RELATIVISTIC COSMIC RAY SPECTRA IN THE FULL NON-LINEAR THEORY OF SHOCK ACCELERATION

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1. Introduction. The test particle theory of diffusive shock acceleration predicts a power law cosmic ray (CR) spectrum in momentum of the form

$$N(p) \propto p^{-s}, \quad s = -(r+2)/(r-1), \tag{1}$$

where N(p) is the differential number density of CRs at momentum p and s is determined solely by the compression ratio, $r = u_1/u_2$. Here, the shock is assumed to be planar, of infinite extent, steady state, discontinuous, piece-wise constant, and u_1 (u_2) is the flow velocity (measured in the shock frame) upstream (downstream) of the shock discontinuity (u_1 is the shock velocity in the lab frame)[2,3,4,6].

Since strong, classical hydrodynamic shocks have a compression ratio of 4, the test particle theory predicts a spectral index of -2, in good agreement with the source spectrum of galactic CRs that is inferred from observations. This account of the spectral index is widely viewed as being superior to previous accounts, which required fine-tuning to get the correct spectrum[1].

The non-linear theory, however, which allows for the fact that the CR pressure may be dynamically significant, accordingly allows the compression ratio to be larger than 4. This follows from the facts that (a) much of the post shock pressure can be in relativistic CRs, and, for a relativistic fluid, the classical compression ratio can be as large as 7, and (b) a significant amount of energy can be lost from the shock front in the form of particles that stream away from the shock before being convected downstream of it, and, r may then be arbitrarily large[6].

High production efficiencies (>10%) appear to be as widespread in astrophysical particle acceleration as does a spectral index between -2 and -3, and the possibility that CRs affect the compression ratio needs to be taken seriously. While one could argue that efficiencies of order 10% are marginally consistent with observation without requiring significant non-linear effects, invoking an injection rate that always gives an efficiency ~ 10% in the test particle theory requires a fine-tuning assumption about the injection, which is probably objectionable.

Here we show that the non-linear theory gives rise to spectra that are very close to a p^{-2} power law, over most of the range of astophysically reasonable shock parameters for supernova blast waves in the hot phase of the interstellar medium (HISM), even though r may differ significantly from 4. The universality of the spectrum, in this view, is related to the nature of the non-linear feedback of the CRs on their own acceleration rate, not to the universality of the compression ratio among shocks.

2. The Basic Calculation. The non-linear, steady-state theory includes an explicit description of the waves that couple the particles to the fluid. The full set of equations is

$$\rho u = C_1, \tag{2}$$

$$\rho u^2 + P_e + P_w + P_g = C_2, \tag{3}$$

$$\frac{dT_w}{dx} = u\,\frac{dP_w}{dx} + v_{ph}\,\frac{dP_e}{dx} - RP_w, \tag{4}$$

$$RP_w = C_1 T\,\frac{ds}{dx}, \tag{5}$$

$$T\,ds = dh - \frac{dP_g}{\rho}, \tag{6}$$

$$-\frac{\partial}{\partial x}(u_s F) + \frac{1}{3}\,\frac{du_s}{dx}\,p\,\frac{\partial F}{\partial p} + \frac{\partial}{\partial x}\,D\,\frac{\partial F}{\partial x} = C_3. \tag{7}$$

Here, ρ is fluid density, u is fluid velocity, u_s is the velocity of the scattering centers, P_e is the pressure in energetic particles, P_w is the wave pressure, P_g is the pressure in gas that has not yet been shocked, T_w is the energy flux due to waves, v_{ph} is the phase velocity of the scattering waves relative to the background medium, i.e. $u-u_s$, R is the damping rate, Tds is the heat per unit mass that enters the fluid via the wave damping, h is the enthalpy per unit mass and C_1, C_2, and C_3 are constant. Equations (2) and (3) express mass and momentum conservation, respectively. Equation (4) is an energy inventory equation for the waves with the terms on the right hand side representing, in order of appearance, amplification of existing waves by compression, growth due to the cyclotron unstable CR gradient, and damping (see 8 and references therein). Equations (5) and (6) describe the energy deposition into the thermal fluid by the wave damping, and (7) is the diffusion equation for $F = 4\pi p^3 f(d\ln p/d\ln E)$, where f(p) is the distribution function and D is the diffusion coefficient. These equations can all be solved using a non-perturbative, non-separable, analytic technique that proves to be extremely accurate in comparison with numerical solutions[5,6]. The numerical solutions referred to are possible only over a limited dynamical energy range. However, the analytic technique exploits the fact that in a real situation there is a very large dynamic energy range - ten decades for galactic CRs - so it should be even more accurate in the latter case than in the comparison with the numerical results.

We have solved the above eqs. for rapid wave damping ($R \to \infty$) and for v_{ph} constant upstream of the shock (v_{ph} is set equal to zero downstream). Results are shown in Fig. 1 where we have plotted partial pressure $\equiv$ (j/3)EF, versus energy, E. Here, j = 1 for relativistic and = 2 for non-relativistic particles. Each curve is the partial pressure profile for a locus of (M_1, v_{ph}) values shown in Fig. 2. Each point on a curve in Fig. 2 produces essentially the same profile as seen in Fig. 1. The range in compression ratios from $v_{ph} = 0$ to $M_1 \to \infty$, is as follows: (a) $7.1<r<7.5$, (b) $4.7<r<5.3$, and (c) $3.9<r<4.6$. The dashed lines in Fig. 1 show energy spectral indices of -2 (horizontal) and -2.7, representative of galactic CR source and observed spectra, respectively. Our solutions are approximately power laws, deviating from the horizontal dashed line by at most a factor of 3 over 6 decades of energy. (That they are not perfect power laws results from the slowing of the flow upstream of the shock by CR pressure). It is clear that a small phase velocity, representative of the HISM, allows an extremely wide range in shock strengths, all producing source spectra essentially indistinguishable from that required to produce the observed CR spectrum. The uncertainties in the CR source spectrum caused by energy dependent leakage exceed the small differences between curves (b), (c), and the horizontal dashed line. Even curve (a) is marginally consistent with the inferred source spectrum.

The ratio of Alfvén to shock velocity is given by, $v_A/u_1 = 2.2\ 10^{11}\ B/(u_1\sqrt{n})$, where B is the magnetic field and n is the number density. For the HISM, we can use B ~ 3μG, n ~ 0.01 per cm^3, and shock velocities ~ 1000 km/s. Therefore, $v_A/u_1 \sim 0.07$, and, as seen in Fig. 2, a very wide range in M_1 will produce spectra similar to those seen in Fig. 1.

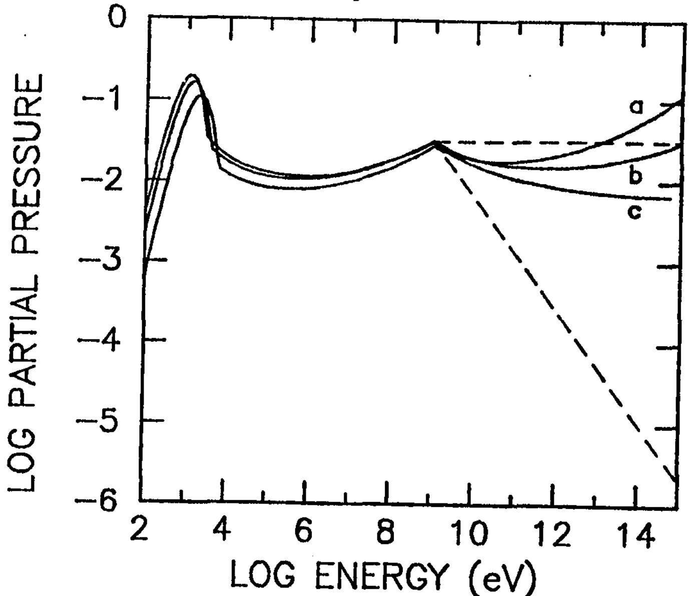

Fig. 1. Postshock partial pressure vs. energy. An energy cutoff = 10^{15}eV has been used. Each label represents a family of curves with a particular u_1/u(GeV).

3. Scaling Laws. The results of the previous section show that the spectrum of relativistic particles is "quasi-universal" in the sense that it depends very weakly on the upstream parameters. The reason for this, we argue, is that the energy content in relativistic particles is a very sensitive function of the spectrum, so that their back-pressure on the flow regulates their spectrum very precisely. This principle is illustrated by the fact that the overall compression ratio of the scatterers is always established in a way such that the compression ratio of mildly relativistic particles is virtually independent of the upstream shock parameters. In Fig. 3 the overall compression ratio of the shock is plotted as a function of Mach number for the limiting case of zero phase velocity. It is seen that beyond modest Mach numbers, $r \propto M^{3/4}$, regardless of the fact that the adiabatic index of the post shock gas (i.e. the relative contribution of relativistic and non-relativistic particle pressure) varies substantially over this range in M_1 . (The classical result for $\gamma = 4/3$ is shown by the dotted line.) This $M^{3/4}$ law can be derived straightforwardly by merely assuming that the Mach number of the flow when it encounters Gev particles, M(GeV), and the compression ratio at 1 GeV, r(GeV), are ~ constant (we find, for $v_{ph} = 0$ and for $10 < M_1 < 100$ that $3.3 < M(\mathrm{GeV}) < 3.6$ and $3.5 < r(\mathrm{GeV}) < 3.75$). The result then follows

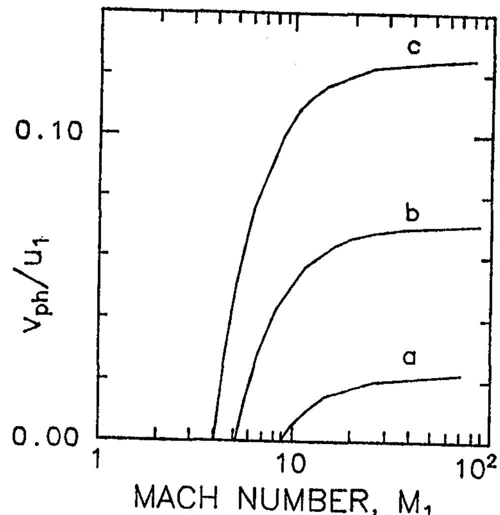

Fig. 2. Labels same as Fig. 1.

from the fact that, because the phase velocity is zero, the upstream fluid behaves adiabatically, i.e. (using eq. [2]),

$$P \propto \rho^{5/3} \Rightarrow [M_1/M(\mathrm{GeV})]^2 = [u_1/u(\mathrm{GeV})]^{8/3}, \qquad (8)$$

Here u(GeV) is the upstream flow velocity felt by GeV particles[6]. Given that M(Gev) and r(GeV) = u(Gev)/u_2 are about constant for $v_{ph} = 0$, r scales as $M^{3/4}$.

As is evident from Fig. 3, the shock compression is dramatically reduced and stabilized by the introduction of even a small v_{ph} for the scattering waves. For an infinite upstream acoustic Mach number, the compression ratio, formally infinite when $v_{ph} = 0$, is reduced to only ~ 7 when $v_{ph}/u_1 \sim 0.03$. This result can be derived analytically to high accuracy; using the same principle invoked to derive the $M^{3/4}$ scaling law when v_{ph} vanishes, one can show that $u_1/u(\mathrm{GeV}) \sim 2$ when $v_{ph}/u_1 \sim 1/[2.7\gamma(\gamma-1)M^2(\mathrm{GeV})]$.

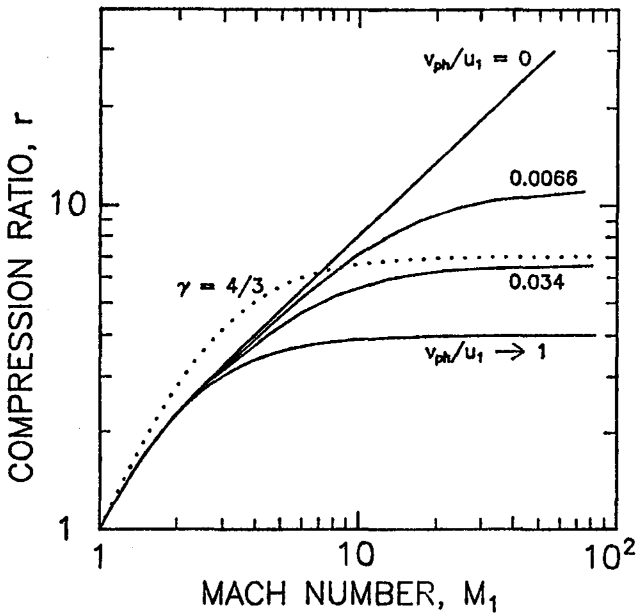

Fig. 3. Dotted line is classical result.

4. Conclusions. The non-linear theory of shock acceleration has been generalized to include wave dynamics. In the limit of rapid wave damping, it is found that a finite wave velocity tempers the acceleration of high Mach number shocks and limits the maximum compression ratio even when energy loss is important. For a given spectrum (see Figs. 1-2), the efficiency of relativistic particle production is essentially independent of v_{ph}. For the three "families" shown in Figs. 1-2, the percentage of kinetic energy flux going into relativistic particles is (a) 72%, (b) 44%, and (c) 26% (this includes the energy lost at the upper energy cuttoff). Even small v_{ph}, typical of the HISM, produce quasi-universal spectra that depend only weakly on the acoustic Mach number. These spectra should be close enough to E^{-2} to satisfy CR source requirements.

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