beta_3 N_3}{\alpha_3(1 - \frac{u_y k_y}{\omega})\,D_1 + \beta_3\beta_2 D_2 - \beta_3 D_3}, \tag{4.2}$$

where (α_i,β_i) denote the cosine and sine of the angle between $\underline{k}$ and the x-axis, and N_i, D_i are suitable generalizations of the functions presented in McKenzie and Westphal, [1968]. The singularities of the function (4.2) correspond to the dispersion equation.

On defining the variables

$$R^2 = 1 + \frac{a_c^2}{c^2}, \quad R^* = 1 + \frac{\gamma-1}{\gamma_c-1}\,\frac{a_c^2}{c^2}, \quad \frac{1}{M^2} = \frac{R^2}{M_s^2}, \quad M_s \equiv \text{gas Mach number},$$

the stability function, F_s, may be written as

$$F_s = \frac{\rho_1}{\rho_2}(1-M_2)\left(2-(\gamma-1)\frac{M_2 s^2}{R_2^*}\left(\frac{\rho_2}{\rho_1}-1\right)\right) - \left(1+(\gamma-1)\frac{M_2 s^2}{R_2^*} - \left(1+(\gamma-1)\frac{R_2^2}{R_2^*}\right)M_2^2\right), \tag{4.3}$$

for waves propagating with the flow, and where the subscript 1(2) indicates ahead (behind) the shock. In the cold plasma approximation, cosmic ray shocks are unstable for all supersonic - subsonic transitions. In the alternate approximation of a tenuous cosmic ray gas ahead of the shock, it may be established that shocks are unstable for all Mach numbers $\gtrsim$ 1,3. The growth rate of these instabilities (for strong shocks) is of the order the compression ratio $\left(\frac{\rho_2}{\rho_1}\right) \times ku_1$.

References

Axford, W.I., Leer, E., Skadron, G.[1977], Proc. 15th Int.Cosmic Ray Conf. (Plovdiv), 11, 132.

Axford, W.I., Leer, E., McKenzie, J.F.,[1982], Astron.Astrophys. 111, 317

D'Iakov, S.P. [1958], Sov.Phys.JETP 6, 729.

Dorfi, E. [1984], Adv. Space Research 4, No.2-3, 205.

Drury, L.O'C., Volk, H.J. [1981], Astrophys.J. 248, 344.

Drury, L.O'C., [1983], Rep. Prog. Phys. 46, 973.

Drury, L.O'C., [1984], Adv. Space Research 4, No.2-3, 185.

Erpenbeck, J.J. [1962a], Phys. Fluids 5, 604.

Erpenbeck, J.J. [1962b], Phys. Fluids 5, 1181

McKenzie, J.F., Westphal, K.O. [1968], Phys. Fluids 11, 2350.

McKenzie, J.F., Völk, H.J. [1982], Astron. Astrophys. 116, 191.

Ptuskin, V.S. [1981], Astrophys. Space Sci. 76, 265.

Völk, H.J., Drury, L. O'C., McKenzie, J.F. [1984], Astron. Astrophys. 130, 19.

Völk, H.J. [1984], Adv. Space Research 4. No. 2-3, 423.